UNE REMARQUE À PROPOS DES FLUCTUATIONS DES MOYENNES ERGODIQUES POUR DES FLOTS SYMBOLIQUES AUTO-INDUITS AVEC UNE VALEUR PROPRE 1.

XAVIER BRESSAUD, ALEXANDER BUFETOV, AND PASCAL HUBERT

Abstract.

1. Contexte

We have a substitution σ on an alphabet A, $|A| = d$. Let M be its matrix. We assume it is diagonalizable and it has an eigenvalue 1. We denote $(\lambda_1, \ldots, \lambda_d)$ and $(e_1, \ldots, e_d)$ the eigenvalues and (normalized) eigenvectors. We shall sometimes denote $\ell = e_1$ (because we think of the Lebesgue measure) and m for that associated to the eigenvalue 1. We construct partitions (tilings) of the real line. First, we have a partition made of intervals labelled by the alphabet and such that the length of the interval labelled a is $\ell(a)$. We assume they are ordered according to a fixed point of the substitution. Naturally, we can replace each of these intervals by a tiling in smaller (rescaled by λ_1^{-1}) labelled intervals, the interval labelled a being replaced by a sequence of intervals chosen according to $\sigma(a)$. We can repeat this operation. At each level we have a partition (Markov partition) of the real line. We call his order the number of time we did the operation. An interval of the level N partition is called a N-cylinder or a cylinder of order N. It has a labell. We sometimes denote $C^N(a)$ for such a cylinder (but of course it has many copies in the tiling). To make things non ambiguous, we also should choose an origin. I do not give more details since this is just a naive way to describe the unstable manifold in the symbolic flow. We should be able to make explicit the connection or to formulate all this in the more formal language of the symbolic flow; this language is recalled in appendix.

1991 *Mathematics Subject Classification.* Primary: 37B10; Secondary: 37E05.

Key words and phrases. substitutive systems, ergodic sums, interval exchange transformations, Markov approximation.

2. Integration par la distrbution qui nous interesse

Let m be the eigenvector corresponding to the eigenvalue 1 of M. For all $n \geq 1$ we define a piecewise density ρ_n measurable with respect to the Markov partition of order n of the line. We set $\rho_n(s) = \lambda_1^n m(a)$ if s belongs to a cylinder (of order n) labelled a. We denote m_n the "measure" $m_n(ds) = \rho_n(s)ds$. The goal is to compute

$$m_n(\varphi) = \int_{\mathbb{R}} \varphi(s)\rho_n(s)ds$$

for some smooth enough φ with compact support and try to give a meaning to the limit when n tends to ∞. Indeed, ρ_n is unbounded; furthermore, here, its "primitive" is not bounded. Although we will be able to integrate a reasonable class of function.

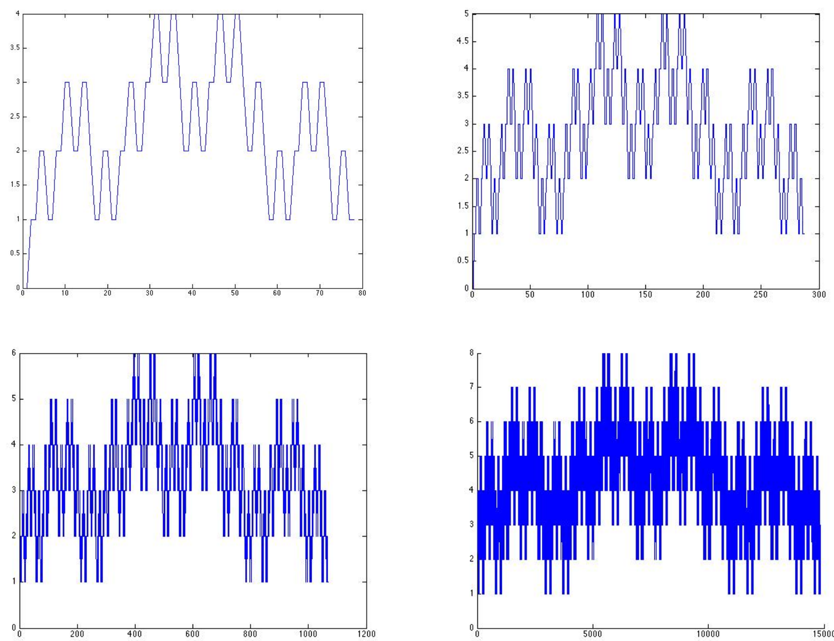

Figure 1. Here we consider the substitution $a \to ac, b \to acbbc, c \to acbc$. This is the stepped line made naively, after 3, 4, 5 and 7 iterations. It should give an idea of the shape and asymptotic behavior of r_n.

