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Abstract. We exhibit a new family of piecewise monotonic expanding interval maps with interesting intermittent-like statistical behaviours. Among these maps, there are uniformly expanding ones for which a Lebesgue-typical orbit spends most of the time close to an "indifferent Cantor set" which plays the role of the usual neutral fixed point. There are also examples with an indifferent fixed point and an infinite absolutely continuous invariant measure. Like in the classical case, the Dirac mass at 0 describes the statistical behaviour at usual time scale while the infinite one tells about the statistical behaviour at larger scales. But, here, there is another invariant measure describing the statistical behaviour of the ergodic sums at a third natural (intermediate) time scale.

To try to understand this last phenomenon, we propose a more general construction that yields an example for which we conjecture there is an infinite number of natural time scales.

1 Introduction

Before stating the main results we discuss the notion of physical measure for hyperbolic dynamical systems, which is not universally defined and we roughly describe the situation in the case of intermittent maps.

1.1 Physical measures

The notion of physical measure is not universally defined. Given a dynamical system with a reference measure (for example the Lebesgue measure, or a Riemannian volume), one wants to call physical measure an invariant measure that has special properties with respect to the reference measure.

For expanding maps under the Lebesgue measure, one can look for invariant probability measures that are absolutely continuous with respect to the Lebesgue measure, a.c.i.p. (See Lasota, Yorke, 1973 ; Walters, 1975). In the case of uniformly hyperbolic area preserving maps of a d dimensional Riemannian manifold, absolute continuity with respect to the Riemannian volume is a relevant property of invariant probability measures (See Bowen, 1975). But when such a map is dissipative, the only property one can expect from a reasonably "natural" measure is absolute continuity with respect to the restriction of the volume to the unstable manifolds. Another important property of such a measure (under reasonable assumptions) is that, almost surely with respect to the reference measure, the orbital measures ($\frac{1}{n}\sum_{k=0}^{n-1}\delta_{T^k x}$) converge to this measure.

For nonuniformly hyperbolic maps, the situation can be slightly more sophisticated. In the case of the model of intermittency introduced by Pommeau and Manneville (1980), as well as in the piecewise affine approximation studied by Wang (1989) (or more generally for classes of expanding maps with an indifferent fixed point at 0), the situation is clear. For some values of the parameter (mainly, the order of the tangency at the fixed point), there are two distinct measures playing a physical role :

- The Dirac mass δ_0 at the fixed point. It is a finite measure. It is the weak limit of the orbital measures for Lebesgue almost every starting point.
- The unique σ-finite (but infinite) invariant measure μ that is absolutely continuous with respect to the Lebesgue measure.

Both are relevant in that they describe asymptotic behaviour of the ergodic sums starting under the Lebesgue measure. The first one is related to the usual time scale. That is, for Lebesgue almost all points,

$$\frac{1}{n}\sum_{k=0}^{n-1}\delta_{T^k x} \to \delta_0.$$

The other one describes the spatial repartition of orbits in areas where they spend only a small fraction of the time. Aaronson's ergodic Theorem (Aaronson, 1997) shows that one cannot expect almost sure results in this direction. However, a standard result (Zweimüller, 1995) is the existence of a renormalizing sequence (α_n) such that, if $f \in L^1_\mu$ — hence, f(0) = 0 — the ergodic sums satisfy, under the Lebesgue measure,

$$\frac{1}{\alpha_n}\sum_{k=0}^{n-1} f\circ T^k \overset{\text{(law)}}{\to} \mu(f)\mathcal{ML}_\alpha,$$

where $\mathcal{ML}_\alpha$ denotes a random variable with the Mittag-Leffer law of index α (see Feller, 1971). The times of passage in sets that are far from the indifferent fixed point are of zero density. So that, to "see" these passages, one must look at the system at another "time scale", given here by (α_n). The infinite measure μ describes the "spatial repartition" at this scale. The random variable $\mathcal{ML}_\alpha$ takes into account statistical fluctuations of this repartition. The time scale depends on the tail of the law of the return time to a subset that is far from the fixed point.

None of the two invariant measures separately really describe the system in a satisfactory way. That is why we suggest to call a physical measure *any* measure such that there is a scale of time for which it is relevant with respect to the behaviour of the ergodic sums, under the reference measure. We shall say that a measure μ is *physical* for some time scale (α_n) as soon as the normalized ergodic sums $\frac{1}{\alpha_n}\sum_{k=0}^{n-1} f\circ T^k$ converge in law under the reference measure to some random variable proportional to $\mu(f)$. More formally,

Definition 1 *We say that μ is a physical measure for some time scale (α_n) if there is a random variable $\mathcal{H}$ such that, for all smooth enough functions $f \in L^1_\mu$,*

$$\frac{1}{\alpha_n}\sum_{k=0}^{n-1} f\circ T^k \overset{\textit{(law)}}{\to} \mu(f)\mathcal{H},$$

under the reference measure.

According to this definition, an ergodic absolutely continuous invariant probability measure is a physical measure with time scale (n) and $\mathcal{H} = \delta_1$. Notice that the situation could be more complicated than in the example above. A system given together with a nonsingular reference measure — that is a differentiable map — could have, a priori, a family of physical measures. Notice also that we do not specify the regularity we require for the class of test functions.

This work was motivated by the following two questions :

- When there are two physical measures, can the finite one be more complicated than just a Dirac mass ? For example, can it support a nontrivial dynamic ?
- Are there systems for which arise naturally more than two physical measures ? And, hence, more than two time scales ?

1.2 Intermittent maps

The Pomeau-Manneville model for intermittency is a class of maps defined on $[0,1]$ with two C^2 expanding onto branches. They are uniformly expanding out of all neighborhoods of 0 and have a neutral (or indifferent) fixed point at 0 ($T'(0) = 1$). An important parameter is the order of the tangency at 0.

What we call a Wang map is a piecewise affine version of the smooth Pommeau-Manneville map (see Section 2.4 for a precise definition). It is constructed in such a way that the return map to the right-most interval is made of affine onto branches. This fact considerably simplifies the analysis since the return times to the right-most interval then form a sequence of identically distributed independent (iid) random variables. Hence, statistical results for occupation times follow directly from classical results about sums of iid random variables. These systems are equivalent to (simple) renewal Markov chain. We will make an intensive use of this simplification.

Roughly speaking the same analysis should hold for smooth cases. But to obtain results, one must have distortion estimates, to control asymptotic independence of these return times. Technically, one has to use more sophisticated tools. Here we shall restrict our analysis to piecewise affine cases.

These examples have been intensively studied. For the values of the parameter where they have an acip, the density of this acip has a singularity at the indifferent fixed point. These systems are interesting in that they are the simplest examples where there is no exponential decay of correlations (even for very regular observables). For estimations on the speed of this decay, see Liverani, Saussol, Vaienti (1999), Hu (1999), Fisher, Lopes (1997), Young (1999), Sarig (2002). For limit theorems, see Fisher, Lopes (1997), Zweimüller (2003). There is a large deviations result (Pollicott, Sharp, Yuri, 1999) in a weak form. This last result is not sharp enough for our purposes. So, we prove a large deviation result for ergodic sums of step functions in this context (Lemma 2.2) which can be of interest independently of further results (see Section 2.4.3).

In the case when the acim is infinite, the main results come after the work of Thaler, from Aaronson's infinite ergodic theory (Aaronson, 1997). The facts we need are well exposed in Zweimüller (1995). We summarize the important facts in Section 2.4.4. An important tool is the study of the return times to a set of finite measure. These return times are naturally related to the occupation rates of the set and hence to the ergodic sums.

1.3 Statement of the main results

Consider the interval $I = [0,1]$ endowed with the Lebesgue measure λ. We are going to define a class of interval maps on I. The study of the asymptotic behaviour of their ergodic sums will provide examples with properties stated in the two following theorems.

Theorem 1.1 *Let $0 < \alpha < 1$ be a real number. There is a piecewise continuous uniformly expanding map T, differentiable on I, except on a countable set, topologically equivalent to $x \mapsto 3x(mod 1)$, with a σ-finite invariant measure $\mu \ll \lambda$ and an invariant probability measure μ_1 with positive entropy $h_{\mu_1}(T) > 0$, such that, for all f continuous,*

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to \mu_1(f), \quad \lambda - a.e.,$$

and, for all $f \in L^1_\mu$ with $\mu(f) > 0$, under λ,

$$\frac{1}{n^\alpha}\sum_{k=0}^{n-1} f \circ T^k \overset{(law)}{\to} \mu(f)\mathcal{ML}_\alpha.$$

It seems that the ideas that yield the construction of this example are very close to those used in the paper by Gora and Schmitt (1989) where they construct a piecewise C^1 uniformly expanding map of the interval which does not have a finite absolutely continuous invariant measure. Our construction is more explicit. For example, we are able to describe explicitly the infinite a.c.i.m. as well as the "attracting" singular probability measure. Moreover, the "intermittent like" behaviour of the maps — while they are expanding — follows from the construction. Our construction also yields examples satisfying the following theorem :

Theorem 1.2 *Let $0 < \alpha < \beta < 1$ be two real numbers with $\alpha + \beta < 1$. There is a piecewise continuous expanding map, differentiable on I, except at a countable set, topologically equivalent to $x \mapsto 3x$ (mod 1) with a σ-finite measure $\mu \ll \lambda$, such that, for all f continuous,*

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to f(0), \quad \lambda - a.e.,$$

and, for all $f \in L^1_\mu$ with $\mu(f) > 0$, under λ,

$$\frac{1}{n^\alpha}\sum_{k=0}^{n-1} f \circ T^k \overset{(law)}{\to} \mu(f)\mathcal{ML}_\alpha.$$

Moreover, there is a σ-finite measure μ_1 and a random variable $\mathcal{H}_{\alpha,\beta}$, such that, for all Lipschitz functions $f \in L^1_{\mu_1}$ with $\mu_1(f) > 0$, under λ,

$$\frac{1}{n^\beta}\sum_{k=0}^{n-1} f \circ T^k \overset{(law)}{\to} \mu_1(f)\mathcal{H}_{\alpha,\beta}.$$

To prove both theorems, we introduce a family of piecewise affine interval maps indexed by two normalized sequences $a = (a_n)$ and $b = (b_n)$ of positive real numbers. Among this family, one can find maps satisfying Theorem 1.1 and maps satisfying Theorem 1.2. They have three full branches defined on three intervals, I_0, I_1 and I_2. They differ only by their slopes. The slopes are constants on intervals of a countable partition of I. They are designed to give prescribed laws of return times to given subsets. The sequences a and b give the laws of these return times. More precisely, let $I = I_0 \cup I_1 \cup I_2$. If the return time to I_2 is long enough — think of it with a distribution whose tail is given by the sequence (a_n) not integrable — then what will appear at usual scales of times is the dynamic restricted to the subsystem $I_0 \cup I_1$. This is more or less what we need for Theorem 1.1. Now, if the dynamic restricted to $I_0 \cup I_1$ is itself intermittent like, with a stochastically smaller return time, then we are in the situation of Theorem 1.2.

These considerations lead to the idea of having more time scales and physical measures. Finally (see Section 4), we construct an example for which we conjecture there is an infinite number of relevant time scales. The way we construct these systems is straightforward and may seem artificial. However, we think that the understanding of the interaction between these distinct scales can help understand some dynamical systems for which there are no a priori natural invariant measures.

1.4 Outline of the paper

Section 2 is devoted to the presentation of preliminary results. In Subsection 2.1, we give a taste of the way a probabilistic point of view can be used to understand the statistical behaviour of a dynamical system. In Subsection 2.2 we recall standard results about asymptotic laws for sums of iid random variables with heavy tails involving stable laws. In Subsection 2.3 we recall the duality between occupation rates and return times needed for results on asymptotic laws for ergodic sums. In Subsection 2.4, we define Wang maps and summarize their properties. We state a large deviations result (proved in Appendix A) in the case where the acim is finite and recall the behaviour of the ergodic sums when the measure is infinite.

Section 3 is devoted to the study of our family of examples. We give an intuitive description of the construction in Subsection 3.1. The family is defined in Subsection 3.2. Its first ergodic properties are summarized in Subsection 3.3. The main results follow from the study of the behaviour of the ergodic sums in Subsection 3.4. Subsection 3.5 shows how to use the latter to prove Theorem 1.1 and Theorem 1.2.

The last section is devoted to a more general construction and is technically independent of the first ones. It also contains further comments and questions.

