ASYMPTOTICS OF ONE-DIMENSIONAL FOREST FIRE PROCESSES

XAVIER BRESSAUD[1], NICOLAS FOURNIER[2]

Abstract. We consider the so-called one-dimensional forest-fire process. At each site of $\mathbb{Z}$, a tree appears at rate 1. At each site of $\mathbb{Z}$ a fire starts at rate $\lambda > 0$, destroying immediately the whole corresponding connected component of trees. We show that when making λ tend to 0, with a correct normalization, the forest-fire process tends to a uniquely defined process, of which we describe precisely the dynamics. The normalization consists of accelerating time by a factor $\log(1/\lambda)$ and of compressing space by a factor $\lambda \log(1/\lambda)$. The limit process is quite simple: it can be built using a graphical construction, and can be perfectly simulated. Finally, we derive some asymptotic estimates (when $\lambda \to 0$) for the cluster-size distribution of the forest-fire process.

Key words: Stochastic interacting particle systems, Self organized criticality, Forest-fire model.

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1. Introduction and main results

1.1. **The model.** Consider two independent families of independent Poisson processes $N = (N_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and $M^\lambda = (M_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$, with respective rates 1 and $\lambda > 0$. Denote by $\mathcal{F}_t^{N,M^\lambda} := \sigma(N_s(i), M_s^\lambda(i), s \leq t, i \in \mathbb{Z})$. For $a, b \in \mathbb{Z}$, with $a \leq b$, we set $[\![a, b]\!] = \{a, ..., b\}$.

Definition 1. *Consider a $\{0,1\}^\mathbb{Z}$-valued $(\mathcal{F}_t^{N,M^\lambda})_{t\geq 0}$-adapted process $(\eta_t^\lambda)_{t\geq 0}$, such that $(\eta_t^\lambda(i))_{t\geq 0}$ is a.s. càdlàg for all $i \in \mathbb{Z}$.*
We say that $(\eta_t^\lambda)_{t\geq 0}$ is a λ-FFP (forest-fire process) if a.s., for all $t \geq 0$, all $i \in \mathbb{Z}$,

$$\eta_t^\lambda(i) = \int_0^t \mathbb{1}_{\{\eta_{s-}^\lambda(i)=0\}} dN_s(i) - \sum_{k\in\mathbb{Z}} \int_0^t \mathbb{1}_{\{k\in C_{s-}^\lambda(i)\}} dM_s^\lambda(k),$$

where $C_s^\lambda(i) = \emptyset$ if $\eta_t^\lambda(i) = 0$, while $C_s^\lambda(i) = [\![l_s^\lambda(i), r_s^\lambda(i)]\!]$ if $\eta_s^\lambda(i) = 1$, with

$$l_s^\lambda(i) = \sup\{k < i; \eta_s^\lambda(k) = 0\} + 1 \quad \text{and} \quad r_s^\lambda(i) = \inf\{k > i; \eta_s^\lambda(k) = 0\} - 1.$$

Formally, saying that $\eta_t^\lambda(i) = 0$ if there is no tree at site i at time t and $\eta_t^\lambda(i) = 1$ else, $C_t^\lambda(i)$ stands for the connected component of occupied sites around i at time t. Thus the forest-fire process starts from an empty initial configuration, trees appear on vacant sites at rate 1 (according to N), and a fire starts on each site at rate $\lambda > 0$ (according to M^λ), burning immediately the corresponding connected component of occupied sites.

This process can be shown to exist and to be unique (for almost every realization of N, M^λ), by using a *graphical construction*. Indeed, to build the process until a given time $T > 0$,

[1]Université Paul Sabatier, Institut de Mathématiques de Toulouse, F-31062 Toulouse Cedex 9, France. E-mail: bressaud@math.univ-toulouse.fr

[2]Université Paris-Est, LAMA, Faculté de Sciences et Technologie, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France. E-mail: nicolas.fournier@univ-paris12.fr

it suffices to work between sites i which are vacant until time T (because $N_T(i) = 0$). Interaction cannot cross such sites. Since such sites are a.s. infinitely many, this allows us to handle a graphical construction. We refer to Van den Berg-Jarai [4], see also Liggett [15] for many examples of graphical constructions. Let us observe that this construction works only in dimension 1.

1.2. **Motivation and references.** The study of self-organized critical (SOC) systems has become rather popular in physics since the end of the 80's. SOC systems are simple models supposed to illuminate temporal and spatial randomness observed in a variety of natural phenomena showing *long range correlations*, like sand piles, avalanches, earthquakes, stock market crashes, forest fires, shapes of mountains, clouds, ... Roughly, the idea, present in Bak-Tang-Wiesenfeld [1] about sand piles, is that of systems *growing* towards a *critical state* and relaxing through *catastrophic* events (avalanches, crashes, fires, ...). The most classical model is the sand pile model introduced in 1987 in [1], but a lot of variants or related models have been proposed and studied more or less rigorously, describing earthquakes (Olami-Feder-Christensen, [16]) or forest fire (Henley [13], Drossel-Schwabl, [8]). For surveys on the subject, see Bak-Tang-Wiesenfeld [1, 2], Jensen [14], and the references therein.

From the point of view of SOC systems, the forest-fire model is interesting in the asymptotic regime $\lambda \to 0$. Indeed fires are less frequent, but when they occur, destroyed clusters may be huge. This model has been subject to a lot of numerical and heuristic studies, see Drossel-Clar-Schwabl [9] and Grassberger [12] for references. But there are few rigorous results. Even existence of the (time-dependent) process for a multidimensionnal lattice and given $\lambda > 0$ has been proved only recently [10, 11], and uniqueness is known to hold only for λ large enough. The existence, uniqueness of an invariant distribution (as well as other qualitative properties) even in dimension 1, have been proved only recently in [5] for $\lambda = 1$. These last results can probably be extended to the case where $\lambda \geq 1$, but the method in [5] completely breaks down for small values of λ.

The asymptotic behaviour of the λ-FFP as $\lambda \to 0$ has been studied numerically and heuristically [8, 9, 7, 12]. To our knowledge, the only mathematical rigorous results are the following.
(a) Van den Berg and Jarai [4] have proved that for $t \geq 3$, $\mathbb{P}[\eta^\lambda_{t\log(1/\lambda)}(0) = 0] \simeq 1/\log(1/\lambda)$, thus giving an idea of the density of vacant sites. This result was conjectured by Drossel-Clar-Schwabl [9].
(b) Van den Berg and Brouwer [3] have obtained some results in the two-dimensional case concerning the behaviour of clusters near the *critical time*. However, these results are not completely rigorous, since they are based on a percolation-like assumption, which is not rigorously proved.
(c) Brouwer and Pennanen [6] have proved the existence of an invariant distribution for each fixed $\lambda > 0$, as well as a precise version of the following estimate, which extends (a): for $\lambda \in (0, 1)$, at equilibrium $\mathbb{P}[\#(C^\lambda(0)) = x] \simeq c/[x \log(1/\lambda)]$ for $x \in \{1, ..., (1/\lambda)^{1/3}\}$. It was conjectured in [9] that this actually holds for $x \in \{1, ..., 1/(\lambda \log(1/\lambda))\}$, but this was rejected in [4].

In this paper, we derive rigorously a limit theorem, which shows that the λ-FFP converges, under rescaling, to some limit forest-fire process (LFFP). We describe precisely the dynamics of the LFFP, and show that it is quite simple: in particular, it is unique, can be built by using a *graphical construction*, and thus can be *perfectly* simulated. Our result allows us to prove a very weak version of (c) for $x \in \{1, ..., (1/\lambda)^{1-\varepsilon}\}$, for any $\varepsilon > 0$, see Corollary 6 below.

1.3. **Notation.** We denote by $\#(I)$ the number of elements of a set I.
For $a, b \in \mathbb{Z}$, with $a \leq b$, we set $[\![a, b]\!] = \{a, ..., b\} \subset \mathbb{Z}$.
For $I = [\![a, b]\!] \subset \mathbb{Z}$ and $\alpha > 0$, we will set $\alpha I := [\alpha a, \alpha b] \subset \mathbb{R}$. For $\alpha > 0$, we of course take the convention that $\alpha\emptyset = \emptyset$.
For $J = [a, b]$ an interval of $\mathbb{R}$, $|J| = b - a$ stands for the length of J, and for $\alpha > 0$, we set $\alpha J = [\alpha a, \alpha b]$.
For $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the integer part of x.

1.4. **Heuristic scales and relevant quantities.** Our aim is to find some time scale for which tree clusters see about one fire per unit of time. But for λ very small, clusters will be very large just before they burn. We thus also have to rescale space, in order that just before burning, clusters have a size of order 1.

Time scale. Consider the cluster $C_t^\lambda(x)$ around some site x at time t. It is quite clear that for $\lambda > 0$ very small and for t not too large, one can neglect fires, so that roughly, each site is occupied with probability $1 - e^{-t}$, and thus $C_t^\lambda(x) \simeq [\![x - X, x + Y]\!]$, where X, Y are geometric random variables with parameter $1 - e^{-t}$. As a consequence, $\#(C_t^\lambda(x)) \simeq e^t$ for t not too large. On the other hand, the cluster $C_t^\lambda(x)$ burns at rate $\lambda\#(C_t^\lambda(x))$ (at time t), so that we decide to accelerate time by a factor $\log(1/\lambda)$. By this way, $\lambda\#(C_{\log(1/\lambda)}^\lambda(x)) \simeq 1$.

Space scale. Now we rescale space in such a way that during a time interval of order $\log(1/\lambda)$, something like one fire starts per unit of (space) length. Since fires occur at rate λ, our space scale has to be of order $\lambda \log(1/\lambda)$: this means that we will identify $[\![0, \lfloor 1/(\lambda \log(1/\lambda)) \rfloor]\!] \subset \mathbb{Z}$ with $[0, 1] \subset \mathbb{R}$.

Rescaled clusters. We thus set, for $\lambda \in (0, 1)$, $t \geq 0$, $x \in \mathbb{R}$, recalling Subsection 1.3,

$$D_t^\lambda(x) := \lambda \log(1/\lambda) C_{t \log(1/\lambda)}^\lambda(\lfloor x/(\lambda \log(1/\lambda)) \rfloor) \subset \mathbb{R}. \tag{1}$$

However, this makes appear an immediate difficulty: recalling that $\#(C_t^\lambda(x)) \simeq e^t$ for t not too large, we see that for all site x, $|D_t^\lambda(x)| \simeq \lambda \log(1/\lambda) e^{t \log(1/\lambda)} = \lambda^{1-t} \log(1/\lambda)$, of which the limit as $\lambda \to 0$ is 0 for $t < 1$ and $+\infty$ for $t \geq 1$.
For $t \geq 1$, there might be fires in effect, and one hopes that this will make finite the possible limit of $|D_t^\lambda(x)|$. But fires can only reduce the size of clusters, so that for $t < 1$, the limit of $|D_t^\lambda(x)|$ will really be 0. Thus, for a possible limit $|D(x)|$ of $|D^\lambda(x)|$, we should observe some paths of the following form: $|D_t(x)| = 0$ for $t < 1$, $|D_t(x)| > 0$ for some times $t \in (1, \tau)$, then it might be killed by a fire and thus come back to 0, then it remains at 0 during a time interval of length 1, and so on...
This cannot be a Markov process because $|D(x)|$ always remains at 0 during a time interval of lenght exactly 1. We thus have to keep in mind more information, in order to control when it exits from 0.

Degree of smallness. As said previously, we hope that for $t < 1$, $|D_t^\lambda(x)| \simeq \lambda^{1-t} \log(1/\lambda) \simeq \lambda^{1-t}$. Thus we will try to keep in mind the degree of smallness. We will denote, for $\lambda \in (0, 1)$, $x \in \mathbb{R}$, $t > 0$,

$$Z_t^\lambda(x) := \frac{\log\left[1 + \#\left(C_{t \log(1/\lambda)}^\lambda(\lfloor x/(\lambda \log(1/\lambda)) \rfloor)\right)\right]}{\log(1/\lambda)} \in [0, \infty). \tag{2}$$

Final description. We will study the λ-FFP through $(D_t^\lambda(x), Z_t^\lambda(x))_{x\in\mathbb{R},t\geq 0}$. The main idea is that for $\lambda > 0$ very small:
(i) if $Z_t^\lambda(x) = z \in (0,1)$, then $|D_t^\lambda(x)| \simeq 0$, and the (rescaled) cluster containing x is microscopic, but we control its smallness, in the sense that $|D_t^\lambda(x)| \simeq \lambda^{1-z}$ in a very unprecise way;
(ii) if $Z_t^\lambda(x) = 1$ (we will show below that $Z_t^\lambda(x)$ will never exceed 1 in the limit $\lambda \to 0$), then automatically the (rescaled) cluster containing x is macroscopic, and has a length equal to $|D_t^\lambda(x)| \in (0,\infty)$.

1.5. **The limit process.** We now describe the limit process. We want this process to be Markov, and this forces us add some variables.
We consider a Poisson measure $M(dt, dx)$ on $[0,\infty)\times\mathbb{R}$, with intensity measure $dtdx$. Again, we denote by $\mathcal{F}_t^M = \sigma(M(A), A \in \mathcal{B}([0,t]\times\mathbb{R}))$. We also denote by $\mathcal{I} := \{[a,b], a \leq b\}$ the set of all closed finite intervals of $\mathbb{R}$.

Definition 2. *A $(\mathcal{F}_t^M)_{t\geq 0}$-adapted process $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ with values in $\mathbb{R}_+ \times \mathcal{I} \times \mathbb{R}_+$ is a limit forest-fire process (LFFP) if a.s., for all $t \geq 0$, all $x \in \mathbb{R}$,*

$$(3)\qquad \begin{cases} Z_t(x) &= \displaystyle\int_0^t \mathbb{1}_{\{Z_s(x)<1\}} ds - \int_0^t \int_{\mathbb{R}} \mathbb{1}_{\{Z_{s-}(x)=1, y\in D_{s-}(x)\}} M(ds, dy), \\ H_t(x) &= \displaystyle\int_0^t Z_{s-}(x) \mathbb{1}_{\{Z_{s-}(x)<1\}} M(ds\times\{x\}) - \int_0^t \mathbb{1}_{\{H_s(x)>0\}} ds, \end{cases}$$

where $D_t(x) = [L_t(x), R_t(x)]$, with

$$L_t(x) = \sup\{y \leq x;\ Z_t(y) < 1 \text{ or } H_t(y) > 0\},$$
$$R_t(x) = \inf\{y \geq x;\ Z_t(y) < 1 \text{ or } H_t(y) > 0\}.$$

A typical path of the finite box-version of the LFFP (see Section 2) is drawn and commented on Figure 2, and a simulation algorithm is explained in the proof of Proposition 8.

Let us explain the dynamics of this process. We consider $T > 0$ fixed, and set $\mathcal{B}_T = \{x \in \mathbb{R}; M([0,T]\times\{x\}) > 0\}$. For each $t \geq 0$, $x \in \mathbb{R}$, $D_t(x)$ stands for the occupied cluster containing x. We call this cluster *microscopic* if $D_t(x) = \{x\}$. We also have $D_t(x) = D_t(y)$ for all y in the interior of $D_t(x)$: if $D_t(x) = [a,b]$, then $D_t(y) = [a,b]$ for all $y \in (a,b)$.

1. Initial condition. We have $Z_0(x) = H_0(x) = 0$ and $D_0(x) = \{x\}$ for all $x \in \mathbb{R}$.

2. Occupation of vacant zones. We consider here $x \in \mathbb{R}\backslash\mathcal{B}_T$. Then we have $H_t(x) = 0$ for all $t \in [0,T]$. When $Z_t(x) < 1$, then $D_t(x) = \{x\}$, and $Z_t(x)$ stands for the *degree of smallness* of the cluster containing x. Then $Z_t(x)$ grows linearly until it reaches 1, as described by the first term on the RHS of the first equation in (3). When $Z_t(x) = 1$, then the cluster containing x is macroscopic, and is described by $D_t(x)$.

3. Microscopic fires. Here we assume that $x \in \mathcal{B}_T$, and that the corresponding mark of M happens at some time t where $z := Z_{t-}(x) < 1$. In such a case, the cluster containing x is microscopic. Then we set $H_t(x) = Z_{t-}(x)$, as described by the first term on the RHS of the second equation of (3), and we leave unchanged the value of $Z_t(x)$. We then let $H_s(x)$ decrease linearly until it reaches 0, see the second term on the RHS of the second equation in (3). At all times where $H_s(x) > 0$, i.e. during $[t, t+z)$, the site x acts like a barrier (see Point 5. below).

4. Macroscopic fires. Here we assume that $x \in \mathcal{B}_T$, and that the corresponding mark of M happens at some time t where $Z_{t-}(x) = 1$. This means that the cluster containing x

is macroscopic, and thus this mark destroys the whole component $D_{t-}(x)$, that is for all $y \in D_{t-}(x)$, we set $D_t(y) = \{y\}$, $Z_t(y) = 0$. This is described by the second term on the RHS of the first equation in (3).

5. Clusters. Finally the definition of the clusters $(D_t(x))_{x\in\mathbb{R}}$ becomes more clear: these clusters are delimited by zones with microscopic sites (i.e. $Z_t(y) < 1$) or by sites where their has (recently) been a microscopic fire (i.e. $H_t(y) > 0$).

1.6. **Main results.** First of all, it is not quite clear that the limit process exists.

Theorem 3. *For any Poisson measure M, there a.s. exists an unique LFFP, recall Definition 2. Furthermore, it can be constructed graphically, and thus its restriction to any finite box $[0, T] \times [-n, n]$ can be perfectly simulated.*

To describe the convergence of the λ-FFP to the LFFP, we need some more notation. Let $\mathbb{D}([0, T], E)$ denote the space of right continuous and left limited functions from the interval $[0, T]$ to a topological space E.

Notation 4. *(i) For two intervals $[a, b]$ and $[c, d]$, we set $\delta([a, b], [c, d]) = |a - c| + |b - d|$. We also set by convention $\delta([a, b], \emptyset) = |b - a|$.*
(ii) For $(x, I), (y, J)$ in $\mathbb{D}([0, T], \mathbb{R} \times \mathcal{I} \cup \{\emptyset\})$, let

$$\delta_T((x, I), (y, J)) = \sup_{[0,T]} |x(t) - y(t)| + \int_0^T \delta(I(t), J(t))dt.$$

We are finally in a position to state our main result.

