# Tribonacci aperiodic tiling as a SFT

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**Abstract**

We give an aperiodic set of tiles for which the tilings of the plane project onto the Tribonacci substitutive aperiodic tiling.

## 1 Introduction

Let $A$ be a finite alphabet and denote $A^{\mathbb{Z}}$ the set of infinite words endowed with the shift map. Classification of subshifts (i.e. closed shift-invariant subsets of $A^{\mathbb{Z}}$ is one of the objects of symbolic dynamics. Complexity of subshifts varies from very low (shifts of a given periodic orbit : a finite number of points) to very high (the full shift : entropy $\log |A|$). Classical intermediate classes are subshift of finite type : subshift made of infinite words avoiding a finite given list of finite factors. When they are not empty nor reduced to periodics orbits, they have high complexity (positive entropy ; contain a lot of periodic orbits) and some "mixing" properties. On the opposite side are substitutive dynamical systems : closure of the orbit of fixed point of a morphism of $A^*$ (without cancellations) ; they usally have low complexity (Pansiot) and under natural conditions linear complexity (which is the lowest possible after finite) and minimality (hence no periodic orbits). [Sofic is a factor of a SFT]

When we turn to tilings the situation is very different. A tiling of $\mathbb{Z}^2$ is an element of $A^{\mathbb{Z}^2}$. The group $\mathbb{Z}^2$ naturally acts by translation. The analog of a subshift is a closed subset of $A^{\mathbb{Z}^2}$ invariant by this action. For coloring of $\mathbb{Z}^2$ analog of SFT is Wang tilings. It is well known that SFT in higher dimension have a very different behaviour. For instance Berger (and Robinson) have exhibited a set of Wang tiles that tiles the plane but has no periodic orbit. The Robinson aperiodic set of tiles yields a somehow self-similar tiling. It appeared later (Moses) that taking a natural definition of a substitution in $A^{\mathbb{Z}^2}$, (a generalized substitution maps a letter to a squared patch, seen as bidimensional word) that indeed all substitutive tilings can be seen as factors of SFT.

Tilings are (also) geometric objects. Action of $\mathbb{R}^2$ instead of $\mathbb{Z}^2$. A set of geometrical tiles may tile the plane or not. We can consider the set of tilings associated to a set of tiles. Observe that a set of wang tiles

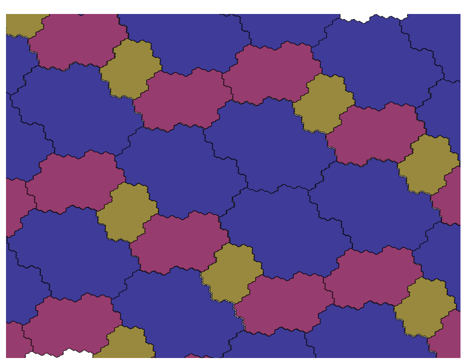

Figure 1: A patch of the renormalised tiling $\tau^{(\infty,\infty)}$

can be seen as a geometrical problem. We can define the notion of (strictly) self-similar tiling of ratio $\mu$. Let $\{T_1, \dots, T_n\}$ be a set of tiles such that for all $T \in \{T_1, \dots, T_n\}$, there is a subsdivision $(S_1, \dots, S_k)$ of $T$ such that for all $i$ there is $j$, $S_i \simeq \mu T_j$. [ou alors : there is a tiling of $\mu^{-1}T$ by $\{T_1, \dots, T_n\}$]. Given such set of tile it is (under mild conditions : polygonal, ...) possible to define a tiling of the plane by inflation and subsdivisions. Goodman-Strauss proved that these tilings are sofic (factor of SFT).
We also mention that approximate version of self similarity exist (Solomyak) and that those quasi-selfsimilar are indeed equivalent to strictly selfsimilar but with possibly fractal boundaries.

The question was whether the so called Tribonacci tiling is sofic or not. Goodman Strauss result does not apply directly since the tiling is not (strictly) selfsimilar. Using Solomyak result we see that the tiling is equivalent to a strictly selfsimilar tiling. But tiles are not polygonal and again this seems to be an obstruction to applying directly the GS result. Another natural idea is to use the presentation of this tiling as a tiling of $\mathbb{Z}^2$, fixed point of a generalized substitution — as defined by Berthe et al. But the substitution the notion of generalized substitution is wider than that used by Moses and the result cannot be used.
Althought ideas developped by Moses and Goodman-Strauss are enough to prove the result and the object of this work is to check the way they must be adapted to this case, and to give an explicit description of a SFT that projects onto the Tribonacci.

The so-called Tribonacci aperiodic tiling appears under various presentations in the litterature. Since this object has a lot of symmetries some definitions may yield slightly different versions of essentially the same object. We use a presentation that makes easy the proof of various geometric statements needed along the study. Namely we define recursively a sequence of compact polygonal domains of the plane — whose limit, if properly renormalized , is the so-called Rauzy fractal. Basically, we start with three tiles $T^{(1)}, T^{(2)}$ and $T^{(3)}$ and set for all $n \geq 3$ :

$$T^{(n+1)} = T^{(n)} \cup \left(T^{(n-1)} + \omega^{(n)}\right) \cup \left(T^{(n-2)} + \omega^{(n)}\right), \qquad (1)$$

where $\omega^{(n)}$ is chosen in such way that $\tau^{(n+1,n-2)} = \{T^{(n)}, T^{(n-1)} + \omega^{(n)}, T^{(n-2)} + \omega^{(n)}\}$ is a tiling of $T^{(n+1)}$. A detailled study of this recursion shows that $\mu^{-n}T^{(n)}$ converges to the Rauzy fractal ; hence in the limit $T^{(n)}$ fills the plane. This study also yields information about the structure of the boundaries of tiles $T^{(n)}$ and combinatorial information on how tiles glue tohgether.
For each $n \geq 3$, we have defined $\tau^{(n+1,n-2)}$. We define recursively $\tau^{(n+1,k)}$ for all $0 \leq k \leq n-3$ by replacing in $\tau^{(n+1,k+1)}$ all occurences of $T^{(k+3)}$ by $\tau^{(k+3,k)}$ (suitably translated). We call Tribonacci tiling of $T^{(n)}$ the tiling $\tau^{(n,1)}$. We observe that the restriction of $\tau^{(n+1,k)}$ to $T^{(n)}$ is exactly $\tau^{(n,k)}$. So that it makes sense to define $\tau^{(\infty,k)}$ as the "limit" of $\tau^{(n,k)}$ when $n \to \infty$. We define the Tribonacci aperiodic tiling space $\mathcal{S}_k$ as the closure of the orbit of $\tau^{(\infty,k)}$ under the action of $\mathbb{R}^2$ by translations.

An important observation is that the combinatorial structure of the tiling is somehow "stable" for $k \geq 5$ ; for instance when $k \geq 5$, multiple points are at most triple points. For this reason we will firstly prove the result for $\tau^{(\infty,5)}$ ; it is then easy to pull it back to $\tau^{(\infty,1)}$ ; this is the object of Section **??**.
We have to understand and describe the hierarchical structure of the tiling.

**Remark 1.** *The symmetries of this particular tiling maybe somehow confusing : two tiles with the same shape may play different roles in the hierarchical structure. So we have to be careful. On the other hand, we take advantage of this symmetry to limit the number of combinatorial situation to analyse and the number of tiles in the SFT.*

Let $k \geq 6$. A tile $T \in \tau^{(\infty,k)}$ divides into three tiles, a small one, a medium one and a big one ( $T^P$, $T^M$ and $T^G$), called its subtiles or children tiles. To stress the hierarchical structure of the tiling, we are going to "draw" a tree in the tiling. We observe that $L_P = T^P \cap T^M$, $L_M = T^M \cap T^G$ and $L_G = T^G \cap T^P$ are three polygonal lines starting at the triple point $v(T) = T^P \cap T^M \cap T^G$ and ending on the boundary of the (meta)tile $T$ respectively at points $v_P(T) = L_P \cap \partial T$, $v_M(T)$ and $v_G(T)$. In this picture, we call $v$ a central vertex and $v_P$, $v_M$, $v_G$ mesovertices. The tripod $L_P \cup L_M \cup L_G$ is called the skeleton of $T$. More specifically, we denote $K^\delta(T) = L_P^\delta(T) \cup L_M(T) \cup L_G^\delta(T)$, where $L_G^\delta = L_G \setminus B_\delta(v_P)$ for some fixed $\delta > 0$ small. The tile $T$ is itself contained in a higher scale metatile $\widetilde{T} = \mathcal{M}(T)$. We observe that, whatever the role of $T$ in $\widetilde{T}$, we have $K^\delta(T) \cap K^\delta(\widetilde{T}) = v_M(T)$ [checkpoint]. It follows that the union of the skeletons of all tiles of all scales is a tree ; reflecting the hierarchical structure of the tiling. The tree does cover the boundaries of all tiles in the tiling (formally if $\delta \to 0$ ; but in the limit it is not a tree). Our proof is based on this picture.

Triple points in the tiling are either central vertices or mesovertices, at least for one (in facts, two) tile(s) of a given scale. It appears (it was not obvious a priori and that is a difference with the polygonal strictly selfsimilar case) that the positions of these triple points on the boundaries of higher scale tiles behave nicely. The hint is the tiling by Rauzy fractals which shows that the triple points can be only at a finite number of positions on the boundaries of the tiles. More specifically, it appears that it is possible to divide the boundaries of the tiles $T^{(n)}$ into seven pieces in such way that central and meso vertices of each scale fall on one of the seven endpoints of these pieces. To prove this fact we provide a recursive definition of the boundaries. This analysis yields properties of the tiles themselves and an exact description of the geometry of their boundaries.

Once we know where the triple point can be on a tile, we try to relate the types of the candidates with the hierarchical situation of the tile in the tiling. Namely, on each tile we have seven "distinguished" vertices. We decide to assign different "states" to each of these vertices with respect to the local configuration of tiles around this vertex and more specifically with respect to the way they behave for the hierarchical structure of the tiling. Let $x$ be a point in the plane corresponding to a distinguished vertex of a tile $T$. If $x$ is not a triple point (only double) we give it state 0 (or empty) on both tile. Otherwise we call $R$ and $S$ the three tiles containing $x$ and we let $K$ be the first scale for which $x$ is not (any longer) a triple point of $\tau^{(\infty,K)}$. The point $x$ is central if at scale $K$ it is in the interior of a tile and meso if it is on the boundary. If it is central then we assign value 2 to the three distinguished vertices of $R$, $S$ and $T$ falling on $x$. Otherwise we assign 0, $-1$ and 1 to the three vertices in the order determined by the trigonometric sens (around $x$) and in such way that the tiles with $\pm 1$ belong to the same tile of scale $K$. Each tile in the tiling is so assigned a sequence of seven states, one per vertex, called vertex configuration. .

It appears that the role played by a tile is encapsulated in the vertex configuration. But there is also some information about the roles the metatiles to which it belongs may be ; this information is encoded in a way we will not analyse specifically (rather sophisticated combinatorics). But a cautious analysis of the way a tile gets cut (and the recursion on the boundary) shows that only some vertex configurations can arise. We denote $\mathcal{V}$ their set ; it is subdivided into $\mathcal{V}_P$, $\mathcal{V}_M$ and $\mathcal{V}_G$ according to the role the tile can play.

From this point let us do a first attempt to produce a SFT. Assume we have a set of tiles with the good shapes (start with $T^{(5)}$, $T^{(6)}$ and $T^{(7)}$). Consider labelled tiles $(T, V)$ i.e. versions of all these tiles with one of the possible vertex configuration. Let us impose local matching rules that force central vertices to get glued together and meso vertices together (with a $-1$ a 1 and a 0. It appears (see Lemma **??**) that small tiles would have to get glued with a medium and a large tile in order to form a meta tile. That is a first step but it is not enough. Indeed, we have no control about the 7 vertices of the so built metatile. In some sense we need to transmit the constraint to the next scale. For this aim we are going to use the "tree" structure.
We come back to the Tribonacci tiling. Let us put some informations on the edges of the tree. Basically each edge belongs to a piece of skeleton of one tile $T$ (of one scale). We put two informations : type of the branch (P, M, G) and vertex configuration of the tile $T$ (i.e. an element of $S \times \mathcal{V}$). Since each skeleton is connected (through the checkpoint) to the skeleton of its meta tile we can ask a "coherence condition". At checkpoints one of the incoming edges is the $M$ edge of the skeleton of a tile $T$ that plays the role $Q$ (=P,M,G) in its metatile $\widetilde{T}$. The other edges are a branch of the skeleton of $\widetilde{T}$. We observe that $T$ and $\widetilde{T}$ have distinguished vertices in common (which ones depend only on the role of $T$ in $\widetilde{T}$) ; they are given the same state. Hence we can ask that the vertex configuration of $T$ is coherent with the vertex configuration of $\widetilde{T}$. If this is satisfied, and the decoration of the skeleton of $\widetilde{T}$ belongs to $\mathcal{V}$ this imposes a "correlation" between the vertex configurations of the three subtiles of $\widetilde{T}$.