1.5 Acknowledgments

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2 Preliminary results

All along the paper, we will use $\mathbf{N}$, $\mathbf{N}^*$, $\mathbf{Z}$ and $\mathbf{R}$ to denote the set of nonnegative integers, of positive integers, of integers and of real numbers, respectively.

2.1 Dynamical systems and random processes.

Consider a probability space $(\Omega, \mathcal{A}, P)$. A real *random variable* X on this probability space is nothing but a measurable map from Ω onto $\mathbf{R}$. Its *law* or *distribution* is the image measure P_X on $\mathbf{R}$ defined by $P_X(A) = P(X^{-1}(A))$ for all Borel sets A in $\mathbf{R}$. It is characterized by the distribution function F_X defined on $\mathbf{R}$ by $F_X(t) = P(X < t) = P(\{\omega \in \Omega, X(\omega) < t\})$. Its expectation $E_P[X]$, or, if not ambiguous, $E[X]$ or $P(X)$, is the integral of this function with respect to the probability measure, $E_P[X] = \int_\Omega X(\omega)dP(w) = \int_{\mathbf{R}} x dP_X(x)$. Two random variables, X and Y, even defined on different spaces can have the same law. We would then write

$$X \overset{\text{(law)}}{=} Y.$$

Now, consider a sequence $(X_n)_{n\in\mathbf{N}}$ of real random variables on this space. It is said to *converge in law* to a random variable X (the definition of X does not matter, only its law) if P_{X_n} converges weakly to P_X, or equivalently, if F_{X_n} converges simply to F_X. It is also equivalent to the convergence of the so-called characteristic functions, $E[e^{itX_n}] \to E[e^{itX}]$. We write,

$$X_n \overset{\text{(law)}}{\to} X.$$

Let $(\mathcal{X}, \mathcal{B}, m)$ be a probability space, and consider the dynamical system $(\mathcal{X}, T)$ on $\mathcal{X}$. Let f be an observable, that is a map from $\mathcal{X}$ to $\mathbf{R}$. In the language of probability, f is a random variable. The sequence $(f \circ T^n)_{n\in\mathbf{N}}$ is a sequence of real random variables. The sequence of ergodic sums $\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$ is also a sequence of real random variables, built as sum of random variables. The entrance time in a measurable subset A of $\mathcal{X}$, defined by $\tau_A(x) = \inf\{n \geq 0, T^n(x) \in A\}$ is an integer-valued random variable.

If the measure m is invariant, then the sequence $(f \circ T^n)$ is identically distributed, i.e. all these random variables have the same law. The ergodic theorem is an almost sure statement on the sequence of ergodic sums. In the ergodic situation, the sequence of random variables converges almost surely to the expected value $m(f) = E_m[f]$ of f. The central limit theorem is a convergence in law of the same sequence, suitably normalized, to the Gaussian law. When it holds, it is a statement about the repartition of the fluctuations of the ergodic sums around the expected value. Formally, it writes, for some σ,

$$m(\left\{ x \in \mathcal{X} : a\sqrt{n} < \sum_{k=0}^{n-1} f \circ T^k(x) - nm(f) < b\sqrt{n} \right\}) \to \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{t^2}{2\sigma}} dt.$$

The observable $f = \mathbf{1}_{\{A\}}$ for $A \in \mathcal{B}$ is of special interest. In the ergodic case, $\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to m(A)$. If R is the induced map on A, $\tau_i = \tau_A \circ R^i$ are the return times to A. Kac's lemma states that $E[\tau_i] = m(A)^{-1}$. We will extensively use the obvious "duality" between the times of passage in A and the number of passages in A formalized by $\left\{\sum_{i=0}^k \tau_i \leq n\right\} = \left\{\sum_{i=0}^n \mathbf{1}_{\{A\}} \circ T^i \geq k\right\}$.

Usually, the random variables $(f \circ T^n)$ are not independent. But mixing properties of the dynamical system can be understood as "asymptotic independence" of f and $f \circ T^n$ when n grows. For smooth uniformly expanding systems, this idea can be quantified for f Hölder, yielding the so-called exponentially mixing properties. Usually, the central limit theorem can be derived from such properties. In the specific case of piecewise affine expanding maps (think for example of the doubling map on the circle, $x \mapsto 2x(mod 1)$) it is possible to observe exact independence of certain natural observables. For this reason, such maps are a nice toy model to understand the statistical behaviour of smooth expanding maps, up to distortion phenomena.

The ideal case is when it is possible to find a set A such that the sequence of return times to A is a sequence of iid random variables. It is the case if the induced map on A is made of affine onto branches (see Subsection 2.3 for definitions and a more precise statement). In this situation, statistical properties of the system may be deduced from an analysis of successive independent excursions out of A.

2.2 Sums of iid random variables and stable laws

We denote by $\mathcal{N}(m,s)$, $\mathcal{G}_\alpha$ and $\mathcal{ML}_\alpha$, the normal law with expectation m and variance s, the one-sided stable law of index $\alpha \in (0,2)$ and the Mittag-Leffer law of index α, respectively. (For precise definitions, see Feller, 1971). We may use the same letters to denote a random variable with the corresponding law. Stable laws appear as laws of sums of random variables, while Mittag-Leffer laws are laws of inverse of stable laws to some power. The first ones have heavy tails, while the latter have all exponential moments. They all have densities with respect to Lebesgue measure on $\mathbf{R}$ and are supported on the positive half line. Although their densities are not explicit in general, they are well known through Fourier-like transforms.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let X be a positive real-valued random variable with law, $F(x) = \mathbf{P}(X < x)$. A sum of iid random variables (X_n) with common law F, satisfies the law of large numbers if the random variables are of finite expectation, $E[X] < +\infty$,

$$\frac{1}{n}\sum_{k=0}^{n-1} X_k \to \mathbf{E}[X], \quad a.e..$$

This result extends to the infinite expectation case, the limit becoming infinite.

If there is $\alpha > 0$ such that $1 - F(x) \sim cx^{-(1+\alpha)}$, then F is in the basin of the one-sided stable law of index α. If $\alpha > 2$, then the sums satisfy a central limit theorem,

$$\frac{1}{\sqrt{n}}\left(\sum_{k=0}^{n-1} X_k - n\mathbf{E}[X_1]\right) \stackrel{\text{(law)}}{\to} \mathcal{N}(0, E[X^2]).$$

If $\alpha \in (1,2)$ then,

$$\frac{1}{n^\alpha}\left(\sum_{k=0}^{n-1} X_k - n\mathbf{E}[X_1]\right) \stackrel{\text{(law)}}{\to} \mathcal{G}_\alpha.$$

If $\alpha \in (0,1)$ then, $\mathbf{E}[X_1] = +\infty$, and,

$$\frac{1}{n^{\frac{1}{\alpha}}} \sum_{k=0}^{n-1} X_k \overset{\text{(law)}}{\to} \mathcal{G}_\alpha. \tag{2.1}$$

For more details and proofs, we refer to Feller (1971). These results are also stated in Zweimüller (1995), Theorem 4.8, page 42.

Remark 1 *For simplicity, we stated results in very particular cases. There are also results for $\alpha = 2$, $\alpha = 1$ and $\alpha = 0$. More generally, distributional convergence also holds for sums of iid random variables when the tail of the law is a regularly varying sequence.*

2.3 Occupation times and Mittag-Leffer laws

Let I be an interval. Let $T : I \to I$ be a piecewise affine map. For a subinterval $B \subset I$, let τ denote the entrance time into B, $\tau(x) = \inf\{k > 0 : T^k(x) \in B\}$ and $R = T^\tau$ be the return map. Assume there is a subdivision of B into intervals $(B_n)_{n\geq 1}$ such that for all $n \geq 1$, τ is constant on B_n and B_n is mapped affinely onto B by R.

In this situation, the law of the entrance time τ is easy to compute in terms of the measures of the B_n. Moreover, the sequence $\tau_i = \tau \circ R^i$ is a sequence of iid random variables. If the law of the entrance time τ to B satisfies,

$$\lambda_B(\tau > n) \sim n^{-\alpha}, \tag{2.2}$$

where $\alpha \in (0,1)$, it is possible to check that there is a unique invariant measure μ absolutely continuous with respect to Lebesgue (we will do this in the specific cases we are going to work with). The induced measure μ_B must be the Lebesgue measure restricted to B, $\lambda_B = \mu_B$.

In addition, under μ_B, $\sum_{i=0}^{n-1} \tau_i$ is a sum of iid random variables. It follows from Statement (2.1) in Section 2.2 that this sum converges in law,

$$\frac{1}{n^{\frac{1}{\alpha}}} \sum_{i=0}^{n-1} \tau_i \overset{\text{(law)}}{\to} \mathcal{G}_\alpha.$$

For all functions f defined on I, we consider the ergodic sums, $\sum_{k=0}^{n-1} f \circ T^k$. Since the measure μ is not finite, there is no hope that these sums have an almost sure limit, even correctly renormalized, according to Aaronson's ergodic theorem (see Aaronson, 1997). But, a direct application of Theorem 11.10, page 135, stated in Zweimüller (1995), (see also Aaronson, 1997), yields

Fact 1 *In this case, for all positive $f \in L^1_\mu$ (with $\mu(f) \neq 0$), and for any initial distribution $\mathbf{P} \ll \mu$,*

$$\frac{1}{n^\alpha} \sum_{k=0}^{n-1} f \circ T^k \overset{\text{(law)}}{\to} \frac{\mu(f)}{\mu(B)} \mathcal{ML}_\alpha,$$

because $\mathcal{ML}_\alpha$ is the law of the inverse of a random variable with stable law of index α to the power α, and n^α is asymptotically inverse to $n^{\frac{1}{\alpha}}$.

2.4 Wang map

In this section, we present the classical example introduced by Gaspard and Wang (1988) to simplify the Pommeau and Manneville (1980) model of intermittency. This model will be our basis for further models. We apply the previous results to describe the asymptotic behaviour of the ergodic sums under the Lebesgue measure and we prove a large deviations result which is not in the literature.

2.4.1 Definition

Let $\overline{I} = [0;1]$ and $\overline{\lambda}$ be the Lebesgue measure on $\overline{I}$. Let (b_n) be a decreasing sequence with $\sum b_n = 1$. Set $c_n = \sum_{k=n}^{+\infty} b_k$ and $\overline{J}_n = [c_{n+1}, c_n)$. We define the *Wang map of sequence* (b_n) on $\overline{I}$ by, $\overline{T}(0) = 0$, and,

$$\overline{T}(x) = \begin{cases} c_n + \frac{b_{n-1}}{b_n}(x - c_{n+1}) & \text{if } n > 0 \text{ and } c_{n+1} \leq x < c_n, \\ \frac{1}{b_0}(x - c_1) & \text{if } c_1 < x < c_0 = 1. \end{cases}$$

2.4.2 First properties

On the one hand, $\overline{T}(c_{n+1}) = c_n$ and $\overline{T}(c_n) = c_{n-1}$ so that $\overline{T}(\overline{J}_{n+1}) = \overline{J}_n$ and, hence, $[0, c_1]$ is mapped bijectlively and continuously onto $[0, 1]$. On the other hand, $\overline{T}(c_1) = 0$ and $\overline{T}(1) = 1$ so that $[c_1, 1]$ is affinely mapped onto $[0, 1]$. So $\overline{T}$ is topologically conjugated to the doubling map. We notice that 0 is an indifferent fixed point as soon as $\frac{b_{n-1}}{b_n} \to 1$.

This conjugacy yields a natural coding sending almost all points in $\overline{I}$ to a one-sided sequence in $\{0,1\}^{\mathbf{N}}$. Let us denote $\overline{I}_0 = [0, c_1)$ and $\overline{I}_1 = [c_1, 1]$. The coding map is defined by $\omega(x) = (\omega_0, \ldots, \omega_n, \ldots) \in \{0,1\}^{\mathbf{N}}$ if and only if, for all $n \geq 0$, $\overline{T}^n x \in \overline{I}_{\omega_n}$. Using this coding, we can denote by $\overline{I}_\omega$, for $\omega \in \{0,1\}^n$ the interval $\overline{I}_\omega = \{x \in \overline{I} : \overline{T}^{m-1} x \in \overline{I}_{\omega_m}, 1 \leq m \leq n\}$. We call *dynamical partition*, the partition $\{\overline{I}_\omega, \omega \in \{0,1\}^N\}$, for some integer N. We set $b_\omega = \overline{\lambda}(\overline{I}_\omega)$. We denote by σ the shift on $\{0,1\}^{\mathbf{N}}$. For $n > 1$ and any finite word $\omega = \omega_1 \cdots \omega_n \in \{0,1\}^n$, we set $\sigma(\omega) = \omega_2 \cdots \omega_n$, so that $\overline{T}(\overline{I}_\omega) = \overline{I}_{\sigma(\omega)}$.