Theorem 5. *Consider, for all $\lambda > 0$, the processes $(Z^\lambda_t(x), D^\lambda_t(x))_{t\geq 0, x\in\mathbb{R}}$ associated to some the λ-FFP, see Definition 1 and (1)-(2). Let $(Z_t(x), D_t(x), H_t(x))_{t\geq 0,x\in\mathbb{R}}$ be a LFFP as in Definition 2.*
(a) For any $T > 0$, any finite subset $\{x_1, ..., x_p\} \subset \mathbb{R}$, $(Z^\lambda_t(x_i), D^\lambda_t(x_i))_{t\in[0,T],i=1,...,p}$ goes in law to $(Z_t(x_i), D_t(x_i))_{t\in[0,T],i=1,...,p}$, in $\mathbb{D}([0, T], \mathbb{R}\times\mathcal{I})^p$, as λ tends to 0. Here $\mathbb{D}([0, \infty), \mathbb{R}\times \mathcal{I})$ is endowed with the distance δ_T, see Notation 4.
(b) For any finite subset $\{(t_1, x_1), ..., (t_p, x_p)\} \subset \mathbb{R}_+ \times \mathbb{R}$, $(Z^\lambda_{t_i}(x_i), D^\lambda_{t_i}(x_i))_{i=1,...,p}$ goes in law to $(Z_{t_i}(x_i), D_{t_i}(x_i))_{i=1,...,p}$ in $(\mathbb{R} \times \mathcal{I})^p$.

Observe that the process H does not appear in the limit, since for each $x \in \mathbb{R}$, a.s., for all $t \geq 0$, $H_t(x) = 0$. (Of course, it is false that a.s., for all $x \in \mathbb{R}$, all $t \geq 0$, $H_t(x) = 0$). We obtain the convergence of D^λ to D only when integrating in time. We cannot hope for a Skorokhod convergence, since the limit process $D(x)$ jumps instantaneously from $\{x\}$ to some interval with positive length, while $D^\lambda(x)$ needs many small jumps (in a very short time interval) to become macroscopic.
As a matter of fact, we will obtain some convergence in probability, using a coupling argument. Essentially, we will consider a Poisson measure $M(dt, dx)$ as in Subsection 1.5, and set, for $\lambda \in (0, 1)$ and $i \in \mathbb{Z}$,

$$M^\lambda_t(i) = M\Big([0, t/\log(1/\lambda)] \times [i\lambda \log(1/\lambda), (i + 1)\lambda \log(1/\lambda))\Big).$$

Then $(M^\lambda_t(i))_{t\geq 0,i\in\mathbb{Z}}$ is an i.i.d. family of Poisson processes with rate λ.
The i.i.d. family of Poisson processes $(N_t(i))_{t\geq 0,i\in\mathbb{Z}}$ with rate 1 can be chosen arbitrarily, but we will decide to choose the same family for all values of $\lambda \in (0, 1)$.

1.7. **Heuristic arguments.** Let us explain here roughly the reasons why Theorem 5 holds. We consider a λ-FFP $(\eta_t^\lambda)_{t\geq 0}$, and the associated process $(Z_t^\lambda(x), D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$. We assume below that λ is very small.

0. Scales. With our scales, there are $1/(\lambda \log(1/\lambda))$ sites per unit of length; about one fire starts per unit of time per unit of length; a vacant site becomes occupied at rate $\log(1/\lambda)$.

1. Initial condition. We have, for all $x \in \mathbb{R}$, $(Z_0^\lambda(x), D_0^\lambda(x)) = (0, \emptyset) \simeq (0, \{x\})$.

2. Occupation of vacant zones. Assume that a zone $[a, b]$ (which corresponds to the zone $[\![\lfloor a/(\lambda\log(1/\lambda))\rfloor, b/(\lambda\log(1/\lambda))\rfloor]\!]$ before rescaling) becomes completely vacant at some time t (or $t\log(1/\lambda)$ before rescaling) because it has been destroyed by a fire.
(i) For $s \in [0, 1)$, and if no fire starts on $[a, b]$ during $[t, t + s]$, we have $D_{t+s}^\lambda(x) \simeq [x \pm \lambda^{1-s}]$ and thus $Z_{t+s}^\lambda(x) \simeq s$ for all $x \in [a, b]$.
Indeed, $D_{t+s}^\lambda(x) \simeq [x-\lambda\log(1/\lambda)X, x+\lambda\log(1/\lambda)Y]$, where X and Y are geometric random variables with parameter $1 - e^{-s\log(1/\lambda)} = 1 - \lambda^s$. This comes from the fact that each site of $[a, b]$ is vacant at time t, and becomes occupied at rate $\log(1/\lambda)$.
(ii) If no fire starts on $[a, b]$ during $[t, t + 1]$, then $Z_{t+1}^\lambda(x) \simeq 1$ and all the sites in $[a, b]$ are occupied (with very high probability) at time $t + 1$. Indeed, we have $(b - a)/(\lambda \log(1/\lambda))$ sites, and each of them is occupied at time $t + 1$ with probability $1 - e^{-\log(1/\lambda)} = 1 - \lambda$, so that all of them are occupied with probability $(1 - \lambda)^{(b-a)/(\lambda \log(1/\lambda))} \simeq e^{-(b-a)/\log(1/\lambda)}$ which goes to 1 as $\lambda \to 0$.

3. Microscopic fires. Assume that a fire starts at some location x (i.e. $\lfloor x/(\lambda\log(1/\lambda))\rfloor$ before rescaling) at some time t (or $t\log(1/\lambda)$ before rescaling), with $Z_{t-}^\lambda(x) = z \in (0, 1)$. Then the possible clusters on the left and right of x cannot be connected during (approximately) $[t, t + z]$, but can be connected after (approximately) $t + z$. In other words, x acts like a barrier during $[t, t + z]$.
Indeed, the fire makes vacant a zone A of approximate length λ^{1-z} around x, which thus contains approximately $\lambda^{1-z}/(\lambda \log(1/\lambda)) \simeq \lambda^{-z}$ sites. The probability that a fire starts again in A after t is very small. Thus, using the same computation as in Point 2-(ii), we observe that $\mathbb{P}[A \text{ is completely occupied at time } t+s] \simeq (1-\lambda^s)^{\lambda^{-z}} \simeq e^{-\lambda^{s-z}}$. When $\lambda \to 0$, this quantity tends to 0 if $s < z$ and to 1 if $s > z$.

4. Macroscopic fires. Assume now that a fire starts at some place x (i.e. $\lfloor x/(\lambda\log(1/\lambda))\rfloor$ before rescaling) at some time t (or $t\log(1/\lambda)$ before rescaling), and that $Z_t^\lambda(x) \simeq 1$. Thus $D_t^\lambda(x)$ is macroscopic (that is its length is of order 1 in our scales). This will thus make vacant the zone $D_t^\lambda(x)$. Such a (macroscopic) zone needs a time of order 1 to be completely occupied, as explained in Point 2-(ii).

5. Clusters. For $t \geq 0$, $x \in \mathbb{R}$, the cluster $D_t^\lambda(x)$ resembles $[x \pm \lambda^{1-z}] \simeq \{x\}$ if $Z_t^\lambda(x) = z \in (0, 1)$. We then say that x is microscopic. Now macroscopic clusters are delimited either by microscopic zones, or by sites where their has been a microscopic fire (see Point 3).

Comparing the arguments above to the rough description of the LFFP, see Subsection 1.5, we hope that the λ-FFP resembles the LFFP for $\lambda > 0$ very small.

1.8. **Decay of correlations.** A by-product of our result is an estimate on the decay of correlations in the LFFP, for finite times. We refer to Proposition 11 below for a precise statement. The main idea is that for all $T > 0$, there are some constants $C_T > 0$, $\alpha_T > 0$ such that for all $\lambda \in (0, 1)$, all $A > 0$, the values of the λ-FFP inside $[-A/(\lambda\log(1/\lambda)), A/(\lambda\log(1/\lambda))]$ are independent of the values outside $[-2A/(\lambda\log(1/\lambda)), 2A/(\lambda\log(1/\lambda))]$ during the time interval $[0, T\log(1/\lambda)]$, up to a probability smaller that $C_T e^{-\alpha_T A}$. In other words, for times of order $\log(1/\lambda)$, the range of correlations is at most of order $1/(\lambda\log(1/\lambda))$.

1.9. **Cluster-size distribution.** We finally give results on the cluster-size distribution, which are to be compared with [4, 6], see Subsection 1.2 above.

Corollary 6. *For each $\lambda > 0$, consider a λ-FFP process $(\eta_t^\lambda)_{t\geq 0}$.*
(i) For some $0 < c < C$, for all $t \geq 5/2$, all $0 \leq a < b < 1$,

$$c(b-a) \leq \lim_{\lambda\to 0} \mathbb{P}\left(\#(C^\lambda_{t\log(1/\lambda)}(0)) \in [\lambda^{-a}, \lambda^{-b}]\right) \leq C(b-a).$$

(ii) For some $0 < c < C$ and some $0 < \kappa_1 < \kappa_2$, for all $t \geq 3/2$, all $B > 0$,

$$ce^{-\kappa_2 B} \leq \lim_{\lambda\to 0} \mathbb{P}\left(\#(C^\lambda_{t\log(1/\lambda)}(0)) \geq B/(\lambda\log(1/\lambda))\right) \leq Ce^{-\kappa_1 B}.$$

Point (i) says approximately that for t large enough (say at equilibrium), for $x << 1/\lambda$ (say for $x \leq (1/\lambda)^{1-\varepsilon}$), choosing $a = \log(x)/\log(1/\lambda)$ and $b = \log(x+1)/\log(1/\lambda)$,

$$\begin{aligned}\mathbb{P}(\#(C^\lambda(0)) = x) &\simeq \mathbb{P}(\#(C^\lambda(0)) \in [x, x+1]) \simeq \mathbb{P}(\#(C^\lambda(0)) \in [\lambda^{-a}, \lambda^{-b}])\\ &\simeq (b-a) \simeq \frac{1}{x\log(1/\lambda)},\end{aligned}$$

Thus it is a very weak form of the result of [6], but it holds for a much wider class of x: here we allow $x \leq 1/\lambda^{1-\varepsilon}$, while $x \leq 1/\lambda^{1/3}$ was imposed in [6]. Another advantage of our result is that we can prove that the limit exists in (i).
Point (ii) describes roughly the cluster-size distribution of macroscopic components, that is of components of which the size is of order $1/(\lambda\log(1/\lambda))$. Here again, rough computations show that for $x > \varepsilon/(\lambda\log(1/\lambda))$, for t large enough (say at equilibrium),

$$\mathbb{P}(\#(C^\lambda(0)) = x) \simeq \lambda\log(1/\lambda)e^{-\kappa x\lambda\log(1/\lambda)}.$$

Thus there is clearly a phase transition near the *critical size* $1/(\lambda\log(1/\lambda))$. See Figure 1 for an illustration.

1.10. **Organization of the paper.** The paper is organized as follows. In Section 2, we give the proof of Theorem 3. We show in Section 3 that in some sense, the λ-FFP can be localized in finite box, uniformly in $\lambda > 0$. Section 4 is devoted to the proof of Theorem 5. Finally, we check Corollary 6 in Section 5.

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2. Existence and uniqueness of the limit process

The goal of this section is to show that the LFFP is well-defined, unique, and that it can be obtained from a graphical construction. First of all, we show that when working on a finite space interval, the LFPP is somewhat discrete.
We consider a Poisson measure $M(dt, dx)$ on $[0,\infty)\times\mathbb{R}$, with intensity measure $dtdx$. We denote by $\mathcal{F}_t^{M,A} = \sigma(M(B), B \in \mathcal{B}([0,t]\times[-A,A]))$.

Definition 7. *A $(\mathcal{F}_t^{M,A})_{t\geq 0}$-adapted process $(Z_t^A(x), D_t^A(x), H_t^A(x))_{t\geq 0, x\in[-A,A]}$ with values in $\mathbb{R}_+ \times \mathcal{I} \times \mathbb{R}_+$ is called a A-LFFP if a.s., for all $t \geq 0$, all $x \in [-A, A]$,*

$$\left\{\begin{array}{lll} Z_t^A(x) & = & \int_0^t \mathbb{1}_{\{Z_s^A(x)<1\}}ds - \int_0^t \int_{[-A,A]} \mathbb{1}_{\{Z_{s-}^A(x)=1, y\in D_{s-}^A(x)\}} M(ds, dy), \\ H_t^A(x) & = & \int_0^t Z_{s-}^A(x) \mathbb{1}_{\{Z_{s-}^A(x)<1\}} M(ds \times \{x\}) - \int_0^t \mathbb{1}_{\{H_s^A(x)>0\}} ds, \end{array}\right.$$

where $D_t^A(x) = [L_t^A(x), R_t^A(x)]$, with

$$\left\{\begin{array}{lll} L_t^A(x) & = & (-A) \vee \sup\{y \in [-A, x];\ Z_t^A(y) < 1 \text{ or } H_t^A(y) > 0\} \\ R_t^A(x) & = & A \wedge \inf\{y \in [x, A];\ Z_t^A(y) < 1 \text{ or } H_t^A(y) > 0\}. \end{array}\right. \tag{4}$$

A typical path of $(Z_t^A(x), D_t^A(x), H_t^A(x))_{t\geq 0, x\in[-A,A]}$ is drawn on Figure 2.

The following proposition is actually almost obvious, but its proof shows the construction of the A-LFFP in an algorithmic way.

Proposition 8. *Consider a Poisson measure $M(dt, dx)$ on $[0, \infty) \times \mathbb{R}$, with intensity measure $dtdx$. For any $A > 0$, there a.s. exists an unique A-LFFP, and it can be perfeclty simulated.*

Proof. We omit the superscript A in this proof. We consider the marks $(T_i, X_i)_{i\geq 1}$ of $M|_{[0,\infty)\times[-A,A]}$, with $0 < T_1 < T_2 < \ldots$. We set $T_0 = 0$ for convenience. We describe the construction through an algorithm, which also shows uniqueness in the sense that there is no choice for the construction.

Step 0. First, we set $Z_0(x) = H_0(x) = 0$ and $D_0(x) = \{x\}$ for all $x \in [-A, A]$.

Step $n+1$. Assume that the process has been built until T_n for some $n \geq 0$, that is we know the values of $(Z_t(x), D_t(x), H_t(x))_{t\in[0,T_n], x\in[-A,A]}$.

We build $(Z_t(x), D_t(x), H_t(x))_{t\in(T_n,T_{n+1}), x\in[-A,A]}$ in the following way: for $t \in (T_n, T_{n+1})$, $x \in [-A, A]$, we set $Z_t(x) = \min(1, Z_{T_n}(x)+t-T_n)$, we set $H_t(x) = \max(0, H_{T_n}(x)-(t-T_n))$ and we define $D_t(x) = [L_t(x), R_t(x)]$ as in (4).

Next we build $(Z_{T_{n+1}}(x), D_{T_{n+1}}(x), H_{T_{n+1}}(x))_{x\in[-A,A]}$.
(i) If $Z_{T_{n+1}-}(X_{n+1}) = 1$, set $H_{T_{n+1}}(x) = H_{T_{n+1}-}(x)$ for all $x \in [-A, A]$ and consider $[a, b] := D_{T_{n+1}-}(X_{n+1})$. Set $Z_{T_{n+1}}(x) = 0$ for all $x \in (a, b)$ and $Z_{T_{n+1}}(x) = Z_{T_{n+1}-}(x)$ for all $x \in [-A, A] \setminus [a, b]$. Set finally $Z_{T_{n+1}}(a) = 0$ if $Z_{T_{n+1}-}(a) = 1$ and $Z_{T_{n+1}}(a) = Z_{T_{n+1}-}(a)$ if $Z_{T_{n+1}-}(a) < 1$, and $Z_{T_{n+1}}(b) = 0$ if $Z_{T_{n+1}-}(b) = 1$ and $Z_{T_{n+1}}(b) = Z_{T_{n+1}-}(b)$ if $Z_{T_{n+1}-}(b) < 1$.
(ii) If $Z_{T_{n+1}-}(X_{n+1}) < 1$, we set $H_{T_{n+1}}(X_{n+1}) = Z_{T_{n+1}-}(X_{n+1})$, we set $Z_{T_{n+1}}(X_{n+1}) = Z_{T_{n+1}-}(X_{n+1})$ and $(Z_{T_{n+1}}(x), H_{T_{n+1}}(x)) = (Z_{T_{n+1}-}(x), H_{T_{n+1}-}(x))$ for all $x \in [-A, A] \setminus \{X_{n+1}\}$.
(iii) Using the values of $(Z_{T_{n+1}}(x), H_{T_{n+1}}(x))_{x\in[-A,A]}$, we finally compute the values of $(D_{T_{n+1}}(x))_{x\in[-A,A]}$. ☐

In case (i) above, we detailed precisely what to do at the boundary of burning macroscopic components. This is not so important: it does not affect the uniqueness statement but corresponds to taking a slightly different definition of the process; we could have made other choices for this.

We now prove a refined version of Theorem 3.

Proposition 9. *Consider a Poisson measure $M(dt, dx)$ on $[0,\infty)\times\mathbb{R}$, with intensity measure $dtdx$. For $A>0$, consider the A-LFFP $(Z^A_t(x), D^A_t(x), H^A_t(x))_{t\geq 0, x\in[-A,A]}$ built in Proposition 8 (using M).*
There a.s. exists an unique LFFP $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ (corresponding to M), and it furthermore satisfies: for all $T>0$, there are some constants $\alpha_T>0$ and $C_T>0$ such that for all $A\geq 2$,

$$\mathbb{P}\Big[(Z_t(x), D_t(x), H_t(x))_{t\in[0,T], x\in[-A/2,A/2]} = (Z^A_t(x), D^A_t(x), H^A_t(x))_{t\in[0,T], x\in[-A/2,A/2]}\Big] \geq 1 - C_T e^{-\alpha_T A}. \tag{5}$$

Proof. We divide the proof into several steps. We fix $T>0$, and work on $[0,T]$.

Step 1. For $a\in\mathbb{Z}$, we define the event Ω_a in the following way (see Figure 3 for an illustration): the Poisson measure M has exactly $3n$ marks in $[0,T]\times[a,a+1]$, for some $n\geq 1$, and it is possible to call them $(T_k, X_k)_{k=1,\dots n}$, $(\tilde{T}_k, \tilde{X}_k)_{k=1,\dots n}$ and $(S_k, Y_k)_{k=1,\dots n}$ in such a way that we have the following properties for all $k=1,\dots,n$ (we set $T_0=\tilde{T}_0=S_0=0$ and $X_0=a$, $\tilde{X}_0=a+1$ for convenience).
(i) T_k and $\tilde{T}_k$ belong to $(S_{k-1}+1/2, S_{k-1}+1)$ and $X_{k-1}<X_k<\tilde{X}_k<\tilde{X}_{k-1}$;
(ii) $S_k\in(S_{k-1}+1, S_{k-1}+2(T_k\wedge\tilde{T}_k-S_{k-1}))$ and $Y_k\in(X_k,\tilde{X}_k)$;
(iii) $S_n>T-1$.