We are now in position to define a set of decorated tiles that is a good candidate. We consider the set of labelled tiles $(T, V) \in \widetilde{\mathcal{T}}_5$, i.e. with $T$ of shape $T^{(5)}$, $T^{(6)}$ or $T^{(7)}$ and $V \in \mathcal{V}$. Then we choose colors on the seven edges (between distinguished vertices) among the couples in $S \times \mathcal{V}$. Further more we ask the color on the edge containing the checkpoint to be coherent with the vertex configuration $V$ of the tile $T$

(in the sense above). This set of decorated tiles is called $\mathcal{D}_5$. We also fix local matching rules : identity of the colors of two edges that get glued, continuity of the colors at distinguished points which are not triple points, compatibility of the states of vertices at triple points (central and meso) and coherence of the colors of incoming edges at a triple point.
Now, when we glue tiles to form a metatile (as in our first attempt above), the coherence constraint (at the three checkpoints) implies that the metatile has a vertex configuration in $\mathcal{V}$. Moreover the condition at the checkpoint of the metatile implies that the new tile itself satisfies the coherence constraint. It follows that starting with a tiling by tiles in $\mathcal{D}_5$ satisfying the local matching rules we can construct a new tiling by glueing all the occurences of small tiles with two neighbours ; the tiling we obtain is a tiling by $T^{(6)}$, $T^{(7)}$ and $T^{(8)}$ coming with a natural labelling ; the new set of decorated tiles we can call $\mathcal{D}_6$ has the same features and the procedure can be applied recursively.

**Remark 2.** *If instead of working with polygonal tiles (these have different shapes at each scale) we were working directly with fractal self similar tiles, the tiling obtained after glueing would be up to a similarity in the same space.*

It follows that all tiles of our tiling of finite type is indeed included in a tile $T^{(n)}$ itself tiled (forgetting the decorations) as is the Tribonacci tiling. To conclude it remains to check that this statement holds true not only for all tiles, but for all patches. It is rather more tricky : indeed it could happen that two tiles belong to different metatiles at all scales. This is a standard difficulty ; it appears here that this phenomenon can occur but that still the patch is a patch of the Tribonacci tiling ; but the proof is slightly less direct and relies on somehow specific properties of the Tribonacci tiling.

Before to get into the technical details we stress the fact that our proof relies on the specific properties of the particular tiling we study ; however the ideas and the adaptation of the Goodman-Strauss/Moses results to this class of tilings seems to remain possible for a rather large class of combinatorial tilings.

# 2 Preliminaries

## 2.1 Tilings

A tile is a domain of the plane homeomorphic to a (closed) disk. We consider a finite set $\mathcal{T}$ of tiles. The group $\mathbb{R}^2$ naturally acts on $\mathcal{T}$ by translations. A tiling of (a part of) the plane $\Delta$ by such a set $\mathcal{T}$ is a *mod* 0 partition of the plane whose atoms are translations of the tiles. More formally, a tiling is determined by a list $\{T_n, n \in I\}$ (where $I$ is either finite either $\mathbb{N}$) of tiles such that
(i) for all $n \in I$, there is $v \in \mathbb{R}^2$ such that $v + T_n \in \mathcal{T}$,
(ii) for all $m, n \in I$, $T_m \cap T_n = \partial T_m \cap \partial T_n$,
(iii) $\cup_{n\in I} T_n = P$.
The group $\mathbb{R}^2$ also acts by translations on the set of tilings. We shall give labels to each tile in $\mathcal{T}$ and then consider labelled tilings (just keeping track of the label in the case two tiles are equivalent). In view of our defintion, the position of tiles in $\mathcal{T}$ in the plane is not crucial. We call *position* of a tile $T$ in a tiling $\tau$ by $\mathcal{T}$ the unique $v \in \mathbb{R}^2$ such that $v + T \in \mathcal{T}$ and *shape* of $T$ its class in $\mathcal{T}/\mathbb{R}^2$. For two tiles $T$ and $T'$, we will denote $T \simeq T'$ if there is $v \in \mathbb{R}^2$ such that $T' = T + v$. We may often use the term "tile" for "tile up to a translation".

**Basic operations on tilings** . Let $\tau = \{T_n, n \in I\}$ be a tiling of $\Delta$ and for all $n$ let $\eta_n = \{S_k^n, k \in I_n\}$ be a tiling of $T_n$. We consider the tiling $\tau^\eta = \{S_k^n, n \in I, k \in I_n\}$. Let $\tau = \{T_n, n \in I\}$ be a tiling of $\Delta$ and for all $T \in \mathcal{T}$ let $\eta_T = \{S_k^T, k \in I_T\}$ be a tiling of $T$. For all tiles $T_n$ in $\tau$, we denote $v_n$ its position, i.e. the vector $v$ such that $T = v + T_n \in \mathcal{T}$ and $\eta_n = \{v_n + S_k^T, k \in I_T\}$. We call the tiling $\tau^\eta = \{S_k^n, n \in I, k \in I_n\}$ refinement of $\tau$ by the tiling of the tiles $\eta_T, T \in \mathcal{T}$. Conversely let $\tau$ be a tiling and consider a list $L = \{T_i^k\}$ of distinct tiles of $\tau$ such that for all $i$ the finite unions $\cup_k T_i^k$ are equivalent (or finite number of classes). We denote $T^L = (\tau \setminus L) \cup \{\cup_k T_i^k\}$ the tiling of $\Delta$ with tiles including the unions.

**Polygonal tiles** We will mainly deal with polygonal tiles. A polygonal tile is a tile whose boundary is a stepped line homeomorphic to a circle. The boundaries of our polygonal tiles will be made of edges chosen in a finite alphabet of vectors : $\mathcal{A} = \{\pm e_a, \pm e_b, \pm e_c\}$ and hence easy to describe by a (reduced) word corresponding to an element of $F(a, b, c)$ the free group with three generators. We can think of $e_a, e_b$ and $e_c$ as the projections on the plane of a canonical basis of $\mathbb{R}^3$. Given a (reduced) word $W$ on $\{a, a^{-1}, b, b^{-1}, c, c^{-1}\}^*$ we write $e_W$ or $e(W)$ for the vector $e(W) = \sum_{i=1}^{|W|} e_{W_i}$ and $L(W)$ for the stepped line $(0, e_{W_1}, e_{W_1W_2}, \ldots, e_{W_1\cdots W_k}, e_W)$. Such stepped line is based at 0. When a stepped line is a simple closed curve, we can define an orientation ; we will describe its boundary by the word corresponding to turning around its interior in the trigonometric sense. The translated stepped line $x + L(W)$ is based at $x \in \mathbb{R}^2$. A polygonal tile is determined by a word $W$ with $e_W = 0$ and a position for the origin.

**Names** Two tiles in a tiling $\tau$ are *adjacent* if their intersection is nonempty and not reduced to one point. In the tiling we will consider this intersection will then be a stepped line. If $T \cap S = x + L(W)$, we say that $T$ an $S$ glue along edge $x + L(W)$, itself called *glueing edge* ; the word $W$ is called *glueing word.* There are multiple points in a tiling. We will more often have to deal with triple points. In this case we always describe the three tiles containing the triple point turning in the trigonometric sense around the triple point. We denote $(a, b, c) \equiv (e, f, g)$ if there is a cycle $s \in \wp_3$ such that $s(a, b, c) = (e, f, g)$.

**Combinatorial structure** Combinatorial structure of a tiling : assume our tiles are marked by points on their boundaries and assume that the intersections are along edges ; more : point to point. Then we consider the graph whose vertices are these vertices and edges are pieces of boundaries (belonging to two tiles) between vertices. If the tiling is easy, then the degree of each vertex is 2 or 3. We can also consider the dual graph. We also should give names to the cells of this planar graph.

## 2.2 Discrete planes

Consider a plane $\Pi$ in $\mathbb{R}^3$ containing the origin. Consider the subset $\mathcal{Z}_\Pi$ of $\mathbb{Z}^3$ defined as

$$\mathcal{Z}_\Pi = \{z \in \mathbb{Z}^3; z + [0,1]^3 \cap \Pi \neq \emptyset\}.$$

The boundaries of the volume $\cup_{z \in \mathcal{Z}_\Pi} z + [0,1]^3$ is made of two "stepped surface", made of squares $\{i\} \times [0,1] \times [0,1]$, $[0,1] \times \{i\} \times [0,1]$ and $[0,1] \times [0,1] \times \{i\}$. Observe that the projections of these squares on the plane $\Pi$ yield a tiling of $\Pi$ by rhombus.

We are specially interested in the example where $\Pi$ is the contracting plane of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

i.e. the plane of equation $\lambda^2 x + \lambda y + z = 0$ where $\lambda$ is the real root of $X^3 = X^2 + X + 1$. We denote $\mu$ (and $\overline{\mu}$) its complex roots and observe that the restriction of $A$ on $\Pi$ is conjugate to a similarity of ratio $\mu$. We denote $\{e_a, e_b, e_c\}$ the projections of the canonical vectors of $\mathbb{R}^3$ onto $\Pi$ (parallel to the expanding direction ?). We can describe the boundaries of the rhombus with these vectors.

## 2.3 Substitutive tilings

There are various definitions of substitutive tilings (and especially of the Tribonacci one). We do not try here to have the more general point of view but a convenient definition suitable for our proof. We will proceed as follows. First we fix three initial tiles. We define recursively a sequence of tiles $\mathcal{T} = \{T^{(n)}, n \geq 1\}$ following $T^{(n+3)} = T^{(n)} \cup T^{(n+1)} \cup T^{(n+2)}$. Of course, we have to specify how the three tiles are glued to form the next one and then we have to check that the rule can be iterated correctly. To do so we define simultaneously the sequence of boundaries of the tiles. This construction immediately yields a tiling $\tau^{(n)}$ of $T^{(n)}$ by the basic tiles. A tiling of the plane is a Tribonacci tiling if all finite patch occur in $\tau^{(n)}$ for some $n$.

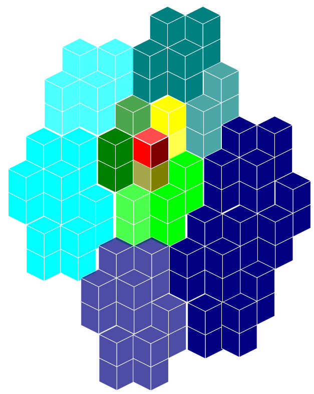

Figure 2: Tiles $T^{(3)} \subset \cdots \subset T^{(11)}$.

## 2.4 SFT

A set of (geometric) tiles may tile the plane. Assume there is a finite number of patches. Now give names (decoration) to copies of the tiles. and prohibit a finite number of patches. (among those geometrically admissibles). The decorated tilings whose patches are admissible are SFT. The geometrical tilings obtained from the SFT by forgetting the decorations are also called SFT (we should say "sofic" for a better analogy with 1-dim). Sofic
A standard way to produce such "matching rules" is to label vertices and boundaries of the geometric tiles and to impose rules on edges (Wang tilings) and/or on vertices

**Labels and colorings**

## 2.5 Tribonacci substitutive tiling

We denote $\omega^{(n)}$ the projection onto $\Pi$ of the vector $-A^n(e_2)$ We denote $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ the basic rhombus and we define recursively for $n \geq 4$,

$$T^{(n+1)} = T^{(n)} \cup \left(T^{(n-1)} + \omega^{(n)}\right) \cup \left(T^{(n-2)} + \omega^{(n)}\right). \tag{2}$$

**Proposition 3.** *For all $n \geq 1$, $T^{(n)}$ is a compact connected domain. Tiles in equation 2 have disjoint interiors.*

This recursion defines a domain of the plane together with a tiling $\tau^{(n)}$ of this domain by tiles $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$. A tiling $\tau$ of the plane by tiles $\{T^{(1)}, T^{(2)}, T^{(3)}\}$ is a *Tribonacci tiling* if all finite patch occur in $\tau^{(n)}$ for some $n$. We denote $\mathcal{S}_1$ the set of Tribonacci tilings by these tiles. Our main result is the following.

**Theorem 4.** *The set $\mathcal{S}_1$ of Trinonacci tilings is sofic.*

The proof is constructive : we give an explicit set of decorated tiles and local matching rules for these tiles such that the SFT produced projects exactly on Tribonacci tilings. We introduce the tiling $\tau^{(n,k)}$ of $T^{(n)}$ by tiles $T^{(k)}$, $T^{(k+1)}$ and $T^{(k+2)}$ and denote $\mathcal{S}_k$ the set of tilings for which all finite patch occur in $\tau^{(n,k)}$ for some $n$.