The study of such a simple map can be done using the return times to the interval $\overline{I}_1$. Let $\overline{\tau}$ denote the first entrance time in $\overline{I}_1$, $\overline{R} = \overline{T}^{\overline{\tau}}$, the induced map and $\overline{\tau}_i = \overline{\tau} \circ \overline{R}^i$, the return times to $\overline{I}_1$. Clearly, the Lebesgue measure restricted to $\overline{I}_1$ is $\overline{R}$-invariant because $\overline{R}$ has full affine branches. Morever, the sequence $(\overline{\tau}_i)$ is a sequence of iid random variables. The law of these return times with respect to the Lebesgue measure is easy to compute. The entrance time is exactly n if $c_{n+1} \leq x < c_n$ (for $n > 0$) because $\overline{T}(\overline{J}_{n+1}) = \overline{J}_n$, so that $\overline{T}^n(\overline{J}_{n+1}) = \overline{J}_0 = \overline{I}_1$. For $n = 0$, it depends on the $\overline{J}_n$ in which falls $\overline{T}(x)$. A simple computation yields,

$$\overline{\lambda}(\overline{\tau} = n) = b_n + b_0 b_{n-1}, \text{ and, for } i > 0,\ \overline{\lambda}(\overline{\tau}_i = n) = b_{n-1}. \tag{2.3}$$

Proposition 2.1 *There is a (unique) $\overline{T}$-invariant measure $\overline{\mu} \ll \overline{\lambda}$. If the sequence $(\frac{b_{n-1}}{b_n})_{n \geq 0}$ as a limit b^*, then, $\overline{T}$ is differentiable at 0 and $\overline{T}'(0) = b^*$.*

(i) If $b^* > 1$, then $\overline{T}$ is uniformly expanding. The measure $\overline{\mu}$ is finite and $h_{\overline{\mu}}(T) > 0$.

(ii) If $b^* = 1$ and $\sum_n nb_n < +\infty$, then, the map has an indifferent fixed point, but $\overline{\mu}$ is finite.

(iii) If $b^* = 1$ and $\sum_n nb_n = +\infty$, then, $\overline{\mu}$ is σ-finite but it is infinite.

If $\overline{\mu}$ is finite, we normalize it, so it is a probability measure. If it is not, we normalize it so that $\overline{\mu}(\overline{I_1}) = 1$.

Proof For all $n \geq 0$ and all $x \in \overline{J}_n$, we set,

$$\overline{\rho}(x) = \frac{1}{b_n} \sum_{k \geq n} b_k,$$

and $\overline{\mu} = \overline{\rho}\overline{\lambda}$, so that $\overline{\mu}(\overline{J}_n) = \sum_{k \geq n} b_k$. This measure is absolutely continuous with respect to $\overline{\lambda}$. It is $\overline{T}$-invariant because the density $\overline{\rho}$ is a fixed point of the Perron Frobenius operator. It is finite on all compact subsets of $\overline{I}$ not containing the fixed point. It is finite if and only if the return time has finite expectation, since the conditions $\sum_n nb_n < +\infty$ and $\sum_n \sum_{k \geq n} b_k < +\infty$ are equivalent. We refer to Wang (1989) for more details. $\square$

2.4.3 Case of a finite measure : a large deviations result

When the measure $\overline{\mu}$ is finite, we will need a result about large deviations, that is a bound on the $\overline{\lambda}\left(|\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k - \overline{\mu}(f)| > \epsilon\right)$, as $n \to +\infty$. When $b^* > 1$ these quantities are known to decay exponentially fast from classical results on uniformly expanding maps (See for example, Waddington, 1996). In the case when (b_n) decays polynomially, since we did not find any result in the literature, we prove,

Lemma 2.2 *Let T be a Wang map with sequence $b_n \leq cn^{-(1+\beta)}$ where $\beta > 1$. For all functions f measurable with respect to the dynamical partition (for some integer N, and hence integrable with respect to $\overline{\mu}$), and, all $\epsilon > 0$, there is a constant $C = C_\epsilon(f)$, such that,*

$$\overline{\lambda}\left(\left|\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k - \overline{\mu}(f)\right| > \epsilon\right) \leq \frac{C}{n^{\beta-1}}.$$

The main idea is to prove the result for the indicator of an element B of the dynamical partition. For such a function the ergodic sums are rates of occupation. Hence by duality we can relate them to the sequence of return times to B which is a sequence of iid random variables. Then, we use a result by Nagaev (1981) about large deviations for sums of iid random variables. A detailed proof is given in Appendix A.

2.4.4 Case of an infinite measure : asymptotic behaviour of ergodic sums

In the case when the invariant measure is not finite, we have,

Proposition 2.3 *If $b_n \sim cn^{-(1+\beta)}$, with $0 < \beta < 1$, then, for all f such that $0 < \overline{\mu}(f) < +\infty$, under $\overline{\lambda}$,*

$$\frac{1}{n^\beta} \sum_{k=0}^{n-1} f \circ T^k \overset{(law)}{\to} \overline{\mu}(f) \mathcal{M}\mathcal{L}_\beta.$$

Proof It is a direct application of the analysis in Section 2.3. The choice of (b_n) and (2.3) shows that (2.2) is fulfilled for $\alpha = \beta$. □

3 A family of interval maps

We will now introduce a family of interval maps indexed by two positive normalized sequences a and b. Firstly, we explain the ideas underlying the construction. Then, we summarize its obvious combinatorial and regularity properties in Proposition 3.1. In Section 3.3, we exhibit two natural invariant measures and estimate the tails of the laws of the return times to different parts of the interval. Then, in Section 3.4, we study the asymptotic behaviour of the ergodic sums with respect to the parameters. Finally, in Section 3.5 we show that we can find, among this family, both, maps satisfying Theorem 1.1 and maps satisfying Theorem 1.2.

3.1 Sketch of the construction

Our idea is to construct a map which is the embedding of two Wang maps one in the other. So we first choose two Wang maps, T_a and T_b, with two different "time scales". They are characterized by their sequences $a = (a_n)$ and $b = (b_n)$.

We start with the piecewise affine map $x \mapsto 3x \mod 1$ on $I = [0, 1]$ endowed with the Lebesgue measure. We split I into three intervals of monotonicity I_0, I_1 and I_2. We want to keep the topological dynamic of this map but to rescale the Lebesgue measure (locally) in order to change the statistical behaviour of typical points. Roughly speaking we want to do it so that, if collapsing I_0 and I_1, one would see only the map T_a (rare occurrences of I_2), and, between two occurrences of I_2, the dynamics on $I_0 \cup I_1$ is the same as of T_b. We are going to do that keeping the map locally affine.

Call J_0 the rightmost interval, $J_0 = I_2$. The remainder can be split according to the number of iterations needed before they meet J_0. Call J_n the subset of $I_0 \cup I_1$ on which this number is exactly n. The subset J_n is made of 2^n intervals (of length 3^{-n-1}) corresponding to different pieces of orbit before reaching I_2. Now, rescale every interval of J_n by $a_n 3^{n+1} 2^{-n}$, so that the total length of J_n is a_n. Since $\sum_{n \geq 0} a_n = 1$, this fits in an interval of length 1. After this manipulation, the map is affine on all the pieces, the slope being given by the ratio $\frac{2a_{n-1}}{a_n}$ in J_n, $n \geq 1$. The law of the entrance time in I_2 is given by the sequence a_n.

Now, we want to take care of the dynamics on $I_0 \cup I_1$. Given n, we have a collection of 2^n intervals corresponding to the 2^n possible n-orbits. After the first rescaling, they all have the same length, that is $a_n 2^{-n}$. To achieve the desired behaviour we shall rescale them so that these n-orbits are distributed as if they were produced by a Wang map T_b. For $\omega \in \{0, 1\}^n$, denote $J_{n,\omega}$ the interval of points whose orbit follows $I_{\omega_1}, \ldots, I_{\omega_n}, I_2$. We want this interval to have (relative) weight $\overline{\lambda}(\overline{I}_\omega)$ (these quantities were defined in Section 2.4).

Since, obviously, $\sum_{\omega\in\{0,1\}^n} \overline{\lambda}(\overline{I}_\omega) = 1$, we can rescale each $J_{n,\omega}$ by $\overline{\lambda}(\overline{I}_\omega)2^n$ and keep the total length of J_n the same ($= a_n$).

The main difficulty to write this analytically is that the positions of the intervals that compose J_n depend on partial sums of the a_k and $b_\omega = \overline{\lambda}(\overline{I}_\omega)$. But there is no deep difficulty.

3.2 Definition and first properties

We fix two decreasing sequences (a_n) and (b_n) of positive real numbers with $\sum a_n = 1$ and $\sum b_n = 1$. We consider the interval $I = [0,1]$ endowed with the Lebesgue measure λ. We are going to define a collection of disjoint intervals $\mathcal{P}_{a,b} = (J_{n,\omega})_{n\in\mathbf{N},\omega\in\{0,1\}^n}$ such that $\lambda(J_0) = a_0$, and, for all $n \geq 1$, $J_n = \cup_{\omega\in\{0,1\}^n} J_{n,\omega}$ is made of 2^n intervals of respective length $\lambda(J_{n,\omega}) = b_\omega a_n$ where $b_\omega = \overline{\lambda}(\overline{I}_\omega)$ in the Wang map of sequence (b_n) (see Section 2.4). We notice that $\lambda(J_n) = \sum_{\omega\in\{0,1\}^n} J_{n,\omega} = a_n$, and, hence, $\sum_{J\in\mathcal{P}_{a,b}} \lambda(J) = 1$.

Given $\omega \in \{0,1,2\}^n$ and $\omega' \in \{0,1,2\}^m$, we denote $\omega\omega' \in \{0,1,2\}^{n+m}$ their concatenation. We say that $\omega \leq \omega'$ in the lexicographical order on finite words of the alphabet $\{0,1,2\}$ if there is a word ω'' such that $\omega' = \omega\omega''$ or if for k the first index with $\omega_k \neq \omega'_k$, we have $\omega_k < \omega'_k$. Note that the empty word is minimal for this order.

Now, for $\omega \in \{0,1\}^n$ and $\omega' \in \{0,1\}^m$, We say that $\omega \preceq \omega'$ if $\omega 2 \leq \omega' 2$ in the lexicographical order on finite words of the alphabet $\{0,1,2\}$ (it is coherent to assume that according to this order, the empty word is maximal, so we put $b_\omega = 1$ if ω is empty).

This corresponds to the order in which we want to put the $J_{n,\omega}$ on the interval. Indeed, think again of the map $T : x \mapsto 3x \bmod 1$ and define, for $n > 0$ and $\omega \in \{0,1,2\}^n$, the interval $I_\omega = \{x \in I : T^{m-1}x \in I_{\omega_m}, 1 \leq m \leq n\}$. For $n > 0$ and $\omega \in \{0,1\}^n$, $J_{n,\omega}$ would be exactly $I_{\omega 2}$. These intervals are disjoint and form a partition (mod 0) of I. They are ordered on I according to the order $\preceq$. Our purpose is now to put abstract intervals in I in the same order, but rescaling their respective length. The position of an interval $J_{n,\omega}$ is determined by the sum of the length of the intervals which must be at its left, that are the intervals $J_{m,\omega'}$ with $\omega' \preceq \omega$.

For all $n \in \mathbf{N}$ and $\omega \in \{0,1\}^n$, we set

$$c_{n,\omega} = \sum_{m\geq 0} \sum_{\substack{\omega' \in \{0,1\}^m \\ \omega' \preceq \omega}} a_m b_{\omega'},$$

and,

$$J_{n,\omega} = [c_{n,\omega}\ ,\ c_{n,\omega} + a_n b_\omega)\,.$$

Almost all points in I are in an element of $\mathcal{P}_{a,b}$. For $\omega \in \{0,1\}^{\mathbf{N}}$, we denote $\omega_1^n \in \{0,1\}^n$ the word made of the first n letters of ω. We set,

$$z_\omega = \lim_{n\to\infty} c_{n,\omega_1^n}\,.$$

The limit z_ω always exists since the sequence is decreasing. The set $Z = \cup_{\omega\in\{0,1\}^{\mathbf{N}}}\{z_\omega\}$ is a Cantor set. We have $I = Z \cup \big(\cup_{J\in\mathcal{P}_{a,b}} J\big)$.