Step 2. Then we observe that if the LFFP exists, then necessarily,

$$\Omega_a\subset\{\forall\, t\in[0,T],\ \exists\, x\in(a,a+1),\ H_t(x)>0 \text{ or } Z_t(x)<1\}.$$

Indeed, $Z_t(x)=t<1$ for all $t\in[0,1)$, all $x\in\mathbb{R}$. Then $H_{T_1}(X_1)=Z_{T_1}(X_1)=T_1$, whence $H_t(X_1)>0$ on $[T_1,2T_1]$, and $H_t(\tilde{X}_1)>0$ on $[\tilde{T}_1,2\tilde{T}_1]$. As a consequence, we know that for all $x\in(X_1,\tilde{X}_1)$, all $t\in[1,S_1)$, $D_t(x)=[X_1,\tilde{X}_1]$. Since now $1<S_1<2(T_1\wedge\tilde{T}_1)$, and since $Y_1\in(X_1,\tilde{X}_1)$, we deduce that $Z_{S_1}(x)=0$ for all $x\in(X_1,\tilde{X}_1)$, and as a consequence, $Z_t(x)=t-S_1<1$ for all $t\in[S_1,S_1+1)$. But now $H_t(X_2)>0$ on $[T_2,T_2+(T_2-S_1))$, and $H_t(\tilde{X}_2)>0$ on $[\tilde{T}_2,\tilde{T}_2+(\tilde{T}_2-S_1))$. As a consequence, we know that for all $x\in(X_2,\tilde{X}_2)$, all $t\in[S_1+1,S_2)$, $D_t(x)=[X_2,\tilde{X}_2]$. Since now $S_1+1<S_2<S_1+2(T_1\wedge\tilde{T}_1-S_1)$, and since $Y_2\in(X_2,\tilde{X}_2)$, we deduce that $Z_{S_2}(x)=0$ for all $x\in(X_2,\tilde{X}_2)$, and thus $Z_t(x)=t-S_2<1$ for all $t\in[S_2,S_2+1)$. And so on...

Step 3. We deduce that for all $a\in\mathbb{Z}$, conditionally on Ω_a, clusters on the left of a are never connected (during $[0,T]$) to clusters on the right of $a+1$. Thus on Ω_a, fires starting on the left of a do not affect the zone $[a+1,\infty)$, and fires starting on the right of $a+1$ do not affect the zone $(-\infty,a]$. Since furthermore Ω_a concerns the Poisson measure M only in $[0,T]\times[a,a+1]$, we deduce that on Ω_a, the processes $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in[a+1,\infty)}$ and $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in(-\infty,a]}$ can be constructed separately.

Step 4. Clearly, $q_T=\mathbb{P}[\Omega_a]$ does not depend on a, by invariance by translation (of the law of M), and obviously $q_T>0$. Thus a.s., there are infinitely many $a\in\mathbb{Z}$ such that Ω_a is realized. This allows a graphical construction: it suffices to work between such a's (i.e. in finite boxes) as in Proposition 8.

Step 5. Using the same arguments, we easily deduce that for $A\geq 2$, the LFFP and the A-LFFP coincide on $[-A/2,A/2]$ during $[0,T]$ as soon as there are $a_1\in[-A,-A/2-1]$ and $a_2\in[A/2,A-1]$ with $\Omega_{a_1}\cap\Omega_{a_2}$ realized. Furthermore, since M is a Poisson measure, Ω_a is independent of Ω_b for all $a\neq b$ (with $a,b\in\mathbb{Z}$). Thus the probability on the LHS of (5) is bounded below, for $A \geq 2$, by

$$1 - \mathbb{P}[\cap_{a\in\mathbb{Z}\cap[-A,-A/2-1]}\Omega_a^c] - \mathbb{P}[\cap_{a\in\mathbb{Z}\cap[A/2,A-1]}\Omega_a^c] \geq 1 - 2(1-q_T)^{A/2-2},$$

whence (5) with $\alpha_T = -\log(1-q_T)/2 > 0$ and $C_T = 2/(1-q_T)^2$. □

3. Localization of the FFP.

We first introduce the (λ, A)-FFP. We consider two independent families of i.i.d. Poisson processes $N = (N_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and $M^\lambda = (M_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$, with respective rates 1 and $\lambda > 0$. For $A > 0$ and $\lambda > 0$, we define

$$A_\lambda := \lfloor A/(\lambda\log(1/\lambda)\rfloor \text{ and } I_A^\lambda := [\![-A_\lambda, A_\lambda]\!], \tag{6}$$

and we set $\mathcal{F}_t^{N,M^\lambda,A} := \sigma(N_s(i), M_s^\lambda(i), s \leq t, i \in I_A^\lambda)$.

Definition 10. *Consider a $(\mathcal{F}_t^{N,M^\lambda,A})_{t\geq 0}$-adapted process $(\eta_t^{\lambda,A})_{t\geq 0}$ with values in $\{0,1\}^{I_A^\lambda}$, such that $(\eta_t^{\lambda,A}(i))_{t\geq 0}$ is a.s. càdlàg for all $i \in I_A^\lambda$.*
We say that $(\eta_t^{\lambda,A})_{t\geq 0}$ is a (λ, A)-FFP if a.s., for all $t \geq 0$, all $i \in I_A^\lambda$,

$$\eta_t^{\lambda,A}(i) = \int_0^t \mathbb{1}_{\{\eta_{s-}^{\lambda,A}(i)=0\}} dN_s(i) - \sum_{k\in I_A^\lambda} \int_0^t \mathbb{1}_{\{k\in C_{s-}^{\lambda,A}(i)\}} dM_s^\lambda(k),$$

where $C_s^{\lambda,A}(i) = \emptyset$ if $\eta_t^{\lambda,A}(i) = 0$, while $C_s^{\lambda,A}(i) = [\![l_s^{\lambda,A}(i), r_s^{\lambda,A}(i)]\!]$ if $\eta_s^{\lambda,A}(i) = 1$, where

$$l_s^{\lambda,A}(i) = (-A_\lambda) \vee \left(\sup\{k < i; \eta_s^{\lambda,A}(k) = 0\} + 1\right),$$
$$r_s^{\lambda,A}(i) = A_\lambda \wedge \left(\inf\{k > i; \eta_s^{\lambda,A}(k) = 0\} - 1\right).$$

For $x \in [-A, A]$ and $t \geq 0$, we introduce

$$D_t^{\lambda,A}(x) = \lambda\log(1/\lambda) C_t^{\lambda,A}(\lfloor x/(\lambda\log(1/\lambda))\rfloor) \subset [-A, A], \tag{7}$$

$$Z_t^{\lambda,A}(x) = \frac{\log\left[1 + \#\left(C_t^{\lambda,A}(\lfloor x/(\lambda\log(1/\lambda))\rfloor)\right)\right]}{\log(1/\lambda)} \geq 0. \tag{8}$$

We now prove the following result, which is similar to Proposition 9 for the λ-FFP.

Proposition 11. *Let $T > 0$ and $\lambda \in (0,1)$. Consider two families of Poisson processes $N = (N_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and $M^\lambda = (M_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$, with respective rates 1 and $\lambda > 0$. Let $(\eta_t^\lambda)_{t\geq 0}$ be the corresponding λ-FFP, and for each $A > 0$, let $(\eta_t^{\lambda,A})_{t\geq 0}$ be the corresponding (λ, A)-FFP. Recall (1)-(2) and (7)-(8). There are some constant $\alpha_T > 0$ and $C_T > 0$, not depending on $\lambda \in (0,1)$, $A \geq 2$, such that, recall (6),*

$$\mathbb{P}\left[(\eta_t^\lambda(i))_{t\in[0,T\log(1/\lambda)], i\in I_{A/2}^\lambda} = (\eta_t^{\lambda,A}(i))_{t\in[0,T\log(1/\lambda)], i\in I_{A/2}^\lambda}\right] \geq 1 - C_T e^{-\alpha_T A},$$
$$\mathbb{P}\left[(Z_t^\lambda(x), D_t^\lambda(x))_{t\in[0,T], x\in[-A/2,A/2]} = (Z_t^{\lambda,A}(x), D_t^{\lambda,A}(x))_{t\in[0,T], x\in[-A/2,A/2]}\right]$$
$$\geq 1 - C_T e^{-\alpha_T A}.$$

Proof. The proof is similar (but more complicated) to that of Proposition 9. Consider the true λ-FFP $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$. Assume for a moment that for $a \in \mathbb{R}$, there is an event Ω_a^λ, depending only on the Poisson processes $N_t(i)$ and $M_t^\lambda(i)$ for $t \in [0, T\log(1/\lambda)]$ and $i \in J_a^\lambda := [\![\lfloor a/(\lambda\log(1/\lambda))\rfloor, \lfloor (a+1)/(\lambda\log(1/\lambda))\rfloor]\!]$, such that
(i) on Ω_a^λ, a.s., for all $t \in [0, T\log(1/\lambda)]$, there is some $i \in J_a^\lambda$ such that $\eta_t^\lambda(i) = 0$;

(ii) there is $q_T > 0$ such that for all $a \in \mathbb{R}$, all $\lambda \in (0,1)$, $\mathbb{P}(\Omega_a^\lambda) \geq q_T$.
Then we conclude using similar arguments to Steps 3, 4, 5 of the proof of Proposition 9.

Fix some $\alpha > 0$ and some $\varepsilon_T > 0$ small enough, say $\alpha = 0.01$ and $\varepsilon_T = 1/(32T)$. Let $\lambda_T > 0$ be such that for $\lambda \in (0, \lambda_T)$, we have $1 < \lambda^{\alpha-1} < \epsilon_T/(\lambda \log(1/\lambda))$.

For $\lambda \in [\lambda_T, 1)$ and $a \in \mathbb{R}$, we set $\Omega_a^\lambda = \{N_{T\log(1/\lambda)}(\lfloor a/(\lambda\log(1/\lambda))\rfloor) = 0\}$, on which of course $\eta_t^\lambda(i) = 0$ for all $t \in [0, T\log(1/\lambda)]$ with $i = \lfloor a/(\lambda\log(1/\lambda))\rfloor \in J_a^\lambda$. Then we observe that $q'_T = \inf_{\lambda\in[\lambda_T,1)} P(\Omega_a^\lambda) = \inf_{\lambda\in[\lambda_T,1)} e^{-T\log(1/\lambda)} = (\lambda_T)^T > 0$.

For $\lambda \in (0, \lambda_T)$ and $a \in \mathbb{R}$, we define the event Ω_a^λ on which points **1, 2, 3** below are satisfied.

1. The family of Poisson processes $(M_t^\lambda(i))_{t\in[0,T\log(1/\lambda)], i\in J_a^\lambda}$ has exactly $3n$ marks, for some $1 \leq n \leq \lfloor T \rfloor$, and it is possible to call them $(T_k^\lambda, X_k^\lambda)_{k=1,\dots n}$, $(\tilde T_k^\lambda, \tilde X_k^\lambda)_{k=1,\dots n}$ and $(S_k^\lambda, Y_k^\lambda)_{k=1,\dots n}$ in such a way that we have the following properties for all $k = 1, ..., n$ (we set $T_0^\lambda = \tilde T_0^\lambda = S_0^\lambda = 0$ and $X_0^\lambda = \lfloor a/(\lambda\log(1/\lambda))\rfloor$, $\tilde X_0^\lambda = \lfloor (a+1)/(\lambda\log(1/\lambda))\rfloor$).
(1a) $X_{k-1}^\lambda < X_k^\lambda < Y_k^\lambda < \tilde X_k^\lambda < \tilde X_{k-1}^\lambda$, with $\min\{X_k^\lambda - X_{k-1}^\lambda, Y_k^\lambda - X_k^\lambda, \tilde X_k^\lambda - Y_k^\lambda, \tilde X_{k-1}^\lambda - \tilde X_k^\lambda\} \geq 4\epsilon_T/(\lambda\log(1/\lambda))$;
(1b) T_k^λ and $\tilde T_k^\lambda$ belong to $[S_{k-1}^\lambda + (\frac{1}{2}+\alpha)\log(1/\lambda), S_{k-1}^\lambda + (1-\alpha)\log(1/\lambda)]$;
(1c) $S_k^\lambda \in [S_{k-1}^\lambda + (1+\alpha)\log(1/\lambda), S_{k-1}^\lambda + 2(T_k^\lambda \wedge \tilde T_k^\lambda - S_{k-1}^\lambda) - \alpha\log(1/\lambda)]$;
(1d) $S_n^\lambda \geq (T - 1 + \alpha)\log(1/\lambda)$.

2. We now set, for $k = 1, ..., n$, $\tau_k^\lambda = (S_k^\lambda - S_{k-1}^\lambda)/(2\log(1/\lambda))$, which belongs to $[(1+\alpha)/2, 1-\alpha]$ due to **1** We consider the intervals

$$
\begin{aligned}
I_k^\lambda &= [\![X_k^\lambda - \lfloor\lambda^{-\tau_k^\lambda}\rfloor, X_k^\lambda + \lfloor\lambda^{-\tau_k^\lambda}\rfloor]\!],\\
I_{k,-}^\lambda &= [\![X_k^\lambda - \lfloor\lambda^{-\tau_k^\lambda}\rfloor - \lfloor\varepsilon_T/\lambda\log(1/\lambda)\rfloor, X_k^\lambda - \lfloor\lambda^{-\tau_k^\lambda}\rfloor - 1]\!],\\
I_{k,+}^\lambda &= [\![X_k^\lambda + \lfloor\lambda^{-\tau_k^\lambda}\rfloor + 1, X_k^\lambda + \lfloor\lambda^{-\tau_k^\lambda}\rfloor + \lfloor\varepsilon_T/\lambda\log(1/\lambda)\rfloor]\!],\\
L_k^\lambda &= [\![X_k^\lambda + \lfloor\lambda^{-\tau_k^\lambda}\rfloor + \lfloor\varepsilon_T/\lambda\log(1/\lambda)\rfloor + 1, \tilde X_k^\lambda - \lfloor\lambda^{-\tau_k^\lambda}\rfloor - \lfloor\varepsilon_T/\lambda\log(1/\lambda)\rfloor - 1]\!],
\end{aligned}
$$

and we consider similar intervals $\tilde I_k^\lambda, \tilde I_{k,-}^\lambda, \tilde I_{k,+}^\lambda$, around $\tilde X_k^\lambda$. For all $k = 1, \dots, n$ the family of Poisson processes $(N_t(i))_{t\geq 0, i\in J_a^\lambda}$ satisfies :
(2a) $\forall i \in I_k^\lambda$, $N_{T_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) > 0$ and $\forall i \in \tilde I_k^\lambda$, $N_{\tilde T_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) > 0$;
(2b) $\exists i \in I_{k,-}^\lambda$, $N_{T_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) = 0$, $\exists i \in I_{k,+}^\lambda$, $N_{T_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) = 0$, $\exists i \in \tilde I_{k,-}^\lambda$, $N_{\tilde T_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) = 0$ and $\exists i \in \tilde I_{k,+}^\lambda$, $N_{\tilde T_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) = 0$;
(2c) $\exists i \in I_k^\lambda$, $N_{S_k^\lambda}(i) - N_{T_k^\lambda}(i) = 0$ and $\exists i \in \tilde I_k^\lambda$, $N_{S_k^\lambda}(i) - N_{\tilde T_k^\lambda}(i) = 0$;
(2d) $\forall i \in L_k^\lambda$, $N_{S_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) > 0$.

3. We finally assume that $\exists i \in L_n^\lambda$, $N_{T\log(1/\lambda)}(i) - N_{S_n^\lambda}(i) = 0$.

To show that on Ω_a^λ, a.s., for all $t \in [0, T\log(1/\lambda)]$, there is some $i \in J_a^\lambda$ such that $\eta_t^\lambda(i) = 0$, we proceed recursively. At time 0 all sites are vacant. Fix $k \in \{1, ..., n\}$. Assume that, for $t \leq S_{k-1}^\lambda$, there is some $i \in J_a^\lambda$ such that $\eta_t^\lambda(i) = 0$ and that, at time S_{k-1}^λ, all sites in the interval L_{k-1}^λ are vacant.
Then, for $S_{k-1}^\lambda \leq t < T_k^\lambda$ (resp. $S_{k-1}^\lambda \leq t < \tilde T_k^\lambda$), (2b) shows that there are vacant sites both in $I_{k,+}^\lambda$ and in $I_{k,-}^\lambda$ (resp. both in in $\tilde I_{k,+}^\lambda$ and in $\tilde I_{k,-}^\lambda$). This together with (2a) shows

that, at time $T_k^\lambda-$ (resp. $\tilde T_k^\lambda-$), all the sites in the intervals I_k^λ and $\tilde I_k^\lambda$ are occupied (no fire may burn those sites, because they are protected by the vacant sites in $I_{k,+}^\lambda, I_{k,-}^\lambda, \tilde I_{k,+}^\lambda, \tilde I_{k,-}^\lambda$). Hence the interval I_k^λ (resp. $\tilde I_k^\lambda$) becomes completely vacant at time T_k^λ (resp. $\tilde T_k^\lambda$). Between time T_k^λ (resp. $\tilde T_k^\lambda$) and time S_k^λ, since I_k^λ (resp. $\tilde I_k^\lambda$) is completely vacant at time T_k^λ (resp. $\tilde T_k^\lambda$), (2c) shows that there is a vacant site in I_k^λ (resp. $\tilde I_k^\lambda$).
At time $S_k^\lambda-$, the interval L_k^λ is completely occupied thanks to (2d) and since it cannot be burnt, because it is protected by vacant sites in $I_{k,+}^\lambda$ (resp. $\tilde I_{k,-}^\lambda$) between S_{k-1}^λ and T_k^λ (resp. $\tilde T_k^\lambda$) and in I_k^λ resp $\tilde I_k^\lambda$ between T_k^λ (resp. $\tilde T_k^\lambda$) and S_k^λ. As a consequence, since $Y_k^\lambda \in L_k^\lambda$, the interval L_k^λ becomes completely vacant at time $S_k^\lambda-$.
All this shows that on Ω_a^λ, there are vacant sites in J_a^λ for all $t \in [0, S_n^\lambda]$, and L_n^λ is completely vacant at time S_n^λ. Finally, **3** implies that there are vacant sites in $L_n^\lambda \subset J_a^\lambda$ during $[S_n^\lambda, T\log(1/\lambda)]$.