We define $r_n(s) = \int_0^s \rho_n(s)ds$ and $R_N(s) = \int_0^s r_n(s)ds$. Let the cylinder C^N have label a. We observe that $R_n^N(a) = \int_{C^N} r_n(s)ds - \int_{C^N} r_N(s)ds$ depends only on N, $n \geq N$ and on the label a of the N-cylinder C^N.

Indeed, the value of r_n at the begining of a cylinder C_N is constant for $n \geq N$. We denote R_n^N this vector. Immediate analysis of the self similar properties of r_n shows that it satisfies, for $n \geq N+1$:

$$R_n^N = MR_n^{N+1}, \tag{2.1}$$

(recall that each cylinder decomposes $C^N(a) = \cup_{b\in\sigma(a)} C^{N+1}(b)$), and, for all $n \geq N$

$$R_{n+1}^{N+1} = \lambda_1^{-1} R_n^N \tag{2.2}$$

Lemma 1.

$$R_{n+1}^N - R_n^N = \lambda_1^{-n} M^{n-N} \Delta$$

where, if $\sigma(a) = a_1 \cdots, a_K$*,*

$$\Delta(a) = \lambda^{-1}\left(\sum_{i=1}^{K}\sum_{j=1}^{i-1} m(a_j)\ell(a_i)\right) + \lambda_1^{-1}\sum_{i=1}^{K} m(a_i)\ell(a_i)/2$$

Proof. We claim that $R_1^0 = R_0^0 + \Delta$. We observe that, on C^0 with label a, r_0 is affine with slope $m(a)$. Assume that r_0 is 0 at the orgin of the cylider. Then, its integral on C^0 is $m(a)\ell(a)/2$. At the next step r_0 is replaced by a piecewise affine map r_1 with the same value at the origin (and at the end). To compute R_1^0, we decompose the cylinder $C^0(a)$ into cylinders of type $C^1(b)$ with $b \in \sigma(a)$. On such a cylinder $C^1(b)$, the integral is equal to the area of the rectangle of height the value of r_1 at the origin of the cylinder and length $\ell(b)$ + the area of the triangle $\lambda_1^{-1} m(b)\ell(b)$. The value of r_1 at the origin of the cylinder is given by the measure m of the union of cylinders whose label form the prefix of $\sigma(a)$ before the occurence of b, i.e. $\sum_{j=1}^{i-1} m(a_j)$. See Figure 2. This proves the claim.

To conclude we use repeatedly (2.1) and (2.2). □

Remark 2. It is important to notice that Δ may very well be non zero. For instance, I imagine we could prove a result like : given a substitution satisfying our assumptions, we can modify the order of the letters in the images of letters (leaving unchanged the matrix) in such way that Δ has all components strictly positive. It should be enough to put first the letters having the greater projection on the eigenspace.

We use the eigen decomposition of M :

$$M^n\Delta = \sum_{k=1}^{d} \lambda_k^n \Pi_k(\Delta).$$

where the Π_k are the projections on the eigendirections e_k ; write $\Pi_k(\Delta) = \pi_k(\Delta)e_k$.