Now we can define our map $T = T_{a,b}$ on I.

$$T(x) = \begin{cases} a_0 x - (1-a_0) & \text{if } x \in J_0 \\ c_{n-1,\sigma(\omega)} + \frac{a_{n-1}b_{\sigma(\omega)}}{a_n b_\omega}(x - c_{n,\omega}), & \text{if } n > 0, \omega \in \{0,1\}^n, \text{ and, } x \in J_{n,\omega} \\ z_{\sigma(\omega)} & \text{if } \omega \in \{0,1\}^{\mathbf{N}} \text{ and } x = z_\omega \end{cases}$$

We set $I_0 = [0, z_{1\underline{0}})$, $I_1 = [z_{1\underline{0}}, c_1)$ and $I_2 = [c_1, c_0] = J_0$, where $1\underline{0} = 10\cdots0\cdots \in \{0,1\}^{\mathbf{N}}$. For $\omega \in \{0,1,2\}^n$, we shall denote the elements of the dynamical partition,

$$I_\omega = \{x \in I : T^i(x) \in I_{\omega_i}, 1 \le i \le n\}.$$

Let us now summarize the properties of the map $T_{a,b}$,

Proposition 3.1 *T is continuous on I_0, I_1 and I_2. On each of these intervals, it is strictly increasing. They are mapped onto I. T is topologically equivalent to $x \to 3x(mod 1)$. It is differentiable on $I \setminus \overline{\cup_{n\ge0,\omega\in\{0,1\}^n}\{c_{n,\omega}\}}$. If the sequences $\left(\frac{a_{n-1}}{a_n}\right)$ and $\left(\frac{b_{n-1}}{b_n}\right)$ have limits a^* and b^*, then T is right differentiable everywhere and the set where it is not differentiable reduces to $\{c_{n,\omega}, n \ge 0, \omega \in \{0,1\}^n\} \cup \{z_{\omega\underline{0}}, n \ge 0, \omega \in \{0,1\}^n\}$ which is countable. If $b^* > 1$, then the map T is uniformly expanding.*

Proof It is clear that $T(J_{n,\omega}) = J_{n-1,\sigma(\omega)}$ so that the possible discontinuity points are the $c_{n,\omega}$'s and the z_ω's. But the map is constructed in such way that points with the same symbolic description are close in the interval.

The map T is differentiable Lebesgue almost everywhere, since, for all $n \ge 0$, all $\omega \in \{0,1\}^n$ and all $x \in J_{n,\omega}$, the derivative is given by the ratios

$$T'(x) = \frac{\lambda(J_{n-1,\sigma(\omega)})}{\lambda(J_{n,\omega})} = \frac{a_{n-1}b_{\sigma(\omega)}}{a_n b_\omega}.$$

Again, the only difficulties are at the $c_{n,\omega}$'s and at the z_ω's. Delicate, but straightforward computations show that when a^* and b^* exist, T is indeed differentiable on Z except on a part of its boundary. Finally, notice that if $b^* > 1$ the left and right derivatives remain uniformly away from 1. $\square$

3.3 Return times, induced maps and invariant measures

3.3.1 Absolutely continuous invariant measure

We define the entrance time τ in I_2, $\tau(x) = \inf\{n > 0 : T^n(x) \in I_2\}$. We also define the induced map $R = T^\tau$ on I_2. It extends to the whole interval. We set $\tau_i = \tau \circ R^i$.

There is a (canonical, mod 0) partition of I_2 such that each interval of this partition is affinely mapped by R onto I_2. This partition is made of the $T^{-1}_{|I_2}(J_{n,\omega})$. On these sets the map is clearly affine. Hence, the Lebesgue measure restricted to I_2, λ_{I_2}, is invariant for the induced map R.

On I_2^c, the entrance time depends on the J_n to which x belongs. Hence, $\lambda(I_2^c \cap \{\tau = n\}) = \lambda(J_n) = a_n$. The interval I_2 is affinely mapped onto $[0,1]$. The return time depends only on the J_n in which x is sent. Hence $\lambda(I_2 \cap \{\tau = n\}) = a_0 \lambda(J_{n-1}) = a_0 a_{n-1}$. Finally, for all $n \geq 1$, we have,

$$\lambda(\tau = n) = a_n + a_0 a_{n-1}. \tag{3.4}$$

Under the Lebesgue measure, the sequence (τ_i) is a sequence of iid random variables. We have, for all $i > 0$, $\lambda(\tau_i = n) = \lambda_{I_2}(\tau = n)$. Hence, for all $i > 0$,

$$\lambda(\tau_i = n) = a_{n-1}. \tag{3.5}$$

Let us set,

$$\rho_{n,\omega} = \frac{a_0}{a_n b_\omega} \sum_{m \geq 0} \sum_{\omega' \in \{0,1\}^m} a_{n+m} b_{\omega' \omega},$$

and, for $x \in J_{n,\omega}$, $\rho(x) = \rho_{n,\omega}$.

Proposition 3.2 *The measure $\mu = \rho\lambda$ is absolutely continuous with respect to the Lebesgue measure λ. It is T-invariant. It is finite if and only if $\lambda(\tau) < +\infty$.*

If μ is finite, we normalize it, so it is a probability measure. If it is not, we normalize it so that $\mu(I_2) = 1$.

Proof The fact that the density ρ preserves the Perron Frobenius operator follows from a heavy but straightforward computation. Since $\mu(J_n) = a_0 \sum_{m \geq 0} a_{n+m}$, μ is finite if and only if $\sum_n n a_n < +\infty$. If μ is not finite, then, we can check that it is finite on all compact sets included in $I \setminus Z$. $\square$

3.3.2 An invariant measure supported on Z

Given a set $A \subset I$ in the dynamical partition, we define its *projection* $\overline{A} \subset \overline{I}$. Write $A = I_\omega$, where $\omega \in \{0,1,2\}^n$. If there is $i \leq n$ such that $\omega_i = 2$, then we set $\overline{A} = \emptyset$, else, we set $\overline{A} = \overline{I}_\omega$. This allows us to define a measure on I by setting, for all $A \subset I$,

$$\lambda_1(A) = \overline{\lambda}(\overline{A}).$$

We also define the projection $\overline{f}$ of any continuous function f on I by $\overline{f}(x) = f(z_{\omega(x)})$. We have $\lambda_1(f) = \overline{\lambda}(\overline{f})$. Notice that $\lambda_1(A) = \lambda_1(\mathbf{1}_{\{A\}}) = \overline{\lambda}(\{x \in \overline{I} : z_{w(x)} \in A\})$. It is clear that the measure $\mu_1(A) = \overline{\mu}(\overline{A})$ is T-invariant and absolutely continuous with respect to λ_1. It is finite or infinite according to whether $\sum n b_n$ is finite of infinite. It is singular with respect to λ and is supported on the Cantor set Z.

Remark 2 *In fact, we simply identified the Cantor set $Z \subset I$ with the interval $\overline{I}$ (of Section 2.4) through the dynamical coding. The measure $\overline{\lambda}$ on $\overline{I}$ maps to a measure on $\{0,1\}^{\mathbf{N}}$ (through the coding of the Wang map with sequence (b_n)). This measure is also a measure on $\{0,1,2\}^{\mathbf{N}}$. We sent it onto the Cantor set Z through the new coding.*

The following lemma shows that these measures arise naturally.

Lemma 3.3 *For all f depending on a finite number m of coordinates, and all $n > m$,*

$$\lambda_{I_2}(f \circ T | \tau = n) = \overline{\lambda}(\overline{f}). \tag{3.6}$$

Proof It is enough to prove the result for $f = \mathbf{1}_{\{I_\omega\}}$ for all $\omega \in \{0,1\}^m$. So, we have to estimate $\lambda_{I_2}(\mathbf{1}_{\{I_\omega\}} \circ T | \tau = n)$. To do so, notice that if ω contains a 2, then, $T^{-1}(I_\omega) \cap \{\tau = n\}$ is empty, while, otherwise, it is exactly $T^{-1}_{|I_2}(I_\omega \cap J_{n-1})$. To conclude, we relate λ and $\overline{\lambda}$ through, $\lambda\left(I_\omega \cap J_{n-1}\right) = a_n \overline{\lambda}(\overline{I}_\omega)$. □

Remark 3 *The idea is that, given $\tau > k$, the system acts exactly as a Wang map T_b. This measure is related to conditionally invariant measures since it is the measure given we did not see the word I_2.*

3.3.3 Return time far from the indifferent fixed point

We define the entrance time in $I_1 \cup I_2$, $\tilde{\tau}(x) = \inf\{n > 0 : T^n(x) \in I_1 \cup I_2\}$, the return map $\tilde{R} = T^{\tilde{\tau}}$ and the successive return times $\tilde{\tau}_i = \tilde{\tau} \circ \tilde{R}^i$. Notice that the successive return times are not iid random variables in this case, since the Lebesgue measure restricted to $I_1 \cup I_2$ is not invariant for the return map $\tilde{R}$. But, the following lemma shows that $\sum_i \tilde{\tau}_i$ is "stochastically" larger than a sum of iid random variables. Recall that $\overline{\tau}$ and τ denote the entrance times in $\overline{I}_1$ for $\overline{T}$ and in I_2 for T, respectively.

Lemma 3.4 *Assume $b^* = 1$. Let $(\tilde{\tau}_i^*)$ be a sequence of iid random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ of common law $\mathbf{P}(\tilde{\tau}_i^* = k) = \overline{\lambda} \otimes \lambda_{I_2}(\min(\overline{\tau}\ ;\ \tau) = k)$. Then, for all $p \geq 0$,*

$$\lambda_{I_2}\left(\sum_{i=0}^{p} \tilde{\tau}_i > n\right) \geq \mathbf{P}\left(\sum_{i=0}^{p} \tilde{\tau}_i^* > n\right).$$

Proof Firstly, we notice that according to Lemma 3.3, on I_2, under λ_{I_2},

$$\tilde{\tau} = \tilde{\tau}\mathbf{1}_{\{\tau \geq \tilde{\tau}\}} + \tau \mathbf{1}_{\{\tau < \tilde{\tau}\}} \overset{\text{(law)}}{=} \min\left(\overline{\tau}\ ;\ \tau\right), \tag{3.7}$$

under $\overline{\lambda} \otimes \lambda_{I_2}$. We denote λ' the probability measure λ given the event $I_2 \cap \{\tau > \sum_{j=0}^{i} \tau_j\}$. Under λ',

$$\tilde{\tau}_{i+1} \overset{\text{(law)}}{=} \min\left(\overline{\tau}_{i+1}\ ;\ \tau - \sum_{j=0}^{i} \overline{\tau}_j\right),$$

under $\overline{\lambda} \otimes \lambda'$. Since $b^* = 1$, for all $k \geq 0$, we have $\lambda_{I_2}(\tau > k + n | \tau > k) \geq \lambda_{I_2}(\tau > n)$. Hence,

$$\overline{\lambda} \otimes \lambda'(\tau > n + \sum_{j=0}^{i} \overline{\tau}_j) = \overline{\lambda} \otimes \lambda_{I_2}(\tau > n + \sum_{j=0}^{i} \overline{\tau}_j \,|\, \tau > \sum_{j=0}^{i} \overline{\tau}_j) \geq \lambda(\tau > n).$$

We conclude noticing that the $\tilde{\tau}_i$ renew when $\tau = \sum_{j=0}^{i} \overline{\tau}_j$, since λ_{I_2} is R-invariant. □

Remark 4 *Assume $a_n \sim n^{-(\alpha+1)}$ and $b_n \sim n^{-(\beta+1)}$. It follows from (3.7) or from a direct computation that the tail of the law of $\tilde{\tau}$ under the Lebesgue measure satisfies,*

$$\lambda(\tilde{\tau} = k) \sim (a_0 + b_0) n^{-(\alpha+\beta+1)}.$$

3.4 Asymptotic behaviour of ergodic sums

In this section, we study the asymptotic behaviour of ergodic sums in different situations.

3.4.1 Convergence (in law) to the absolutely continuous invariant (infinite) measure

Proposition 3.5 *If $a_n \sim n^{-(1+\alpha)}$ with $\alpha < 1$, then for all $f \in L^1_\mu$, with $\mu(f) > 0$,*

$$\frac{1}{n^\alpha}\sum_{k=0}^{n} f \circ T^k \overset{(law)}{\to} \mu(f)\mathcal{M}\mathcal{L}_\alpha,$$

under the Lebesgue measure λ.