It remains to prove that there is $q_T''>0$ such that for all $a\in\mathbb{R}$, all $\lambda\in(0,\lambda_T)$, $\mathbb{P}(\Omega_a^\lambda)\ge q_T''$. We treat separately the conditions **1** on M^λ and **2** on N (conditionally on M^λ) and use independence of these two families of Poisson processes to conclude.

Firstly, for $\lambda\in(0,\lambda_T)$, we observe that we can construct M^λ using a Poisson measure M on $[0,\infty)\times\mathbb{R}$ with intensity $dtdx$, by setting, for all $i\in\mathbb{Z}$:

$$M_t^\lambda(i) = M\Big([0,t/\log(1/\lambda)]\times[i\lambda\log(1/\lambda),(i+1)\lambda\log(1/\lambda))\Big).$$

Hence (since $\varepsilon_T/(\lambda\log(1/\lambda))>1$) the event on which M^λ satisfies **1** contains the event Ω_a' on which M has exactly $3n$ marks in $[0,T]\times[a,a+1]$, for some $1\le n\le\lfloor T\rfloor$ which can be called $(T_k,X_k)_{k=1,\dots n}$, $(\tilde T_k,\tilde X_k)_{k=1,\dots n}$ and $(S_k,Y_k)_{k=1,\dots n}$ in such a way that we have the following properties (we set $T_0=\tilde T_0=S_0=0$ and $X_0=a$, $\tilde X_0=a+1$ for convenience) for all $k=1,...,n$:
- $\min(\{X_k-X_{k-1}, Y_k-X_k, \tilde X_k-Y_k, \tilde X_{k-1}-\tilde X_k\})>5\epsilon_T$.
- T_k and $\tilde T_k$ belong to $(S_{k-1}+1/2+\alpha, S_{k-1}+1-\alpha)$
- $S_k\in(S_{k-1}+1+\alpha, S_{k-1}+2(T_k\wedge\tilde T_k-S_{k-1})-\alpha)$.
- $S_n\ge(T-1)+\alpha$.

Then we have $\mathbb{P}(\Omega_a')>0$ (as in the proof of Proposition 9 and since ε_T and α are sufficiently small), and this probability does not depend on a (by invariance of the law of M by translation) nor on $\lambda\in(0,\lambda_T)$ (since it concerns only M).

Then, we use basic computations on i.i.d. Poisson processes with rate 1 to show that there is a (deterministic) constant $c>0$ such that for all $k=1,\dots,n$, all $\lambda\in(0,\lambda_T)$, conditionally on M^λ, (we write $\mathbb{P}_M$ for the conditional probability w.r.t. M^λ),
- since $T_k^\lambda - S_{k-1}^\lambda \ge (\tau_k^\lambda+\alpha/2)\log(1/\lambda)$ due to (1c) and since $\#(I_k^\lambda)=2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1$,

$$\begin{aligned}\mathbb{P}_M(\forall i\in I_k^\lambda, N_{T_k^\lambda}(i)-N_{S_{k-1}^\lambda}(i)>0) &= \Big(1-e^{-(T_k^\lambda-S_{k-1}^\lambda)}\Big)^{2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1}\\ &\ge \Big(1-\lambda^{\tau_k^\lambda+\alpha/2}\Big)^{2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1} \ge c\end{aligned}$$

(it tends to 1 as $\lambda\to0$) and the same computation works for $\tilde I_k^\lambda$;

• since $T_k^\lambda - S_{k-1}^\lambda \le (1-\alpha)\log(1/\lambda)$ by (1b), and since $\#(I_{k,+}^\lambda) = \lfloor \varepsilon_T/(\lambda\log(1/\lambda))\rfloor$,

$$\mathbb{P}_M(\exists i \in I_{k,+}^\lambda, N_{T_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) = 0) = 1 - \Big(1 - e^{-(T_k^\lambda - S_{k-1}^\lambda)}\Big)^{\lfloor \varepsilon_T/\lambda\log(1/\lambda)\rfloor} \ge 1 - \big(1-\lambda^{1-\alpha}\big)^{\lfloor \varepsilon_T/(\lambda\log(1/\lambda))\rfloor} \ge c$$

and the same computation works for $I_{k,-}^\lambda, \tilde I_{k,+}^\lambda, \tilde I_{k,-}^\lambda$;

• since $S_k^\lambda - T_k^\lambda \le (\tau_k^\lambda - \alpha/2)\log(1/\lambda)$ due to (1c) (use that $S_k^\lambda \le 2T_k^\lambda - S_{k-1}^\lambda - \alpha\log(1/\lambda)$, whence $2S_k^\lambda \le 2T_k^\lambda + S_k^\lambda - S_{k-1}^\lambda - \alpha\log(1/\lambda) = 2T_k^\lambda + 2(\tau_k^\lambda - \alpha/2)\log(1/\lambda)$), and since $\#(I_k^\lambda) = 2\lfloor\lambda^{-\tau_k^\lambda}\rfloor + 1$,

$$\mathbb{P}_M(\exists i \in I_k^\lambda, N_{S_k^\lambda}(i) - N_{T_k^\lambda}(i) = 0) = 1 - \Big(1 - e^{-(S_k^\lambda - T_k^\lambda)}\Big)^{2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1} \ge 1 - \Big(1 - \lambda^{\tau_k^\lambda - \alpha/2}\Big)^{2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1} \ge c$$

and this also holds for $\tilde I_k^\lambda$;

• since $S_k^\lambda - S_{k-1}^\lambda \ge (1+\alpha)\log(1/\lambda)$ thanks to (1c), and since $\#(L_k^\lambda) \le \lfloor(1/\lambda\log(1/\lambda))\rfloor$,

$$\mathbb{P}_M(\forall i \in L_k^\lambda, N_{S_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) > 0) = \Big(1 - e^{-(S_k^\lambda - S_{k-1}^\lambda)}\Big)^{\#(L_k^\lambda)} \ge \big(1-\lambda^{1+\alpha}\big)^{\lfloor 1/\lambda\log(1/\lambda)\rfloor} \ge c;$$

• since $T\log(1/\lambda) - S_n^\lambda \le (1-\alpha)\log(1/\lambda)$ by (1d) and $\#(L_n^\lambda) \ge 4\varepsilon_T/(\lambda\log(1/\lambda))$ by (1a),

$$\mathbb{P}_M(\exists i \in L_n^\lambda, N_{T\log(1/\lambda)}(i) - N_{S_n^\lambda}(i) = 0) = 1 - \Big(1 - e^{-(T\log(1/\lambda) - S_n^\lambda)}\Big)^{\#(L_n^\lambda)} \ge 1 - \big(1-\lambda^{1-\alpha}\big)^{4\varepsilon_T/(\lambda\log(1/\lambda))} \ge c.$$

We observe that the domains $I_k^\lambda \times (S_{k-1}^\lambda, T_k^\lambda]$, $\tilde I_k^\lambda \times (S_{k-1}^\lambda, \tilde T_k^\lambda]$, $I_{k,+}^\lambda \times (S_{k-1}^\lambda, T_k^\lambda]$, $I_{k,-}^\lambda \times (S_{k-1}^\lambda, T_k^\lambda]$, $\tilde I_{k,+}^\lambda \times (S_{k-1}^\lambda, \tilde T_k^\lambda]$, $\tilde I_{k,-}^\lambda \times (S_{k-1}^\lambda, \tilde T_k^\lambda]$, $I_k^\lambda \times (T_k^\lambda, S_k^\lambda]$, $\tilde I_k^\lambda \times (\tilde T_k^\lambda, S_k^\lambda]$, $L_k^\lambda \times (S_{k-1}^\lambda, S_k^\lambda]$, for $k = 1, \dots, n$, and $L_n^\lambda \times (S_n^\lambda, T\log(1/\lambda)]$ are pairwise disjoint thanks to **1** and to the smallness of ε_T and λ_T: we have $\lfloor\lambda^{-\tau_k^\lambda}\rfloor \le \lambda^{\alpha-1} \le \varepsilon_T/(\lambda\log(1/\lambda))$.

Since $n \le T$, we deduce from all the previous estimates the existence of $q_T'' > 0$ such that for all $a \in \mathbb{R}$, all $\lambda \in (0, \lambda_T)$, $\mathbb{P}(\Omega_a^\lambda) \ge q_T''$. We conclude choosing $q_T = \min(q_T', q_T'')$. □

4. Convergence proof

The goal of this section is to check Theorem 5.

4.1. Coupling.

We introduce a coupling between the λ-FFP, the LFFP, and their localized versions.

Notation 12. *We consider a Poisson measure $M(dt, dx)$ on $[0,\infty)\times\mathbb{R}$ with intensity measure $dtdx$. We consider an independent family of Poisson processes $(N_t(i))_{t\ge0, i\in\mathbb{Z}}$ with rate 1. For $\lambda \in (0,1)$ and $i \in \mathbb{Z}$, we set*

$$M_t^\lambda(i) = M\Big([0, t/\log(1/\lambda)] \times [i\lambda\log(1/\lambda), (i+1)\lambda\log(1/\lambda))\Big).$$

Then $(M_t^\lambda(i))_{t\ge0, i\in\mathbb{Z}}$ is a family of independent Poisson processes with rate λ. We consider, for all $\lambda \in (0,1)$, the λ-FFP $(\eta_t^\lambda)_{t\ge0}$ (see Definition 1), and for all $A > 0$, the

(λ, A)-FFP $(\eta_t^{\lambda,A})_{t\geq 0}$ (see Definition 10) built with N, M^λ. We also introduce the processes $(Z_t^\lambda(x), D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$ as in (1)-(2) and $(Z_t^{\lambda,A}(x), D_t^{\lambda,A}(x))_{t\geq 0, x\in[-A,A]}$ as in (7)-(8). We denote by $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ the LFFP built with M (see Definition 2), and by $(Z_t^A(x), D_t^A(x), H_t^A(x))_{t\geq 0, x\in[-A,A]}$ the A-LFFP built with M (see Definition 7).

4.2. **Localization.** Assume for a moment that the following result holds.

Proposition 13. *Adopt Notation 12 as well as Notation 4.*
(a) For any $T>0$, any $A>0$, any $x_0\in(-A,A)$, in probability, as $\lambda\to 0$,

$$\delta_T\left((Z^{\lambda,A}(x_0), D^{\lambda,A}(x_0)), (Z^A(x_0), D^A(x_0))\right) \text{ tends to } 0.$$

(b) For any $t\in[0,\infty)$, any $A>0$, any $x_0\in(-A,A)$, in probability, as $\lambda\to 0$,

$$|Z_t^{\lambda,A}(x_0) - Z_t^A(x_0)| + \delta\left(D_t^{\lambda,A}(x_0)), D_t^A(x_0)\right) \text{ tends to } 0.$$

Then we are in a position to give the

Proof of Theorem 5. We only prove point (a), (b) being similarly checked. Let $T>0$ and $\{x_1,...,x_n\}\subset[-B,B]\subset\mathbb{R}$ be fixed. Consider the coupling introduced in Notation 12. Proposition 13 ensures us that for any $\varepsilon>0$, any $A>B$,

$$\lim_{\lambda\to 0}\mathbb{P}\left[\sum_1^n \delta_T\left((Z^{\lambda,A}(x_i), D^{\lambda,A}(x_i)), (Z^A(x_i), D^A(x_i))\right) > \varepsilon\right] = 0.$$

Let now

$$\Omega_{A,T}^\lambda := \Big\{\forall\, i=1,...,n,\ \forall\, t\in[0,T],\ (Z_t^\lambda(x_i), D_t^\lambda(x_i)) = (Z_t^{\lambda,A}(x_i), D_t^{\lambda,A}(x_i)) \text{ and } (Z_t(x_i), D_t(x_i)) = (Z_t^A(x_i), D_t^A(x_i))\Big\}.$$

Now for all $A>2B$,

$$\Omega_{A,T}^\lambda \subset \Big\{(Z_t^\lambda(x), D_t^\lambda(x))_{t\in[0,T], x\in[-A/2,A/2]} = (Z_t^{\lambda,A}(x), D_t^{\lambda,A}(x))_{t\in[0,T], x\in[-A/2,A/2]} \text{ and } (Z_t(x), D_t(x))_{t\in[0,T], x\in[-A/2,A/2]} = (Z_t^A(x), D_t^A(x))_{t\in[0,T], x\in[-A/2,A/2]}\Big\}.$$

But Propositions 9 and 11 yield that $\mathbb{P}[(\Omega_{A,T}^\lambda)^c] \leq 2C_T e^{-\alpha_T A}$. Thus for any $A>2B$,

$$\limsup_{\lambda\to 0}\mathbb{P}\left[\sum_1^n \delta_T\left((Z^\lambda(x_i), D^\lambda(x_i)), (D(x_i), Z(x_i))\right) > \varepsilon\right] \leq 0 + 2C_T e^{-\alpha_T A}.$$

Making A tend to infinity, we deduce that $\sum_{i=1}^n \delta_T((Z^\lambda(x_i), D^\lambda(x_i)), (D(x_i), Z(x_i)))$ tends to 0 in probability as $\lambda\to 0$, whence the result. ☐

4.3. **Heart of the proof.** The aim of this subsection is to prove Proposition 13. We fix $T>0$ and $A>0$. We consider the (λ, A)-FFP and the A-LFFP coupled as in Notation 12 and we use the notation introduced in (6). Along this proof we will omit the superscript A, and we do not take into account the possible dependences in A and T.

For $J=(a,b)$ an open interval of $(-A,A)$, $\lambda\in(0,1)$ and $\mu\in(0,1]$, we consider

$$J_{\lambda,\mu} = \left[\!\!\left[\left\lfloor\frac{a}{\lambda\log(1/\lambda)} + \frac{\mu}{\lambda\log^2(1/\lambda)}\right\rfloor, \left\lfloor\frac{b}{\lambda\log(1/\lambda)} - \frac{\mu}{\lambda\log^2(1/\lambda)}\right\rfloor\right]\!\!\right] \subset\mathbb{Z}, \tag{9}$$

$$\tilde{Z}_t^{\lambda,\mu}(J) = 1 - \frac{\log(1+\#\{k\in J_{\lambda,\mu},\ \eta^\lambda_{t\log(1/\lambda)}(k)=0\})}{\log(1+\#(J_{\lambda,\mu}))}.$$

Observe that $\tilde{Z}_t^{\lambda,\mu}(J) = 1$ if and only if all the sites of $J_{\lambda,\mu}$ are occupied at time $t\log(1/\lambda)$. The quantity $\tilde{Z}_t^{\lambda,\mu}(J)$ is a function of the density of vacant clusters in the (rescaled) zone J. Under some exhangeability properties, it should be closely related to the size of occupied clusters in that zone, i.e. to $Z_t^\lambda(x)$, for $x \in J$.

For $x \in (-A, A)$, $\lambda \in (0,1)$ and $\mu \in (0,1]$, we introduce

$$x_{\lambda,\mu} = \left[\!\!\left[\left\lfloor \frac{x}{\lambda\log(1/\lambda)} - \frac{\mu}{\lambda\log^2(1/\lambda)} \right\rfloor + 1, \left\lfloor \frac{x}{\lambda\log(1/\lambda)} + \frac{\mu}{\lambda\log^2(1/\lambda)} \right\rfloor - 1 \right]\!\!\right] \subset \mathbb{Z}, \tag{10}$$

$$\tilde{H}_t^{\lambda,\mu}(x) = \frac{\log(1 + \#\{k \in x_{\lambda,\mu},\ \eta^\lambda_{t\log(1/\lambda)}(k) = 0\})}{\log(1 + \#(x_{\lambda,\mu}))}.$$

Here again, $\tilde{H}_t^{\lambda,\mu}(x) = 0$ if and only if all the sites of $x_{\lambda,\mu}$ are occupied at time $t\log(1/\lambda)$. Assume that a microscopic fires starts at some x. Then the process $\tilde{H}_t^{\lambda,\mu}(x)$ will allow us to quantify the duration for which this fire will be in effect.

Observe that we always have $\log(1 + \#(x_{\lambda,\mu})) \sim \log(1 + \#(J_{\lambda,\mu})) \sim \log(1/\lambda)$ as $\lambda \to 0$. Observe also that if $\tilde{Z}_t^{\lambda,\mu}(J) = z$, then there are $(1+\#(J_{\lambda,\mu}))^{1-z} - 1 \simeq \lambda^{z-1}$ vacant sites in $J_{\lambda,\mu}$ at time $t\log(1/\lambda)$. By the same way, $\tilde{H}_t^{\lambda,\mu}(x) = h$ says that there are $(1+\#(x_{\lambda,\mu}))^h - 1 \simeq \lambda^{-h}$ vacant sites in $x_{\lambda,\mu}$ at time $t\log(1/\lambda)$.

We work conditionally to M. We denote $\mathbb{P}_M$ the conditional probability given M. We recall that conditionally to M, $(Z_t(x), D_t(x), H_t(x))_{t\in[0,T],x\in[-A,A]}$ is deterministic. We denote by $n = M([0,T] \times [-A; A])$, which is a.s. finite. We set $T_0 = 0$ and consider the marks $(X_q, T_q)_{1\leq q\leq n}$ of M, ordered in such a way that $T_0 < T_1 < ... < T_n < T$.

We set $\mathcal{B}_0 = \emptyset$, and for $q = 1, \dots, n$, we consider $\mathcal{B}_q = \{X_1, \dots, X_q\}$, as well as the set $\mathcal{C}_q$ of connected components of $(-A, A) \setminus \mathcal{B}_q$ (sometimes referred to as *cells*).

Observe that by construction, we have, for $c \in \mathcal{C}_q$ and $x, y \in c$, $Z_t(x) = Z_t(y)$ for all $t \in [0, T_{q+1})$, thus we can introduce $Z_t(c)$.

We consider $\lambda_\mu > 0$ (which depends on M) such that for all $\lambda \in (0, \lambda_\mu)$, $(X_i)_{\lambda,\mu} \neq \emptyset$ and $(X_i)_{\lambda,\mu} \cap (X_j)_{\lambda,\mu} = \emptyset$ for all $i \neq j$ with $i, j \in \{1, ..., n\}$.
Then we observe that for $\lambda \in (0, \lambda_\mu)$, for each $q = 0, ..., n$, $\{x_{\lambda,\mu}, x \in \mathcal{B}_q\} \cup \{c_{\lambda,\mu}, c \in \mathcal{C}_q\}$ is a partition of $[\![-\tilde{A}_{\lambda,\mu}, \tilde{A}_{\lambda,\mu}]\!]$, where $\tilde{A}_{\lambda,\mu} = \lfloor A/(\lambda\log(1/\lambda)) - \mu/(\lambda\log^2(1/\lambda))\rfloor$.

