A Classification of Bijective Polygonal Piecewise Isometries

X. Bressaud G. Poggiaspalla

Abstract

We aim to give a classification of euclidian bijective polygonal piecewise isometries with a finite number of compact polygonal atoms. We rely on a specific type of triangulation process which enables us to describe a notion of combinatorial type similar to its one-dimensional counterpart for interval exchange maps. Moreover, it is possible to handle all the possible piecewise isometries, given two combinatorial types. We show that most of the examples treated in the literature of piecewise isometries can be retrieved by systematic computations. We also exhibit a new class of maps, the piecewise similarities, which fit nicely in this framework and whose behaviour is shown to be highly non-trivial.

1 Introduction

Piecewise isometries (P.W.I.) are simply defined objects constituted by a partition of a domain of $\mathbb{R}^d$ with an isometry attached to each piece. Such simple maps may yield sophisticated dynamics though of zero entropy (Buzzi, [5]). Of particular interest is the case when the map is bijective, i.e. the partition is mapped onto another partition (maybe up to the boundaries of the atoms). In one dimension, we have the Interval Translation Maps and Interval Exchange Maps (I.E.T.) which have been intensively studied ([4, 21]), with heavy use of renormalization techniques. With higher dimension domains, the situation is much more complicated for several reasons.

First, there is obviously much more freedom in the choice of the partitions than in the one-dimensional case. We will restrict our investigation to the two dimensional cases with polygonal partitions. Then, the isometry group is bigger, unlike the one-dimensional case, we have rotations. A translation of the torus is an easy example of P.W.I. of higher dimension and has been extensively studied. Still, this example does not involve any rotations. To our knowledge, the first examples involving both translations and rotations appeared in engineering problems related to overflow in digital filters (Chua, Lin, see [7]). Starting from these observations, Adler, Kitchens and Tresser [1] introduced a one parameter family of P.W.I. of the rhombus. An interesting feature of these maps is the coexistence of numerous "periodic islands" and of a minimal dynamics. For few values of the parameter they were able to exhibit self-similarity and hence to describe the dynamics. For all other values though, they left more questions than answers. Independently Goetz and Boshernitzan tackled the case of two half-planes and developed the fruitful idea of self-similarity in other examples (see [11]). Poggiaspalla and Goetz [10] constructed and studied yet another family of examples (towers) and found partial self-similarity. P.W.I. also appeared naturally in a more arithmetic context after work by Vivaldi and Lowenstein, see [15, 16, 17] on discretized rotations.

One of the difficulty to understand the important phenomena is related to the lack of large interesting classes and the relatively poor number of "typical" examples. This is specially true for bijective P.W.I. Indeed, even in the polygonal case, it is not easy to find bijective P.W.I. Whether or not, given a set of polygons, it is possible to arrange them in two distinct ways to draw the same figure (as in a tangram) is in itself a nice combinatorial problem. A first classification of P.W.I. was proposed by Ashwin and Fu [2] but was not aimed to be systematic. We propose a systematic way to describe *all* the P.W.I. with polygonal domains and partitions. P.W.I. are naturally embedded in the larger class of piecewise similarities (P.W.S.). To our knowledge, those maps have not been studied although they are a possible generalization of the so called affine interval exchange maps (see for example [6]). We provide an algorithmic and geometric description of the set of P.W.S.

The point of view we adopted mimics the road-map followed for I.E.T. We try to distinguish the "combinatorial aspect" and the "real parameter" aspect. To be more specific, recall that an I.E.T. is easily described by:

- the permutation of the intervals,
- the lengths of the intervals.

For any permutation, each set of lengths yields an I.E.T., this point of view is fruitful as soon as we deal with renormalization. Induction on a well chosen interval yields a new I.E.T. (i.e. a new permutation and a new set of lengths). The dynamics of this renormalization is interesting from the combinatorial point of view (e. g. Rauzy classes, see [21]) and more generally since it is closely related to generalizations of continued fractions algorithms.