## 2.6 SFT

We consider the set of tiles $\mathcal{T}_5 = \{T^{((5)}, T^{(6)}, T^{(7)}\}$. On each tile we fix 7 vertices marked $1, \ldots, 7$, as shown on Figure **??**. To each vertex we shall give a *state* among $\{0, -1, 1, 2\}$. To each tile we associate

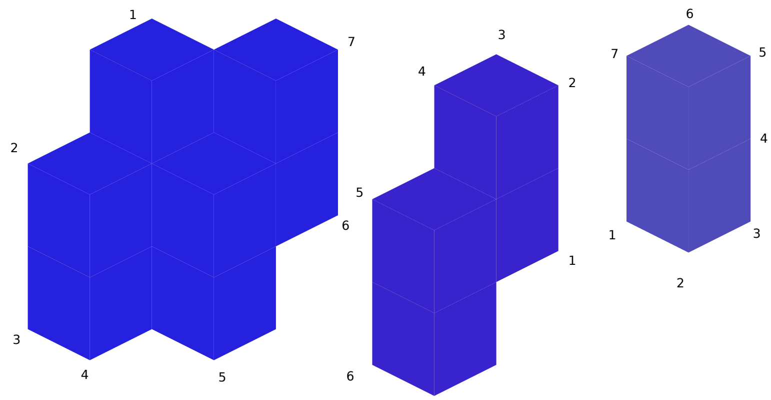


Figure 3: The set of tiles $\mathcal{T}_5$.

a *vertices configuration*, the list of the states of its vertices, which is a word of length 7 on the alphabet $\{0, -1, 1, 2\}$. We fix a subset of *admissible* vertices configurations $\mathcal{V} = \mathcal{V}_P \cup \mathcal{V}_M \cup \mathcal{V}_G \subset \{0, -1, 1, 2\}^7$ :

$$\mathcal{V}_P = \{ \begin{array}{rrrrrrr} 0 & 0 & 1 & 0 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 & 2 & 0 & -1 \\ 1 & 0 & 1 & 0 & 2 & 0 & -1 \end{array} \},$$

$$\mathcal{V}_M = \{ \begin{array}{rrrrrrr} 1 & 0 & 1 & 0 & 2 & -1 & 1 \\ 1 & 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & 1 & 0 & 2 & -1 & -1 \\ 0 & 1 & 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 & 2 & -1 & 0 \end{array} \},$$

$$\mathcal{V}_G = \{ \begin{array}{rrrrrrr} -1 & 2 & 0 & 1 & 2 & -1 & 2 \\ 0 & -1 & 0 & 1 & 2 & -1 & 2 \\ -1 & 1 & 0 & 1 & 2 & -1 & 2 \\ -1 & 0 & 0 & 1 & 2 & -1 & 2 \\ -1 & -1 & 0 & 1 & 2 & -1 & 2 \end{array} \}.$$

We define the set of labelled tiles.

$$\widetilde{\mathcal{T}}_5 = \{T^{((5)}, T^{(6)}, T^{(7)}\} \times \mathcal{V}_P \cup \{T^{(6)}, T^{(7)}\} \times \mathcal{V}_M \cup \{T^{(7)}\} \times \mathcal{V}_G$$

The exhaustive list of labelled tiles for $n = 5$ is shown on Figure 6.
An edge of a tile is a piece of boundary between two distinguished vertices. To each edge, we associate a *color* which consists in a symbol among $S = \{P/M, M/G, G/P\}$ (the skeleton part) and an element of $\mathcal{V}$ (the type part); i.e. a symbol in $S \times \mathcal{V}$. Now, to each tile we associate a *boundary coloring*, the list of colors of its 7 edges, described as a word in $(S \times \mathcal{V})^7$.

**Definition 5** ($c$ reflects $U$)**.** *Let $U$ be the vertices configuration of $T$ and $c = (s, V)$ be the color of edge between $v_4$ and $v_5$. We say that the coloring is compatible with the vertices configuration or that $c$ reflects $U$, if,*

- *If $U \in \mathcal{V}_P$, then $s = P/M$ and $U_1 = V_1$.*
- *If $U \in \mathcal{V}_M$, then $s = M/G$ and $U_1 = V_3, U_2 = V_4$ and $U_7 = V_2$.*
- *If $U \in \mathcal{V}_G$, then $s = G/P$ and $U_1 = V_6, U_2 = V_7$ and $U_7 = V_5$.*

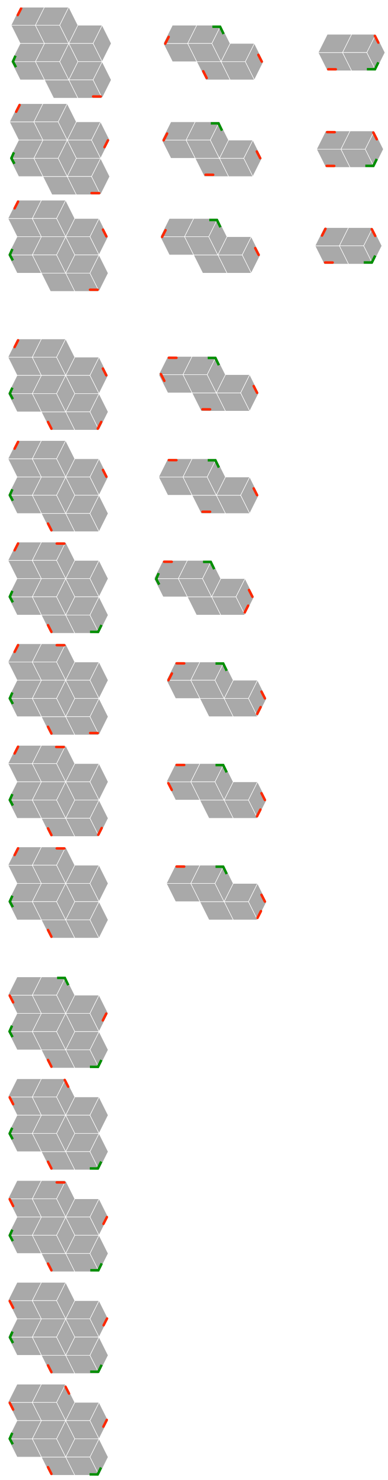

Figure 4: Set of labelled tiles $\widetilde{\mathcal{T}}_5$. Vertices configurations are represented by symbols at vertices : nothing fo 0, red going back for $-1$, red in the positive sense for $+1$ and double green for 2.

We define the set $\mathcal{D}_5$ of decorated tiles (of scale 5)

$$\mathcal{D}_5 = \{((T,U),(c_1,\dots,c_7)) \in \widetilde{\mathcal{T}}_5 \times (S \times \mathcal{V})^7; c_4 \text{ reflects } U\}.$$

**Remark 6.** *In view of the matching rules to be defined later, it is possible to reduce the (huge) number of decorated tiles ; indeed, the local matching rule we are going to define will prohibit the use of certain decorated tiles. For instance, the color will be the same along the boundary of a tile between two vertices which have non zero state (that is to say : if $v_i = 0$, then $c_{i-1} = c_i$). Moreover, condition on $c_4$ will also determine the color $c_5$ : $c_5 = c_4$ and $s_5 = s_4 + 1$.*

**Remark 7.** *We observe that if three tiles, one of each type, reflect a configuration $U$, then $U$ is an admissible configuration ; exactly the one that will split into the three configurations of the subtiles.*

## 2.7 Local matching rules

We are going to consider tilings of the plane with tiles $\mathcal{D}_5$.

The tiling must indeed be a tiling : tiles must fit along boundaries. The colors of two adjacent edges must be the same. Multiple points are triple points and must coincide with a distinguished vertiex of at least two of the involved tiles. The states of the three vertices must be either $(2,2,2)$ (central) or $(0,-1,1)$ in this cyclic order (meso). If the vertex is central, then the skeleton part of the three colors (on the incoming edges) must be $P/M$, $M/G$ and $G/P$ (up to cyclic permutation). The type part of the colors of the three edges must be equal. If the vertex is meso, then say $c_1$ and $c_2$ belong to the tile with empty vertex. We must have $c_1 = c_2$. If the skeleton part of $c_3$ is $P/M$ or $G/P$ then the type parts of $c_1$ and $c_3$ are independent. Althought if the skeleton part of $c_3$ is $M/G$, then $c_1 = c_2$ must reflect the configuration in the type prt of $c_3$. We will say that a tiling $\tau$ with tiles in $\mathcal{D}_n$ is admissible if the local matching rules are satisfied at all boundaries and all vertices and will denote $\mathcal{S}_5^*$ the subshift containing all admissible tilings.

To give a more formal definition of these matching rules, we introduce a few more notations. If $x$ belongs to the boundary of a tile $T$ then $i_x(T)$ denotes the index of edge to which $x$ belongs in $T$ : $i_x(T) = i$ if $x \in L_i(T)$. If $x$ is a distinguished vertex, $i_x(T)$ is the index of the vertex : $v_{i_x(T)} = x$. We let $V_x(T)$ be the state of the vertex of $T$ that coincide with $x$ : $V_x(T) = V_{i_x(T)}$ (if ever $x$ does not coincide with a distinguished vertex, then we set $V_x(T) = 0$). We also let $c_x(T)$ be the color of the boundary $c_{i_x(T)}(T)$. If $x$ is a vertex, we define $c_{x^-}(T) = c_{i-1}(T)$ and $c_{x^+}(T) = c_i(T)$ if $x = v_i(T)$ to be the colors of the previous and next edges. We write $c_x = (s_x, U_x)$.

**Definition 8.** *A tiling $\tau$ of the plane by tiles in $\mathcal{D}_5$ is in $\mathcal{S}_5^*$ if*

*(0) $\tau$ is a tiling ; multiple points are at most triple point.*

*(1) Two adjacent tiles have the same color on their common boundary. For all $T$ and $S$ with $T \cap S \neq \emptyset$ and all $x \in T \cap S$, $c_x(T) = c_x(S)$ (except vertices).*

*(2) For all tile $T$ and all $i = 1,\dots,7$, let $x = v_i(T)$ and $e = V_i(T)$.*

*(a) If $e = 0$ then $c_{i-1}(T) = c_{i+1}(T)$.*

*(b) If $e = 2$ then $x$ is a triple point : there are two tiles $R$ and $S$ with $x = R \cap S \cap T$ (in cyclic order $R, S, T$) ; they satisfy $V_x(R) = V_x(S) = 2$, $U_i(T) = U_{x^+}(R) = U_{x^+}(S)$ and $(s_i(T), s_{x^+}(R), s_{x^+}(S)) \equiv (P/M, M/G, G/P)$.*

*(c) If $e = \pm 1$, then $(V_i(T), V_x(R), V_x(S)) \equiv (0,-1,1)$. Moreover, denoting, for $\epsilon = 0, \pm$ by $Q_\epsilon$ that of the three tiles with $V_x(Q_\epsilon) = \epsilon$, if $s_{x_-}(Q_-) = M/G$ then $c_{x^-}(Q_0)$ reflects $U_{x_-}(Q_-)$.*

## 2.8 Results

Our central result is

**Theorem 9.**

$$\mathcal{S}_5 = \mathcal{S}_5^*.$$

To prove this result, we need a more specific analysis of of the tiles of the Tribonacci substitutive tiles and tilings.

# 3 Tribonacci substitutive tiling

## 3.1 The set of tiles

Starting with a set of tile $\{T^{(1)}, T^{(2)}, T^{(3)}\}$ we have constructed recursively a family of tiles $\mathcal{T} = \{T^{(n)}, n \geq 1\}$ by taking the unions : $T^{(n+1)} = T^{(n-2)} \cup T^{(n-1)} \cup T^{(n)}$ in a specific sense (see **??**). For each $n \geq 1$, the (polygonal) boundary of the tile $T^{(n)}$ is determined by a word $W^{(n)}$, itself decomposed into 7 stepped lines : $W^{(n)} = W_1^{(n)} \cdots W_7^{(n)}$. We define the sequence $(W^{(n)})_{n\geq 1}$ recursively. We start with the following decomposition of the boundaries of the three elementary rhombus :

$$\begin{cases} (W_1^{(1)}, \ldots, W_7^{(1)}) = (\emptyset, c, \emptyset, a, c^{-1}, \emptyset, a^{-1}), \\ (W_1^{(2)}, \ldots, W_7^{(2)}) = (\emptyset, a, \emptyset, b, a^{-1}, \emptyset, b^{-1}), \\ (W_1^{(3)}, \ldots, W_7^{(3)}) = (b, \emptyset, \emptyset, c, b^{-1}, \emptyset, c^{-1}). \end{cases}$$

We set :

$$\begin{cases} W_1^{(n+1)} = W_1^{(n-2)} W_2^{(n-2)} W_6^{(n-1)} \\ W_2^{(n+1)} = W_7^{(n-1)} \\ W_3^{(n+1)} = W_1^{(n-1)} \\ W_4^{(n+1)} = W_2^{(n-1)} W_6^{(n)} \\ W_5^{(n+1)} = W_7^{(n)} \\ W_6^{(n+1)} = W_1^{(n)} \\ W_7^{(n+1)} = W_2^{(n)} W_3^{(n)} W_7^{(n-2)}. \end{cases} \tag{3}$$

We give names to the end points of stepped lines $L(W_i^{(n)})$ : $v_1^{(n)}, \ldots, v_7^{(n)}$ (and say $v_8 = v_1$). The stepped line between $v_i^{(n)}$ and $v_{i+1}^{(n)}$ is $L(W_i^{(n)})$. Observe that $v_i - v_1 = \sum_{1 \leq k < i} e_{W_k}$. We claim that

**Proposition 10.** *For all $n \geq 1$, the boundary of $T^{(n)}$ is a (closed) stepped line described by the word $W^{(n)}$. Decomposition 2 is a tiling of $T^{(n+1)}$ by tiles in $\mathcal{T}_{n-2}$. Interesection between tiles are given by*

$$\begin{aligned} T^{(n-2)} \cap T^{(n-1)} &= W_3^{(n-2)} W_4^{(n-2)} = \overline{W}_5^{(n-1)}, \\ T^{(n-1)} \cap T^{(n)} &= W_3^{(n-1)} W_4^{(n-1)} = \overline{W}_5^{(n)}, \\ T^{(n)} \cap T^{(n-2)} &= W_4^{(n)} = \overline{W}_6^{(n-2)} \overline{W}_5^{(n-2)}. \end{aligned}$$

*The triple point is $v_5^{(n-1)} = v_5^{(n-2)} = v_5^{(n-3)}$ ; moreover, $v_3^{(n-3)} = v_6^{(n-2)}$, $v_3^{(n-2)} = v_6^{(n-1)}$ and $v_4^{(n-1)} = v_7^{(n-3)}$.*

In view of this lemma we can glue $T^{(n-2)} \cup T^{(n-1)} \cup T^{(n)}$ along the three pieces of boundaries as shown in Figure **??**. It is straightforward to check that the boundary of the metatile is given by $W^{(n+1)}$. Such glueing of the three tiles is called a *correct glueing*. The three pieces of boundaries along which is the tiles are glued are called the *skeleton* of tile $T^{(n)}$ (tiled that way). The intersection point is the central vertex. The three other extremal points are called meso vertex. The union of the boundaries of the three tiles decomposes into the skeleton and the boundary of tile $T^{(n)}$.