With our coupling, for the (λ, A)-FFP $(\eta_t^\lambda)_{t\geq 0}$, for each $i = 1, \dots, n$, a fire starts at the site $\lfloor X_i/(\lambda\log(1/\lambda))\rfloor$ at time $T_i\log(1/\lambda)$, and this describes all the fires during $[0, T\log(1/\lambda)]$.

The lemma below shows some exchangeability properties inside cells (connected components of $(-A, A) \setminus \mathcal{B}_q$). This will allow us to prove that for c a cell and $x \in c$, the size of occupied cluster around x (described by $Z^\lambda(x)$) is closely related to the global density of occupied clusters in c (described by $\tilde{Z}^{\lambda,\mu}(c)$).

Lemma 14. *For $\lambda \in (0,1)$ and $\mu \in (0,1]$, set $\mathcal{E}_0^{\lambda,\mu} = \Omega$, and for $q = 1, ..., n$, consider the event (recall Definition 10 and (9))*

$$\mathcal{E}_q^{\lambda,\mu} = \Big\{ \forall\, i = 1, ..., q,\ \forall\, c \in \mathcal{C}_i,\ \textit{either } c_{\lambda,\mu} \subset C^\lambda_{T_i\log(1/\lambda)-}(X_i)$$
$$\textit{or } \eta^\lambda_{T_i\log(1/\lambda)-}(k) = 0 \textit{ for some } \max c_{\lambda,\mu} < k < \min C^\lambda_{T_i\log(1/\lambda)-}(X_i)$$
$$\textit{or } \eta^\lambda_{T_i\log(1/\lambda)-}(k) = 0 \textit{ for some } \max C^\lambda_{T_i\log(1/\lambda)-}(X_i) < k < \min c_{\lambda,\mu} \Big\}.$$

Conditionally to M and $\mathcal{E}_q^{\lambda,\mu}$, for all $c\in\mathcal{C}_q$, the random variables $(\eta^\lambda_{T_q\log(1/\lambda)}(k))_{k\in c_{\lambda,\mu}}$ are exchangeable.

Proof. Let $c\in\mathcal{C}_q$, let σ be a permutation of $c_{\lambda,\mu}$, and set for simplicity $\sigma(i)=i$ for $i\in I_A^\lambda\setminus c_{\lambda,\mu}$ (recall (6)).
Consider the (λ,A)-FFP process $(\eta^\lambda_t)_{t\geq 0}$ built with M and the family of Poisson processes $(N(i))_{i\in I_A^\lambda}$. Consider also the (λ,A)-FFP process $(\tilde\eta^\lambda_t)_{t\geq 0}$ built with M and the family of Poisson processes $(\tilde N(i))_{i\in I_A^\lambda}$ defined by $\tilde N(i)=N(\sigma(i))$.
Observe that $\mathcal{E}_{k+1}^{\lambda,\mu}\subset\mathcal{E}_k^{\lambda,\mu}$. For all $k=0,...,q$, $c\subset c_k$ for some $c_k\in\mathcal{C}_k$. We will show the following claims, by induction on $k=0,...,q$.
(i) If $\tilde{\mathcal{E}}_k^{\lambda,\mu}$ is the same event as $\mathcal{E}_k^{\lambda,\mu}$ corresponding to $(\tilde\eta^\lambda_t)_{t\geq0}$, then $\tilde{\mathcal{E}}_k^{\lambda,\mu}=\mathcal{E}_k^{\lambda,\mu}$.
(ii) On $\mathcal{E}_k^{\lambda,\mu}$, for all $t\in[0,T_k\log(1/\lambda)]$, $\tilde\eta^\lambda_t(i)=\eta^\lambda_t(\sigma(i))$ for all $i\in I_A^\lambda$ (in particular, $\tilde\eta^\lambda_t(i)=\eta^\lambda_t(i)$ for all $i\notin c_{\lambda,\mu}$).
Of course, (i) and (ii) with $k=q$ imply the Lemma. Indeed, let $\varphi:\{0,1\}^{\#(c_{\lambda,\mu})}\mapsto\mathbb{R}$. We have $\mathbb{E}_M[\mathbb{1}_{\mathcal{E}_q^{\lambda,\mu}}\varphi((\eta^\lambda_{T_q\log(1/\lambda)}(i))_{i\in c_{\lambda,\mu}}))]=\mathbb{E}_M[\mathbb{1}_{\tilde{\mathcal{E}}_q^{\lambda,\mu}}\varphi((\tilde\eta^\lambda_{T_q\log(1/\lambda)}(i))_{i\in c_{\lambda,\mu}}))]$. Then using (i) and (ii), we deduce that

$$\mathbb{E}_M[\mathbb{1}_{\mathcal{E}_q^{\lambda,\mu}}\varphi((\eta^\lambda_{T_q\log(1/\lambda)}(i))_{i\in c_{\lambda,\mu}}))]=\mathbb{E}_M[\mathbb{1}_{\mathcal{E}_q^{\lambda,\mu}}\varphi((\eta^\lambda_{T_q\log(1/\lambda)}(\sigma(i)))_{i\in c_{\lambda,\mu}}))],$$

which proves the Lemma.

First, (i) and (ii) with $k=0$ are obviously satisfied. Assume now that for some $k\in\{0,...,q-1\}$, we have (i) and (ii). Then on $\mathcal{E}_k^{\lambda,\mu}$, for all $t\in[0,T_{k+1}\log(1/\lambda))$, $\tilde\eta^\lambda_t(i)=\eta^\lambda_t(\sigma(i))$ for all $i\in I_A^\lambda$. Indeed, they are equal on $[0,T_k\log(1/\lambda)]$ by assumption and they use the same Poisson process $\tilde N(i)=N(\sigma(i))$ on the time interval $[T_k\log(1/\lambda),T_{k+1}\log(1/\lambda)))$.
We now check that $\mathcal{E}_{k+1}^{\lambda,\mu}=\tilde{\mathcal{E}}_{k+1}^{\lambda,\mu}$. We know that $\mathcal{E}_k^{\lambda,\mu}=\tilde{\mathcal{E}}_k^{\lambda,\mu}$, and the additional condition (at time $T_{k+1}\log(1/\lambda)-$) concerns:
• sites outside $c_{\lambda,\mu}$, for which the values of η^λ and $\tilde\eta^\lambda$ at time $T_{k+1}\log(1/\lambda)-$ are the same;
• the event $c_{\lambda,\mu}\subset C^\lambda_{T_{k+1}\log(1/\lambda)-}$, which is the same for η^λ and $\tilde\eta^\lambda$, (it can be realized only if there are no vacant sites in $c_{\lambda,\mu}$, which occurs or not simultaneously for η^λ and $\tilde\eta^\lambda$).

We now conclude that (ii) remains true at time $T_{k+1}\log(1/\lambda)$, since the zone subject to fire
• either is disjoint of $c_{\lambda,\mu}$, so that the values of $\eta^\lambda,\tilde\eta^\lambda$ are left invariant in $c_{\lambda,\mu}$, while they are modified in the same way outside $c_{\lambda,\mu}$;
• or contains the whole zone $c_{\lambda,\mu}$, which is thus destroyed simultaneously for η^λ and $\tilde\eta^\lambda$, and the values of $\eta^\lambda,\tilde\eta^\lambda$ are modified in the same way outside $c_{\lambda,\mu}$. ☐

The next Lemma shows in some sense that if a cell is *almost* completely occupied at time t, then it will be *really* completely occupied at time $t+$; and if the effect of a microscopic fire is *almost* ended at time t, then it will be *really* ended at time $t+$.

Lemma 15. *Let $\mu\in(0,1]$. Consider $k\in\{0,...,n\}$, $c\in\mathcal{C}_k$, $x\in\mathcal{B}_k$, and $t\in[T_k,T_{k+1})$.*
(i) Assume that for all $\varepsilon>0$, $\lim_{\lambda\to0}\mathbb{P}_M(\tilde Z_t^{\lambda,\mu}(c)<1-\varepsilon)=0$. Then for all $s\in(t,T_{k+1})$, $\lim_{\lambda\to0}\mathbb{P}_M(\tilde Z_s^{\lambda,\mu}(c)=1)=1$.
(ii) Assume that for all $\varepsilon>0$, $\lim_{\lambda\to0}\mathbb{P}_M(\tilde H_t^{\lambda,\mu}(x)>\varepsilon)=0$. Then for all $s\in(t,T_{k+1})$, $\lim_{\lambda\to0}\mathbb{P}_M(\tilde H_s^{\lambda,\mu}(x)=0)=1$.

Proof. The proofs of (i) and (ii) are similar. Let us for example prove (i). Let thus $T_k\leq t<t+\varepsilon=s<T_{k+1}$. We start with

$$\mathbb{P}_M(\tilde Z_{t+\varepsilon}^{\lambda,\mu}(c)=1)\geq\mathbb{P}_M(\tilde Z_{t+\varepsilon}^{\lambda,\mu}(c)=1\mid\tilde Z_t^{\lambda,\mu}(c)>1-\varepsilon/2)\mathbb{P}_M(\tilde Z_t^{\lambda,\mu}(c)>1-\varepsilon/2),$$

so that it suffices to check that $\lim_{\lambda\to 0}\mathbb{P}_M(\tilde Z^{\lambda,\mu}_{t+\varepsilon}(c)=1 \mid \tilde Z^{\lambda,\mu}_t(c)>1-\varepsilon/2)=1$. Call $v^{\lambda,\mu}_t$ the number of vacant sites in $c_{\lambda,\mu}$ (for $\eta^\lambda_{t\log(1/\lambda)}$). Then $\tilde Z^{\lambda,\mu}_{t+\varepsilon}(c)=1$ is equivalent to $v^{\lambda,\mu}_{t+\varepsilon}=0$, and one easily checks that $\tilde Z^{\lambda,\mu}_t(c)>1-\varepsilon/2$ implies that $v^{\lambda,\mu}_t \le (1+\#(c_{\lambda,\mu}))^{\varepsilon/2}\le (1+2A/(\lambda\log(1/\lambda)))^{\varepsilon/2}$.
Since $M((t,s]\times[-A,A])=0$ by assumption, we deduce that $M^\lambda_{s\log(1/\lambda)}(i)=M^\lambda_{t\log(1/\lambda)}(i)$ for all $i\in I^\lambda_A$: no fire starts during $(t\log(1/\lambda),s\log(1/\lambda)]$. Hence each occupied site at time $t\log(1/\lambda)$ remains occupied at time $s\log(1/\lambda)$, and each vacant site at time $t\log(1/\lambda)$ becomes occupied at time $s\log(1/\lambda)$ with probability $1-e^{(t-s)\log(1/\lambda)}=1-\lambda^\varepsilon$. Thus

$$\mathbb{P}_M(\tilde Z^{\lambda,\mu}_{t+\varepsilon}(c)=1 \mid \tilde Z^{\lambda,\mu}_t(c)>1-\varepsilon/2)\ge (1-\lambda^\varepsilon)^{(1+2A/(\lambda\log(1/\lambda)))^{\varepsilon/2}},$$

which tends to 1 as $\lambda\to 0$. $\square$

We end preliminaries with a last lemma, which concerns estimates about the time needed to occupate vacant zones.

Lemma 16. *Let $\mu\in(0,1]$. Let $(\zeta^\lambda_0(i))_{i\in I^\lambda_A}\in\{0,1\}^{I^\lambda_A}$, and consider a family of i.i.d. Poisson processes $(P^\lambda_t(i))_{t\ge 0,i\in I^\lambda_A}$, with rate $\log(1/\lambda)$, independent of ζ^λ_0. Set $\zeta^\lambda_t(i)=\min(\zeta^\lambda_0(i)+P^\lambda_t(i),1)$.*

1. Let $J=(a,b)\subset(-A,A)$ and $h\in[0,1]$. Set $v^{\lambda,\mu}_t=\#\{i\in J_{\lambda,\mu},\zeta^\lambda_t(i)=0\}$. Assume that

$$\forall\,\varepsilon>0,\quad \mathbb{P}\left(\left|\frac{\log(1+v^{\lambda,\mu}_0)}{\log(1+\#(J_{\lambda,\mu}))}-h\right|\ge\varepsilon\right)=0.$$

(a) Then for all $T>0$, all $\varepsilon>0$,

$$\lim_{\lambda\to0}\mathbb{P}\left(\sup_{[0,T]}\left|\frac{\log(1+v^{\lambda,\mu}_t)}{\log(1+\#(J_{\lambda,\mu}))}-(h-t)_+\right|\ge\varepsilon\right)=0.$$

(b) If the family $(\zeta^\lambda_0(i))_{i\in J_{\lambda,\mu}}$ is exchangeable, then for all $x\in J$, all $T>0$, all $\varepsilon>0$,

$$\lim_{\lambda\to0}\mathbb{P}\left(\sup_{[0,T]}\left|\frac{\log(1+\#(G^\lambda_t(x)))}{\log(1/\lambda)}-(1-(h-t)_+)\right|\ge\varepsilon\right)=0,$$

where $G^\lambda_t(x)$ is the connected component of occupied sites around $\lfloor x/\lambda\log(1/\lambda)\rfloor$ in ζ^λ_t.

2. Let $x\in(-A,A)$, and $h\in[0,1]$. Set $v^{\lambda,\mu}_t=\#\{i\in x_{\lambda,\mu},\zeta^\lambda_t(i)=0\}$.
Assume that

$$\forall\,\varepsilon>0,\quad \mathbb{P}\left(\left|\frac{\log(1+v^{\lambda,\mu}_0)}{\log(1+\#(x_{\lambda,\mu}))}-h\right|\ge\varepsilon\right)=0$$

Then for all $T>0$, all $\varepsilon>0$,

$$\lim_{\lambda\to0}\mathbb{P}\left(\sup_{[0,T]}\left|\frac{\log(1+v^{\lambda,\mu}_t)}{\log(1+\#(x_{\lambda,\mu}))}-(h-t)_+\right|\ge\varepsilon\right)=0.$$

Proof. The proof of 2. is the same as that of 1-(a), because $\log(1+\#(J_{\lambda,\mu}))\sim\log(1+\#(x_{\lambda,\mu}))\sim\log(1/\lambda)$ as $\lambda\to0$. We thus prove only 1, and we replace everywhere $\log(1+\#(x_{\lambda,\mu}))$ by $\log(1/\lambda)$ without difficulty. By assumption, we have, for all $\varepsilon>0$, $\lim_{\lambda\to0}\mathbb{P}(v^{\lambda,\mu}_0\in(\lambda^{\varepsilon-h}-1,\lambda^{-\varepsilon-h}))=1$. We call $h_t=(h-t)_+$, $V^{\lambda,\mu}_t=\log(1+v^{\lambda,\mu}_t)/\log(1/\lambda)$, and finally $\Gamma^\lambda_t=\log(1+\#(G^\lambda_t(x)))/\log(1/\lambda)$.

Step 1. Let $t \geq 0$ be fixed. We first show that for all $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}\left(|V_t^{\lambda,\mu} - h_t| \geq \varepsilon\right) = 0$. Conditionally on $v_0^{\lambda,\mu}$, the random variable $v_t^{\lambda,\mu}$ follows a Binomial distribution $B(v_0^{\lambda,\mu}, \lambda^t)$, because each vacant site at time 0 remains vacant with probability $e^{-t\log(1/\lambda)} = \lambda^t$.

Case $h_t > 0$. Let $\varepsilon \in (0, h_t)$. We have to prove that $\mathbb{P}\left(v_t^{\lambda,\mu} \in (\lambda^{\varepsilon-h_t}, \lambda^{-\varepsilon-h_t})\right) \to 1$. We know that $\lim_{\lambda\to 0} \mathbb{P}(v_0^{\lambda,\mu} \in (\lambda^{\varepsilon/2-h}, \lambda^{-\varepsilon/2-h})) = 1$. The Bienaymé-Chebyshev inequality implies

$$\begin{aligned} P[|v_t^{\lambda,\mu} - v_0^\lambda \lambda^t| \leq (v_0^{\lambda,\mu}\lambda^t)^{2/3} \mid v_0^{\lambda,\mu} \in (\lambda^{\varepsilon/2-h}, \lambda^{-\varepsilon/2-h})] \\ \geq 1 - \mathbb{E}[v_0^{\lambda,\mu}\lambda^t(1-\lambda^t)(v_0^{\lambda,\mu}\lambda^t)^{-4/3} \mid v_0^{\lambda,\mu} \in (\lambda^{\varepsilon/2-h}, \lambda^{-\varepsilon/2-h})] \\ \geq 1 - \mathbb{E}[(v_0^{\lambda,\mu}\lambda^t)^{-1/3} \mid v_0^{\lambda,\mu} \in (\lambda^{\varepsilon/2-h}, \lambda^{-\varepsilon/2-h})] \geq 1 - (\lambda^{\varepsilon/2-h+t})^{-1/3}, \end{aligned}$$

which tends to 1 since $h_t = h - t > \varepsilon$.
But the events $|v_t^{\lambda,\mu} - v_0^{\lambda,\mu}\lambda^t| \leq (v_0^{\lambda,\mu}\lambda^t)^{2/3}$ and $v_0^{\lambda,\mu} \in (\lambda^{\varepsilon/2-h}, \lambda^{-\varepsilon/2-h})]$ imply that $v_t^{\lambda,\mu} \in (\lambda^{\varepsilon/2-h_t} - (\lambda^{-\varepsilon/2-h_t})^{2/3}, \lambda^{-\varepsilon/2-h_t} + (\lambda^{-\varepsilon/2-h_t})^{2/3}) \subset (\lambda^{\varepsilon-h_t}, \lambda^{-\varepsilon-h_t})$ for λ small enough, whence the result.

Case $h_t = 0$. We have to show that for all $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}(v_t^{\lambda,\mu} > \lambda^{-\varepsilon}) = 0$, and it suffices to check that $\lim_{\lambda\to 0} \mathbb{P}(v_t^{\lambda,\mu} > \lambda^{-\varepsilon} \mid v_0^{\lambda,\mu} < \lambda^{-\varepsilon/2-h}) = 0$. But

$$\begin{aligned} \mathbb{P}(v_t^{\lambda,\mu} > \lambda^{-\varepsilon} \mid v_0^{\lambda,\mu} < \lambda^{-\varepsilon/2-h}) \leq \lambda^\varepsilon \mathbb{E}[v_t^{\lambda,\mu} \mid v_0^{\lambda,\mu} < \lambda^{-\varepsilon/2-h}] = \lambda^\varepsilon \mathbb{E}[v_0^{\lambda,\mu}\lambda^t \mid v_0^{\lambda,\mu} < \lambda^{-\varepsilon/2-h}] \\ \leq \lambda^{\varepsilon+t}\lambda^{-\varepsilon/2-h} = \lambda^{\varepsilon/2+t-h}, \end{aligned}$$

which tends to 0, since $t - h \geq 0$ by assumption.