In this case, the return time τ to I_2 is not integrable, so that the a.c.i.m. is infinite. We see what happens on the interval I_2 only at large scales of time.

Proof This result is a direct consequence of Fact 1 stated in Section 2.3. The time scale is given by the tail of the return time to a set of finite measure, here the set I_2. □

3.4.2 Convergence to the measure on the subsystem

Proposition 3.6 *If $a_n \sim n^{-(1+\alpha)}$ and $b_n \leq n^{-(1+\beta)}$, with $\alpha < 1$ and $\beta > 3$, then, for all f continuous,*

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \to \mu_1(f), \quad \lambda - a.e..$$

In this case, the return time τ to I_2 is not integrable, while, the Wang map of sequence (b_n) has an a.c.i.p. and, hence, the measure μ_1 is finite. A typical orbit spends most of the time in the interval $I_0 \cup I_1$. During this time, its dynamic is driven by the Wang map of sequence (b_n), so that what is seen at the standard time scale is the physical invariant measure supported on this subsystem.

Technically, the ergodic sums decompose in independent trials $(\sum_{k=0}^{\tau} f \circ T^k) \circ R^i$ which are close to $\mu_1(f)$ each time τ is large enough (and in general, τ is large). To control the fluctuations about $\mu_1(f)$, we use our large deviations result (Lemma 2.2).

Proof Let f be a continuous function such that $\mu_1(f) = 0$. In a first stage, we assume that f depends only on the first coordinate (of the coding). Let $q_n = \sum_{i=0}^{n-1} \mathbf{1}_{\{I_2\}} \circ T^i$, denote the number of passages in I_2, and, $M_n = \sum_{i=0}^{q_n} \tau \circ R^i$ denote the time of the last passage in I_2 before n.

We decompose,

$$\sum_{k=0}^{n-1} f \circ T^k = \sum_{k=0}^{\tau-1} f \circ T^k + \sum_{k=\tau}^{M_n-1} f \circ T^k + \sum_{k=M_n}^{n-1} f \circ T^k. \tag{3.8}$$

The first term is apart because at time 0, we may not be in I_2. It is easily bounded by,

$$\sum_{k=0}^{\tau-1} f \circ T^k \leq \tau ||f||_\infty. \tag{3.9}$$

The second term is the most important contribution. We will rewrite it in order to see it as a sum of independent pieces of trajectories remaining in $I_0 \cup I_1$.

$$\begin{aligned} \left| \sum_{k=\tau}^{M_n-1} f \circ T^k \right| &= \left| \sum_{i=1}^{q_n} \left(\sum_{k=0}^{\tau-1} f \circ T^k \right) \circ R^i \right| \\ &= \left| \sum_{i=1}^{q_n} \left(\sum_{k=1}^{\tau-1} f \circ T^k \right) \circ R^i + \sum_{i=1}^{q_n} f \circ R^i \right| \\ &\leq \left| \sum_{i=1}^{q_n} \left(\sum_{k=1}^{\tau-1} f \circ T^k \right) \circ R^i \right| + q_n ||f||_\infty. \end{aligned} \tag{3.10}$$

The last term corresponds to the last "excursion" out of I_2. It is a priori smaller than a whole excursion, but to obtain an almost sure control is technical because the length of this last excursion could be of the same order as the concatenation of all the previous ones (property of long tail distributions). We write it,

$$\begin{aligned} \sum_{k=M_n}^{n-1} f \circ T^k &\leq ||f||_\infty + \epsilon(n - M_n)||f||_\infty + \left(\sum_{k=M_n+1}^{n-1} f \circ T^k - \epsilon(n-M_n)||f||_\infty \right) \\ &\leq ||f||_\infty + \epsilon n ||f||_\infty + H_{q_n}, \end{aligned} \tag{3.11}$$

where, for all $q \geq 0$,

$$H_q = \sup_{1 \leq t \leq \tau_q} \left(\left(\sum_{k=1}^{t} f \circ T^k - \epsilon ||f||_\infty t \right) \circ R^q \right).$$

Finally,

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \right| \leq \epsilon ||f||_\infty + \frac{1}{n}(\tau+1)||f||_\infty + \frac{1}{n} q_n ||f||_\infty + \left| \sum_{i=1}^{q_n} \left(\sum_{k=1}^{\tau-1} f \circ T^k \right) \circ R^i \right| + \frac{1}{n} H_{q_n}. \tag{3.12}$$

To prove the result, we just have to prove that, almost surely, the limsup is smaller than ϵ for all ϵ. So we fix $\epsilon > 0$. It is obvious that $\frac{1}{n}(\tau+1)||f||_\infty$ tends to 0 almost surely. We are going to prove separately that all the other terms in the upper bound converge to 0 almost surely.

Firstly, we claim that

$$\lim_{n \to +\infty} \frac{1}{n} q_n = 0, \quad \lambda - \text{almost surely.} \tag{3.13}$$

Notice that $q_n = \inf\{k \geq 0 : \sum_{i=0}^{k} \tau \circ R^i > n\}$. Here, the sequence of return times $(\tau_i)_{i \geq 1}$ is a sequence of iid random variables. Hence, by the law of large numbers for a sum of iid random variables with infinite expectation, we see that,

$$\frac{1}{q} \sum_{i=1}^{q} \tau \circ R^i \to \infty, \quad \lambda - \text{almost surely.}$$

We notice that when $n \to \infty$, $q_n \to \infty$ λ-almost surely. Hence,

$$\frac{n}{q_n} \geq \frac{1}{q_n} \sum_{i=1}^{q_n} \tau \circ R^i \to \infty, \quad \lambda - \text{almost surely}, \tag{3.14}$$

which proves Claim (3.13).

To prove the other two convergences, we must understand better what happens to the ergodic sums between two occurrences of I_2. This is easier in the case when f depends only of one coordinate and that is why we first did this assumption. We recall that the return map $R = T^\tau$ leaves the Lebesgue measure on I_2 invariant. So that, if f depends only on the first coordinate, the sequence

$$\left(\left(\sum_{k=1}^{\tau-1} f \circ T^k \right) \circ R^i \right)_{i \geq 1}$$

is a sequence of iid random variables if x is chosen according to the Lebesgue measure. Hence, under the Lebesgue measure, the process

$$\left(\sum_{i=1}^{q_n} \left(\sum_{k=1}^{\tau-1} f \circ T^k \right) \circ R^i \right)_{n \geq 1}$$

is a sum of iid random variables.

Let then $\overline{f}$ be the "projection" of f on $\overline{I}$ (for definition, see Section 3.3.2). We notice that $\overline{\mu}(\overline{f}) = \mu_1(f) = 0$. We consider the process defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P}) = (I \times \overline{I}^{\mathbf{N}}, \mathcal{B} \times \overline{\mathcal{B}}^{\mathbf{N}}, \lambda \otimes \overline{\lambda}^{\otimes \mathbf{N}})$ by

$$\Sigma_n(x, (y_j)_{j \in \mathbf{N}}) = \sum_{i=1}^{q_n(x)} \sum_{k=1}^{\tau \circ R^i(x) - 1} \overline{f} \circ \overline{T}^k(y_i).$$

To simplify the notations, let us define a random sequence on $(\overline{I}, \overline{\lambda})$, To simplify the notations, let us write

$$\Sigma_n = \sum_{i=1}^{q_n} V_i(\tau_i).$$

where $V_i(m)(x, (y_j)_{j \in \mathbf{N}}) = V(m)(y_i)$ and $V(m)(y) = \sum_{k=1}^{m-1} \overline{f} \circ \overline{T}^k(y)$. We can think of the V_i as a countable number of independent copies of V. We claim that,

$$\left(\sum_{i=1}^{q_n} \left(\sum_{k=1}^{\tau-1} f \circ T^k \right) \circ R^i \right)_{n \geq 1} \overset{\text{(law)}}{=} \left(\sum_{i=1}^{q_n} V_i(\tau_i) \right)_{n \geq 1}, \tag{3.15}$$

where the left hand side is distributed according to λ and the right-hand side according to $\mathbf{P}$. Both processes are sums of iid random variables. So we just have to check that these random variables have the same law. If f depends on one coordinate, then $F = \sum_{k=1}^{m-1} f \circ T^k$ depends on $m-1$ coordinates (again, in the sense of the coding) and $\overline{F} = \sum_{k=1}^{m-1} \overline{f} \circ \overline{T}^k$. But, according to Lemma 3.3, if

F depends on $m-1$ coordinates, and $\overline{F}$ is its projection on $\overline{I}$, then, $\overline{\lambda}\left(\overline{F}\in A\right)=\lambda_{I_2}\left(F\in A|\tau=m\right)$. Hence, for all $i>0$,

$$\begin{aligned}
\mathbf{P}(V(\tau_i)\in A) &= \sum_{m\geq 1}\mathbf{P}(V(m)\in A)\mathbf{P}(\tau_i=m) \\
&= \sum_{m\geq 1}\overline{\lambda}\left(\sum_{k=1}^{m-1}\overline{f}\circ\overline{T}^k\in A\right)\lambda_{I_2}(\tau=m) \\
&= \sum_{m\geq 1}\lambda_{I_2}\left(\sum_{k=1}^{m-1}f\circ T^k\in A|\tau=m\right)\lambda_{I_2}(\tau=m) \\
&= \lambda_{I_2}\left(\sum_{k=1}^{\tau-1}f\circ T^k\in A\right).
\end{aligned}$$

Hence, Claim (3.15) is proved.

Remark 5 *The claim is that if we look at our dynamical system T under the Lebesgue measure, it is the same as to look at independent trials of the dynamical system $\overline{T}$ each of them under the Lebesgue measure, during the right amount of time.*

We claim that,

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{q_n}V_i(\tau_i)=0,\ \mathbf{P}-\text{a. s.}. \tag{3.16}$$

Splitting the sum according to whether $\{V_i(\tau_i)\leq\epsilon\tau_i\}$ or not, we obtain, for all $\epsilon>0$,

$$\left|\sum_{i=1}^{q_n}V_i(\tau_i)\right| \leq \epsilon\sum_{i=1}^{q_n}\tau_i+||f||_\infty\sum_{i=1}^{q_n}\mathbf{1}_{\{V_i(\tau_i)>\epsilon\tau_i\}}\tau_i.$$

The first term is smaller than ϵn. Using Lemma 2.2 to see that $\mathbf{P}(V(k)>\epsilon k)\leq c_\epsilon k^{-\beta+1}$ and the assumption $\mathbf{P}(\tau=k)\sim k^{-\alpha-1}$, we deduce that, as soon as $\alpha+\beta>2$, the ratio $\frac{1}{q}\sum_{i=1}^{q}\mathbf{1}_{\{V_i(\tau_i)>\epsilon\tau_i\}}\tau_i$ is essentially bounded, uniformly in q. According to (3.13), $q_n\to+\infty$ and $\frac{q_n}{n}\to 0$ a.s. when $n\to+\infty$. This proves Claim (3.16).

Now we claim that,

$$\lim_{n\to+\infty}\frac{1}{n}H_{q_n}=0,\quad \lambda-a.s.. \tag{3.17}$$

The sequence H_q is a sequence of iid random variables. Using (3.15), we see that

$$H_q\overset{(\text{law})}{=}\sup_{1\leq t\leq\tau_q}\left(V(t)-\epsilon||f||_\infty t\right),$$

It follows from Lemma 2.2 that,

$$\mathbf{P}(V(t)\geq\epsilon||f||_\infty t)\leq\frac{c}{t^{\beta-1}}.$$

Hence,

$$\mathbf{P}(\exists t > k : V(t) \geq \epsilon ||f||_\infty t) \leq \sum_{t \geq k} \frac{c}{t^{\beta-1}} \leq \frac{c}{k^{\beta-2}}.$$

But, on the event $\{\forall t > k, V(t) \leq \epsilon ||f||_\infty t\}$, we have, $\sup_{t\geq 0}(V(t) - \epsilon||f||_\infty t) \leq k||f||_\infty$, so that,

$$\mathbf{P}\left(\sup_{t\geq 0}(V(t) - \epsilon||f||_\infty t) \geq k||f||_\infty\right) \leq \frac{c}{k^{\beta-2}}.$$

Since $\frac{1}{n}H_{q_n} \leq \frac{H_{q_n}}{\sum_{i=1}^{q_n}\tau_i}$, we apply the Borel Cantelli Lemma to see that, if $\beta > 3$, for all $\epsilon > 0$, the number of indices q such that $H_q > \epsilon\sum_{i=1}^{q}\tau_i$ is λ-a.s. finite. This proves Claim (3.17).