In two dimensions, the first difficulty is to decide what would play the role of the "combinatorial type" (i.e. of the permutation). It is not a restriction to limit the analysis to triangulations of a triangular domain. But still, it is not obvious to decide what a combinatorial type should be. To overcome this difficulty, we introduce a specific type of triangulations, namely, triangulations by bisections (so called nice triangulations). They are "nice" because all bijective P.W.I. with polygonal domain and partition can be defined using such partitions (see proposition 1) and they prove relatively easy to manipulate. Each such partition is described by a set of combinatorial data, we call it the combinatorial type, and a set of continuous parameters. Notice that it is fairly simple to enumerate all the combinatorial types. It is thus possible to describe by linear equations all the bijective P.W.S., and thus recover all the P.W.I. compatible with a given pair of partition types.

The paper will be structured as follows:

In Section 2, we introduce the notations and definitions used to manipulate partitions and P.W.I.

In Section 3, we propose a framework using the idea of triangulation by successive bisections. The idea is the following. We start with an initial triangle with unspecified angles. Then we choose one of its three vertices and cut the triangle from this vertex to the opposite side. The construction itself is associated with a continuous (three-dimensional) family of parameters, namely the angles of the two triangles given by the splitting. All these parameters are not independent from each other, we have two degrees of freedom for the triangle itself and one more for the bisector, cf. figure 1. As the two atoms of the new partition are triangles, the

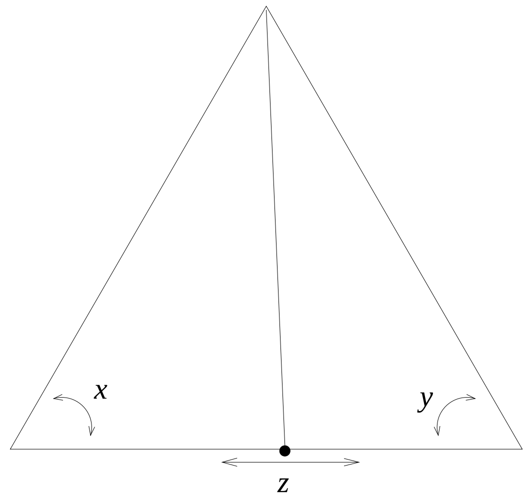

Figure 1: By splitting a triangle from a vertex, we are led to a system with 3 degrees of freedom, x, y and z on the figure.

procedure can be iterated in each atom. Finally a type of partition is described by a sequence (or a tree) of bisections. Roughly speaking it is a way to organize a triangular partition. We give a formal definition and explain how to make an intuitive use of it through lists of triangles. We also stress the fact that this object is not a partition of a triangle, it describes a continuous family of partitions (for n triangles we have $n+1$ parameters). If most partitions are of only one type, it can happen that a partition belongs to several types. An alternative way to specify completely a partition in a given type is to give the angles of all the triangles under consistency constraints. Indeed, to a type with n triangles we can associate an $n+1$ dimensional simplex in $\mathbb{R}^{3n}$ (cf. lemma 2).

In section 4, we consider two types of partition with the same number of triangles, and a combinatorial description of how the two partitions should be sent one onto the other. That is to say which triangle to which triangle, and for each triangle, which vertex on which vertex. From that, we derive the linear system of equations that the angles must satisfy in order for the map to be a P.W.S. We investigate the form of the solutions and try to give hints for a rough classification.
We naturally distinguish P.W.I. among the P.W.S. and it is simple to decide from the combinatorial data if the solution will preserve orientation or not. Another important feature of a solution is whether or not it is isolated in the parameter space. We call such solutions fixed solutions. In this case, arithmetic properties are natural to consider, we have rational angles. Notice that conversely, if the angles of a partition are not rational, then the P.W.I. is included in a continuous family (this answers a question formulated by A. Goetz). Finally, if the solution is not isolated, we stress the dimension of the simplex.

In section 5, we illustrate this formalism with explicit computations for partitions with two and three triangles. Even for such low numbers of atoms, the number of solutions is amazingly high and a huge amount of non-trivial behaviors arise. In fact, our method appears to be a valuable source of new examples. We decided, in subsection 5.1 to do an exhaustive study of the two-triangle cases. The result is that nothing really surprising arises. Then, in subsection

5.2 we selected some cases with three triangles, among which appear "old friends" and new examples. The analysis carried on in this section is computationally intensive and requires the use of a computer. We chose the computer algebra software *Mathematica* release 5 to help us. A tool-box of Mathematica functions were developed in order to handle most of the process. For the sake of conciseness, we will not include the listings of the functions, but the interested reader may find the Mathematica notebook at the following URL: [8].