Step 2. We now prove that for all $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}(|\Gamma_t^\lambda - (1-h_t)| \geq \varepsilon) = 0$. It suffices to check that $\lim_{\lambda\to 0} \mathbb{P}(\#(G_t^\lambda(x)) \in (\lambda^{\varepsilon+h_t-1} - 1, \lambda^{-\varepsilon+h_t-1})) = 1$. But we know from Step 1 that there are around $(1/\lambda)^{h_t}$ vacant sites in $J_{\lambda,\mu}$, and $\#(J_{\lambda,\mu}) \simeq (1/\lambda\log(1/\lambda))$. We also know that the family $(\zeta_t^\lambda(i))_{i\in J_{\lambda,\mu}}$ is exchangeable, so that the vacant sites are uniformly distributed in $J_{\lambda,\mu}$ (this is sligthly false: there cannot be two vacant sites at the same place). We conclude that $\#(G_t^\lambda(x)) \simeq (1/\lambda\log(1/\lambda))/(1/\lambda)^{h_t} \simeq \lambda^{h_t-1}$. This can be done rigorously without difficulty.

Step 3. We now prove 1-(a), which relies on Step 1 and an *ad hoc* version of Dini Theorem. Let $\varepsilon > 0$. Consider a subdivision $0 = t_0 < t_1 < ... < t_l = T$, with $t_{i+1} - t_i < \varepsilon/2$. Using Step 1, we have $\lim_{\lambda\to 0} \mathbb{P}[\max_{i=0,...l} |V_{t_i}^{\lambda,\mu} - (h-t_i)_+| > \varepsilon/2] = 0$.
Observe now that $t \mapsto V_t^{\lambda,\mu}$ and $t \mapsto (h-t)_+$ are a.s. nonincreasing, and that $t \mapsto (h-t)_+$ is Lipschitz continuous with Lipschitz constant 1.
We deduce that $\sup_{[0,T]} |V_t^{\lambda,\mu} - (h-t)_+| \leq \varepsilon/2 + \max_{i=0,...,l}\{|V_{t_i}^{\lambda,\mu} - (h-t_i)_+|\}$. Thus $\mathbb{P}(\sup_{[0,T]} |V_t^{\lambda,\mu} - (h-t)_+| > \varepsilon) \leq \mathbb{P}[\max_{i=0,...l} |V_{t_i}^{\lambda,\mu} - (h-t_i)_+| > \varepsilon/2]$, which concludes the proof of 1-(a).

Step 4. Point 1-(b) is deduced from Step 2 exactly as Point 1-(a) is deduced from Step 1, using that $t \mapsto \Gamma_t^\lambda$ and $t \mapsto 1 - h_t$ are a.s. nondecreasing. □

We finally may handle the

Proof of Proposition 13.
For $x \in (-A, A)$ and $t \geq 0$, we introduce $Z_t(x-) = \lim_{y\to x, y<x} Z_t(y)$ and $Z_t(x+) = \lim_{y\to x, y>x} Z_t(y)$, which represent the values of Z_t in the cells on the left and right of

x. If $x \in \mathcal{B}_n$, it is at the boundary of two cells $c_-, c_+ \in \mathcal{C}_n$, and then $Z_t(x-) = Z_t(c_-)$ and $Z_t(x+) = Z_t(c_+)$.

For $x \in \mathcal{B}_n$ and $t \geq 0$ we set $\tilde{H}_t(x) = \max(H_t(x), 1-Z_t(x), 1-Z_t(x-), 1-Z_t(x+))$. Observe that for the LFFP, x is *microscopic* (or *acts like a barrier*) if and only if $\tilde{H}_t(x) > 0$, and if so, it will remain microscopic during exactly $[t, t+\tilde{H}_t(x))$. Note that, in facts, $Z_t(x)$ always equals either $Z_t(x-)$ or $Z_t(x+)$.

We consider the set of times $\mathcal{K} := \{t \in \{0, T\}$, there is $x \in (-A, A)$, $\tilde{H}_t(x) = 0$ but $\tilde{H}_{t-\varepsilon}(x) > 0$ for all $\varepsilon > 0$ small enough$\}$. By construction, we see that $\mathcal{K} \subset \{1, T_i+1, T_i + Z_{T_i-}(X_i), i = 1, ..., n\} \subset \{1, T_i+1, T_i+(T_i-T_j), 0 \leq j < i \leq n\}$.

We work conditionally to M, by induction on $q = 0, ..., n$. Consider the assumption

($\mathcal{H}_q$): (i) For all $0 < \mu \leq 1$, all $c \in \mathcal{C}_q$, all $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(|\tilde{Z}^{\lambda,\mu}_{T_q}(c) - Z_{T_q}(c)| > \varepsilon) = 0$.

(ii) For all $x \in \mathcal{B}_q$, all $0 < \mu \leq 1$, all $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(|\tilde{H}^{\lambda,\mu}_{T_q}(x) - \tilde{H}_{T_q}(x)| > \varepsilon) = 0$.

(iii) For all $0 < \mu \leq 1$, $\lim_{\lambda\to 0} \mathbb{P}_M(\mathcal{E}^{\lambda,\mu}_q) = 1$ (recall Lemma 14).

First, ($\mathcal{H}_0$) is obviously satisfied, because $T_0 = 0$, $\mathcal{C}_0 = (-A, A)$, $\tilde{Z}^{\lambda,\mu}_0((-A,A)) = 0 = Z_0((-A,A))$, $\mathcal{B}_0 = \emptyset$, and $\mathcal{E}^{\lambda,\mu}_0 = \Omega$.

The proposition will essentially be proved if we check that for $q = 0, ..., n-1$, ($\mathcal{H}_q$) implies

(a) for $c \in \mathcal{C}_q$, $0 < \mu \leq 1$, $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(\sup_{[T_q, T_{q+1})} |\tilde{Z}^{\lambda,\mu}_t(c) - Z_t(c)| > \varepsilon) = 0$;
(b) for $x \in (-A, A) \setminus \mathcal{B}_q$, $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(\sup_{[T_q, T_{q+1})} |Z^{\lambda}_t(x) - Z_t(x)| > \varepsilon) = 0$;
(c) for $x \in \mathcal{B}_q$, $t \in [T_q, T_{q+1})$, $0 < \mu \leq 1$, $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(|\tilde{H}^{\lambda,\mu}_t(x) - \tilde{H}_t(x)| > \varepsilon)$;
(d) for $x \in (-A, A) \setminus \mathcal{B}_q$, $t \in (T_q, T_{q+1}) \setminus \mathcal{K}$, $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(\delta(D^{\lambda}_t(x), D_t(x)) > \varepsilon) = 0$;
(e) for $x \in (-A, A) \setminus \mathcal{B}_q$, $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(\int_{T_q}^{T_{q+1}} \delta(D^{\lambda}_t(x), D_t(x))dt > \varepsilon) = 0$;
(f) ($\mathcal{H}_{q+1}$) holds.

We thus assume ($\mathcal{H}_q$) for some $q \in \{0, ..., n-1\}$ fixed, and prove points (a), ..., (f). We repeatedly use below that on the time interval $[T_q, T_{q+1})$, there are no fires at all in $(-A, A)$ for the LFFP, and no fires at all during $[T_q \log(1/\lambda), T_{q+1} \log(1/\lambda))$ for the λ-FFP.

Set $\zeta^{\lambda}_0(i) = \eta^{\lambda}_{T_q \log(1/\lambda)}(i)$, and consider the i.i.d. Poisson processes $P^{\lambda}_t(i) = N_{(T_q+t)\log(1/\lambda)}(i) - N_{T_q \log(1/\lambda)}(i)$ with rate $\log(1/\lambda)$. Then for $t \in [T_q, T_{q+1})$, $\eta^{\lambda}_{t \log(1/\lambda)}(i) = \min(\zeta_0(i) + P^{\lambda}_{t-T_q}(i), 1)$.

Point (a). Let $0 < \mu \leq 1$. Let $c \in \mathcal{C}_q$. Observe that ($\mathcal{H}_q$)-(i) says exactly that with $h = 1 - Z_{T_q}(c) \in [0,1]$, $\log(1 + \#\{k \in c_{\lambda,\mu}, \zeta^{\lambda}_0(k) = 0\}) / \log(1 + \#(c_{\lambda,\mu}))$ tends to h in probability (for $\mathbb{P}_M$). Applying Lemma 16-1-(a) (with $J = c$), we get that $\sup_{[T_q, T_{q+1})} |1 - \tilde{Z}^{\lambda,\mu}_t(c) - (h - (t-T_q))_+|$ tends to 0 in probability (for $\mathbb{P}_M$). But for $t \in [T_q, T_{q+1})$, we have $Z_t(c) = \min(Z_{T_q}(c) + (t - T_q), 1) = \min(1 - h + (t - T_q), 1) = 1 - (h - (t - T_q))_+$. Point (a) follows.

Point (b). Let now $x \in (-A, A) \setminus \mathcal{B}_q$. Then $x \in c$, for some $c \in \mathcal{C}_q$. Due to Lemma 14, we know that $(\zeta^{\lambda}_0(i))_{i \in c_{\lambda,\mu}}$ are exchangeable on $\mathcal{E}^{\lambda,1}_q$. The previous reasonning, using Lemma 16-1-(b) instead of Lemma 16-1-(a) shows that for all $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(\mathcal{E}^{\lambda,1}_q \cap \{\sup_{[T_q, T_{q+1})} |Z^{\lambda}_t(x) - Z_t(x)| > \varepsilon\}) = 0$. We conclude using ($\mathcal{H}_q$)-(iii) for $\mu = 1$.

Point (c). Let $0 < \mu \leq 1$. Let $x \in \mathcal{B}_q$, and set $h = \tilde{H}_{T_q}(x)$. We know by $\mathcal{H}_q$-(ii) that $\tilde{H}^{\lambda,\mu}_{T_q}(x)$ tends to $\tilde{H}_{T_q}(x) = h$ in probability (for $\mathbb{P}_M$). Using now Lemma 16-2-(a), we deduce that $\sup_{[T_q,T_{q+1})} |\tilde{H}^{\lambda,\mu}_t(x) - (h-(t-T_q))_+|$ tends to 0 in probability (for $\mathbb{P}_M$). We conclude observing that by construction, $\tilde{H}_t(x) = (h-(t-T_q))_+$ for $t \in [T_q, T_{q+1})$.

Point (d). Let $x \in (-A, A) \setminus \mathcal{B}_q$ and $t \in (T_q, T_{q+1}) \setminus \mathcal{K}$ be fixed.

Case $Z_t(x) < 1$. Then $D_t(x) = \{x\}$, so that $\delta(D_t(x), D^\lambda_t(x)) = |D^\lambda_t(x)|$. But we get from (1)-(2) that $|D^\lambda_t(x)| \leq \lambda^{1-Z^\lambda_t(x)} \log(1/\lambda)$. Since we know from (b) that $Z^\lambda_t(x)$ goes to $Z_t(x) < 1$ in probability (for $\mathbb{P}_M$), we easily deduce that $|D^\lambda_t(x)|$ goes to 0 in probability (for $\mathbb{P}_M$).

Case $Z_t(x) = 1$. Then $D_t(x) = [a, b]$ for some $a, b \in \mathcal{B}_q \cup \{-A, A\}$. We assume that $-A < a < b < A$ for simplicity, the other cases being treated in a similar way. We thus have $Z_t(c) = 1$ for all $c \in \mathcal{C}_q$ with $c \subset (a, b)$, $\tilde{H}_t(y) = 0$ for all $y \in \mathcal{B}_q \cap (a, b)$, and $\tilde{H}_t(a)\tilde{H}_t(b) > 0$.

On the one hand, we prove that for any $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(D^\lambda_t(x) \subset [a-\varepsilon, b+\varepsilon]) = 1$. Let us consider e.g. the left boundary a, and prove that $\lim_{\lambda\to 0} \mathbb{P}_M(D^\lambda_t(x) \subset [a-\varepsilon, A]) = 1$.

We have $\tilde{H}_t(a) = h_a > 0$. We deduce from (c) that $\lim_{\lambda\to 0} \mathbb{P}_M(\tilde{H}^{\lambda,1}_t(a) \geq h_a/2) = 1$, which implies that there are vacant sites in $a_{\lambda,1}$, that is $\lim_{\lambda\to 0} \mathbb{P}_M(\exists\, i \in a_{\lambda,1}, \eta_{t\log(1/\lambda)}(i) = 0) = 1$. Recalling the definition of $a_{\lambda,1}$ (see (10)), we see that this implies that $\lim_{\lambda\to 0} \mathbb{P}_M(D^\lambda_t(x) \subset [a-1/\log(1/\lambda), A]) = 1$, whence $\lim_{\lambda\to 0} \mathbb{P}_M(D^\lambda_t(x) \subset [a-\varepsilon, A]) = 1$ for any $\varepsilon > 0$.

On the other hand, we prove that $\lim_{\lambda\to 0} \mathbb{P}_M((a+1/\log(1/\lambda), b-1/\log(1/\lambda)) \subset D^\lambda_t(x)) = 1$. Since $t \notin \mathcal{K}$, we deduce that there is $s \in (T_q, t)$ such that $Z_s(c) = 1$ for all $c \in \mathcal{C}_q$ with $c \subset (a, b)$ and $\tilde{H}_s(y) = 0$ for all $y \in \mathcal{B}_q \cap (a, b)$. We deduce from (a) that for all $c \in \mathcal{C}_q$ with $c \subset (a, b)$, $\lim_{\lambda\to 0} \mathbb{P}_M(\tilde{Z}^{\lambda,1}_s(c) > 1-\varepsilon) = 0$, whence, by Lemma 15-(i) $\lim_{\lambda\to 0} \mathbb{P}_M(\tilde{Z}^{\lambda,1}_t(c) = 1) = 1$. Similarly, we deduce from (c) that for all $y \in \mathcal{B}_q$ with $y \in (a, b)$, $\lim_{\lambda\to 0} \mathbb{P}_M(\tilde{H}^{\lambda,1}_s(y) > \varepsilon) = 0$, whence, by Lemma 15-(ii) $\lim_{\lambda\to 0} \mathbb{P}_M(\tilde{H}^{\lambda,1}_t(y) = 0) = 1$. As a consequence, $\lim_{\lambda\to 0} \mathbb{P}_M((a+1/\log(1/\lambda), b-1/\log(1/\lambda)) \subset D^\lambda_t(x)) = 1$.

This concludes the proof of Point (d).

Point (e). Point (e) follows from (d). Indeed, observe that $\delta(I, J) \leq 2A$ for any intervals $I, J \subset (-A, A)$. Thus for $x \in (-A, A) \setminus \mathcal{B}_q$, (d) implies that for $t \in [T_q, T_{q+1}) \setminus \mathcal{K}$, $\lim_{\lambda\to 0} \mathbb{E}_M(\delta(D^\lambda_t(x), D_t(x))) = 0$. Since now $\mathcal{K}$ is finite, we deduce from the Lebesgue dominated convergence Theorem that $\lim_{\lambda\to 0} \int_{T_q}^{T_{q+1}} \mathbb{E}_M(\delta(D^\lambda_t(x), D_t(x)))dt = 0$, from which (e) follows.

Point (f). We show here that $(\mathcal{H}_{q+1})$ holds. We set $z := Z_{T_{q+1}-}(X_{q+1})$, and treat separately the cases $z \in (0, 1)$ and $z = 1$. We a.s. never have $z = 0$, because $Z_{T_{q+1}-}(X_{q+1}) = \min(Z_{T_q}(X_{q+1}) + (T_{q+1} - T_q), 1)$, with $Z_{T_q}(X_{q+1}) \geq 0$ and $T_{q+1} > T_q$.

Case $z \in (0, 1)$. We fix $\mu \in (0, 1]$. In that case $D_{T_{q+1}-}(X_{q+1}) = \{X_{q+1}\}$, and for all $c \in \mathcal{C}_{q+1}$ (thus $c \subset \tilde{c}$ for some $\tilde{c} \in \mathcal{C}_q$), $Z_{T_{q+1}}(c) = Z_{T_{q+1}-}(c)$. We have $\tilde{H}_{T_{q+1}}(X_{q+1}) = \max(z, 1-z)$, and for all $x \in \mathcal{B}_q$, $\tilde{H}_{T_{q+1}}(x) = \tilde{H}_{T_{q+1}-}(x)$. Consider the event $\Omega^\lambda_\alpha = \{Z^\lambda_{T_{q+1}-}(X_{q+1}) \leq z + \alpha\}$, for some $\alpha \in (0, 1-z)$. Point (b) implies that $\lim_{\lambda\to 0} \mathbb{P}_M(\Omega^\lambda_\alpha) = 1$ (because $X_{q+1} \notin \mathcal{B}_q$).

- On Ω_α^λ, we have $\#(C^\lambda_{T_{q+1}\log(1/\lambda)-}(X_{q+1})) \leq (1/\lambda)^{z+\alpha}$ (see (2)). Since $z+\alpha<1$, we deduce that on Ω_α^λ, $\#(C^\lambda_{T_{q+1}\log(1/\lambda)-}(X_{q+1})) < \mu/(2\lambda\log^2(1/\lambda))$, (for all μ, provided $\lambda>0$ is small enough). Thus on Ω_α^λ, for all $c\in\mathcal{C}_{q+1}$, there is a vacant site (strictly) between $c_{\lambda,\mu}$ and $C^\lambda_{T_{q+1}\log(1/\lambda)-}(X_{q+1})$. Hence $\mathcal{E}_q^{\lambda,\mu}\cap\Omega_\alpha^\lambda\subset\mathcal{E}_{q+1}^{\lambda,\mu}$. Using $\mathcal{H}_q$-(iii), we deduce that $\lim_{\lambda\to 0}\mathbb{P}_M(\mathcal{E}_{q+1}^{\lambda,\mu})=1$.
- This also implies that on Ω_α^λ, for all $c\in\mathcal{C}_{q+1}$, $\tilde Z^{\lambda,\mu}_{T_{q+1}}(c)=\tilde Z^{\lambda,\mu}_{T_{q+1}-}(c)$, and thus Point (a) and $\lim_{\lambda\to0}\mathbb{P}_M(\Omega_\alpha^\lambda)=1$ imply that $\lim_{\lambda\to0}\mathbb{P}_M(|\tilde Z^{\lambda,\mu}_{T_{q+1}}(c)-Z_{T_{q+1}}(c)|\geq\varepsilon)=0$ for all $\varepsilon>0$.
- For $x\in\mathcal{B}_{q+1}\setminus\{X_{q+1}\}=\mathcal{B}_q$, still on Ω_α^λ, we also have $\tilde H^{\lambda,\mu}_{T_{q+1}}(x)=\tilde H^{\lambda,\mu}_{T_{q+1}-}(x)$, thus point (c) allows us to conclude that $\mathcal{H}_{q+1}$-(ii) holds for those points x.