Remark 6 *Of course it is possible to say something when $\beta < 3$ and even when $\alpha + \beta < 2$. But one has to enter more into details. That is not our goal here.*

Inequality (3.12) together with (3.13), (3.16) and (3.17) concludes the proof if f depends only on the first coordinate.

If f depends on a finite number of coordinates of the coding, then we must get rid of the cases when τ is smaller than this dependence and of the first terms of the sum. But all this is almost surely small compared to n. So the problem reduces to the previous case. Formally, if f depends on m coordinates,

$$\left|\sum_{k=\tau}^{M_n-1} f\circ T^k\right| \quad \leq \quad \left|\sum_{i=1}^{q_n}\left(\mathbf{1}_{\{\tau>m\}}\sum_{k=1}^{\tau-m} f\circ T^k\right)\circ R^i\right| + 2q_n m||f||_\infty.$$

Using the same notations as for one coordinate, we can construct the random sequences associated with the "projection" $\overline{f}$ of f. We see that the only thing to prove is that $\frac{1}{n}\sum_{i=1}^{q_n}\mathbf{1}_{\{\tau_i>m\}}V_i(\tau_i)$ tends to 0 almost surely. Since,

$$\left|\sum_{i=1}^{q_n}\mathbf{1}_{\{\tau_i>m\}}V_i(\tau_i)\right| \leq \left|\sum_{i=1}^{q_n}V_i(\tau_i)\right|,$$

we conclude by the same arguments as before. The large deviation argument still holds since $\overline{f}$ depends on finitely many coordinates.

Finally, if f is continuous, we approximate f uniformly by a sequence $(f_m)_{m\geq 0}$ of step functions — i.e. depending on a finite number m of coordinates. We write

$$\frac{1}{n}|\sum_{k=0}^{n-1} f\circ T^k - \mu_1(f)| \leq \frac{1}{n}|\sum_{k=0}^{n-1} f_m\circ T^k - \mu_1(f_m)| + 2||f - f_m||_\infty,$$

and we apply the previous argument to f_m, to see that almost surely the limsup is smaller than $||f - f_m||_\infty$ which is arbitrarily small. □

3.4.3 Convergence to the Dirac mass at the indifferent fixed point

Proposition 3.7 *If $a_n \sim n^{-(1+\alpha)}$ and $b_n \sim n^{-(1+\beta)}$, with $\alpha+\beta<1$ and $\alpha<\beta$, then, for all f continuous,*

$$\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k \to f(0), \quad \lambda - a.e..$$

In this case, the return time $\tilde{\tau}$ to $I_1 \cup I_2$ is not integrable, so that a typical orbit spends most of the time close to the fixed point 0. The proof is almost the same as it would be for a Wang map with a nonintegrable return time. The main difference is that the sequence of return times we consider is not a sequence of iid random variables.

Remark 7 *Notice that this result does not follow directly from the fact that the measure is infinite. Indeed, the measure becomes infinite close to the Cantor set Z whereas, for these values of the parameters, only the fixed point 0 is attracting the statistical behaviour of the system.*

Proof Let $\tilde{q}_n = \sum_{i=0}^{n-1} \mathbf{1}_{\{I_1\cup I_2\}}\circ T^i$, denote the number of passages in $I_1\cup I_2$. The $\tilde{\tau}_i$'s are not a sequence of iid random variables, but according to Lemma 3.4, the sum stochastically dominates a sum of iid random variables $\tilde{\tau}_i^*$. Since $\alpha+\beta<1$, Remark 4 shows that the $\tilde{\tau}_i^*$ have infinite expectations. Hence, by the law of large numbers for a sum of iid random variables with infinite expectation, we see that

$$\frac{1}{q}\sum_{i=1}^{q} \tilde{\tau}\circ\tilde{R}^i \to \infty, \quad \lambda - \text{almost surely.}$$

We notice that when $n\to\infty$, $\tilde{q}_n\to\infty$ λ-almost surely. Hence,

$$\frac{n}{\tilde{q}_n} \geq \frac{1}{\tilde{q}_n}\sum_{i=1}^{\tilde{q}_n} \tilde{\tau}\circ\tilde{R}^i \to \infty, \quad \lambda - \text{almost surely.} \tag{3.18}$$

Let

$$\tilde{M}_n = \sum_{i=0}^{\tilde{q}_n} \tilde{\tau}\circ\tilde{R}^i$$

denote the time of the last passage before n.

Let f be a continuous function. We assume without loss of generality that, $f(0)=0$. For all $\epsilon>0$, there is an integer k_0 such that, for all $k>k_0$, $\sup_{x\in I_{0^k}} |f(x)|<\epsilon$. Hence, for all $x\in I_1\cup I_2$,

$$\sum_{k=0}^{\tilde{\tau}-1} |f\circ T^k| \leq \sum_{k=0}^{\tilde{\tau}-1} \sup_{y\in I_{0^k}} |f(y)| \leq k_0 ||f||_\infty + \epsilon\,\tilde{\tau}.$$

We decompose,

$$\sum_{k=0}^{n-1} f\circ T^k = \sum_{k=0}^{\tilde{\tau}-1} f\circ T^k + \sum_{k=\tilde{\tau}}^{\tilde{M}_n-1} f\circ T^k + \sum_{k=\tilde{M}_n}^{n-1} f\circ T^k.$$

We have,

$$\left|\sum_{k=\tilde{\tau}}^{\tilde{M}_n-1} f\circ T^k\right| = \left|\sum_{i=1}^{\tilde{q}_n}\sum_{k=0}^{\tilde{\tau}\circ R^i-1} f\circ\tilde{R}^i\circ T^k\right| \le \tilde{q}_n k_0||f||_\infty + \epsilon\tilde{M}_n,$$

$$\left|\sum_{k=\tilde{M}_n}^{n-1} f\circ T^k\right| \le \left|\sum_{k=0}^{n-\tilde{M}_n-1} f\circ\tilde{R}^{\tilde{q}_n}\circ T^k\right| \le k_0||f||_\infty + \epsilon n,$$

and,

$$\left|\sum_{k=0}^{\tilde{\tau}-1} f\circ T^k\right| \quad\le\quad ||f||_\infty\tilde{\tau}.$$

Hence,

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k\right| &\le ||f||_\infty\frac{\tilde{\tau}}{n} + k_0||f||_\infty\frac{\tilde{q}_n}{n} + \frac{\epsilon}{n}\sum_{i=0}^{\tilde{q}_n}\tilde{\tau}\circ\tilde{R}^i + \frac{k_0||f||_\infty}{n} + \epsilon\frac{n}{n}\\ &\le M\frac{\tilde{\tau}}{n} + K\frac{\tilde{q}_n}{n} + \epsilon.\end{aligned}$$

Using (3.18), we conclude that for λ-almost all x, the limsup is smaller than any ϵ. □

3.4.4 An intermediate time scale

Proposition 3.8 *If $a_n \sim n^{-(1+\alpha)}$ and $b_n \sim n^{-(1+\beta)}$, with $\alpha < \beta < 1$, then, there is a random variable $\mathcal{H}_{\alpha,\beta}$, such that, for all Lipschitz functions f in $L^1_{\mu_1}$ with $\mu_1(f) > 0$, under λ,*

$$\frac{1}{n^\beta}\sum_{k=0}^{n-1} f\circ T^k \overset{(law)}{\rightarrow} \mu_1(f)\,\mathcal{H}_{\alpha,\beta}.$$

Proof Firstly, we assume that f depends only on the first coordinate. The decomposition (3.8) used above is still relevant. The first and the last term go to zero in probability, essentially because $n - M_n$ is smaller than τ in probability. For the same reason, $\frac{1}{n}\sum_{i=1}^{q_n}\tau_i$ converges to 1 in probability, since $n = \sum_{i=1}^{q_n}\tau_i + (n - M_n)$. So the problem is reduced to the study of the asymptotic behaviour of

$$\frac{\sum_{i=1}^{q_n}(\sum_{k=1}^{\tau-1} f\circ T^k)\circ R^i}{(\sum_{i=1}^{q_n}\tau_i)^\beta}.$$

Again, following (3.15), we notice that the sequence of random variables $(\sum_{k=1}^{\tau-1} f\circ T^k)\circ R^i$ under λ have the same law as $\sum_{k=1}^{\tau_i}\overline{f}\circ\overline{T}^k(y_i)$ under $\mathbf{P}$. Since $q_n\to\infty$ a.s. with n, we have to describe the asymptotic behaviour as $q\to+\infty$ of

$$\frac{\sum_{i=1}^{q} V_i(\tau_i)}{(\sum_{i=1}^{q}\tau_i)^\beta}.$$

According to Proposition 2.3, and noticing that $\mu_1(f) = \overline{\mu}(\overline{f})$, we see that $\frac{1}{k}V_i(k)$ converges in law to $\mu_1(f)\mathcal{M}\mathcal{L}_\beta$ as $k\to+\infty$ (independently of i). Hence, the law of $\tau_i^{-\beta} V_i(\tau_i)$ given $\{\tau_i > k\}$ converges, as $k \to +\infty$, to a copy of a $\mu_1(f)\mathcal{M}\mathcal{L}_\beta$ independent of the other involved random variables. The contribution of small τ_i goes to 0 a.s. since $\sum_{i=1}^q V_i(\tau_i)\mathbf{1}_{\{\tau_i \le k\}} \le qk||f||_\infty$ and $\frac{q_n}{n} \to 0$ a.s.. Let us denote $\mathcal{M}_i$ a sequence of iid random variables with common law $\mathcal{M}\mathcal{L}_\beta$. We have $\sum_{i=1}^q \tau_i^\beta \mathcal{M}_i \mathbf{1}_{\{\tau_i \le k\}} \le k^\beta \sum_{i=1}^q \mathcal{M}_i$. The $\mathcal{M}_i$ having all moments, the contribution of this sum goes to 0 a.s.. Finally, we can claim that $\sum_{i=1}^q V_i(\tau_i)$ and $\mu_1(f)\sum_{i=1}^q \tau_i^\beta \mathcal{M}_i$. have the same limiting law once suitably renormalized.

Finally, the point is to understand the asymptotic behaviour (as $q \to \infty$) of

$$\frac{\sum_{i=1}^q \tau_i^\beta \mathcal{M}_i}{(\sum_{i=1}^q \tau_i)^\beta}$$

We are now very close to the situation studied by Logan, Mallows, Rice and Sheep (1973). They prove that

Fact 2 (LMRS) *There is a random variable, $\mathcal{Z}_{\alpha,\beta}$ such that,*

$$\frac{\sum_{i=1}^q \tau_i^\beta}{\left(\sum_{i=1}^q \tau_i\right)^\beta} \overset{(law)}{\to} \mathcal{Z}_{\alpha,\beta}.$$

Following their lines, we notice that the asymptotic law of our ratio would be the same as if τ_i had stable laws of index α, instead being only in the basin of such stable law. Assuming this, we follow their approach to compute the characteristic function of the joint law,

$$\begin{aligned}
\xi_q(s,t) &= E[\exp(is\frac{1}{q^{\frac{\beta}{\alpha}}}\sum_{i=1}^q \tau_i^\beta \mathcal{M}_i + it\frac{1}{q^{\frac{1}{\alpha}}}\sum_{i=1}^q \tau_i)] \\
&= E[\exp(i\frac{s}{q^{\frac{\beta}{\alpha}}}\tau^\beta \mathcal{M} + i\frac{t}{q^{\frac{1}{\alpha}}}\tau)]^q \\
&= \left(1 + E\left[\int_0^{+\infty} \left(\exp(is\mathcal{M}\theta^\beta + it\theta) - 1\right) q^{\frac{1}{\alpha}} g_\alpha(q^{\frac{1}{\alpha}}\theta) d\theta\right]\right)^q \\
&= \left(1 + E[\psi_q(t, s\mathcal{M})]\right)^q,
\end{aligned}$$

where g_α denotes the density of the stable law of index α. In the proof in LMRS (1973), they show that $q\psi_q(t,s) \to \psi(t,s)$. So, if m_β denotes the density of the Mittag-Leffer law of index β,

$$\begin{aligned}
\xi_q(s,t) &\sim \left(1 + \frac{1}{q}\int_0^{+\infty} \psi(t,sx) m_\beta(x) dx\right)^q \\
&\to \exp \int_0^{+\infty} \psi(t,sx)) m_\beta(x) dx.
\end{aligned}$$

Hence it converges to the characteristic function of the joint distribution of normalized $A_q = \frac{1}{q^{\frac{\beta}{\alpha}}}\sum_{i=1}^q \tau_i^\beta \mathcal{M}_i$ and $B_q = \frac{1}{q^{\frac{1}{\alpha}}}\sum_{i=1}^q \tau_i$. We conclude that the ratio $\frac{A_q}{B_q^\beta}$ converges in distribution.