Finally, in Section 6, we give a few hints for further investigations.

2 Notations and Definitions

We denote $\mathbb{R}^2$ the Euclidean plane. Given three points a, b and c, the segment $[ab]$ is the convex hull of $\{a, b\}$; the triangle $[abc]$ is the convex hull of $\{a, b, c\}$. We denote by (abc) the interior of $[abc]$. The boundary of the triangle $[abc]$ is the set $\partial[abc] = [abc] \setminus (abc) = [ab] \cup [bc] \cup [ca]$.

Definition 1 *A polygonal domain (or polygon) is a compact subset of $\mathbb{R}^2$ whose boundary is a finite union of segments. It is (simply) connected if its interior is (simply) connected. For simplicity we assume that it is always the case. It is non degenerate if it is the closure of its interior.*

Let $\mathcal{P} = (P_1, \ldots, P_n)$ be a finite collection of polygons. We say it is an *essential partition* of the polygon P if $P = \bigcup_{i=1}^n P_i$ and, for all $i \neq j$, $(P_i) \cap (P_j) = \emptyset$. We say it is a *triangulation* if, for all i, P_i is a triangle.
As a consequence, the intersections $P_i \cap P_j \subset \partial P_i \cap \partial P_j$ are finite unions of segments. We denote by $Seg(\mathcal{P})$ the minimal list of segments such that $\bigcup_{s \in Seg(\mathcal{P})} s = \bigcup_{i=1}^n \partial P_i$. We denote by $|\mathcal{P}|$ the number of polygons in the collection and by $s(\mathcal{P})$ the number of segments in $Seg(\mathcal{P})$.

Let $\mathcal{P} = (P_1, \ldots, P_n)$ be an essential partition of a polygon. If they have a segment in common, we can *glue* two elements P_i and P_j to obtain a new partition:

$$\mathcal{P} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{j-1}, P_{j+1}, \ldots, P_n, P_i \cup P_j).$$

If $(Q_1, \ldots, Q_m)$ is an essential partition of P_i, then, we can *cut* P_i to obtain a new partition, $(P_1, \ldots, P_{i-1}, Q_1, \ldots, Q_m, P_{i+1}, \ldots, P_n)$. Both operations preserve the property of being an essential partition.

Definition 2 *A piecewise isometry (resp. similarity, affine map) of a polygon P is a map f from P to P such that there is an essential partition of P in polygons, called atoms, $\mathcal{P} = (P_1, \ldots, P_n)$ and a list $(f_1, \ldots, f_n)$ of isometries (resp. similarity, affine map) such that, for all $i = 1, \ldots, n$, the restriction of f on the interior of P_i is f_i , i.e.*

$$f_{|(P_i)} = f_i.$$

Standard definitions of piecewise isometries (cf. for example [1, 9, 10]) usually include the boundaries of the atoms. In the present work, we will not be interested in the behaviour of the map on the boundary segments. The dynamics on the images of the singular set have shown very interesting behaviour though, but our aim is, at least as a starting point, to consider the most possible global concepts in an attempt to classify the maps themselves rather than to give a detailed study of each of them.

Definition 3 *We say that a polygonal piecewise isometry (resp. similarity) is* essentially bijective *if and only if the image of the initial partition is itself an essential partition.*

We call $\mathcal{S}$ the set of all the essentially bijective polygonal piecewise similarities, and $\mathcal{I}$ the subset of all the essentially-bijective polygonal piecewise isometries. Moreover, if for $f \in \mathcal{S}$ and $i \neq j$, $f_i = f_j$ and $P_i \cap P_j$ contains a segment, we could glue P_i and P_j, without changing the map. In the following, we may or may not make the identification depending on the context. The aim of this paper is to give an algorithmic way to classify the elements of the set $\mathcal{S}$. In order to do so, we will switch to a formalism where all the polygons are triangles, given by a specific triangulation scheme.