  We now show that $\lim_{\lambda\to0}\mathbb{P}_M(|\tilde H^{\lambda,\mu}_{T_{q+1}}(X_{q+1})-\tilde H_{T_{q+1}}(X_{q+1})|\geq\varepsilon)=0$ for all $\varepsilon>0$, which implies that $\mathcal{H}_{q+1}$-(ii) holds for $x=X_{q+1}$. Recall that $\tilde H_{T_{q+1}}(X_{q+1})=\max(z,1-z)$. Consider $c\in\mathcal{C}_q$ such that $X_{q+1}\in c$, and call $v_t^{\lambda,\mu}$ the number of vacant sites in $x_{\lambda,\mu}$ at time $t\log(1/\lambda)$. Point (a) implies that at time $T_{q+1}\log(1/\lambda)-$, there are around $(1/\lambda)^{1-z}$ vacant sites in $c_{\lambda,\mu}$. Thus by exchangeability of the family $(\eta^\lambda_{T_{q+1}\log(1/\lambda)-}(i))_{i\in c_{\lambda,\mu}}$, (on the event $\mathcal{E}_q^{\lambda,\mu}$, see Lemma 14), since $x_{\lambda,\mu}\subset c_{\lambda,\mu}$, and since $\#(x_{\lambda,\mu})/\#(c_{\lambda,\mu})\simeq 1/\log(1/\lambda)$, we deduce that $v^{\lambda,\mu}_{T_{q+1}-}\simeq(1/\lambda)^{1-z}/\log(1/\lambda)\simeq(1/\lambda)^{1-z}$ on $\mathcal{E}_q^{\lambda,\mu}$. On the other hand, recalling (2), we have $\#(C^\lambda_{T_{q+1}\log(1/\lambda)-}(X_{q+1}))\simeq(1/\lambda)^z$. At time $T_{q+1}\log(1/\lambda)$, this component is destroyed. Thus, still on $\mathcal{E}_q^{\lambda,\mu}$, $v^{\lambda,\mu}_{T_{q+1}}=v^{\lambda,\mu}_{T_{q+1}-}+\#(C^\lambda_{T_{q+1}\log(1/\lambda)}(X_{q+1}))\simeq(1/\lambda)^{1-z}+(1/\lambda)^z\simeq(1/\lambda)^{\max(z,1-z)}$. We conclude that $\tilde H^{\lambda,\mu}_{T_{q+1}}(X_{q+1})=\log(1+v^{\lambda,\mu}_{T_{q+1}})/\log(\#((X_{q+1})_{\lambda,\mu}))\simeq\max(z,1-z)=\tilde H_{T_{q+1}}(X_{q+1})$. All this can be done rigorously without difficulty, and we deduce that for $\varepsilon>0$ and all $\mu\in(0,1]$, $\lim_{\lambda\to0}\mathbb{P}_M(|\tilde H^{\lambda,\mu}_{T_{q+1}}(X_{q+1})-\tilde H_{T_{q+1}}(X_{q+1})|\geq\varepsilon)=0$.

Case $z=1$. Let $a,b\in\mathcal{B}_q\cup\{-A,A\}$ such that $D_{T_{q+1}-}(X_{q+1})=[a,b]$. We assume that $a,b\in\mathcal{B}_q$, the other cases being treated in a similar way. We thus have $h_a:=\tilde H_{T_{q+1}-}(a)>0$, $h_b:=\tilde H_{T_{q+1}-}(b)>0$. We also have $\tilde H_{T_{q+1}}(x)=\tilde H_{T_{q+1}-}(x)$ for all $x\in\mathcal{B}_q\backslash[a,b]$, $\tilde H_{T_{q+1}}(x)=1$ for all $x\in\mathcal{B}_q\cap(a,b)$, $Z_{T_{q+1}}(c)=Z_{T_{q+1}-}(c)$ for all $c\in\mathcal{C}_{q+1}$ with $c\cap(a,b)=\emptyset$, and $Z_{T_{q+1}}(c)=0$ for all $c\in\mathcal{C}_{q+1}$ with $c\subset(a,b)$.

Let $\mu\in(0,1]$. Consider here $\tilde\Omega^{\lambda,\mu}$ the event that for all $c\in\mathcal{C}_q$ such that $c\subset(a,b)$, we have $\tilde Z^{\lambda,\mu}_{T_{q+1}-}(c)=1$, that $\tilde H^{\lambda,\mu}_{T_{q+1}-}(a)>0$, that $\tilde H^{\lambda,\mu}_{T_{q+1}-}(b)>0$, and that for all $x\in\mathcal{B}_q\cap(a,b)$, $\tilde H^{\lambda,\mu}_{T_{q+1}-}(x)=0$. Then (a), (c) and Lemma 15 imply that $\lim_{\lambda\to0}\mathbb{P}_M(\tilde\Omega^{\lambda,\mu})=1$ for all $\mu\in(0,1]$.

- We easily check that $\mathcal{E}_q^{\lambda,\mu}\cap\tilde\Omega^{\lambda,\mu}\subset\mathcal{E}_{q+1}^{\lambda,\mu}$ (because for $c\in\mathcal{C}_{q+1}$ with $c\subset[a,b]$, we have $c_{\lambda,\mu}\subset C^\lambda_{T_{q+1}\log(1/\lambda)-}(X_{q+1})$, while for $c\in\mathcal{C}_{q+1}$ with $c\cap[a,b]=\emptyset$, the vacant sites in $a_{\lambda,\mu}$ and $b_{\lambda,\mu}$ separate $c_{\lambda,\mu}$ from $C^\lambda_{T_{q+1}\log(1/\lambda)-}(X_{q+1})$). As a consequence, $\mathcal{H}_{q+1}$-(iii) holds for all $\mu\in(0,1]$.

- On $\tilde\Omega^{\lambda,\mu}$, we have $\tilde Z^{\lambda,\mu}_{T_{q+1}}(c) = 0 = Z_{T_{q+1}}(c)$ for all $c \in \mathcal{C}_{q+1}$ with $c \subset [a,b]$, and $\tilde Z^{\lambda,\mu}_{T_{q+1}}(c) = \tilde Z^{\lambda,\mu}_{T_{q+1}-}(c)$ for $c \in \mathcal{C}_{q+1}$ with $c \cap (a,b) = \emptyset$, from which $\mathcal{H}_{q+1}$-(i) easily follows (using (a)).
- We also have, still on $\tilde\Omega^{\lambda,\mu}$, $\tilde H^{\lambda,\mu}_{T_{q+1}}(x) = 1 = \tilde H_{T_{q+1}}(x)$ for all $x \in \mathcal{B}_{q+1}$ with $x \in (a,b)$, and $\mathcal{H}_{q+1}$-(ii) follows for those x. For $x \in \mathcal{B}_{q+1}$ with $x \notin [a,b]$, we have $\tilde H^{\lambda,\mu}_{T_{q+1}}(x) = \tilde H^{\lambda,\mu}_{T_{q+1}-}(x)$, whence $\mathcal{H}_{q+1}$-(ii) by point (c).

  Finally, we have to check that $\mathcal{H}_{q+1}$-(ii) holds for $x = a$ and $x = b$. Consider e.g. the case of a. We are here in the situation where $Z_{T_{q+1}}(a+) = 0$, so that of course, $\tilde H_{T_{q+1}}(a) = 1$. Let c be the cell containing $a+$. We know that $\tilde Z^{\lambda,\mu/2}_{T_{q+1}-}(c) = 1$ which, on $\tilde\Omega^{\lambda,\mu/2}$, implies that all sites between $a + \frac{\mu}{2\log(1/\lambda)}$ and $a + \frac{\mu}{\log(1/\lambda)}$, i.e. on an interval of length $\frac{\mu}{2\log(1/\lambda)}$ are empty at time T_{q+1}, showing that a fixed proportion of $a_{\lambda,\mu}$ is empty. Recalling that $\lim_{\lambda\to 0} \mathbb{P}_M(\tilde\Omega^{\lambda,\mu/2}) = 1$, it readily follows that, for all $\varepsilon > 0$, $\lim_{\lambda\to0} \mathbb{P}_M(\tilde H^{\lambda,\mu}_{T_{q+1}}(a) > 1-\varepsilon) = 1$. Recalling that $\tilde H^{\lambda,\mu}_{T_{q+1}}(a) \le 1$ we conclude that $\mathcal{H}_{q+1}$-(ii) holds for $x = a$.

Conclusion. Using points (b) and (e) above (with $q = 0,...,n$), plus very similar arguments on the time interval $(T_n, T]$ (during which there are no fires), we deduce that for all $x_0 \in (-A,A) \setminus \mathcal{B}_n$, all $\varepsilon > 0$,

$$\lim_{\lambda\to0} \mathbb{P}_M \left(\sup_{[0,T]} |Z^\lambda_t(x_0) - Z_t(x_0)| + \int_0^T \delta(D^\lambda_t(x_0), D_t(x_0))dt \ge \varepsilon \right) = 0.$$

But of course, for $x_0 \in (-A,A)$, we have $\mathbb{P}(x_0 \in \mathcal{B}_n) = 0$, so that

$$\lim_{\lambda\to0} \mathbb{P} \left(\sup_{[0,T]} |Z^\lambda_t(x_0) - Z_t(x_0)| + \int_0^T \delta(D^\lambda_t(x_0), D_t(x_0))dt \ge \varepsilon \right) = 0.$$

It remains to prove that for $t \in [0,T]$ and $x_0 \in (-A,A)$, $\lim_{\lambda\to0} \mathbb{P}(\delta(D^\lambda_t(x_0), D_t(x_0))) = 0$.

Case $t \neq 1$. We deduce from point (d) above that if $x_0 \notin \mathcal{B}_n$ and $t \notin \mathcal{K}$, then we have $\lim_{\lambda\to0} \mathbb{P}_M(\delta(D^\lambda_t(x_0), D_t(x_0))) = 0$. Since $\mathbb{P}(x_0 \in \mathcal{B}_n) = 0$ and since $\mathbb{P}(t \in \mathcal{K}) = 0$ (because $t \neq 1$, recall the definition of $\mathcal{K}$), we easily conclude.

Case $t = 1$. Then $t \in \mathcal{K}$, but the result still holds. Observe that $Z_1(x_0) = 1$ by construction. Consider $q \in \{0,...,n\}$ such that $T_q < 1 < T_{q+1}$ (with the convention $T_0 = 0$, $T_{n+1} = T$), and consider $a, b \in \mathcal{B}_q \cup \{-A, A\}$ such that $D_1(x_0) = [a,b]$. Then using the same arguments as in the proof of (d) (see Step 1), we easily check that $\lim_{\lambda\to0} \mathbb{P}_M(D^\lambda_1(x_0) \subset [a-\varepsilon, b+\varepsilon]) = 1$ for all $\varepsilon > 0$ (the set $\mathcal{K}$ was not considered there). We also check as in the proof of (d) (see Step 2) that for all $y \in \mathcal{B}_q$ with $y \in (a,b)$, $\lim_{\lambda\to0} \mathbb{P}_M(H^{\lambda,1}_1(y) = 0) = 1$ (the set $\mathcal{K}$ was under consideration there, but the time 1 was not usefull, since 1 is a.s. not a time where some $H(x)$ reaches 0 for the first time). Finally, we just have to prove that for all $c \in \mathcal{C}_q$ with $c \subset (a,b)$, $\lim_{\lambda\to0} \mathbb{P}_M(\tilde Z^{\lambda,1}_1(c) = 1) = 1$. Let thus $c \in \mathcal{C}_q$ with $c \subset (a,b)$, and recall that $\lim_{\lambda\to0} \mathbb{P}_M(\mathcal{E}^{\lambda,1}_q) = 1$. But on $\mathcal{E}^{\lambda,1}_q$, there are no death event in c_λ during the time interval $[0, \log(1/\lambda)]$, so that each site of $c_{\lambda,1}$ is occupied at time $\log(1/\lambda)$ with probability $1-\lambda$, whence all the sites of $c_{\lambda,1}$ are occupied with probability $(1-\lambda)^{\#(c_{\lambda,1})}$. Since $\#(c_{\lambda,1}) \le 2A/(\lambda\log(1/\lambda))$, we get $\mathbb{P}_M(\tilde Z^{\lambda,1}_1(c) = 1 | \mathcal{E}^{\lambda,1}_q) \ge (1-\lambda)^{2A/(\lambda\log(1/\lambda))}$, which

tends to 1 as λ tends to 0. Since we know that $\lim_{\lambda\to 0}\mathbb{P}_M(\mathcal{E}_q^{\lambda,1}) = 1$, we deduce that $\lim_{\lambda\to 0}\mathbb{P}_M([a+1/\log(1/\lambda), b-1/\log(1/\lambda)] \subset D_1^\lambda(x_0)) = 1$.
Finally, $\lim_{\lambda\to 0}\mathbb{P}_M(\delta(D_1^\lambda(x_0), D_1(x_0)) \geq \varepsilon) = 0$ for all $\varepsilon > 0$, which was our goal. □

5. Cluster-size distribution

The aim of this section is to prove Corollary 6. We will use Theorem 5, which asserts that the λ-FFP behaves as the LFFP for $\lambda > 0$ small enough. We start with preliminary results.

Lemma 17. *Consider a LFFP $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$.*
(i) For any $t \in (1,\infty)$, any $x \in \mathbb{R}$, any $z \in [0,1)$, $\mathbb{P}[Z_t(x) = z] = 0$.
(ii) For any $t \in [0,\infty)$, any $B > 0$, any $x \in \mathbb{R}$, $P[|D_t(x)| = B] = 0$.
(iii) There are some constants $C > 0$ and $\kappa_1 > 0$ such that for all $t \in [0,\infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\mathbb{P}[|D_t(x)| \geq B] \leq Ce^{-\kappa_1 B}$.
(iv) There are some constants $c > 0$ and $\kappa_2 > 0$ such that for all $t \in [3/2,\infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\mathbb{P}[|D_t(x)| \geq B] \geq ce^{-\kappa_2 B}$.
(v) There exist some constants $0 < c < C$ such that for all $t \geq 5/2$, all $0 \leq a < b < 1$, all $x \in \mathbb{R}$, $c(b-a) \leq \mathbb{P}(Z_t(x) \in [a,b]) \leq C(b-a)$.

Proof. By invariance by translation, it suffices to treat the case $x = 0$.

Point (i). By Definition 2, we see that for $t \in [0,1]$, we have a.s. $Z_t(0) = t$. But for $t > 1$ and $z \in [0,1)$, $Z_t(0) = z$ implies that the cluster containing 0 has been killed at time $t - z$, so that necessarily $M(\{t-z\} \times \mathbb{R}) > 0$. This happens with probability 0, since $t - z$ is deterministic.

Point (ii). Recalling Definition 2, we see that for any $t \in [0,T]$, $|D_t(0)|$ is either 0 or of the form $|X_i - X_j|$ (with $i \neq j$), where $(T_i, X_i)_{i\geq 1}$ are the marks of the Poisson measure M. We easily conclude as previously that for $B > 0$, $\mathbb{P}(|D_t(0)| = B) = 0$.

Point (iii). First if $t \in [0,1)$, we have a.s. $|D_t(0)| = 0$, and the result is obvious. Next consider $t \geq 1$. Recalling Definition 2, we see that $|D_t(0)| = |L_t(0)| + R_t(0)$. Clearly, $|L_t(0)|$ and $R_t(0)$ have the same law. For $B > 0$, $\{R_t(0) > B\} \subset \{M([t-1/4,t] \times [0,B]) = 0\}$. Indeed, on $\{M([t-1/4,t] \times [0,B]) > 0\}$, denote by $(\tau, X) \in [t-1/4,t] \times [0,B]$ a mark of M.
• Either $Z_{\tau-}(X) = 1$, thus this mark makes start a macroscopic fire, so that $Z_\tau(X) = 0$ and $Z_s(X) = s - \tau < 1$ for all $s \in [\tau, \tau+1)$. Since $\tau \in [t-1/4,t]$, we clearly have $t \in [\tau,\tau+1))$, so that $Z_t(X) < 1$. As a consequence, $R_t(0) \leq X \leq B$.
• Or $Z_{\tau-}(X) \in (1/4,1]$, so that $H_\tau(X) = Z_{\tau-}(X)$, and thus $H_s(X) = Z_{\tau-}(X) - (s-\tau) > 0$ for all $s \in [\tau, \tau + Z_{\tau-}(X))$. Since $\tau \in [t-1/4,t]$ and $Z_{\tau-}(X) > 1/4$, we have $t \in [\tau, \tau + Z_{\tau-}(X))$. Thus $H_t(X) > 0$, whence $R_t(0) \leq X \leq B$.
• Or finally $Z_{\tau-}(X) \leq 1/4$, and in such a case $Z_s(X) = Z_{\tau-}(X) + (s-\tau) < 1$ for all $s \in (\tau, \tau + 1 - Z_{\tau-}(X))$ and in particular $Z_t(X) < 1$, whence $R_t(0) \leq X \leq B$.
As a conclusion, for all $t \geq 1$, $\mathbb{P}[R_t(0) > B] \leq \mathbb{P}[M([t-1/4,t] \times [0,B]) = 0] = e^{-B/4}$, whence $\mathbb{P}[|D_t(0)| > B] \leq \mathbb{P}[|L_t(0)| > B/2] + \mathbb{P}[R_t(0) > B/2] \leq 2e^{-B/8}$.

Point (iv). We first observe that for all (t_0, x_0) such that $M(\{t_0, x_0\}) = 1$, we have $\max(1 - Z_t(x_0), H_t(x_0)) > 0$ for all $t \in [t_0, t_0 + 1/2)$.
Indeed, if $Z_{t_0-}(x_0) = 1$, then $Z_{t_0+s}(x_0) \leq s < 1$ for all $s \in [0,1)$. If now $z = Z_{t_0-}(x_0) < 1$, then $Z_{t_0+s}(x_0) = s + z < 1$ for $s \in [0, 1-z)$ and $H_{t_0+s}(x_0) = z - s > 0$ for $s \in [0,z)$, so that $\max(1 - Z_{t_0+s}(x_0), H_{t_0+s}(x_0)) > 0$ for all $s \in [0,1/2)$.