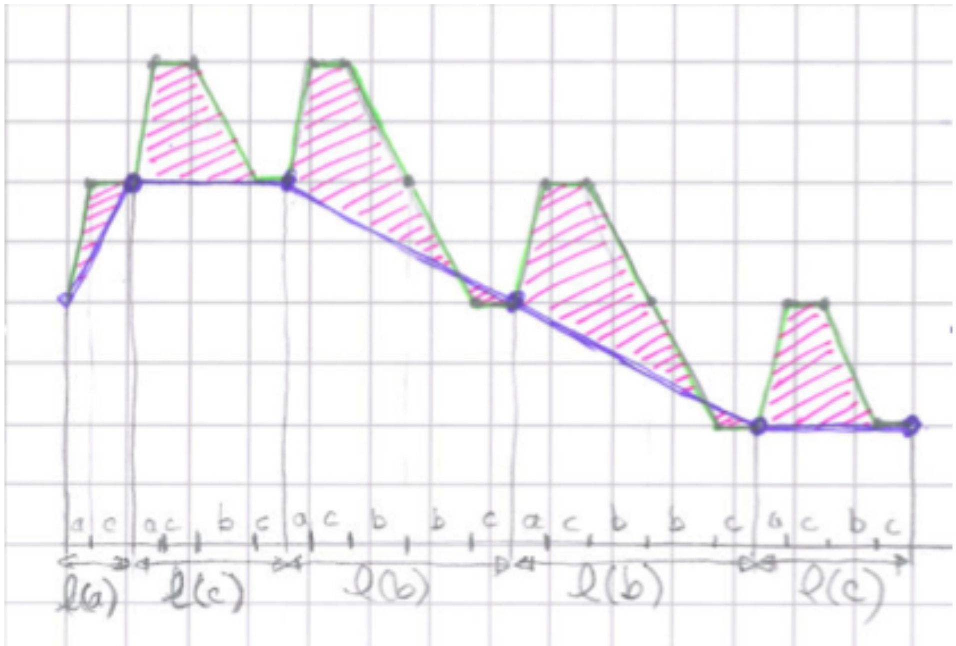

FIGURE 2. Here we consider the substitution $a \to ac, b \to acbbc, c \to acbc$. Its matrix is M. Main eigenvalue is $2+\sqrt{3}$ with eigenvector $(1, 2+\sqrt{3}, 1+\sqrt{3})$ and the eigenvalue 1 has eigenvector $(1,-1,0)$. The picture shows the difference between $R_1^1(b)$ and $R_2^1(b)$. Since a, b and c appear in $\sigma(b)$, one can also see $R_1^0(a)$, $R_1^0(b)$, and $R_1^0(c)$.

Lemma 3.

$$R_n^N = \left((n-N)\pi_1(\Delta) + m\ell + \sum_{k=2}^{d} \frac{1-\left(\lambda_k\lambda_1^{-1}\right)^{n+1}}{1-\lambda_k\lambda_1^{-1}} \pi_k(\Delta) \right) \lambda_1^{-N}$$

Proof. It follows from Lemma 1 that

$$
\begin{aligned}
R_n^N &= \sum_{p=N+1}^{n} \lambda_1^{-p} M^{p-N} \Delta \\
&= \sum_{p=N+1}^{n} \lambda_1^{-p} \sum_{k=1}^{d} \lambda_k^{p-N} \Pi_k(\Delta) \\
&= \lambda_1^{-N} \sum_{p=N+1}^{n} \Pi_1(\Delta) + \sum_{k=2}^{d} \lambda_k^{-N} \sum_{p=N+1}^{n} (\lambda_k \lambda_1^{-1})^{-p} \Pi_k(\Delta) \qquad (2.3)\\
&= \lambda_1^{-N} (n-N) \Pi_1(\Delta) + \sum_{k=2}^{d} \lambda_k^{-N} \sum_{p=N+1}^{n} (\lambda_k \lambda_1^{-1})^{-p} \Pi_k(\Delta) \\
&= \left((n-N)\Pi_1(\Delta) + \sum_{k=2}^{d} \frac{1-\left(\lambda_k\lambda_1^{-1}\right)^{n+1}}{1-\lambda_k\lambda_1^{-1}} \Pi_k(\Delta) \right) \lambda_1^{-N}
\end{aligned}
$$

□

Next, we try to integrate with respect to m_n a *chapeau* map which is piecewise affine. It appears that the main term disappears due to a compensation between increasing and decreasing pieces. ($\int_{\mathbb{R}} \varphi' = 0$) and we can consider the limit.