3 Triangulation by Bisection and Combinatorial Types

3.1 Triangulation by bisection

To tackle the problem of the classification of polygonal piecewise similarities in a computationally reasonable way, we consider only triangulations, and moreover a specific type of triangulation.

Definition 4 *We call a* nice triangulation *a triangulation $\mathcal{P}$ of a triangle T in n triangles with the following property:*

There is a sequence of triangulations $(\mathcal{P}_i)_{i=0,\dots,n}$ such that,

$$\mathcal{P}_0 = (T), \mathcal{P}_n = \mathcal{P} \text{ and, for all } 0 \leq i < n,$$

$\mathcal{P}_i$ is obtained from $\mathcal{P}_{i+1}$ by gluing two elements along a common side.

Notice that the point is that we ask that at each step, all the elements of the partition remain triangles. Two triangles that are glued in the sequence will sometimes be called *twins* in the following.

Definition 5 *We call* path of gluing *or* gluing chain *such a sequence of triangulations.*

To obtain such a partition, one can also follow the steps of the chain of partitions from the original triangle (T), each time cutting one of the triangles into two triangular pieces, i.e. along a bisector. In this case, we will call this sequence a *path of bisections.* Both processes are equivalent, or more precisely they are inverse. Figure 2 shows some paths of gluing if we start from the top partitions and it shows paths of bisections if we start from the bottom partitions. Figure 2 also shows that a nice triangulation may have several paths of gluing.

We call $\mathcal{S}_T$ (resp. $\mathcal{I}_T$) the sets of polygonal piecewise similarities (resp. isometries) such that the initial partitions and their images are nice triangulations in the sense above.

It is important to see that any polygonal partition has a refinement which is a triangulation by bisection. But it is crucial to notice that it is not enough since we have to deal with the dynamics of piecewise similarities defined onto them. Indeed, the properties of the triangulation have to be somehow compatible with the dynamical process.

The following result ensures that we have no loss of generality considering such unusual triangulations.

Proposition 1 *We have $\mathcal{S} = \mathcal{S}_T$.*

This result holds, provided that we naturally glue together two contiguous atoms if the same map is defined on both.

Proof of the proposition

Notice first that a polygon can always be included in a triangle and that any partition can be triangulated. We will need the following lemma.

Lemma 1 *For all family S of segments in a triangle T there is an essential partition by bisections (i.e. a nice triangulation) $\mathcal{P}$ such that $S \subset Seg(\mathcal{P})$.*

Proof of the lemma

We proceed by induction on the number of segments. Given a segment, we can prolong it until we reach the boundary of T. If it lands on a vertex, we are done. Otherwise, we have to draw a segment from the opposite vertex to the landing point of the prolonged segment and cover the latter.

Now let us suppose we have $N+1$ segments and that the statement is true for N. Then, let s be one of these $N+1$ segments, either we can prolong s to reach a vertex of the triangle T or not. If we can, then we are left with two triangles containing at most N segments and the induction applies.

If the prolonged segment lands on point x_0 on a side of T, then we split T by joining x_0 to the opposite vertex. We are led to two triangles each of them containing at most $N+1$ segments. If both triangles contain less than $N+1$ segments, we are done. Only the triangle containing s could have $N+1$ segments. If it is the case, we know that s can be continued to reach a vertex of the triangle, thus the above argument applies which completes the proof of the lemma. $\square$

Now we know that any polygonal partition can be refined to become a nice triangulation. Thus, any polygonal piecewise similarity can be viewed as defined on a nice triangulation, by cutting atoms and adding redundancy. But the image of such a partition may not be nice.