If f depends on a finite number of coordinates, the same argument as below can be used to adapt the proof. It is also simple to check that the proof does not change if f is measurable with respect to the partition $\mathcal{P}_{a,b}$ because in this case we can still replace f with $\overline{f}$ between passages in I_2. For smooth f the situation is slightly more complicated. It is not possible to use a uniform approximation because it would yield a constant error term n times while we normalize with n^β. We take advantage of the Lipschitz property noticing that we can approximate f with a function $\hat{f}$, $\mathcal{P}_{a,b}$-measurable, satisfying $\mu_1(f) = \mu_1(\hat{f})$, and such that on all intervals of J_n, $|f(x) - \hat{f}(x)| \leq$ const. a_n. This is enough to guarantee the uniform summability of the approximation error along a piece of orbit which does not enter I_2. For all n and all x such that $\{T^i(x), i = 1, \dots, n\} \cap I_2 = \emptyset$,

$$\sum_{k=1}^{n} |f \circ T^k(x) - \hat{f} \circ T^k(x)| \leq const. \sum_{k=1}^{n} a_k < const..$$

Hence, the difference between the ergodic sums of f and of $\hat{f}$ is bounded (up to *const.*) by the number q_n of such pieces of orbit before time n. We conclude by noting that $\frac{q_n}{n^\beta} \to 0$ in probability. □

Remark 8 *At this stage, we could identify the limiting distribution through properties of its characteristic function. It also seems possible to be more explicit using ideas developed in Pitman and Yor (1991). We conjecture the limiting law can be represented as the law of weighted sum of the jumps of a stable subordinator. Let (T_s) be a stable subordinator of index α. Let (Δ_i) be the infinite sequence of its jumps before it reaches* 1 *(the sequence of lengths of the maximal open subintervals of range(T_s)$\cap(0,1)$) and $(\mathcal{M}_i)$ a sequence of iid Mittag-Leffer random variables of index β independent of T. The conjecture can be written*

$$\mathcal{H}_{\alpha,\beta} \stackrel{(law)}{=} \sum_{i\geq 1} \mathcal{M}_i \Delta_i^\beta.$$

3.5 Proofs of Theorem 1.1 and 1.2

It is now straightforward to prove both Theorem 1.1 and 1.2. Firstly, let $a_n = n^{-(1+\alpha)}$ and $b_n = (1-\beta)^{-1}\beta^{1+n}$, with $\alpha < 1$ and $\beta < 1$. Since $\frac{a_{n-1}}{a_n} \to 1$ and $\frac{b_{n-1}}{b_n} \to \frac{1}{\beta} > 1$, Proposition 3.1 shows that the map $T_{a,b}$ is continuous, topologically equivalent to $x \mapsto 3x(mod 1)$, differentiable except on a countable set of points, and, uniformly expanding.

Remark 9 *In this particular case, we can compute the $c_{n,\omega}$,*

$$c_{n,\omega} = \sum_{i=1}^{n} \omega_i \beta^i - \sum_{m=0}^{n-1} a_m \sum_{i=m+1}^{n} \omega_i \beta^i.$$

Since $\sum_n na_n = +\infty$, Proposition 3.2 shows that there is an a.c.i.m. μ which is not finite. The measure μ_1 is well defined and is finite since $\sum_n nb_n < +\infty$. We conclude applying Proposition 3.6 and Proposition 3.5. □

Then, to prove Theorem 1.2, we let $a_n = n^{-(1+\alpha)}$ and $b_n = n^{-(1+\beta)}$, with $\alpha + \beta < 1$, and, $\alpha < \beta < 1$. Since $\frac{a_{n-1}}{a_n} \to 1$ and $\frac{b_{n-1}}{b_n} \to 1$, Proposition 3.1 shows that, in this case, the map $T_{a,b}$ is continuous, topologically equivalent to $x \to 3x(mod 1)$, and differentiable except on a countable set of points. It has an indifferent fixed point at 0, and $T' > 1$ elsewhere (if it is defined). Since $\sum_n na_n = +\infty$, Proposition 3.2 shows that there is an a.c.i.m. μ which is not finite. The measure μ_1 is well defined. It is infinite since $\sum_n nb_n = +\infty$. We conclude applying Proposition 3.7 , Proposition 3.5 and Proposition 3.8. $\square$

4 A more general construction

In this section we try to take advantage of the ideas involved for the construction of interval maps. We construct an explicit example for which we conjecture there is an infinite number of mutually singular physical measures. To simplify, we will not construct an interval map, but a measure on $\{0,1\}^{\mathbf{N}}$ as a limit. The difficulty is to prove that it is nonsingular with respect to the shift. So that it is possible to think of it as the Lebesgue measure for a piecewise affine interval map, following the spirit of the latter section. Then we give some comments and ask further questions.

4.1 An example

Let us specify some notations. A word ω in $\{0,1\}^n$ will be considered as a subset $\{0,1\}^{\mathbf{N}}$ and sometimes called a *cylinder*. The word $\omega\omega'$ is the concatenation of the two words ω and ω'. The length of the word ω is denoted by $|\omega|$. The word 1_k is the word made of k consecutive 1. For a finite word ω, we denote $|\omega|_n$ the number of occurrences of 1_n in ω.

We choose a sequence of sequences $(a^{(n)})_{n\geq 1}$, $a^{(n)} = (a_k^{(n)})_{k\geq 0}$ such that, for all $n \geq 1$, $\sum_{k\geq 0} a_k^{(n)} = 1$. We denote $X = \{0,1\}^{\mathbf{N}}$ and for all $n \geq 0$,

$$X_n = \{\omega \in X : \forall k \geq 0, \omega_k \cdots \omega_{k+n} \neq 1 \cdots 1\}.$$

We are going to construct measures λ_n on X, supported on the subshifts of finite type X_n.

First of all, we set, for all $k \geq 0$,

$$\lambda_1(0_k 1) = a_k^{(1)}.$$

We set $\lambda_{n+1}(\omega) = 0$ if ω contains the word 1_{n+2}, that is if $\omega \not\subset X_{n+1}$. Let ω be a word in X_{n+1}. If ω writes $\omega = \omega_1 1_{n+1} \omega_2$, where, $\omega_1 \subset X_n$ and $\omega_2 \subset X_{n+1}$, then, we set,

$$\lambda_{n+1}(\omega) = \lambda_n(\omega_1 1_n) a_{|\omega_1|_n}^{(n+1)} \lambda_{n+1}(\omega_2). \tag{4.19}$$

We notice that all finite words in X_{n+1} contain a finite number of occurrences of 1_{n+1} and that all words in X_{n+1} with no occurrence of 1_{n+1} can be written as a countable union of words with only one occurrence of 1_{n+1} at the end. So that this recurrence rule should be enough to define λ_n for all $n \geq 0$.

Proposition 4.1 *The sequence (λ_n) is a sequence of probability measures on $\{0,1\}^{\mathbf{N}}$ supported on X_n. It has a weak limit λ_∞, which is a probability measure and is nonsingular with respect to the shift.*

Proof Firstly, we check that λ_1 is a well defined probability measure. The only point to check is that the measure of words containing only 0 is defined. We have

$$\lambda_1(0_m) = \lambda_1(\cup_{k\geq m} 0_m 1) = \sum_{k\geq m} \lambda_1(0_m 1) = \sum_{k\geq m} a_m^{(1)}.$$

Let $\omega \subset X_n$ and set

$$\Omega_n(\omega) = \{\omega' : \omega\omega' \in X_n, \exists \omega^*, \omega\omega' = \omega^* 1_n\}.$$

As a word in X_{n+1}, ω can be written

$$\omega = \bigcup_{\omega' \in \Omega_n(\omega)} \omega\omega' 1.$$

So that we can write $\lambda_{n+1}(\omega)$ for all $\omega \subset X_n$,

$$\lambda_{n+1}(\omega) = \sum_{\omega' \in \Omega_n(\omega)} \lambda_{n+1}(\omega\omega' 1) = \sum_{\omega' \in \Omega_n(\omega)} \lambda_n(\omega\omega') a^{(n+1)}_{|\omega\omega'|_n - 1}$$

$$= \sum_{k\geq 1} a^{(n+1)}_{k-1} \sum_{\omega' \in \Omega_n(\omega):|\omega\omega'|_n = k} \lambda_n(\omega\omega').$$

But, $\omega \subset X_n$ writes, for all $k \geq 0$, $\omega = \bigcup_{\omega' \in \Omega_n(\omega):|\omega\omega'|_n = k} \omega\omega'$, and, hence,

$$\sum_{\omega' \in \Omega_n(\omega):|\omega\omega'|_n = k} \lambda_n(\omega\omega') = \lambda_n(\omega).$$

Finally,

$$\lambda_{n+1}(\omega) = \lambda_n(\omega). \tag{4.20}$$

Now, to go to the limit, we notice that the length of the longest sequence of 1 in ω is smaller than the length of ω. Hence, in virtue of (4.20),

$$\lambda_\infty(\omega) = \lambda_{|\omega|}(\omega)$$

This last term is computable recursively. So the limit is defined for all ω. The weak limit exists.

The limit is a probability measure. It is nonsingular with respects to the shift. In effects, it is easy to see that the set of $\omega \in X$ that can be written in the form, $\omega = 1_n 0 \omega' 0 1_{n+1} \omega''$, where $n \geq 0$ is the length of the first sequence of 1's and $\omega' \subset X_n$ is of full λ_∞ measure. Now, for $l \geq |\omega'| + 2n + 3$, the (unique) cylinder ω_l of length l that contains ω can be written $\omega_l = 1_n 0 \omega' 0 1_{n+1} \omega'''$. Hence,

$$\frac{\lambda_\infty(\sigma(\omega_l))}{\lambda_\infty(\omega_l)} = \frac{\lambda_\infty(1_{n-1} 0 \omega' 0 1_{n+1} \omega''')}{\lambda_\infty(1_n 0 \omega' 0 1_{n+1} \omega''')} = \frac{\lambda_{n+1}(1_{n-1} 0 \omega' 0 1_{n+1})}{\lambda_{n+1}(1_n 0 \omega' 0 1_{n+1})},$$

is independent of l, if l is large enough. The point is that the number of occurrences of 1_k for $k \geq n+1$ is the same in ω_l and $\sigma(\omega_l)$. So the derivative is locally constant. $\square$

We conjecture that this is a simple example of a case where there would be infinitely many invariant measures associated with infinitely many time scales. The idea is that the choice of the sequences $a^{(n)}$ can be made such that the return times to the cylinders 1_n are longer and longer. More precisely, let $\tau^{(n)}$ denote the return time to the word 1_n, $S^{(n)} = T^{\tau^{(n)}}$ and T_n be the first entrance time of $S^{(n)}$ to the word 1_{n+1}. These random variables are related by,

$$\tau^{(n+1)} = \sum_{k=1}^{T_n} \tau^{(n)} \circ S^{(n)},$$

and the law of T_n is given by

$$\lambda_\infty(T_n = k) = a_{k-1}^{(n)}.$$

4.2 Comments and questions

- Our family of maps contains maps having an acip. It is the case when $\beta > \alpha > 1$. What is the speed of mixing in this case ? What kind of large deviations results could be derived for these systems ?

- It would be interesting to produce a C^1 version of this example. It sounds possible but, as we already mentioned, the proofs would be much more complicated.

- The situation described in Theorem 1.1 appears naturally for product maps (product of an intermittent map with any smooth uniformly expanding map). It was more difficult to see how it can arise for an interval map.

- The situation with two physical measures also appears in other contexts. For example some uniformly piecewise expanding interval maps with an infinite number of pieces may have a finite "statistically attracting" measure and an infinite absolutely continuous invariant measure.

- Using the ideas introduced here, it seems possible to construct an interval map for which the finite statistically attracting measure is a sturmian measure, the dynamics on its support being semi conjugated to a rotation.