Proposition 4. *Let* $\varphi(s) = \sum_{\ell=1}^{L} \mathbb{1}_{C_\ell}(s)(\alpha_\ell s + \beta_\ell)$ *be a continuous, piecewise affine, map with constant slope on cylinders of order* N *and compact support. Then, if for all* $1 \le \ell \le L$, a_ℓ *is the label of cylinder* C^ℓ*,*

$$\lim_{n\to\infty} \int \varphi \rho_n = \lambda_1^{-N} \sum_{\ell=1}^{L} \alpha_\ell R_\infty(a_\ell),$$

where

$$R_\infty = m\ell + \sum_{k=2}^{d} \frac{\pi_k(\Delta)}{1-\lambda_k\lambda_1^{-1}} e_k.$$

Proof. We integrate by parts :

$$\int_{\mathbb{R}} \varphi(s)\rho_n(s)ds = \sum_{\ell=1}^{L} \alpha_\ell \int_{C_\ell} r_n(s)ds.$$

Hence,

$$m_n^N(\varphi) := R_n^N(\varphi') := \int_{\mathbb{R}} \varphi(s)(\rho_n(s) - \rho_N(s))ds = \sum_{\ell=1}^{L} \alpha_\ell R_n^N(a_\ell)$$

So,

$$m_n^N(\varphi) = \sum_{\ell=1}^{L} \alpha_\ell \left((n-N)\Pi_1(\Delta)(a_\ell) + \sum_{k=2}^{d} \frac{1-\left(\lambda_k\lambda_1^{-1}\right)^{n+1}}{1-\lambda_k\lambda_1^{-1}} \Pi_k(\Delta)(a_\ell) \right) \lambda_1^{-N}$$

Recalling that $\pi_1(\Delta)$ is proportional to $\ell = e_1$, we compute

$$\begin{aligned}
\sum_{\ell=1}^{L} \alpha_\ell (n-N)\Pi_1(\Delta)(a_\ell)\lambda_1^{-N} &= (n-N)\pi_1(\Delta)\left(\sum_{\ell=1}^{L} \alpha_\ell \ell(a_\ell)\lambda_1^{-N}\right) \\
&= (n-N)\pi_1(\Delta)\int_{\mathbb{R}} \varphi'(s)ds \\
&= 0
\end{aligned}$$

Hence,

$$m_n^N(\varphi) = \sum_{\ell=1}^{L} \alpha_\ell \sum_{k=2}^{d} \frac{1-\left(\lambda_k\lambda_1^{-1}\right)^{n+1}}{1-\lambda_k\lambda_1^{-1}} \Pi_k(\Delta)(a_\ell)\lambda_1^{-N}$$

or,

$$m_n^N(\varphi) = \lambda_1^{-N} \sum_{k=2}^{d} \frac{1-\left(\lambda_k\lambda_1^{-1}\right)^{n+1}}{1-\lambda_k\lambda_1^{-1}} \pi_k(\Delta) \sum_{\ell=1}^{L} \alpha_\ell e_k(a_\ell) \tag{2.4}$$

Now, letting n tend to infinity,

$$\lim_{n\to\infty} m_n^N(\varphi) = \lambda_1^{-N} \sum_{k=2}^{d} \frac{\pi_k(\Delta)}{1-\lambda_k\lambda_1^{-1}} \sum_{\ell=1}^{L} \alpha_\ell e_k(a_\ell)$$

or

$$\lim_{n\to\infty} m_n(\varphi) = \lambda_1^{-N}(\sum_{\ell=1}^{L} \alpha_\ell(m(a_\ell)\ell(a_\ell) + \sum_{k=2}^{d} \frac{\pi_k(\Delta)}{1-\lambda_k\lambda_1^{-1}} e_k(a_\ell))$$

If we set,

$$R_\infty = m\ell + \sum_{k=2}^{d} \frac{\pi_k(\Delta)}{1-\lambda_k\lambda_1^{-1}} e_k$$

then,

$$\lim_{n\to\infty} \int \varphi\rho_n = \lambda_1^{-N} \sum_{\ell=1}^{L} \alpha_\ell R_\infty(a_\ell)$$

□

Lemma 5. *Let φ be C^2 with compact support. Then $\lim_{n\to\infty}\int_{\mathbb{R}} \varphi(s)\rho_n(s)ds$ exists.*

Proof. Let us write $\varphi'(s) = \sum_{N=1}^{\infty} \alpha^{(N)}(s)$ where $\alpha^{(n)}$ is measurable with respect to the Markov partition of order N. It should be clear that we can force $||\alpha^{(N)}||_\infty \leq ||\varphi''||_\infty \lambda_1^{-N}$ and $\int_{\mathbb{R}} \alpha^{(N)} ds = 0$. We integrate by parts and decompose along the cylinders of order N for each N:

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(s)\rho_n(s)ds &= \int_{\mathbb{R}} \varphi'(s) r_n(s)ds \\
&= \sum_{N=1}^{\infty} \int_{\mathbb{R}} \alpha^{(N)}(s) r_n(s)ds \\
&= \sum_{N=1}^{\infty} \sum_{\ell=1}^{L_N} \int_{C_\ell^N} \alpha^{(N)}(s) r_n(s)ds \\
&= \sum_{N=1}^{\infty} \sum_{\ell=1}^{L_N} \alpha_\ell^{(N)} \int_{C_\ell^N} r_n(s)ds
\end{aligned}$$

Using a uniform convergence argument, we can interchange the limits and use Proposition 4 in:

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(s)\rho_n(s)ds &= \lim_{n\to\infty} \sum_{N=1}^{\infty} \int_{\mathbb{R}} \alpha^{(N)}(s) r_n(s)ds \\
&= \sum_{N=1}^{\infty} R_\infty(\alpha^{(N)}) \\
&= \sum_{N=1}^{\infty} \lambda_1^{-N+1} \sum_{\ell=1}^{L_N} \alpha_\ell^{(N)} R_\infty(a_\ell^N)
\end{aligned}$$

with

$$\begin{aligned}
|\sum_{\ell=1}^{L_N} \alpha_\ell^{(N)} \int_{C_\ell^N} r_n(s)ds| &\leq \lambda_1^{-N} L_N ||\alpha^{(N)}||_\infty ||R_\infty||_1 \\
&\leq \lambda_1^{-N} \lambda_1^N |\mathrm{supp}(\varphi)| \lambda_1^{-N} ||\varphi''|| ||R_\infty||_1 \\
&\leq \lambda_1^{-N} |\mathrm{supp}(\varphi)| ||\varphi''|| ||R_\infty||_1
\end{aligned}$$

$\square$

3. Appendix

Let $\mathcal{A}$ be a finite alphabet, let σ be a morphism of the free monoid $\mathcal{A}^*$ and let M_σ denote its abelianization matrix. It is determined by the images of letters in $\mathcal{A}$. If no letter have empty image, then the action of σ can be naturally extended to the (full) shift $\mathcal{A}^{\mathbb{Z}}$. If the matrix M_σ is primitive (we also say that the substitution is primitive), then we denote $X_\sigma \subseteq \mathcal{A}^{\mathbb{Z}}$ the smallest σ-invariant subshift. We study here the asymptotic behavior of ergodic sums for the minimal dynamical system (X_σ, T) in the (non-hyperbolic) case when the matrix M_σ has an eigenvalue of modulus one (here T denotes the shift map). As we shall see later these objects are to be related with holonomy flows of Anosov and pseudo Anosov maps.

3.1. **Vershik automorphisms and suspension flows.** We refer to Section 2, 3, 5 of [**?**] for details. Given an oriented graph Γ with m vertices, let $\mathcal{E}(\Gamma)$ be the set of edges of Γ. For $e \in \mathcal{E}(\Gamma)$ we denote $I(e)$ its initial vertex and $F(e)$ its terminal vertex. To the graph Γ we assign a non-negative $m \times m$ non-negative matrix $A(\Gamma)$ by the formula

$$A = A(\Gamma)_{i,j} = \sharp\{e \in \mathcal{E}(\Gamma) \ : \ I(e) = i, F(e) = j\}$$

We assume that A is a primitive matrix.
We define the Markov compactum:

$$Y = \{y = y_1 \dots y_n \dots : y_n \in \mathcal{E}(\Gamma), F(y_{n+1}) = I(y_n)\}$$

The shift on Y is denoted by $\mathfrak{S}$. Assume that there is an order on the set of edges starting from a given vertex. This partial order extends to a partial oder on Y: we write $y < y'$ if there exists $l \in \mathbb{N}$ such that $y_l < y'_l$ and $y_n = y'_n$ for $n > l$. The Vershik automorphism T^Y is the map from Y to itself defined by

$$T^Y y = \min_{y' > y} y'.$$

As A is primitive, there is a unique probability measure invariant under T^Y denoted by μ_Y.
Generalizing classical ideas for IET and translation surfaces, a suspension flow over (Y, T^Y) is defined as follow: Let H be the Perron-Frobenius eigenvector of A normalized in a suitable way. h_t is the special flow over (Y, T^Y) with roof $\tau(y) = h_{I(y_1)}$. The phase space of the flow is

$$Y(\tau) = \{(y, t) : y \in Y, 0 \le t < \tau(y)\}.$$

The measure μ_Y induces a probability measure ν_Γ on Y_τ.