In order to prove this proposition we give an alternative presentation of boundary words more suitable to understanding its combinatorics. We consider the substitution on the alphabet $\{A, B, C, D\}$ defined by :

$$\sigma : \begin{array}{ccc} A & \to & B \\ B & \to & CDC \\ C & \to & A \\ D & \to & \overline{C} \end{array}$$

We observe that $\sigma$ has a periodic orbit of period 3 with periodic points starting with $A$, $B$ and $C$ and a periodic orbit of period 3 with periodic points starting with $\overline{A}$, $\overline{B}$ and $\overline{C}$. We define a morphism $\varphi$ by

$$\begin{array}{llll} \varphi: & A & \to & \overline{b}\overline{b} \\ & B & \to & \overline{c}b\overline{c} \\ & C & \to & a \\ & D & \to & \overline{c} \end{array}$$

**Lemma 11.** *For all $n \geq 5$,*

$$W^{(n+2)} = \big(\varphi(\sigma^n(\overline{B})), \varphi(\sigma^n(A)), \varphi(\sigma^n(\overline{C})), \varphi(\sigma^n(\overline{DC})), \varphi(\sigma^n(B)), \varphi(\sigma^n(\overline{A})), \varphi(\sigma^n(CDC))\big)$$

*Proof.* We easily check by hand that the result holds for $n = 5, 6, 7$ and proceed recursively to show that it holds for all $n \geq 5$ : in view of the recursion defining $W^{(n+1)}$ we can write

$$\begin{aligned} W^{(n+3)} &= \big(\sigma^{n-2}(\overline{B}A)\sigma^{n-1}(\overline{A}), \sigma^{n-1}(CDC), \sigma^{n-1}(\overline{B}), \sigma^{n-1}(A)\sigma^n(\overline{A}), \sigma^n(CDC), \sigma^n(\overline{B}), \sigma^n(A\overline{C})\sigma^{n-2}(CDC)\big) \\ &= \big(\sigma^{n-2}(\overline{B}A\overline{B}), \sigma^{n-2}(A\overline{C}A), \sigma^{n-2}(\overline{CDC}), \sigma^{n-2}(B\overline{CDC}), \sigma^{n-2}(B\overline{A}B), \sigma^{n-2}(\overline{A}C\overline{A}), \sigma^{n-2}(CDC\overline{B}CDC)\big) \\ &= \big(\sigma^{n+1}(\overline{B}), \sigma^{n+1}(A), \sigma^{n+1}(\overline{C}), \sigma^{n+1}(\overline{D}\,\overline{C}), \sigma^{n+1}(B), \sigma^{n+1}(\overline{A}), \sigma^{n+1}(CDC)\big) \end{aligned}$$

□

**Remark 12.** *It readily follows that, for all $n \geq 5$, $W^{(n)} = \varphi(\sigma^n(\overline{CDA}CDA))$.*

*Proof of Proposition 10.* We observe that $v_5^{(3)} = 0$. Then we observe that $v_5^{(n+1)} - v_5^{(n)} = c^{(n)}$. Assume $v_5^{(n)}$ is the triple point at level $n$. we write $v_5^{(n+1)} - v_5^{(n)} + e(W_5^{(n)}) + e(W_6^{(n)}) = e(\varphi(\sigma^n(B))) + e(\varphi(\sigma^n(\overline{A}))) = M^n(e_b - 2e_c)$, where $M$ is the restriction of the (abelianised) matrix of $\sigma$ on the plane defined by the projection $\varphi$. [It should be proved that it corresponds to the similarity of ratio $\mu$.because it is the inverse] When clear that $= c^{(n)}$, then immediatly implies that it is the intersection point of the next level. Equalities between glueing boundaries then shows identity of the other distinguished vertices.
For all $n \geq 1$, $e(W^{(n)}) = 0$ and $L(W^{(n)})$ is a simple closed curve. Moreover, equalities

$$W_3^{(n-2)} W_4^{(n-2)} = \overline{W}_5^{(n-1)},\ W_3^{(n-1)} W_4^{(n-1)} = \overline{W}_5^{(n)},\ W_4^{(n)} = \overline{W}_6^{(n-2)} \overline{W}_5^{(n-2)}. \tag{4}$$

follow from identifications :

$$\begin{aligned} A &= \overline{W}_4^P \overline{W}_3^P = W_5^M = W_7^P = \overline{W}_1^M = W_2^G = \overline{W}_6^G &(5)\\ B &= \overline{W}_4^M \overline{W}_3^M = W_5^G = W_7^M = \overline{W}_1^G &(6)\\ C &= W_5^P = \overline{W}_1^P = W_2^M = \overline{W}_6^M = \overline{W}_3^G &(7)\\ D &= W_6^P = \overline{W}_2^P = W_3^M &(8)\\ CD &= W_5^P W_6^P = \overline{W}_4^G &(9)\\ CDC &= W_7^G = \overline{W}_4^G \overline{W}_3^G &(10) \end{aligned}$$

□

Now we state and prove a few properties of the tiles for later use.

**Proposition 13** (Higher scales)**.** *The tile $T^{(n)}$ does not intersect the boundary of the tile $T^{(n+3)}$.*

This result implies that $T^{(n)}$ is increasing and that for all $M > 0$ there is $N$ such that $T^{(N)}$ contains the ball of radius $M$.

*Proof.* It is straightforward to check that the seven stepped line $L(W_i^{(n)})$ get glued. See Figure 2. □

We denote $u_X = \lim_{n\to\infty} \varphi(\sigma^{3n}(X))$ for $X = A, B, C$.

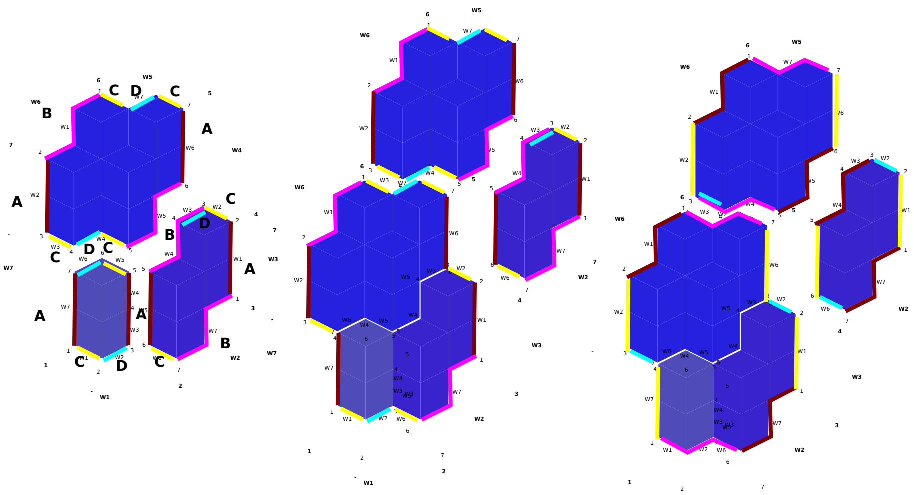

Figure 5: Boundaries of tiles.

**Proposition 14** (Shape of the skeleton)**.** *The three pieces of the skeleton of $T^{(n)}$ starting from the central vertex are prefixes of one of the three words $u^A$, $u^B$ and $u^C$ ; their length is increasing with $n$. The boundaries close to $v_7$ and $v_2$ have similar geometry.*

**Proposition 15** (Checkpoint)**.** *For all $m = n-1, n-2, n-3$, the M/G part of the skeleton tile $T^{(m)}$ "inside" tile $T^{(n)}$ touches the skeleton of $T^{(n)}$ along the piece $W_4^{(m)}$ (and hence does not intersect with the boundary of $T^{(n)}$).*

**Proposition 16** (Triple points)**.** *For all $k \geq 5$, the the multiple points of $\tau^{(\infty,k)}$ are at most triple points.*

*Proof.* Indeed, the angles on the boundary is always $\geq 2\pi/3$. □

For all $n \geq 1$, we have defined the set of tiles $\mathcal{T}_n$. Each tile $T \in \mathcal{T}_n$ comes with a set of vertices $\mathcal{V}(T) = \{v_i^{(n)}, i = 1, \ldots, 7\}$ and stepped lines described by a sequence of words $\mathcal{W}(T) = \{W_i^{(n)}, i = 1, \ldots, 7\}$.

## 3.2 Hierarchical structure of a Tribonacci tiling

Let $T \in \tau^{(\infty,k)}$. The tile $T$ may be small, medium or big (relative to $k$) according to whether $T \simeq T^{(k)}$, $T \simeq T^{(k+1)}$ or $T \simeq T^{(k+2)}$. Observe that for all $m \geq 0$, it is included in a tile $T'$ of $\tau^{(n,k+m)}$. We consider the smallest integer $m$ such that $T' \neq T$, call *metatile* of $T$ and denote $\mathcal{M}(T)$ the tile $T'$ corresponding to this $m$. We say that $T$ play the role $P$, $M$ or $G$ according to whether $m = 1, 2$ or 3. We observe that small tiles always play the role $P$, medium tiles may play role $P$ or $M$ and big tiles can play the three roles $P, M$ or $G$. We define recursively $\mathcal{M}^{i+1}(T)$ the metatile of $\mathcal{M}^i(T)$ and observe that there is $i_0$ such that $\mathcal{M}^{i_0}(T) = T^{(n)}$. We call *hierarchical name* of the tile $T$ in $\tau^{(n,k)}$ the finite sequence of roles of $T$, $\mathcal{M}(T)$, $\mathcal{M}(\mathcal{M}(T)), \ldots, \mathcal{M}^{i_0-1}(T)$. Let $T \in \tau^{(\infty,k)}$. We call *subtiles* the three tiles $T'$ in $\tau^{(\infty,k')}$ such that $\mathcal{M}(T') = T$ (with $k' = k-1, k-2$ or $k-3$); we denote $S(T) = (T^P, T^M, T^G)$ the list of subtiles ordered according to their role (note that it is a tiling of $T$).

## 3.3 Vertices and triple points

Let $n > k \geq 5$. Consider the tiling $\tau = \tau^{(\infty,k)}$. We recall that the intersection of three distinct tiles in $\tau$ is either empty or reduced to one point. We denote $\mathcal{P}(\tau)$ the set of non empty such intersections (triple points) of the tiling and $\mathcal{V}(\tau) = \cup_{T\in\tau}\mathcal{V}(T)$ the set of points that coincide with a distinguished vertices of a tile. We stress that multiple (triple) points are points in the plane ; they belong to (at least) three tiles. Distinguished vertices are to be seen as points of a tile. As so, they are points of the plane ; but of course more than one may project on the same point. We recall the notations : if $x$ belongs to the boundary of

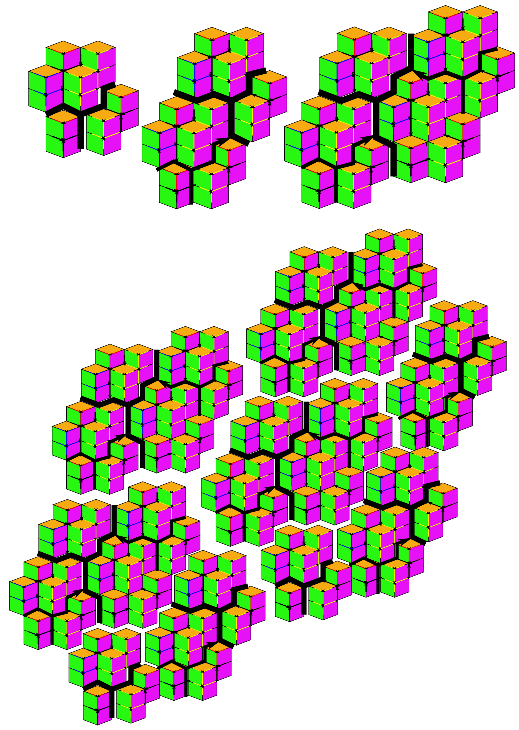

Figure 6: Patches of tilings in SUBST : $\tau^{(8)}, \tau^{(9)}, \tau^{(10)}$ and $\tau^{(13)}$.

a tile $T$ then $i_x(T)$ denotes the index of edge to which $x$ belongs in $T$ : $i_x(T) = i$ if $x \in L_i(T)$. If $x$ is a distinguished vertex, $i_x(T)$ is the index of the vertex : $v_{i_x(T)}(T) = x$.

**Definition 17** (Central vertex)**.** *Let $n \geq 5$ and $\tau = \tau^{(\infty,k)}$. A triple point $v \in \mathcal{P}(\tau)$ is a* central vertex *if there are $K \geq k$ and $T \in \tau^{(\infty,K)}$, such that $v$ is on the boundary of $T$ and in the interior of $\mathcal{M}(T)$.*

Observe that a central vertex is the point $v_5^{(k)}$ of the tile $T \in \tau^{(\infty,K)}, T \simeq T^{(k)}$ to which it belongs ($k = K, K+1$ or $K+2$) ; indeed the unique triple point with this property in any metatile.