Let P be the polygonal partition of a polygonal piecewise similarity f and let Q be its image. Let P' be a refinement of P such that P' is nice; by the lemma, we can always find such a partition. Then the image Q' of P' is a triangular partition refining Q. It may not be nice but we can find a nice partition Q'' refining Q'. Then the preimage P'' of the partition Q'' is triangular and refines the nice partition P'. Let P'_i be an atom of P', it is partitioned in triangles, and this sub-partition G_i is in correspondence up to only one similarity f_i with a sub-partition of an atom of Q'. This latter atom has been partitioned when taking Q'' in such a way that $f_i(G_i)$ is nice. Hence G_i is nice. The argument holds for all possible G_i, we conclude that P'' is nice as it is a refinement of a nice partition P' by refining all the triangles in a nice way. The map f extended on P'' remains essentially the same but maps a nice triangulation onto a nice triangulation. $\square$

Remark 1 *Notice that it is possible to give a bound on the number of triangles needed in the nice triangulation in terms of the number of segments needed to describe the initial partition.*

3.2 Combinatorial types

Given a path of gluing, we now consider not only a specific partition given by this path but *all* the possible partitions that can be constructed with the same path. From now, all the triangles will be oriented counter-clockwise, and for the sake of clarity, we name $(1, 2, 3)$ the vertices of the initial triangle. With no loss of generality, we can assume that the vertices 1 and 2 are the points $(0, 0)$ and $(1, 0)$ of the real plane respectively, the point 3 remaining free in the upper half plane.

Let us look at a simple example, the triangle labelled $(1, 2, 3)$, bisected by a segment starting from the vertex 1 and landing on the opposite side, thus creating a fourth vertex, which we call 4. Then, we choose to bisect the triangle $(1, 4, 3)$ with a segment starting at 4 and landing on the side (13). We would like to stress the fact that this description is "combinatorial". We did not mention the continuous informations needed to actually describe a partition of a triangle. In other words, there is a continuous family of partitions associated with this description. In order to describe a particular one we would have to specify the initial triangle (i. e. the position of the point labelled 3, or the angles at the points 1 and 2), the angle between the segments [12] and [14] ; and finally the angles between the segments [43] and [45].
Notice that the list $((1, 2, 4)(1, 4, 5)(5, 4, 3))$ provides all the informations needed to trace the path of gluing of the triangles (the underlines show the common sides):

$$(1, 2, 4), (1, \underline{4, 5}), (\underline{5, 4}, 3) \to (1, 2, 4), (1, 4, 3)$$

$$(\underline{1}, 2, \underline{4}), (\underline{4}, 3, \underline{1}) \to (1, 2, 3).$$

The list thus corresponds to a continuous family of partitions. Note also that we can choose a different set of parameters, for instance the 9 angles linked by a linear system of equations.

We will now formalize this notion. We want to get rid of the continuous parameters, we will then work with "combinatorial triangles", that are merely lists of vertices. A bisection of a triangle [123] can be described by a list of two triangles. If we call 4 the "new" vertex, we have three cases, depending on whether 4 belongs to (13), (12) or (23). The list will be ([234], [412]), ([314], [423]) or ([124], [431]), respectively.

A path of bisection of a partition of a triangle [123] corresponds to a growing list of triangles. At each stage, we have the names of the triangles in $\mathcal{P}_i$, where the new vertex created at stage i is called $i + 3$. A sequence of $n + 1$ bisections provides a list of n triangles, the names of the vertices range from 1 to $n + 2$. Given the final list of triangles, it is easy to recover the path by gluing the two triangles containing the vertex with highest index, and so on, as seen on the example above.

We will say that two paths of bisections are *combinatorially* the same if they produce the same sequence of lists $(\mathcal{P}_i)_{i=0\cdots n}$, or equivalently, if they produce the same final list.

To a combinatorial path of bisections, yielding n triangles, we associate a map t from $A_n = \{1, 2, \dots, 3n\}$ onto $V_n = \{1, 2, 3, 4, \dots, n + 2\}$ which describes the final list of triangles:

$$(\,(t(1), t(2), t(3)), \dots, (t(3n - 2), t(3n - 1), t(3n))\,).$$

It will be convenient to identify a combinatorial path of bisections with such a map. Notice that not all such maps correspond to a path of bisection. A map will be called *admissible* if it is the case.

Remark 2 *The number of paths of bisections yielding n triangles is bounded by $3^n(n-1)!$. Indeed, at each step, the next bisection is determined by the choice of the triangle and of one of its vertices.*

We will say that two paths of bisection are *equivalent* if they correspond to the same partitions when the continuous parameters vary. More precisely, they are equivalent if for each partition obtained with the first path and a fixed set of parameters, it is possible to choose the parameters of the other one to obtain the same partition.