- In the spirit of what is done here, one could try to generalize systems with two or more indifferent fixed points. Is there a general way to classify these systems ?

- One could also imagine systems with "nonstationary" changes in the global behaviour. For example, a typical orbit could oscillate between typical behaviour of two distinct finite measures during nonintegrable periods of time. We hope that this work can give ideas to tackle this kind of problems.

- A more statistical point of view can be interesting. Given a source of symbols, one can try to estimate by which Gibbs measure it was produced. Is it possible to guess if the underlying system is in fact a system with a nonsingular measure and no invariant measure ?

- Our systems can be defined symbolically in the thermodynamic formalism. It would be interesting to understand the corresponding systems in terms of statistical mechanics. It seems that ideas involved here are related to Fisher's model (Fisher, 1972).
- It is possible to specify the regularity of the examples with respect to the symbolic metric induced by the (natural) coding. In the situation of Theorem 1.1, we can check that $var_{n+1}(-\log|T'|) \leq \frac{1+\alpha}{n}$, where

$$var_n(f) = \sup_{\omega \in \{0,1,2\}^n} \sup_{x,y \in I_\omega} |f(x) - f(y)|.$$

A Proof of the large deviations estimate for Wang maps.

We will give a proof of Lemma 2.2. Firstly, we state a large deviations result. We found it in Nagaev (1981). Its form is slightly stronger than what we will need.

Fact 3 (Nagaev) *Suppose that* $\mathbf{E}[X] = 0$ *and*

$$1 - F(x) \leq x^{-\alpha} L(x),$$

as $x \to +\infty$*, where* $\alpha > 1$ *and* L *is slowly varying. Then, for all* $c > 0$ *and* $x \geq cn$*,*

$$\mathbf{P}(\sum_{k=0}^{n-1} X_k \geq x) \leq n(1 - F(x))(1 + \epsilon_n),$$

as $n \to +\infty$*, where* $\epsilon_n \to 0$*.*

Proof of Lemma 1 For all f, we denote $S_n(f) = \sum_{k=0}^{n-1} f \circ T^k$. Let first $f = \mathbf{1}_{\{B\}}$ be the characteristic function of the interval $B = \overline{I}_1$. Let $[x]$ denote the integer part of the real number x, and set $p_n = 1 + [n(\overline{\mu}(f) - \epsilon)]$ and $q_n = [n(\overline{\mu}(f) + \epsilon)]$. Then,

$$\overline{\lambda}\left(|\frac{1}{n} S_n(f) - \overline{\mu}(f)| > \epsilon\right) \leq \overline{\lambda}\left(S_n(f) \geq p_n\right) + \overline{\lambda}\left(S_n(f) \leq q_n\right).$$

Recall that $\overline{\tau}$ is the first entrance time in $\overline{I}_1$ and $\overline{\tau}_i$ is the ith return time to $\overline{I}_1$. Since the map $\overline{R}$ is piecewise affine with onto branches, the sequence $(\overline{\tau}_i)_{i \geq 1}$ is a sequence of iid random variables of common law,

$$\overline{\lambda}(\overline{\tau}_i = k) = \overline{\lambda}_B(\overline{\tau} = k),$$

and of common mean $\theta = \overline{\lambda}(\overline{\tau}_i) = \overline{\lambda}_B(\overline{\tau}) = \overline{\mu}_B(\overline{\tau}) = \frac{1}{\overline{\mu}(B)} = \frac{1}{\overline{\mu}(f)}$ by Kac's Lemma. We set

$$\mathcal{T}_n = \sum_{i=1}^{n} \overline{\tau}_i - n\theta.$$

The sum $\sum_{i=1}^{q} \overline{\tau}_i$ counts the time between the first (after 0) occurrence of B and the $(q+1)$th. But $S_n(f) \geq q$ means that there are more than q occurrences of B between time 0 and time $n-1$. Hence, $S_n(f) \geq q \Rightarrow \sum_{i=1}^{q-2} \overline{\tau}_i \leq n-1 \Rightarrow \mathcal{T}_{q-2} \leq n-1-(q-2)\theta$. Since $\theta\overline{\mu}(f) = 1$, there is a constant c_0 such that $n-1-\theta(q_n-2) \geq c_0 - \theta\epsilon n$. We deduce,

$$\overline{\lambda}(S_n(f) \geq q_n) \leq \overline{\lambda}(\mathcal{T}_{q_n} \leq c_0 - \theta\epsilon n).$$

Since $\overline{\lambda}(\overline{\tau}_i - \theta < -t) = 0$ as soon as t is large enough, we can apply a standard large deviations result to see that $\overline{\lambda}(\mathcal{T}_{q_n} \leq c_0 - \theta\epsilon n)$ decays exponentially fast. Hence, there are constants c_1 and c_2 such that,

$$\overline{\lambda}(S_n(f) \leq q_n) \leq c_1 e^{-c_2 n}. \tag{A.21}$$

The same analysis shows that $S_n(\mathbf{1}_{\{B\}}) \leq p \Rightarrow \overline{\tau} + \mathcal{T}_p \geq n$. Since $\theta\overline{\mu}(f) = 1$, there is a constant c_3 such that $n - p_n\theta \leq c_3 + \theta\epsilon n$. We deduce,

$$\overline{\lambda}(S_n(\mathbf{1}_{\{B\}}) \geq p) \leq \overline{\lambda}(\overline{\tau} + \mathcal{T}_{p_n} \geq c_3 + \theta\epsilon n).$$

For a sum of two positive terms to be larger than n, one of the terms must be larger than $\frac{n}{2}$. Hence,

$$\overline{\lambda}(\overline{\tau} + \mathcal{T}_{q_n} \geq c_3 + \theta\epsilon n) \leq \overline{\lambda}(\overline{\tau} \geq \frac{1}{2}(c_3 + \theta\epsilon n)) + \overline{\lambda}(\mathcal{T}_{q_n} \geq \frac{1}{2}(c_3 + \theta\epsilon n)).$$

For the first term, we use directly the assumption, $\overline{\lambda}(\overline{\tau} > t) \leq ct^{-\beta}$, so,

$$\overline{\lambda}(\overline{\tau} \geq \frac{1}{2}(c_3 + \theta\epsilon n)) \leq c_4 n^{-\beta}. \tag{A.22}$$

For the second term, we can apply Nagaev's result to the sequence of iid random variables $(\overline{\tau}_i - \theta)_{i\geq 1}$. We have $\overline{\lambda}(\overline{\tau}_i - \theta) = 0$ and $\overline{\lambda}(\overline{\tau}_i - \theta > t) \leq \overline{\lambda}(\overline{\tau}_i > t) = \overline{\lambda}_{I_1}(\overline{\tau} \geq t) \leq ct^{-\beta}$. Hence,

$$\begin{aligned}
\overline{\lambda}\left(\mathcal{T}_{p_n} \geq \frac{1}{2}(c_3 + \theta\epsilon n)\right) &\leq c_5 p_n \, \overline{\lambda}_{I_1}\left(\overline{\tau} \geq \frac{1}{2}(c_3 + \theta\epsilon n)\right) \\
&\leq c_5\left(1 + [n(\overline{\mu}(B) - \epsilon)]\right)\left(\frac{1}{2}(c_3 + \theta\epsilon n)\right)^{-\beta} \\
&\leq c_6 n^{1-\beta} \\
&\leq \frac{c_6}{n^{\beta-1}}. \qquad (A.23)
\end{aligned}$$

Combining (A.21), (A.22) and (A.23), we get the result for $f = \mathbf{1}_{\{\overline{I}_1\}}$.

Let now $f = \mathbf{1}_{\{B\}}$ be the characteristic function of an element B of the dynamical partition that does not contain the fixed point 0. Let $\tau^B = \inf\{n > 0 : \overline{T}^n(x) \in B\}$ be the first entrance time in B and τ_i^B be the ith return time to B. Since B does not contain the fixed point 0, the map $R^B = T^{\tau^B}$ is piecewise affine with onto branches and hence, the sequence $(\tau_i^B)_{i\geq 1}$ is a sequence of iid random variables of common law,

$$\overline{\lambda}(\tau_i^B = k) = \overline{\lambda}_B(\tau^B = k).$$

and of common mean $\theta = \overline{\lambda}(\tau_i^B) = \overline{\mu}(\tau_i^B) = \frac{1}{\overline{\mu}(B)}$ by Kac's Lemma. We must estimate the tails of the laws of τ^B and τ_i^B. Firstly, we notice that for all sets A with $\overline{\lambda}(A) > 0$, $\overline{\lambda}_A(\tau^B > t) \leq c_A \overline{\lambda}(\tau^B > t)$, where $c_A = \overline{\lambda}(A)^{-1}$. So that it is enough to control the tail of τ^B under $\overline{\lambda}$. Then we notice that, if $B \subset [0, c_1]$, then, it has a preimage B' in $[c_1, 1]$, so that $\tau^B \leq \tau^{B'} + 1$. Hence, we can assume without loss of generality that $B \subset [c_1, 1]$. In this case, the function τ^B itself can be seen as a sum,

$$\tau^B = \sum_{i=0}^{K} \overline{\tau}_i,$$

where $K(x) = \inf\{i > 0 : \overline{R}^i(x) \in B\}$ is a random variable. Clearly, K has a geometrical law (depending on B). We know $\overline{\lambda}(\overline{\tau} > t) \leq ct^{-\beta}$. We deduce that $\overline{\lambda}(\overline{\tau}_i > t | K > i) = \overline{\lambda}_{I_2 \cap B^c}(\overline{\tau}_i > t) \leq c'_B t^{-\beta}$. Using the fact that for a sum of k terms to be larger than t, at least one of the terms must be larger than $\frac{t}{k}$, it is straightforward to conclude that there is a constant C_B such that,

$$\overline{\lambda}(\tau^B > t) \leq C_B t^{-\beta}. \tag{A.24}$$

Hence we can apply exactly the same arguments to conclude.

Now, let f be a function depending on a finite number of coordinates and equal to 0 in a neighborhood of 0. We write f as finite sum of indicators, $f = \sum_{k=1}^{K} a_k \mathbf{1}_{\{B_k\}}$, where the B_k's are elements of the dynamical partition, none of them containing 0. We write,

$$\begin{aligned}
\overline{\lambda}(|S_n(f) - n\overline{\mu}(f)| > n\epsilon) &= \overline{\lambda}(|\sum_{k=1}^{K} a_k (S_n(\mathbf{1}_{\{B_k\}}) - \overline{\mu}(B_k)| > n\epsilon) \\
&\leq \overline{\lambda}(\sum_{k=1}^{K} a_k |S_n(\mathbf{1}_{\{B_k\}}) - \overline{\mu}(B_k)| > n\epsilon) \\
&\leq \sum_{k=1}^{K} \overline{\lambda}(a_k |S_n(\mathbf{1}_{\{B_k\}}) - \overline{\mu}(B_k)| > n\frac{\epsilon}{K}) \\
&\leq K \max_{1 \leq k \leq K} \overline{\lambda}(|S_n(\mathbf{1}_{\{B_k\}}) - \overline{\mu}(B_k)| > n\frac{\epsilon}{K a_k}).
\end{aligned}$$

Then we apply the previous argument to each of this finite number of terms.

To treat the case $f = \mathbf{1}_{\{B\}}$ where B is an element of the dynamical partition containing 0, we write $g = 1 - f$ and notice that we have already proved the result for such a g. It is straightforward to conclude for any function depending on a finite number of coordinates. □

Remark 10 *It is simple to extend this result to a large deviation result under the invariant measure. We can prove that, for all functions f measurable with respects to the dynamical partition, and, all $\epsilon > 0$, there is a constant $C = C_\epsilon(f)$, such that,*

$$\overline{\mu}\left(|\frac{1}{n}S_n(f) - \overline{\mu}(f)| > \epsilon\right) \leq \frac{C}{n^{\beta-1}}.$$

To adapt the proof given below, one must notice that

$$\overline{\mu}(\overline{\tau} = k) \leq C \sum_{n \geq k} b_n \leq C \frac{1}{k^\beta}.$$

So that we loose 1 *in the exponent of the tail of the law of* $\overline{\tau}$*, as well as of this of* τ^B*. But nothing is lost in the tail of the* $\overline{\tau}_i$*, since* $\overline{\mu}(\overline{\tau}_i = k) = \overline{\lambda}(\overline{\tau}_i = k) = \overline{\lambda}_B(\overline{\tau} = k)$*,. So that Nagaev's result applies in the same conditions.*

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