**Definition 18** (Meso vertex)**.** *Let $n \geq 5$ and $\tau = \tau^{(\infty,k)}$. A triple point $v \in \mathcal{P}(\tau)$ is a* meso vertex *if there are $K \geq k$, $T$ and $T' \in \tau^{(\infty,K)}$ with $\mathcal{M}(T) = \mathcal{M}(T')$, such that $v$ belongs to $T$ and $T'$ while it is on the boundary of $\mathcal{M}(T)$.*

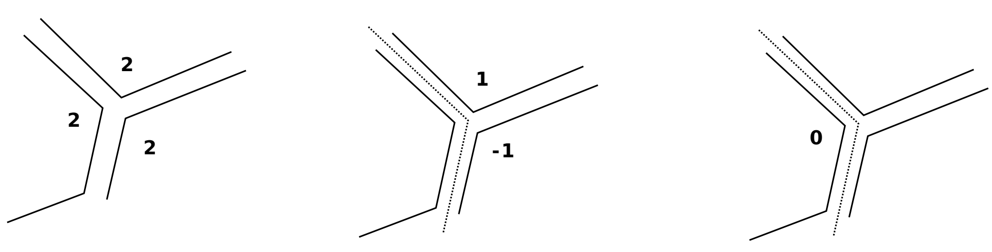


Figure 7: There are different types of vertices : either (1) at the intersection of three tiles of the same scale (central), or (2) at the intersection of two tiles of the same scale and another one of larger scale (meso) or (3) the vertex of one tile corresponding to a "flat" point of the adjacent tile (at this scale ; maybe corresponding to a vertex of type 2 at lower scale) (flat)

**Remark 19.** *We observe that if $v \in \mathcal{P}(\tau^{(\infty,k)})$ is a central (resp. meso) vertex, then it is also a central (resp. meso) vertex as a vertex of $\mathcal{P}(\tau^{(\infty,k')})$ for all $1 \leq k' \leq k$. Let then $K$ be the maximal $k$ such that $v$ is central (resp. meso) for $\tau^{(n,k)}$ ; we say that $v$ is* used *at scale $K$. Observe that for $k > K$, $v$ is not in $\mathcal{P}(\tau^{(\infty,k)})$ (if meso, it may still be on the boundary of a (two) tiles, but it is not even a distinguished vertex).*
*We observe that $\mathcal{P}(\tau^{(n,k+1)}) \subset \mathcal{P}(\tau^{(n,k)})$ and that $\mathcal{V}(\tau^{(n,k+1)}) \subset \mathcal{V}(\tau^{(n,k)})$.*

**Lemma 20.** *(i) All triple points in $\tau$ are either central or meso vertex. (ii) $\mathcal{P}(\tau) \subset \mathcal{V}(\tau)$.*

*Proof.* Consider a triple point $x = T_1 \cap T_2 \cap T_3$. Let $K$ be the smallest scale at which there is a tile $T \in \tau^{(\infty,K)}$ and $i \neq j$ such that $T_i \cup T_j \subset T$. Let $S(T) = (T^P, T^M, T^G)$. Let $T_i \subset T^Q$ and $T_j \subset T^R$ for some $Q$ and $R$ in $\{P, M, G\}$. Let $k \neq i, j$. Either we have also $T_k \subset T$ and the triple point $x$ is central, either $T_k \cap T = \emptyset$ so $x$ is on the boundary of $T$ and it is a mesovertex : in both cases it is a vertex of $T^R$ and $T^Q$ and hence a vertex of $T_i$ and $T_j$. (i) is proved and (ii) then follows from Proposition 10 which shows that central and meso vertices are distinguished vertices. □

**Remark 21.** *A tile $T$ is in a metatile. The first remark is that its triple points implying the other subtiles of the common metatile (the sisters) fall on distinguished vertices. For each tile : one is central and two are meso. The other ones in the skeleton are empty. Those on the boundary of the metatile are not determined. [The meso on the boundary could a priori coincide with mesos of other scales. Prove later. Indeed impossible since there are no quadruple points]. Now at upper scale only the distinguished of the meta can play a role. What about the other ones ? Can they be triple ? Only if meso au cran du dessus. See.*

**Remark 22.** *Fix $n$ and $k$ and consider the tiling $\tau^{(n,k)}$ of $T^{(n)}$ by tiles in $\mathcal{T}_k$. We observe that as soon as $k \geq 5$, the words $W_i^{(n)}$ are non empty for all $i = 1, \ldots, 7$. It follows that the combinatorial structure of $\tau^{(n,k)}$ is the same as the combinatorial structure of $\tau^{(n+m,k+m)}$ for all $m \geq 0$ in the sense that the graph is the same. HERE: is this obvious ?*

### 3.4 Combinatorics of vertices

A vertex of a tile $T$ in $\tau^{(\infty,k)}$ inherits the properties of being central or meso from the corresponding vertex in the tiling. Hence it can be central, meso or empty.

**Definition 23.** *Let $T$ be a tile in $\tau^{(\infty,k)}$ and $x$ be a vertex of $T$. The* state *$V_x(T)$ of $x$ (in $T$) is a symbol among $\{0, -1, 1, 2\}$ defined as follows :*
*If $x$ is a central vertex, then $V_x(T) = 2$.*
*If $x$ is a mesovertex then we let $K \geq k$, $Q_0, Q_{-1}$ and $Q_1$ in $\tau^{(\infty,K)}$ with $\mathcal{M}(Q_{-1}) = \mathcal{M}(Q_1)$ be such that $x = Q_0 \cap Q_{-1} \cap Q_1$ and $x$ is on the boundary of $\mathcal{M}(Q_1)$. We set $V_x(T) = \epsilon$ for if $T \subset Q_\epsilon$.*
*Otherwise we set $V_x(T) = 0$.*

We observe that $V_x(T)$ may be 0 either if it is not a triple point either if it is a mesovertex.

**Definition 24.** *A tile $T$ in $\tau^{(\infty,k)}$ has 7 vertices. We call* vertices configuration *or for short* v-configuration *(of $T$) the list of states of its vertices.*

It is a priori a 7-letters word on the alphabet $\{0, -1, 1, 2\}$. The object of this subsection is to specify the $v$-configurations which really occur in the Tribonacci tiling.

**Lemma 25.** *For all $T \in \tau^{(\infty,k)}$, $V(T) \in \mathcal{V}$.*

*Proof.* We decompose the proof into 4 steps.

**Step 1.** Let us consider a tile $T \in \tau^{(\infty,k)}$, and its three subtiles $T^P, T^M$ and $T^G$. We want to compare the $v$-configuration $V$ of $T$ with the $v$-configurations $V^P$, $V^M$ and $V^G$ of its subtiles. It follows from the recursive definition of $W^{(n)}$ that the seven distinguished vertices of $T$ are $v_1^P v_7^M v_1^M v_2^M v_7^G v_1^G v_2^G$. Observe that the meso vertices of the subtiles are not distinguished vertices of $T$. Observe also that $v_2^P$ and $v_3^G$ are not distinguished vertices of $T$ while they are on its boundary. The vertices configuration of $T$ is indeed :

$$V = V_1^P V_7^M V_1^M V_2^M V_7^G V_1^G V_2^G. \tag{11}$$

**Step 2.** Conversely, the vertex configuration of the three subtiles of a tile is determined by the v-configuration of the tile itself. Indeed, we claim that

$$\left\{\begin{array}{lcccccccc} V^P & = & V_1 & 0 & 1 & 0 & 2 & 0 & -1 \\ V^M & = & V_3 & V_4 & 1 & 0 & 2 & -1 & V_2 \\ V^G & = & V_6 & V_7 & 0 & 1 & 2 & -1 & V_5 \end{array}\right. \tag{12}$$

It follows from the definitions that $v_5^{(k)}, v_5^{(k+1)}$ and $v_5^{(k+2)}$ are central (name 2), that $v_4^{(k)}, v_6^{(k)}, v_4^{(k+1)}$ are empty (name 0) and that $v_3^{(k)}$, $v_3^{(k+1)}$, $v_4^{(k+2)}$, $v_7^{(k)}$, $v_6^{(k+1)}$ and $v_7^{(k+2)}$ are meso vertices (name $-1$ and $+1$). The vertices $v_4^P, v_6^P, v_4^M$ are obviously in state 0 since they do not correspond to triple points. The new vertices on the boundary are in state 0 : indeed, either they correspond to double points, either they correspond triple points ; in the latter case we observe that they necessarily correspond to the higher scale part of meso vertices since they are not distinguished points of $T$ : in both cases their state is empty (0).

**Step 3.** We define a labelled automaton $\mathcal{G}$ : the set of states is the set of v-configurations and the set of labels is the set of roles $\{P, M, G\}$ ; there is an edge $U \xrightarrow{Q} V$ (from $U$ to $V$ labelled $Q$) if there is a tile $T$ with configuration $U$ whose subtile with role $Q$ has v-configuration $V$. By construction all states have outgoing degree 3. We denote $\mathcal{V}$ the recurent component of this graph. We decompose $\mathcal{V} = \mathcal{V}_P \cup \mathcal{V}_M \cup \mathcal{V}_G \subset \{0, -1, 1, 2\}^7$. We can check by computation using (12) that (see also Figure 9),

| $V$ | $\rightarrow$ | $V^P, V^M, V^G$ |
|---|---|---|
| $P1$ | $\rightarrow$ | $P1, M2, G2$ |
| $P2$ | $\rightarrow$ | $P2, M2, G2$ |
| $P3$ | $\rightarrow$ | $P3, M2, G2$ |
| $M1$ | $\rightarrow$ | $P3, M2, G3$ |
| $M2$ | $\rightarrow$ | $P3, M2, G4$ |
| $M3$ | $\rightarrow$ | $P1, M1, G1$ |
| $M4$ | $\rightarrow$ | $P1, M1, G5$ |
| $M5$ | $\rightarrow$ | $P1, M1, G3$ |
| $M6$ | $\rightarrow$ | $P1, M1, G4$ |
| $G1$ | $\rightarrow$ | $P2, M3, G1$ |
| $G2$ | $\rightarrow$ | $P1, M4, G1$ |
| $G3$ | $\rightarrow$ | $P2, M5, G1$ |
| $G4$ | $\rightarrow$ | $P2, M6, G1$ |
| $G5$ | $\rightarrow$ | $P2, M4, G1$ |

We observe that if we start with a tile $T$ with any $v$-configuration, and consider a subtile $S$ (of level $k$) with no common vertices then $V(S) \in \mathcal{V}$.

**Step 4.** Consider a tile $T$ in $\tau^{(\infty,k)}$. Let $n$ be such that $T \subset T^{(n)}$. By Proposition 13, $T$ does not intersect the boundary of $T^{(n+3)}$. Let $N$ be such that $\mathcal{M}^N(T) = T^{(n+3)}$. Applying $N$ times (12) we wee that $V(T) \in \mathcal{V}$.

□

## 3.5 Higher level SFT

As in the definition of $\mathcal{S}_5^*$ we define a set of decorated tiles corresponding to each scale. We consider $\mathcal{T}_n = \{T^{(n)}, T^{(n+1)}, T^{(n+2)}\}$ the set of (geometrical) tiles (at scale $n$). The tiles are labelled with a *vertex configuration* in $\mathcal{V}$.

**Definition 26** (Labelled tiles)**.** *We define the set of labelled tiles.*

$$\widetilde{\mathcal{T}}_n = \{T^{(n)}, T^{(n+1)}, T^{(n+2)}\} \times \mathcal{V}_P \cup \{T^{(n+1)}, T^{(n+2)}\} \times \mathcal{V}_M \cup \{T^{(n+2)}\} \times \mathcal{V}_G$$

To each tile we now associate a sequence of coloration of its 7 edges. So each piece of boundary has a color which consists in a symbol among $S = \{P/M, M/G, G/P\}$ (the skeleton part) and an element of $\mathcal{V}$ (the type part); i.e. a symbol in $S \times \mathcal{V}$. Formally, the piece $W_i$ has color $c_i(T)$.

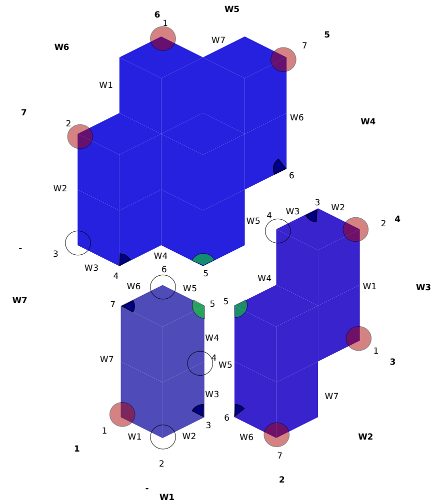


Figure 8: Meta-tile $T^{(8)}$ with informations on marked vertices.