This equivalence relation takes into account two technical points. Firstly, if during the sequences of bisections we get two non-overlapping triangles (T') and (T''), we can bisect them separately in any order. Different orders will lead to final lists with the same structure, changing only the names of the vertices which are in relation with the moment they are created. Secondly, if a triangle is cut twice (or more) from the same vertex then the order in which the splittings are done does not matter. Nonetheless, it will change the names of the vertices and may also affect the order in which the triangles are listed.

Definition 6 *A* combinatorial type of partition *is an equivalence class of combinatorial paths of bisections. We will denote by $|\tau|$ the number of triangles involved in the combinatorial type τ. In τ we choose once for all a representative list map denoted by t_τ.*

If t is an admissible map corresponding to a path of bisection of type τ, we may write $t \in \tau$, but for conciseness, we may also call "combinatorial type of partition" the map associated to any representative of a combinatorial type and preferably the selected representative t_τ.

Hence, a partition type may have several gluing paths. An easy example is given by the following list $((1,2,4)(1,4,5)(1,5,3))$. We can choose to glue the two upper triangles first and then the remaining one or we can glue the two lower triangles first. Both paths of gluing are valid and lead to the same type, cf. figure 2.

It is also worth noticing that a partition may be given by more than one combinatorial type, as shown in figure 3. Figure 13 provides an exhaustive list of combinatorial types with 3 triangles.

In the description of a combinatorial type through a formal list of triangles, the names of the vertices (except $1, 2$ and 3) and the order in which the triangles are listed do not matter. It is not very difficult to check that given such a list it is possible to recover a path of bisection. At each stage we glue two triangles which have two common vertices, one of them being distinct from 1, 2 and 3 and appearing in no other triangle of the list. In the following, it will be convenient for us to associate to each type τ a particular list of triangles written in the order given by t_τ.

3.3 Alternative Description

The number of combinatorial types with n triangles is bounded by the number of combinatorial paths of bisections, i.e. $3^n(n-1)!$. To enumerate all the combinatorial types it is certainly more efficient to use the following point of view.

Let $\mathcal{T}$ be the set of finite planar rooted trees G with set of vertices V, set of edges E and root $r \in V$. As a rooted tree, G has no vertex of degree 2 and its root r has degree more than 1. Call $L \subset V$ the subset of leaves (i. e. vertices with degree 1) and set $n = |L|$. We say that G is a labelled tree if it comes with a map that associates a value in $\{1, 2, 3\}$ to the root and a value in $\{-1, +1\}$ to each vertex in $V \setminus (L \cup \{r\})$.
We claim that there is a bijection between the combinatorial types with n triangles and the labelled trees with n leaves. This representation is simply a way to avoid the redundancy described by the equivalence relation introduced above.
We do not want to enter into the details though. Let us just say that, roughly speaking, the root represents the triangle [123]; the label of the root tells us from which vertex of the triangle the first bisection is done. Each edge starting from the root goes to a vertex that represents a triangle. There can be more than two edges when the triangle is cut in more than two triangles, from the same vertex.
Then starting from each of these triangles, the tree describes how the triangle is itself cut. Notice that at each vertex except the root we can decide to do the next bisection from only two of the three vertices as it is not allowed to use the same vertex again.
For the purposes of this paper, it is not necessary to develop this formalism any further.