For each $e \in \mathcal{E}(\Gamma)$, the set

$$\{(y,t) : y \in Y, y_1 = e,\ 0 \leq t < h_{I(y_1)}\}$$

is called a *rectangle* in the sequel.
The space $X = \{x = \dots x_{-n} \dots x_n \cdots : x_n \in \mathcal{E}(\Gamma), F(x_{n+1}) = I(x_n)\}$ is the natural extension of (Y, S). X and $Y(\tau)$ are canonically isomorphic as measurable spaces. Thus, the flow h_t can be defined on X and satisfies the important relation:

$$\mathfrak{S} \circ h_t = h_{\exp(\theta_1 t)} \circ \mathfrak{S} \tag{3.1}$$

where $\exp(\theta_1)$ is the Perron-Frobenius eigenvalue of A.
We now connect these notations with the language of substitution dynamical system. Consider the alphabet $\mathcal{A} = \{1, \dots, m\}$ as the set of vertices of Γ. For all $a \in \mathcal{A}$, we denote $\sigma(a)$ the sequence $\{F(e) : I(e) = a\}$ ordered with the partial order on $\{e : I(e) = a\}$. The dynamical system (X_σ, T) is a topological factor of the Vershik automorphism (Y, T^Y). The semi-conjugacy is given by the prefix-suffix decomposition. Almost every $u \in X_\sigma$ can be written in the form

$$u = \cdots \sigma^n(P_n) \cdots \sigma(P_1) P_0 . a_0 S_0 \sigma(P_1) \cdots \sigma^n(P_n) \cdots ,$$

where, for all $n \in \mathbb{N}$, $\sigma(a_{n+1}) = P_n a_n S_n$. Thus, it is clear that such point correspond to the unique path in Y such that, for all $n \geq 0$, $a_n = F(y_{n+1})$ (and $= I(y_n)$ for $n > 0$) and y_{n+1} has exactly $|P_n|$ predecessors in the partial order around $I(y_{n+1})$. We recall that the semi-conjugacy is not always a topological conjugacy because there may be multiple writings but it is a measurable conjugacy.
When σ has constant length, the roof function of the suspension flow is constant because ${}^t(1, \dots, 1)$ is an eigenvector for A.
We observe that roughly $\sigma \approx \mathfrak{S}^{-1}$ and that the matrix M_σ of the substitution is $M_\sigma = A^T$.

3.2. **Ergodic means.** Let $\varphi \in C_0^\infty(\mathbb{R})$. We are interested in computing the limit in distribution when n tends to infinity of

$$\int_{\mathbb{R}} \varphi(s \exp(-\theta_1 n)) f \circ h_s(x) ds,$$

where x is chosen according to the invariant measure ν_Γ.
The first observation is that for all $n \in \mathbb{Z}$, the law of x is the same as the law of $\mathfrak{S}^n(x)$ by $\mathfrak{S}$-invariance of the measure ν_Γ. Hence in law this sequence of random variables is the same as

$$\int_{\mathbb{R}} \varphi(s \exp(-\theta_1 n)) f \circ h_s(\mathfrak{S}^{-n}(x)) ds.$$

In view of (3.1), [A corriger]

$$\int_{\mathbb{R}} \varphi(s) f \circ h_s(\mathfrak{S}^{-n}(x)) ds.$$

We choose a function f constant on rectangles (defined in Subsection 3.1). We are interested in the case when this function "corresponds" to the coordinates of an eigenvector associated with eigenvalue 1, i.e. an invariant vector (ie $f(\sigma(a)) = f(a)$ for every letter a).

Université Paul Sabatier, Institut de Mathématiques de Toulouse, 118 route de Narbonne, F-31062 Toulouse Cedex 9, France
E-mail address: bressaud@math.univ-toulouse.fr

Rice University, Houston, Texas
E-mail address: aib1@rice.edu

Laboratoire Analyse, Topologie et Probabilités, Case cour A, Faculté des Sciences de Saint-Jerôme, Avenue Escadrille Normandie-Niemen, 13397 Marseille Cedex 20, France.
E-mail address: hubert@cmi.univ-mrs.fr