**Definition 27** (Decorated tiles)**.** *The set $\mathcal{D}_n$ of decorated tiles (of scale $n$) is*

$$\mathcal{D}_n = \{((T,U),(c_1,\dots,c_7)) \in \widetilde{\mathcal{T}}_n \times (S \times \mathcal{V})^7; c_4 \textit{ reflects } U\}.$$

**Remark 28.** *We observe that the local matching rule we are going to define will prohibit the use of certain decorated tiles. For instance, the color will be the same along the boundary of a tile between two vertices which have non zero state (that is to say : if $v_i = 0$, then $c(W_{i-1}) = c(W_i)$). Moreover, condition on $W_4$ will also determine the color of other parts of the boundary in view of the nature of the vertices of the tile. [But it seems that we do not have to impose this now].*

At scale $n$ we use similar notation as in the case $n = 5$ to relate combinatorial data associated to a tile to the position of the tile in space. We let $V_x(T)$ be the state of the vertex of $T$ that coincide with $x$ : $V_x(T) = V_{i_x(T)}$ (if ever $x$ does not coincide with a distinguished vertex, then we set $V_x(T) = 0$). We also let $c_x(T)$ be the color of the boundary $c_{i_x(T)}(T)$. If $x$ is a vertex, we define $c_{x^-}(T) = c_{i-1}(T)$ and $c_{x^+}(T) = c_i(T)$ if $x = v_i(T)$ to be the colors of the previous and next edges. We write $c_x = (s_x, U_x)$.

**Definition 29** (SFT)**.** *A tiling with tiles in $\mathcal{D}_n$ is in $\mathcal{S}_n^*$ if it satisfies de local matching rules of Definition 8.*

# 4 SUBST ⊂ SFT

**Proposition 30.** *There is a tiling in $\mathcal{S}_5^*$ that projects on the Tribonacci basic tiling $\tau^{(\infty,5)}$.*

*Proof.* Let $\tau$ be the basic Tribonacci tiling. We construct a tiling $\widetilde{\tau}$ on $\mathcal{D}_5$ by giving to each tile a decoration, i.e. labels and colorings. This step does not affect the geometry of the tiling ; hence $\widetilde{\tau}$ projects on $\tau$. Then to show that $\widetilde{\tau}$ is in $\mathcal{S}_5^*$ we check that local matching rules are satisfied.

**Step 1.** We consider a scale $k$ and a tile $T \in \tau^{(\infty,k)}$. We determine the state of each vertex following Definition 24. The list $V(T)$ of these states is the label of $T$. In virtue of Lemma 25, $V(T) \in \mathcal{V}$.

**Step 2.** We determine colors on the edges of the tilings. We proceed recursively. Consider a tile $T \in \tau^{(\infty,k)}$. Let $V$ be its v-configuration and assume its boundary coloring $c = (c_1,\dots,c_7)$ is given. We consider the subtiles $S(T) = (T^P, T^M, T^G)$ as a (sub)tiling of $T$. We set

$$\begin{aligned} c(T^P) &= (c_1, c_1, (P/M, V), (P/M, V), (G/P, V), (G/P, V), c_7) && (13)\\ c(T^M) &= (c_3, c_4, (M/G, V), (M/G, V), (P/M, V), c_1, c_2) && (14)\\ c(T^G) &= (c_6, c_7, c_7, (G/P, V), (M/G, V), c_4, c_5) && (15) \end{aligned}$$

We observe that the coloring of the skeleton of $T$ is determined independently of the coloring of the boundary and that the colors of the boundaries of the subtiles which also belong to the boundary of $T$ are coherent with the color of the boundary of $T$ itself (in the sense that if a piece of boundary $E_1$ of a tile $T_1$ in $\tau^{(\infty,k)}$ is included in a piece of boundary $E_2$ of a tile $T_2$ with $T_1 \subset T_2$, then $c_{E_1}(T_1) = c_{E_2}(T_2)$). Also observe that two edges that are glued are given the same color.
Henceforth, the boundary coloring $c(T)$ of a tile $T$ can be determined by starting from a large enough metatile $T^{(n)}$ (large enough so that $T$ is does not intersect the boundary of $T^{(n)}$, see Proposition13) with arbitrary boundary coloring since it will not depend of this arbitrary coloring. For all $k$ and all tile $T \in \tau^{(\infty,k)}$, we have determined a decorated tile $((T, V(T)), c(T)) \in \widetilde{\mathcal{T}}_k \times (S \times \mathcal{V})^7$.

**Step 3.** We check that the decorated tile is indeed in $\mathcal{D}_5$ (i.e. the condition $c_4$ reflects $U$ is fullfilled). Consider a tile $T \in \tau^{(\infty,5)}$ and its metatile $\mathcal{M}(T)$. By definition of the coloring $c_4 = (S, V)$ where $V$ is the v-configuration of $\mathcal{M}(T)$ and $S$ depends on the role played by $T$ in $\mathcal{M}(T)$. ($S = P/M$ if role $P$, $S = M/G$ if role $M$ and $S = G/P$ if role $G$). We check the different situations :

- If $T$ plays the role $P$ in $\mathcal{M}(T)$, then its vertex $v_1(T) = v_1(\mathcal{M}(T))$ (and $V_2(T) = 0$ since it is not a vertex of $\mathcal{M}(T)$).
- If $T$ plays the role $M$ in $\mathcal{M}(T)$, then its vertices $v_7(T) = v_2(\mathcal{M}(T))$, $v_1(T) = v_3(\mathcal{M}(T))$ and $v_2(T) = v_4(\mathcal{M}(T))$.
- If $T$ plays the role $G$ in $\mathcal{M}(T)$, then its vertices $v_7(T) = v_5(\mathcal{M}(T))$, $v_1(T) = v_6(\mathcal{M}(T))$, $v_3(T) = v_7(\mathcal{M}(T))$ (and $V_3(T) = 0$ since it is not a vertex of $\mathcal{M}(T)$).

Hence the color of edge $c_4(T)$ reflects the v-configuration of $T$ : the decorated tiles are in $\mathcal{D}_5$.

We define the tiling $\widetilde{\tau} = \{((T, U(T)), c(T)); T \in \tau^{(\infty,5)}\}$.

**Step 4.** It remains to check that the local matching rules are satisfied :

(0) The tiling $\widetilde{\tau}$ obviously satisfies the geometrical constraints.

(1) Let $T$ and $S$ be two adjacent tiles in $\widetilde{\tau}$. Let $k$ be the scale at which they are being glued (i.e. the smallest integer such that there is $Q \in \tau^{(\infty,k)}$ with $T \subset Q$ and $S \subset Q$). There are two subtiles $Q_T$ and $Q_S$ of $Q$ with $T \subset Q_T$ and $S \subset Q_S$ ; the glueing edge $T \cap S$ between $T$ and $S$ is included in the glueing edge between $Q_T$ and $Q_S$. But these last ones have the same color (since they coincide to form a branch of the skeleton of $Q$).

(2) For all tile $T \in \widetilde{\tau}$ and all $i = 1, \ldots, 7$, let $x = v_i(T)$ and $e = V_i(T)$.

(a) If $e = 0$ then there is a scale $k$ and two tiles $Q$ and $R$ in $\tau^{(\infty,k)}$ with $T \subset Q$ such that $x$ belongs only to $Q$ and $R$ (not a triple point). Let then $K$ be the scale at which $Q$ and $R$ get glued (smallest integer such that there is a tile $S \in \tau^{(\infty,K)}$ with $Q \subset S$ and $R \subset S$). The color of the edge $Q \cap R$ is uniform and given by the skeleton of $S$. It implies $c_{i-1}(T) = c_{i+1}(T)$.

(b) If $e = 2$ then $x$ is a triple point : Let $K$ be the scale at which there is $Q \in \tau^{(\infty,K)}$ such that $x$ is in the interior of $Q$. Let $Q'$, $Q''$ and $Q'''$ be the three subtiles of $Q$ ; $x = Q' \cap Q'' \cap Q'''$. Let $R$ and $S$ in $\tau$ be such that $x = R \cap S \cap T$, and $T \subset Q'$, $S \subset Q''$, and $S \subset Q'''$. Since $V_x(Q') = V_x(Q'') = V_x(Q''') = 2$ we also have $V_x(R) = V_x(S) = 2$. We also observe that $c_i(T) = c_{x^+}(Q')$, $c_{x^+}(R) = c_{x^+}(Q'')$ and $c_{x^+}(S) = c_{x^+}(Q''')$. It follows that $U_i(T) = U_{x^+}(R) = U_{x^+}(S)$ because they are equal to $U_{x^+}(Q') = U_{x^+}(Q'') = U_{x^+}(Q''')$ and $(s_i(T), s_{x^+}(R), s_{x^+}(S)) \equiv (P/M, M/G, G/P)$. [Observe that $i_x(Q') = i_x(Q'') = i_x(Q''') = 5$]

(c) If $e = \pm 1$, then $x$ is a triple point. Assume for instance $e = +1$ ; the case $e = -1$ is symmetric. Let $K'$ be the (smallest) scale at which there are $Q', Q'' \in \tau^{(\infty,K')}$ such that $x$ belongs only to $Q'$ and $Q''$ (not a triple point any longer) and $K$ the smallest scale at which $Q'$ and $Q''$ belong to a common tile

$Q \in \tau^{(\infty,K)}$. Assume for instance $T \subset Q'$. Let $R \in \tau$ be the other tile in $Q'$ containing $x$ and $S \in \tau$ the tile in $Q''$ containing $x$. We observe that $(V_i(T), V_x(R), V_x(S)) \equiv (0, -1, 1)$.
We denote, for $\epsilon = 0, \pm$ by $Q_\epsilon$ that of the three tiles with $V_x(Q_\epsilon) = \epsilon$. Observe that $s_{x_-}(Q_-)$ is the skeleton part of the color of the boundary between $T$ and $S$ and $c_{x^-}(Q_0)$ is the color of the boundary between $Q'$ and $Q''$. Our main point is that if $s_{x_-}(Q_-) = M/G$, the subtiles of $Q'$ are $Q^P, Q^M$ and $Q^G$ and $T \subset Q^M$ while $S \subset Q^G$ (of course) and that then, $K = K' + 1$ because $Q$ is the metatile of $Q'$. It follows that the type part of the color between $Q'$ and $Q''$ is the v-configuration of $\mathcal{M}(Q')$ while the color between $Q^G$ and $Q^M$ is the v-configuration of $Q'$. It readily follows that then $c_{x^-}(Q_0)$ reflects $U_{x_-}(Q_-)$.

Hence we have shown that the tiling $\tau^{(\infty,5)}$ is the projection of a tiling $\widetilde{\tau}$ in $\mathcal{S}_5$. □

# 5 SFT ⊂ SUBST

## 5.1 Admissible glueings

We observe (or recall) that boundaries have uniform colors between two non empty vertices. Indeed, the vertex rules allow changes of colors only at triple points and not at mesovertex for the tile marked 0. We state as a lemma an easy remark which will be crucial for the key step of the proof of the result.
Let $n \geq 5$ be an integer. We consider a tiling $\tau \in \mathcal{S}_n^*$. We denote $W_x^+(T)$ the word $W_i(T) \cdots W_{j-1}(T)$ where $j$ is the smaller integer larger than $i$ such that $V_j(T) \neq 0$. Hence, it is the shape of the edge of $T$ between two successive non empty distinguished vertices.

**Lemma 31.** *Let $T$ and $T' \in \tau$ be two adjacent tiles. Let $x \in T \cap T'$ and $i = i_x(T)$ and $i' = i_x(T')$. Assume $V_i(T) = 2$. Let $\widetilde{W}$ be the word coding the polygonal line $T \cap T'$, starting from $x$ and denote $y$ its other end. Let $W = W_i^\epsilon(T)$ and $W' = W_{i'}^{-\epsilon}(T')$. We have $V_x(T') = 2$ and $W_x^\epsilon(T)$ and $W_x^{-\epsilon}(T')$ are prefixes of $\widetilde{W}$. Moreover one of the three alternatives hold :*
*(i) $W_x^\epsilon(T) = W_x^{-\epsilon}(T') = \widetilde{W}$ and $V_y(T) = -V_y(T') = \epsilon$*
*(ii) $W_x^\epsilon(T) = \widetilde{W}$ and $V_y(T) = -\epsilon$, $V_y(T') = 0$*
*(iii) $W_x^{-\epsilon}(T') = \widetilde{W}$ and $V_y(T) = 0$, $V_y(T') = \epsilon$*

*Proof.* Since $V_x(T) = 2$ we have $V_x(T') = 2$. We observe that $\widetilde{W}$ is a common prefix of $W$ and $W'$.
Firstly assume that $W = \widetilde{W} \neq W'$ . Then $V_y(T') = 0$ and $y$ is a mesovertex. For orientation reason we must have $V_y(T) = -\epsilon$ and we are in situation (ii). Conversely if $W' = \widetilde{W} \neq W$, then $V_y(T) = 0$, and again $y$ is a mesovertex ; but here orientation imposes $V_y(T') = \epsilon$ and we are in situation (iii). Finally if $W = W' = \widetilde{W}$, we observe that $y$ cannot be a central vertex (there is always a $\pm 1$ between two 2 in any vertex configuration) ; hence $y$ is a mesovertex ; since $V_y(T)$ and $V_y(T')$ are non empty, orientation imposes $V_y(T) = -V_y(T') = \epsilon$ ; we are in situation (i). □

## 5.2 Metatiles

Consider a tiling by tiles in $\mathcal{D}_n$. We can glue together some tiles (take the union). The resulting tiling (a priori using more shapes) is still a tiling. It makes sense to ask if local rules are still satisfied (because of the structure of our local rules) I mean at the next level they are automatically satisfied (No ?). We observe that if we do the natural glueing, the supertile has exactly the same set of labels and decorations as any tiles.