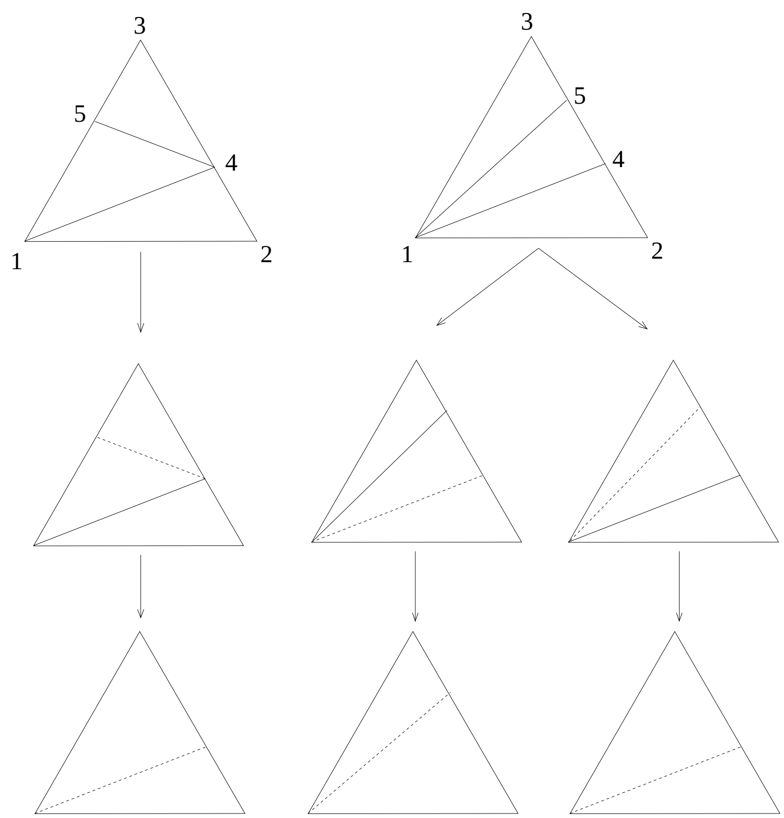

Figure 2: Unlike the first triangulation on the left, in the triangulation on the right, which illustrates the triangles list $((1, 2, 4)(1, 4, 5)(1, 5, 3))$, we see that we have two possible gluing paths to recover the initial triangle. Each gluing is figured on the pictures as dashed lines.

To summarize, we have three different objects that can be considered as abstraction layers of an intuitive concept:

- We consider a nice triangulation of a triangle called [123] up to similarities. However, in

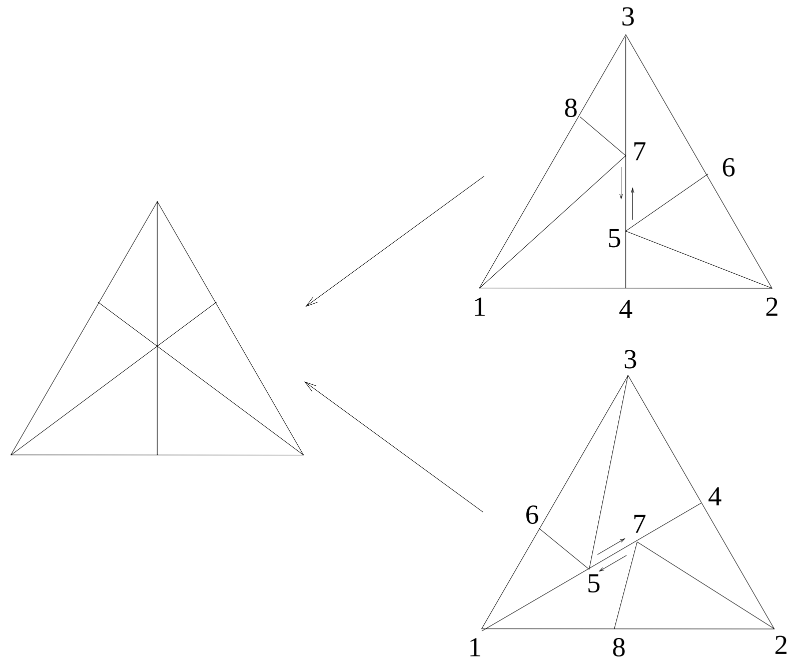

Figure 3: The partition on the left can be given by any of the two combinatorial types on the right.

the following, we chose to assume that 1 and 2 are fixed and that 3 is in the upper half plane, for the sake of clarity.

- Then we consider a combinatorial type of the path of bisections generating the nice triangulation. It is represented by a list map, which formalizes an equivalence up to a set of continuous parameters (for instance the angles).

- Finally, as a given partition can be described by different paths of bisections, we have another equivalence relation among the paths of bisections and a combinatorial type is represented by one of the above lists (or alternatively by a labelled tree).