Once this is seen, fix $t \geq 3/2$, Consider the event $\tilde\Omega_{t,B} = \tilde\Omega^1_{t,B} \cap \tilde\Omega^2_t \cap \tilde\Omega^3_{t,B}$, where
• $\tilde\Omega^1_{t,B} = \{M([t-3/2,t]\times[0,B])=0\}$;
• $\tilde\Omega^2_t$ is the event that in the box $[t-3/2,t]\times[-1,0]$, M has exactly four marks $(S_i,Y_i)_{i=1,\dots,4}$ with $Y_4<Y_3<Y_2<Y_1$ and $t-3/2<S_1<t-1$, $S_1<S_2<S_1+1/2$, $S_2<S_3<S_2+1/2$, $S_3<S_4<S_3+1/2$, and $S_4+1/2>t$.
• $\tilde\Omega^3_{t,B}$ is the event that in the box $[t-3/2,t]\times[B,B+1]$, M has exactly four marks $(\tilde S_i,\tilde Y_i)_{i=1,\dots,4}$ with $\tilde Y_1<\tilde Y_2<\tilde Y_3<\tilde Y_4$ and $t-3/2<\tilde S_1<t-1$, $\tilde S_1<\tilde S_2<\tilde S_1+1/2$, $\tilde S_2<\tilde S_3<S_2+1/2$, $\tilde S_3<\tilde S_4<\tilde S_3+1/2$, and $\tilde S_4+1/2>t$.
We of course have $p:=\mathbb{P}(\tilde\Omega^2_t)=\mathbb{P}(\tilde\Omega^3_{t,B})>0$, and this probability does not depend on $t\geq 3/2$ nor on $B>0$. Furthermore, $\mathbb{P}(\tilde\Omega^1_{t,B})=e^{-3B/2}$. These three events being independent, we conclude that $\mathbb{P}(\tilde\Omega_{t,B})\geq p^2e^{-3B/2}$. To conclude the proof of (iv), it thus suffices to check that $\tilde\Omega_{t,B}\subset\{[0,B]\subset D_t(0)\}$. But on $\tilde\Omega_{t,B}$, using the arguments described at the beginning of the proof of Point (iv), we observe that:
• the fire starting at (S_2,Y_2) can not affect $[0,B]$, because at time $S_2\in[S_1,S_1+1/2)$, $H_{S_2}(Y_1)>0$ or $Z_{S_2}(Y_1)>0$, with $Y_2<Y_1<0$;
• then the fire starting at (S_3,Y_3) can not affect $[0,B]$, because at time $S_3\in[S_2,S_2+1/2)$, $H_{S_3}(Y_2)>0$ or $Z_{S_3}(Y_2)>0$, with $Y_3<Y_2<0$;
• then the fire starting at (S_4,Y_4) can not affect $[0,B]$, because at time $S_4\in[S_3,S_3+1/2)$, $H_{S_4}(Y_3)>0$ or $Z_{S_4}(Y_3)>0$, with $Y_4<Y_3<0$;
• furthermore, the fires starting on the left at -1 during $(S_1,t]$ cannot affect $[0,B]$, because for all $t\in(S_1,t]$, there is always a site $x_t\in\{Y_1,Y_2,Y_3,Y_4\}\subset[-1,0]$ with $H_t(x_t)>0$ or $Z_t(x_t)<1$;
• the same arguments apply on the right of B.
As a conclusion, the zone $[0,B]$ is not affected by any fire during $(S_1\vee\tilde S_1,t]$. Since the length of this time interval is greater than 1, we deduce that for all $x\in[0,B]$, $Z_t(x)=\min(Z_{S_1\vee\tilde S_1}+t-S_1\vee\tilde S_1,1)\geq\min(t-S_1\vee\tilde S_1,1)=1$ and $H_t(x)=\max(H_{S_1\vee\tilde S_1}-(t-S_1\vee\tilde S_1),0)\leq\max(1-(t-S_1\vee\tilde S_1),0)=0$, whence $[0,B]\subset D_t(0)$.

Point (v). We observe, recalling Definition 2, that for $0\leq a<b<1$ and $t\geq1$, we have $Z_t(0)\in[a,b]$ if and only there is $\tau\in[t-b,t-a]$ such that $Z_\tau(0)=0$. This happens if and only if $X_{t,a,b}:=\int_{t-b}^{t-a}\int_{\mathbb{R}}\mathbb{1}_{\{y\in D_{s-}(0)\}}M(ds,dy)\geq1$. We deduce that

$$\mathbb{P}\left(Z_t(0)\in[a,b]\right)=\mathbb{P}\left(X_{t,a,b}\geq1\right)\leq\mathbb{E}\left[X_{t,a,b}\right]=\int_{t-b}^{t-a}\mathbb{E}[|D_s(0)|]ds\leq C(b-a),$$

where we used point (iii) for the last inequality.
Next, we have $\{M([t-b,t-a]\times D_{t-b}(0))\geq1\}\subset\{X_{t,a,b}\geq1\}$: it suffices to note that a.s., $\{X_{t,a,b}=0\}\subset\{X_{t,a,b}=0,D_{t-b}(0)\subset D_s(0)\text{ for all }s\in[t-b,t-a]\}\subset\{M([t-b,t-a]\times D_{t-b}(0))=0\}$. Now since $D_{t-b}(0)$ is $\mathcal{F}^M_{t-b}$-measurable, we deduce that for $t\geq5/2$

$$\begin{aligned}\mathbb{P}\left(Z_t(0)\in[a,b]\right)&\geq\mathbb{P}\left[M((t-b,t-a]\times D_{t-b}(0))>0\right]\\&\geq\mathbb{P}\left[|D_{t-b}(0)|\geq1\right](1-e^{-(b-a)})\geq c(1-e^{-(b-a)}),\end{aligned}$$

where we used Point (iv) (here $t-b\geq3/2$) to get the last inequality. This concludes the proof, since $1-e^{-x}\geq x/2$ for all $x\in[0,1]$. ☐

We now may handle the

Proof of Corollary 6. We thus consider, for each $\lambda > 0$, a λ-FFP $(\eta_t^\lambda)_{t\geq 0}$. Let also $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ be a LFFP.

Point (i). Using Lemma 17-(v) we only need to prove that for all $0 \leq a < b < 1$, all $t \geq 5/2$,

$$\lim_{\lambda\to 0} \mathbb{P}\left(\#(C^\lambda_{t\log(1/\lambda)}(0)) \in [\lambda^{-a}, \lambda^{-b}]\right) = \mathbb{P}\left(Z_t(0) \in [a,b]\right).$$

Recalling (2), we observe that

$$\mathbb{P}\left(\#(C^\lambda_{t\log(1/\lambda)}(0)) \in [\lambda^{-a}, \lambda^{-b}]\right) = \mathbb{P}\left(Z_t^\lambda(0) \in [a+\varepsilon(a,\lambda), b+\varepsilon(b,\lambda)]\right),$$

where $\varepsilon(z,\lambda) = \log(1+\lambda^z)/\log(1/\lambda) \to 0$ as $\lambda \to 0$ (if $z \geq 0$).
We conclude using Theorem 5 (which asserts that $Z_t^\lambda(0)$ goes in law to $Z_t(0)$) and Lemma 17-(i) (from which $\mathbb{P}(Z_t(0) = a) = \mathbb{P}(Z_t(0) = b) = 0$).

Point (ii). Using Lemma 17-(iii)-(iv) and recalling (1), it suffices to check that for all $t \geq 3/2$, all $B > 0$,

$$\lim_{\lambda\to 0} \mathbb{P}\left[|D_t^\lambda(0)| \geq B\right] = \mathbb{P}\left[|D_t(0)| \geq B\right].$$

This follows from Theorem 5 and the fact that $\mathbb{P}(|D_t(0)| = B) = 0$ thanks to Lemma 17-(ii). □

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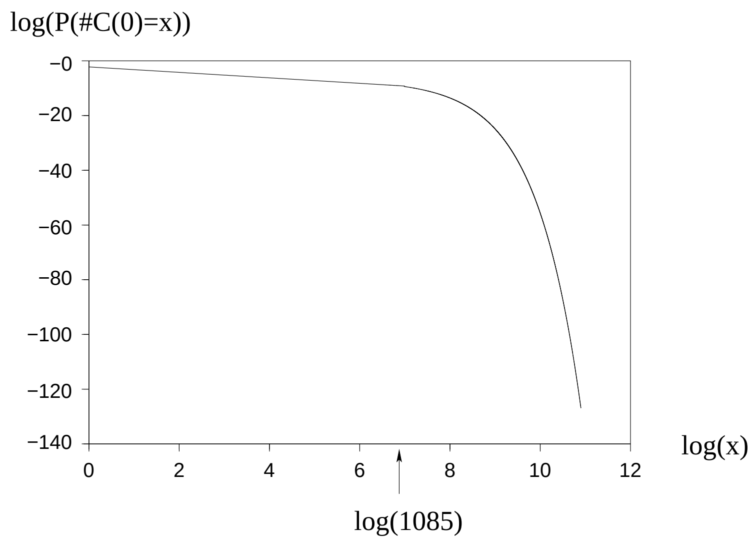
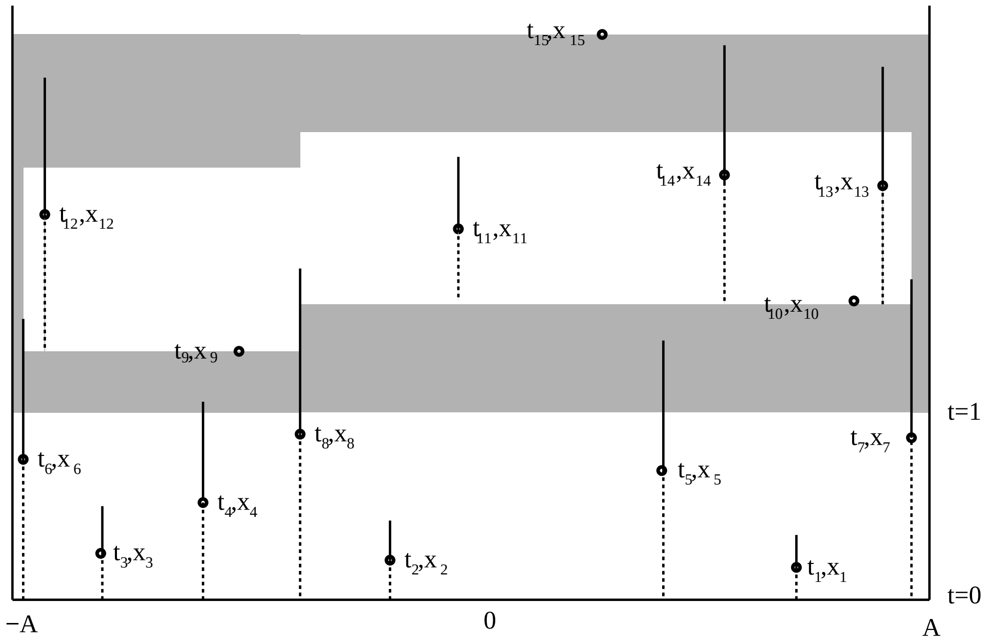
Figure 1: Shape of the cluster-size distribution



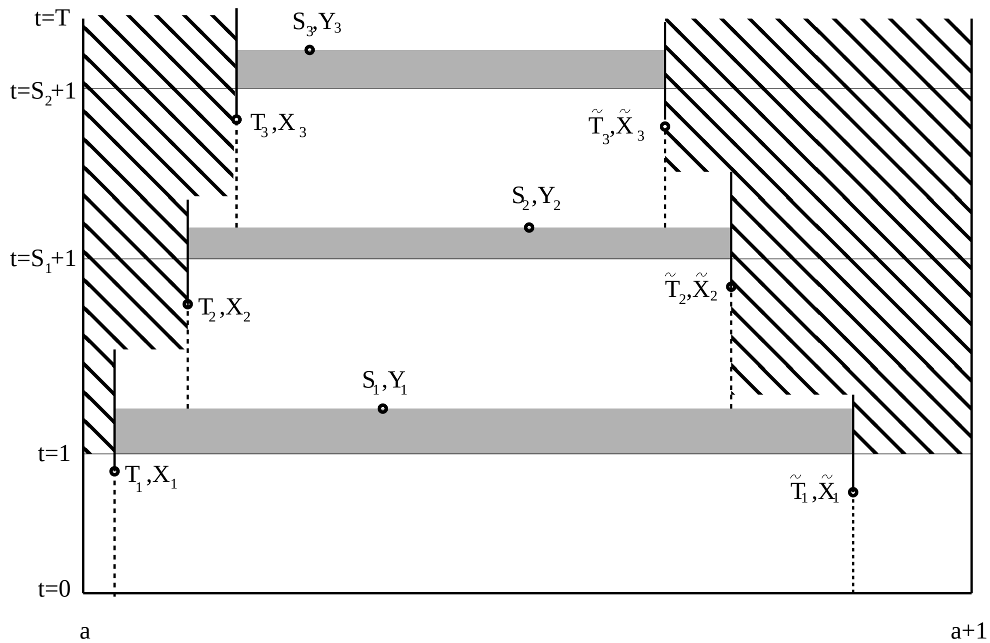
Here $\lambda = 0.0001$, and the critical size is thus $1/(\lambda \log(1/\lambda)) \simeq 1085$. We have drawn the approximate value (computed roughly just after Corollary 6) of $\log(\mathbb{P}(\#(C^\lambda(0)) = x))$ as a function of $\log(x)$, for $x = 1, ..., 54250$. We have made the curve continuous around $x = 1085$ (without justification). The curve is linear for $x = 1, ..., 1085$, and nonlinear for $x \geq 1085$.

Figure 2: Limit forest-fire process in a finite box



The filled zones represent zones in which $Z_t^A(x) = 1$ and $H_t^A(x) = 0$, that is macroscopic clusters. The plain vertical segments represent the sites where $H_t^A(x) > 0$. In the rest of the space, we always have $Z_t^A(x) < 1$. Until time 1, all the particles are microscopic. The 8 first marks of the Poisson measure fall in that zone. As a consequence, at each of these marks, the process H^A starts. Their life-time is equal to the instant where they have started (for example the segment above t_1, x_1 ends at time $2t_1$). At time 1, all the clusters where there has been no mark become macroscopic and merge together. But this is limited by vertical segments. Here we have at time 1 the clusters $[-A, x_6]$, $[x_6, x_4]$, $[x_4, x_8]$, $[x_8, x_5]$, $[x_5, x_7]$ and $[x_7, A]$. The segment above (t_4, x_4) ends at time $2t_4$, and thus at this time the clusters $[x_6, x_4]$ and $[x_4, x_8]$ merge into $[x_6, x_8]$. The 9-th mark falls in the (macroscopic) zone $[x_6, x_8]$, and thus destroys it immediately. This zone $[x_6, x_8]$ will become macroscopic again only at time $t_9 + 1$. Then a process H^A starts at x_{12} at time t_{12}. Since $Z^A_{t_{12}-}(x_{12}) = t_{12} - t_9$ (because $Z^A_{t_9}(x_{12})$ has been set to 0), the segment above (t_{12}, x_{12}) will end at time $2t_{12} - t_9$. On the other hand, the segment $[x_8, x_7]$ has been destroyed at time t_{10}, and thus will remain microscopic until $t_{10} + 1$. As a consequence, the only macroscopic clusters at time $t_9 + 1$ are $[-A, x_{12}]$, $[x_{12}, x_8]$ and $[x_7, A]$. Then the zone $[x_8, x_7]$ becomes macroscopic (but their has been marks at x_{13}, x_{14}), so that at time $t_{10} + 1$, we get the macroscopic clusters $[-A, x_{12}]$, $[x_{12}, x_{14}]$, $[x_{14}, x_{13}]$ and $[x_{13}, A]$. These clusters merge by pairs, at times $2t_{12} - t_9$, $2t_{13} - t_{10}$ and $2t_{14} - t_{10}$, so that we have an unique cluster $[-A, A]$ just before time t_{15}, where a mark falls and destroys the whole cluster $[-A, A]$.
With this realization, we have $0 \in (x_{11}, x_{15})$, and thus $Z_t^A(0) = t$ for $t \in [0, 1]$, $Z_t^A(0) = 1$ for $t \in [1, t_{10})$, then $Z_t^A(0) = t - t_{10}$ for $t \in [t_{10}, t_{10} + 1)$, then $Z_t^A(0) = 1$ for $t \in [t_{10} + 1, t_{15})$,... We also see that $D_t^A(0) = \{0\}$ for $t \in [0, 1)$, $D_t^A(0) = [x_8, x_5]$ for $t \in [1, 2t_5)$, $D_t^A(0) = [x_8, x_7]$ for $t \in [2t_5, t_{10})$, $D_t^A(0) = \{0\}$ for $t \in [t_{10}, t_{10}+1)$, $D_t^A(0) = [x_{12}, x_{14}]$ for $t \in [t_{10}+1, 2t_{12}-t_9)$, $D_t^A(0) = [-A, x_{14}]$ for $t \in [2t_{12}-t_9, 2t_{14}-t_{10})$, ... Of course, $H_t^A(0) = 0$ for all $t \geq 0$, but for example $H_t^A(x_{11}) = 0$ for $t \in [0, t_{11})$, $H_t^A(x_{11}) = 2t_{11} - t_{10} - t$ for $t \in [t_{11}, 2t_{11} - t_{10})$, and then $H_t^A(x_{11}) = 0$ for $t \in [2t_{11} - t_{10}, \infty)$.

Figure 3: The event Ω_a (proof of Theorem 3)



In hachured zones, we cannot say the values of the LFFP, because one would need to know what happens outside $[a, a+1]$.
Microscopic fires start at (T_1, X_1) and $(\tilde{T}_1, \tilde{X}_1)$. Hence at time S_1- the connected component $[X_1, \tilde{X}_1]$ is macroscopic, because $S_1 \geq 1$, and because during $[1, S_1)$, this component has not been subject to fires starting outside $[a, a+1]$: it is protected by X_1 and $\tilde{X}_1$ until time $2\min(T_1, \tilde{T}_1) \geq S_1$. As a consequence, the component $[X_1, \tilde{X}_1]$ is entirely killed by (S_1, Y_1). Then we iterate the arguments until we reach the final time T.
With such a configuration, there are always *microscopic* sites in $[a, a+1]$ during $[0, T]$. Indeed, during $[0, 1)$, all the sites are microscopic, during $[1, S_1)$, the sites X_1 and $\tilde{X}_1$ are microscopic, during $[S_1, S_1+1)$, all the sites in $[X_1, \tilde{X}_1]$ are microscopic, ...