Let $n \geq 5$ be an integer. We consider a tiling $\tau \in \mathcal{S}_n^*$. We show that

**Lemma 32.** *Let $T$ be a tile in $\tau$. If $T \simeq T^{(n)}$, then there are two tiles $T' \simeq T^{(n+1)}$ and $T'' \simeq T^{(n+2)}$ in $\tau$ with $V(T') \in \mathcal{V}_M$ and $V(T'') \in \mathcal{V}_G$ such that $\widetilde{T} = T \cup T' \cup T" \simeq T^{(n+3)}$ and $v_5(T) = v_5(T') = v_5(T'')$.*

In other words, all small tile belong to a patch to which we will think of as "metatile".

*Proof.* We consider a tile $T \simeq T^{(n)}$ in $\tau$. We observe that $V = V(T) \in \mathcal{V}_P$, hence $V_3(T) = -1$, $V_4(T) = 0$, $V_5(T) = 2$, $V_6(T) = 0$ and $V_7(T) = 1$. We let $x = v_5(T)$. Since $V_5(T) = 2$, $x$ is a triple point : there are two tiles $T'$ and $T''$ in $\tau$ adjacent to $T$ and containing $x$. In virtue of Proposition 14 the words describing the (polygonal) boundaries $T\cap T'$, $T'\cap T''$ and $T" \cap T$ are prefixes of the three fixed points.

We first deal with $T'$. We let $\widetilde{W}$ be the word such that $T\cap T' = x + L(\widetilde{W})$. In view of the boundary of $T$, $\overline{W}_4^{(n)}$ is a prefix of $\widetilde{W}$. Hence, according to Proposition 10, $\widetilde{W}$ starts with $\varphi(\sigma^n(C))$. Since $V_4(T) = 0$ and $V_3(T) = -1$ we are in situation (i) or (iii) of Lemma 31. It follows that $T'$ must satisfy : $V_x(T') = 2$ and $\varphi(\sigma^n(C))$ is a prefix of $W_x^+(T')$.
Let $S \in \mathcal{D}_n$, $V \in \mathcal{V}$, $k \in \{0, 1, 2\}$ and $i \in \{1, \ldots, 7\}$ be such that $S \simeq T^{(n+k)}$, $V(S) = V$ and $T' = x - v_i(S) + S$. We have $V_i(S) = 2$ and $\varphi(\sigma^n(C))$ is a prefix of $W_i^+(S) = W_i^{(n+k)}$. We check that among vertices of tiles in $\mathcal{D}_n$, only $v_5^{(n)}$, $v_5^{(n+1)}$, $v_7^{(n+1)}$, $v_5^{(n+2)}$, $v_7^{(n+2)}$ or $v_2^{(n+2)}$ may be in state 2. In view of Lemma 10, $\varphi(\sigma^n(C))$ is a prefix of $W_i^{(n+k)}$ only if $(i,k) = (5,1)$ or $(i,k) = (2,2)$. If $(i,k) = (2,2)$, then $V_3(S) = 0$ while $v_3(T') = v_3(T)$ which is prohibited (since $V_3(T) = -1$) in view of Lemma 31. Hence $(i,k) = (5,1)$. Now we observe that $v_6(T') = v_3(T)$ so that $V_6(S)$ must be equal to 1 ; it follows that $V(S) \in \mathcal{V}_M$. We conclude that $\sim(T') \in \{T^{(n+1)}\} \times \mathcal{V}_M$.

Now we check the possibilities for $T''$. Of course $V_x(T'') = 2$. Then, geometric considerations around the boundaries $T\cap T"$" and $T'\cap T''$ relying on Proposition 10 show that $\varphi(\sigma^n(A))$ is a prefix of $W_i^-(T'')$ and $\varphi(\sigma^n(B))$ is a prefix of $W_i^+(T'')$.
Let $S \in \mathcal{D}_n$, $V \in \mathcal{V}$, $k \in \{0, 1, 2\}$ and $i \in \{1, \ldots, 7\}$ be such that $S \simeq T^{(n+k)}$, $V(S) = V$ and $T'' = x - v_i(S) + S$. We have $V_i(S) = 2$, t $\varphi(\sigma^n(A))$ prefix of $W_i^-(S)$ and $\varphi(\sigma^n(B))$ prefix of $W_i^+(S)$. It follows that $(i,k) = (2,1)$, $(i,k) = (5,2)$. The first option is excluded because then $v_1(T'') = v_3(T')$ while $V_3(T') = -1$ and $V_1(S) = 0$. Hence $(i,k) = (5,2)$. Since we must have $V_4(T'') = -1$ and $V_6(T'') = +1$, we deduce that $V(S) \in \mathcal{V}_G$. We conclude that $\sim(T') \in \{T^{(n+2)}\} \times \mathcal{V}_G$.
We conclude by observing that the tiling of the tile $T \cup T' \cup T"$ by $T, T'$ and $T''$ is (up to a translation) $\tau^{(n+3,n)}$ (in other words, the glueing is correct). Indeed, the common point is $v_5$. □

We also need the converse statement :

**Lemma 33.** *Let $T$ be a tile in $\tau$. If $T \simeq T^{(n+1)}$ and $\mathcal{V}(T) \in \mathcal{V}_M$, then there are two tiles $T' \simeq T^{(n)}$ and $T'' \simeq T^{(n+2)}$ in $\tau$ such that $\widetilde{T} = T \cup T' \cup T" \simeq T^{(n+3)}$ and if $T \simeq T^{(n+2)}$ and $\mathcal{V}(T) \in \mathcal{V}_G$, then there are two tiles $T' \simeq T^{(n)}$ and $T'' \simeq T^{(n+1)}$ in $\tau$ such that $\widetilde{T} = T \cup T' \cup T" \simeq T^{(n+3)}$.*

*Proof.* Conversely : if $T \simeq T^{(n+1)}$ and $\mathcal{V}(T) \in \mathcal{V}_M$, $V_5(T) = 2$, $V_6(T) = -1$ and $W_i^+(T) = W_5^M$ can not be glued at any other place [Some other would fit geometrically like $\overline{W}_6^G$ but when $V_7 = 2$, then $V_6 = -1$ and glueing is forbiden.]. So $\sim(T') \in \{T^{(n)}\} \times \mathcal{V}_P$. The same analysis as above shows that $\sim(T'') \in \{T^{(n+2)}\} \times \mathcal{V}_G$.
Now if $T \simeq T^{(n+2)}$ and $\mathcal{V}(T) \in \mathcal{V}_M$, $V_5(T) = 2$, and $W_i^-(T) = \overline{W}_4^G$. We deduce that $\sim(T') \in \{T^{(n)}\} \times \mathcal{V}_P$. But we know that close to $T'$ there must be $\sim(T'') \in \{T^{(n+1)}\} \times \mathcal{V}_M$. □

## 5.3 De-substitution

Let $n \geq 5$ be an integer. We define a map $g_n$ from $\mathcal{S}_n^*$ to the space of tilings with $\mathcal{D}_{n+1}$ as follows. Let $\tau \in \mathcal{S}_n^*$. We classify the tiles of $\tau$ : for all tile $T \in \tau$ with $T \simeq T^{(n)}$, we denote $M_T \in \tau$ and $G_T \in \tau$ the tiles $M_T \simeq T^{(n+1)}$ and $G_T \simeq T^{(n+2)}$ such that $v_5(T) = v_5(M_T) = v_5(G_T)$ exhibited in Lemma 32 and $\widetilde{T} = T \cup M_T \cup G_T$. We consider the tile $\widetilde{T}$ together with the following v-configuration and boundary coloring :

$$\widetilde{V} := V_1^P V_7^M V_1^M V_2^M V_7^G V_1^G V_2^G \tag{16}$$

$$\widetilde{c} := c_1^P c_7^M c_1^M c_2^M c_7^G c_1^G c_2^G \tag{17}$$

We define $g_n(\tau)$ by replacing in $\tau$ all occurrences of $T \simeq T^{(n)}$ and its two neighbors by the corresponding $\widetilde{T}$. Formally,

$$g_n(\tau) = \{(\widetilde{T}, \widetilde{V}); T \in \tau, T \simeq T^{(n)}\} \cup \{(T, V(T)), T \in \tau,\ T \simeq T^{(n+1)}, V(T) \in \mathcal{V}_P \text{ or } T \simeq T^{(n+2)}, V(T) \in \mathcal{V}_M \cup \mathcal{V}_P\}.$$

We claim that

**Lemma 34.**

$$g_n(\tau) \in \mathcal{S}^*_{n+1}.$$

*Proof.* Firstly we observe that $g_n(\tau)$ is indeed a tiling : all tile in $\tau$ is either a tile in $g_n(\tau)$ or appears exactly once in a tile of $g_n(\tau)$.

Then we claim that all tile $T$ in $g_n(\tau)$ are in $\mathcal{D}_{n+1}$. It follows from Lemma 33 that if tile $T \simeq T^{(n+1)}$, then $V(T) \in \mathcal{V}_P$ and that if tile $T \simeq T^{(n+2)}$, then $V(T) \in \mathcal{V}_M$ or $V(T) \in \mathcal{V}_P$ (otherwise they would belong to some $\widetilde{T}$). For $\widetilde{T} \in g_n(\tau)$, $\widetilde{T} \simeq T^{(n+3)}$, let us consider the tile $T$ such that $\widetilde{T} = ((\widetilde{T}, \widetilde{V}), \widetilde{c})$.
We notice that (since $T, M_T, G_T \in \mathcal{D}_n$, the colors of edges $c^P_4$, $c^M_4$ and $c^G_4$ reflect $V^P$, $V^M$ and $V^G$ respectively. Then, we recall that $v_5(T) = v_5(M_T) = v_5(G_T)$ and $V_5(T) = 2$. Since $\tau \in \mathcal{S}^*_n$ the local matching rule at central vertex $v_5(T)$ is satisfied ; hence $(c^P_4, c^M_4, c^G_4) = (P/M, U), (M/G, U), (G/P, U)$ for some $U \in \mathcal{V}$. But we observe that in view of the definition [$c$ reflects $U$] we have $V^P_1 = U_1$, $V^M_1 = U_3$, $V^M_2 = U_4$, $V^M_7 = U_2$, $V^G_1 = U_6$, $V^G_2 = U_7$ and $V^G_7 = U_5$ which means that $U = \widetilde{V}$ ; hence $\widetilde{V} \in \mathcal{V}$. For all $i = 1, \ldots, 7$, $\widetilde{c}_i \in S \times \mathcal{V}$ ; it remains to check that $\widetilde{c}_4$ reflects $\widetilde{V}$. To do so, we recall that $(c^P_4, c^M_4, c^G_4) = (P/M, \widetilde{V}), (M/G, \widetilde{V}), (G/P, \widetilde{V})$. Then, using the local matching rule for colors of the boundary between $M_T$ and $G_T$, we deduce that $c^G_5 = c^M_3 = c^M_4 = (M/G, \widetilde{V})$. Noting that at the mesovertex $v^M_3 = v^G_6$ the local matching rule implies (stress $M/G$) $c^M_2 = c^G_6$ reflects $\widetilde{V}$. Since by definition $\widetilde{c}_4 = c^M_2$, we have proved the claim.

Hence $g_n(\tau)$ is a tiling by decorated tiles in $\mathcal{D}_{n+1}$. Now, we check that the local matching rules are fullfilled since they were fullfilled by $\tau$. Indeed :
Condition (0) is obviously satisfied.
Condition (1) : We observe that in view of the vertex condition at vertices $v^P_2$ and $v^G_3$ (with state 0) and at mesovertices $v^P_3 = v^M_6$, $v^M_3 = v^G_6$ and $v^G_4 = v^P_7$ (state $\pm 1$), the following equalities hold : $\tilde{c}_1 := c^P_1 = c^P_2 = c^M_6, \tilde{c}_2 := c^M_7, \tilde{c}_3 := c^M_1, \tilde{c}_4 := c^M_2 = c^G_6, \tilde{c}_5 := c^G_7, \tilde{c}_6 := c^G_1, \tilde{c}_7 := c^G_2 = c^G_3 = c^P_7$. They show that at all point $x \in \partial\widetilde{T}$ the color $c_x(\widetilde{T})$ is equal to $c_x(T^Q)$ whenever $x \in \partial T^Q$. Since in $g_n(\tau)$ there are less boundaries of tiles (than in $\tau$) and the colors on the remaining boundaries have not been changed, we have $c_x(T) = c_x(S)$ whenever $x \in T \cap S$ is not a distinguished vertex [be careful with old vertices]
Condition (2) : for all distinguished vertices $x$ of $\widetilde{T}$, there is a tile $T \in \tau$ such that $V_x(\widetilde{T}) = V_x(T)$, $T \subset \widetilde{T}$ and $x \in T$. If $x$ is a triple point for $\tau$ then we denote $R$ and $S$ (in $\tau$) such that $x = R \cap S \cap T$. We observe that $c_{x^-}(T) = c_{x^-}(\widetilde{T})$ and $c_{x^+}(T) = c_{x^+}(\widetilde{T})$ since distinguished points of $\widetilde{T}$ do not belong to intersections of subtiles. We also observe that if $x$ is a triple point for $g_n(\tau)$ then it is a triple point for $\tau$ and the picture is the same locally (in the sense $c_{x^{\pm\epsilon}}(\widetilde{S}) = c_{x^{\pm\epsilon}}(S)$ if $x \in S \subset \widetilde{S}$).
(a) if $e = 0$, $c_{i-1}(T) = c_i(T)$
(b) if $e = 2$, then $x$ is a triple point for $\tau$. We observe that $\widetilde{R} \neq \widetilde{S}$ (because otherwise $v_5(R) = v_5(S)$ [To convince we must analyse more closely correct glueings]. It follows that $x$ is a triple point for $g_n(\tau)$ and the names and colorings are locally the same around $x$ as for $\tau$.
(c) is $e = \pm 1$ then $x$ is a triple point for $\tau$ and that $(V_x(R), V_x(S), V_x(T)) = (0, -1, 1)$ or $(1, 0, -1)$. We observe that $\widetilde{R} \neq \widetilde{S}$ (this would be possible only if $(V_x(R), V_x(S), V_x(T)) = (-1, 1, 0)$ [why ?]. □

## 5.4 Conclusion

We have shown that tile $T^{(n)}$ with role $P$, tiles $T^{(n+1)}$ with role $M$ and tiles $T^{(n+2)}$ with role $G$ always appear in a common metatile $T^{(n+3)}$ ; if we glue them, we recover a tiling made only of tiles $T^{(n+1)}$ with role $P$, tiles $T^{(n+2)}$ with roles $M$ or $P$ and tiles $T^{(n+3)}$ (with any role). Formally :

**Corollary 35.** *The tiling $\widetilde{\tau}$ obtained from $\tau$ by glueing together all tiles $T \simeq T^{(n)}$ with the tiles $T'$ and $T$" (as above) into a meta tile $\widetilde{T}$ yields a tiling in $\mathcal{S}_{n+1}$.*

From this result on, taking the projection, we should be able to conclude. Let $\tau \in \mathcal{S}^*_5$. and consider a patch $Q$ in $\tau$. We can apply recursively result 35. If we are lucky, there is $n$ such that $Q$ is included in a tile of $g_n(g_{n-1}(\cdots g_2(g(\tau))\cdots))$ ; this immediately implies that the patch $Q$ projects onto a patch of $\tau^{(n,5)}$. Otherwise situation is more tricky. Since $Q$ is finite there is $N$ and a partition $(Q_1, \ldots, Q_d)$of $Q$ such that for all $n \geq N$ the trace of $g^n(\tau)$ on $Q$ is the same : all patches $Q_1, \ldots, Q_d$ are included to a distinct tile of $g^n(\tau)$. All patches $Q_1, \ldots, Q_d$ project on patches of $\tau^{(n,5)}$. If there is a triple point ($d \geq 3$) then we check that there are only two possibilities for the shapes of the boundaries and that both are allowed in the language of $\tau^{(\infty,5)}$. The most tricky case is when $d = 2$.

For all $n > N$ there are two tiles $Q_1^{(n)}$ and $Q_2^{(n)}$ in $\alpha^{(n)} = g_n(g_{n-1}(\cdots g_2(g(\tau))\cdots))$ (translated of tiles in $\mathcal{T}_n$) such that $Q_1 \subset Q_1^{(n)}$ and $Q_2 \subset Q_2^{(n)}$. We consider $R_1 = Q_1^{(N+3)}$ and $R_2 = Q_2^{(N+3)}$. we observe that $L = R_1 \cap R_2$ is a stepped line "crossing" $Q$. We denote $x$ and $y$ the triple points in $L$ (left and right, positive sense on $Q_1$).

1. They are in state $0, -1$ or $1$ (not 2). one of our tiles is marked 0 (otherwise glue at some scale ?)
2. Choose one, say $x$. Assume $V_x(R_1) = 0$ and $V_x(R_2) = -1$ : there is a scale at which this triple point is "used" (otherwise triple point situation). Let $K$ be this scale and consider $S_1 = Q_1^{(K)}$ and $S_2 = Q_2^{(K)}$ ($x$ is not any longer a triple point at scale $K$)
3. We observe that $x = v_i(Q_2^{(K-1)}$ for $i$ a future mesovertex is the glueing at $K$. More precisely, $S_2 = S_2^P \cup S_2^M \cup S_2^G$ and $Q_2^{(K-1)} = S_2^E$ with $E = P, M$ or $G$. follows $i = 3, 3$ or $4$ respectively. We deduce the shape of the boundary of $S_2$ around $x$. We deduce the shape of the boundary of $S_1$ around $x$.
4. From the shape of the boundary around $x$ we see that there is a finite number of possibilities among the tiles in $T \in \mathcal{T}_K$ and vectors $V$ such that $S_2 = v + T$. Indeed, once given $E$, there are only two possibilities.
5. We consider all the possibilities. Then, we check that all the pairs of tiles of level $K - 3$ that can appear (after decomposition of $S_1$ and $S_2$ are pairs of tiles that appear in $\tau^{(\infty,K-3)}$. Recal that $Q_1$ and $Q_2$ are included in tiles of scale $N < K - 3$. We conclude that $Q$ is a patch of $\tau^{(\infty,5)}$.

# 6 Proof of the theorem

It is very simple to conclude (if we do not care the number of tiles used. We consider all rhombus that appear in the labelled and decorated tiles at all possible positions with different names. The rule is that a rhombus can only be glued with its neighbors in the tile. For those touching the boundary and the distinguished vertices, we ask the local rules to be fulfilled. And that is it.

# 7 Perspectives

**Estimation of the number of tiles needed**

**A more combinatorial point of view : Bedaride and Hillion**

**The fractal tiling $\tau^{(\infty,\infty)}$.**

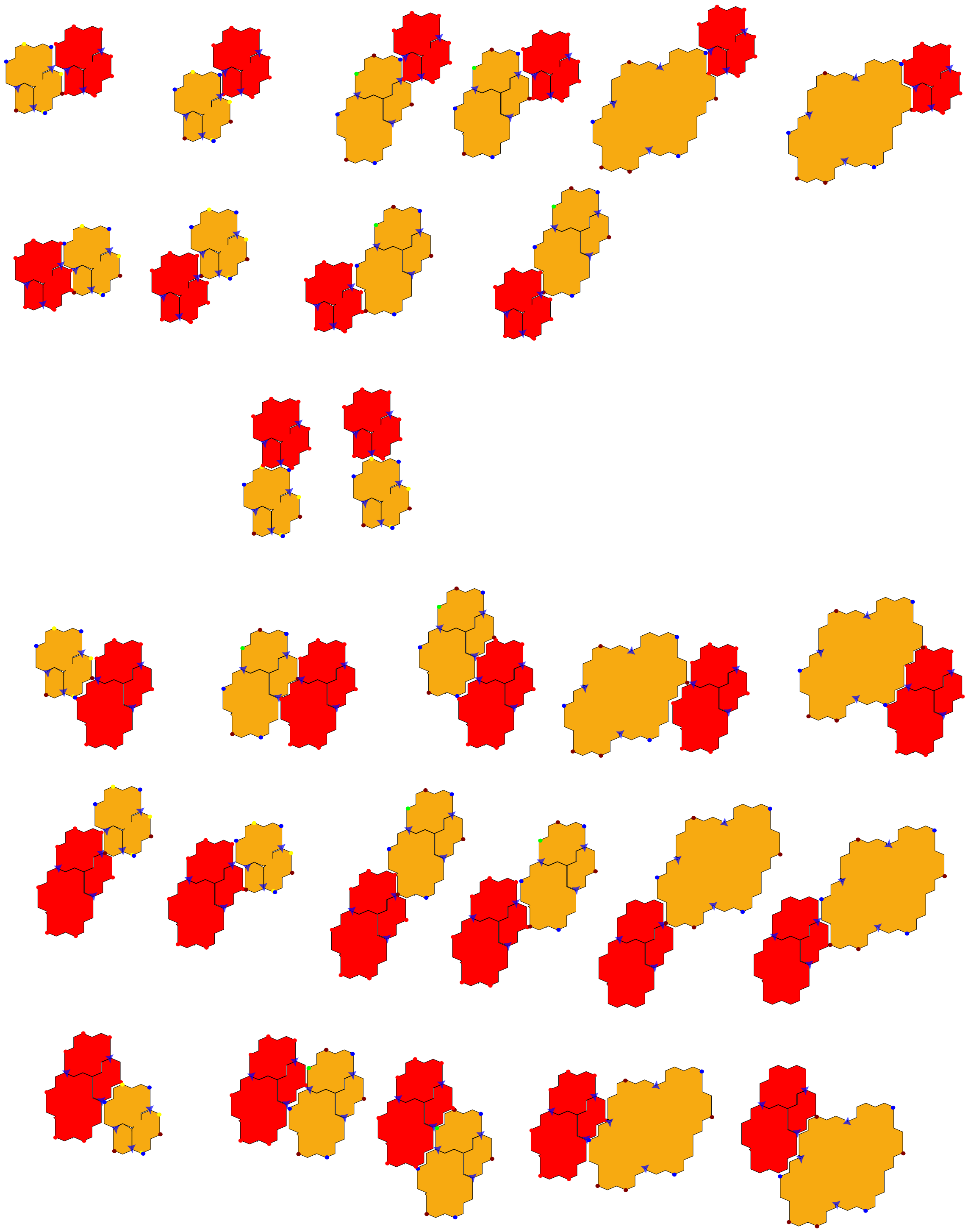

Figure 9: Pairs close to a mesovertex.

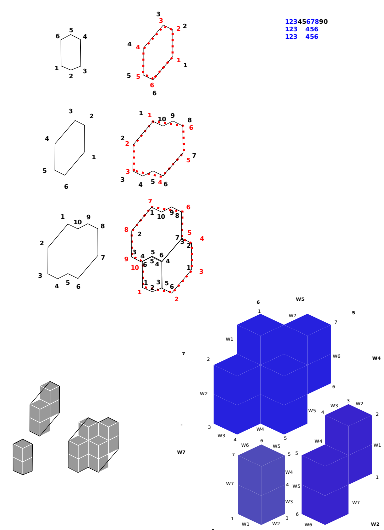

Figure 10: Combinatorial substitution corresponding to Tribo

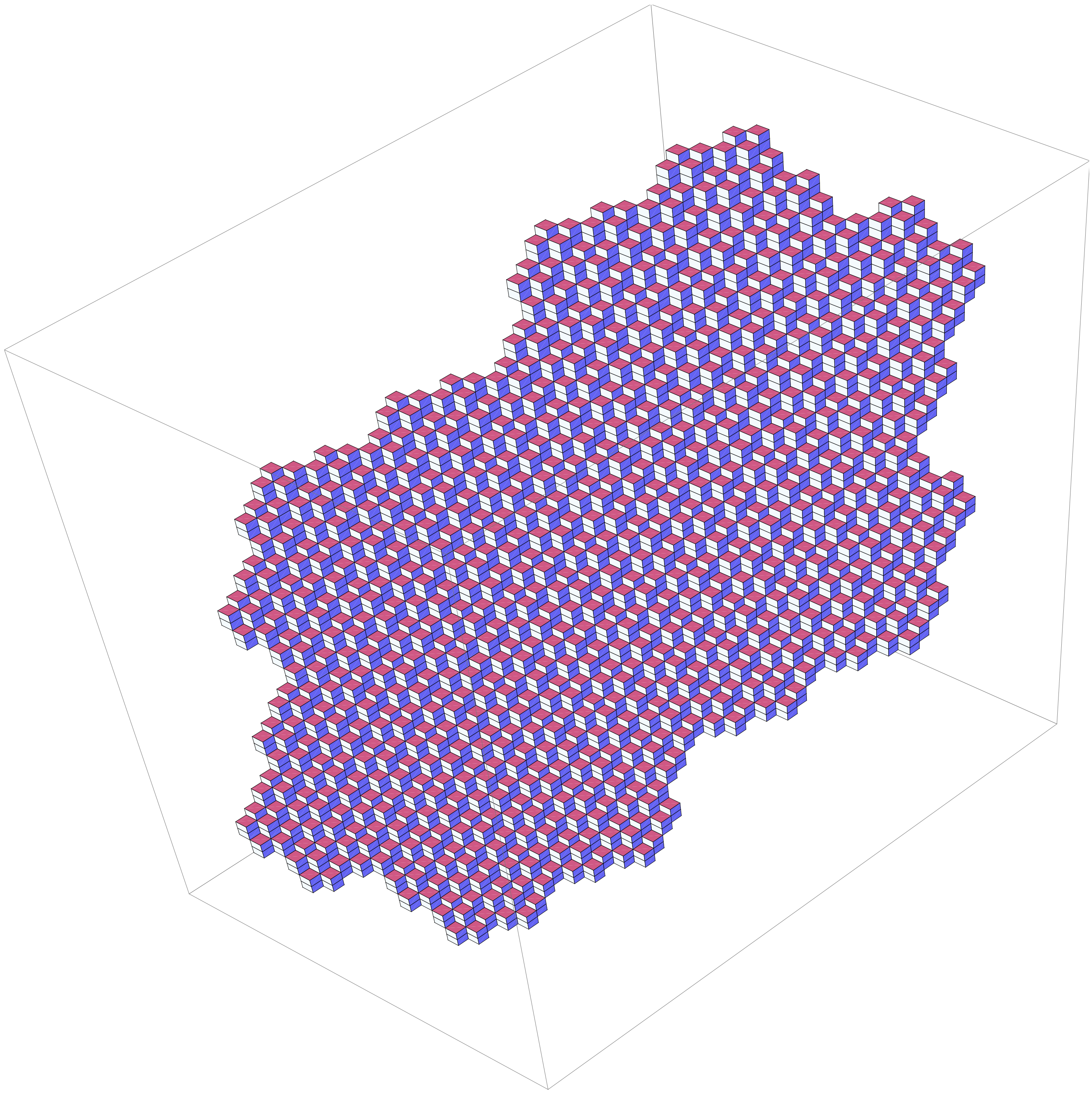

Figure 11: The tiling