3.4 Partitions in a combinatorial type

As by definition of a type, for a given combinatorial type τ, the set of partitions following this type does not depend on the path of bisection chosen, it can be parameterized by the angles of the $n = |\tau|$ triangles. There are n triangles and hence, $3n$ angles, under linear constraints. Given a partition, we can consider the angle vector $A \in]0, \pi[^{3|\tau|}$. Its coordinates are ordered according the map $t \in \tau$ as the order of the list provides an order on the vertices and hence on the angles. The choice of the order is in itself unimportant but must be made once and for all before any further computations.
Given list map $t \in \tau$ we call $A(\tau)$ the subset of $]0, \pi[^{3|\tau|}$ of the angles attained by all the partitions following t. We have:

Lemma 2 *$A(\tau)$ is a convex subset of $]0, \pi[^{3|\tau|}$ of dimension $|\tau| + 1$.*

Proof of the lemma
This proof is easily done via an induction. We put $n = |\tau|$. If we have only one triangle, we

need 2 parameters to describe it. Now suppose we have n triangles and by induction hypothesis $n+1$ parameters. To add one more triangle, we have to split one of the existing ones. Thus, we let all the $n+1$ parameters be fixed and choose one triangle to bisect. When we cut a triangle, we have only one degree of freedom which is the position of the landing point of the bisector. We then have $n+1$ triangles and $n+2$ parameters, which completes the induction. □

We now write all the equations the angles have to fulfil. First, we have the consistency condition for each of the n triangles, for all $j = 0, \ldots, n-1$ if we call α_{3j+k} the angles at the vertex k in the triangle j, we have:

$$\sum_{k=1}^{3} \alpha_{3j+k} = \pi, \quad j \in \{0, \ldots, n-1\}. \tag{1}$$

All these equations are clearly independent, as each deals with a separate set of angles.

Moreover, each created vertex v lies on a side, the sum of the angles around it must be π:

$$\sum_{i:t(i)=v} \alpha_i = \pi, \quad v \in \{4, \ldots, n+2\}. \tag{2}$$

The equations of this set are independent as well. Each involves a separate set of two angles. If two angles are in an equation, then they cannot be in another one as a different equation deals with a different point.

We express these conditions using matrices. Condition (1) is expressed by the $(n+1) \times 3n$ matrix $C(n)$:

$$C_{i,j}(n) = \begin{cases} 1 & \text{if } j = 3i-2, 3i-1, 3i \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The $(n-1) \times 3n$ matrix $V(\tau)$ will express condition (2):

$$V_{i,j}(\tau) = \begin{cases} 1 & \text{if } t(i) = j+3, \quad j \in \{1, \ldots, n-1\} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

It is easy to see that no line of the matrix (3), which has always three contiguous ones, can be expressed in terms of a combination of lines of the matrix (4). Indeed, each line of the latter contains 2 ones and the matrix never has 2 ones in the same column (that is, each angle is only used once). We thus have $2n-1$ independent equations, by the lemma, it is enough to describe the system.

We can write the constraints in the compact form:

$$A > 0, \quad CA = \pi\mathbf{1}, \quad \text{and} \quad VA = \pi\mathbf{1}.$$

In the following, we will need to ensure that the exterior triangle $(1,2,3)$ remains unchanged by the P.W.S. If we call its angles (α, β, γ), then certainly:

$$\alpha = \sum_{i:t(i)=1} \alpha_i, \quad \beta = \sum_{i:t(i)=2} \alpha_i \quad \text{and} \quad \gamma = \sum_{i:t(i)=3} \alpha_i. \tag{5}$$

We introduce an additional matrix E giving two of these angles. Indeed, the consistency of the triangle $(1,2,3)$ being already encoded in the matrix C, the third angle of $(1,2,3)$ does not give any information. The matrix E is $2 \times 3n$ and is defined by

$$E_{i,j}(\tau) = \begin{cases} 1 & \text{if } t(i) = j \quad j \in \{1, 2\} \\ 0 & \text{otherwise} \end{cases}$$

Thus:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = EA.$$

Finally, we remark that taking any other t' in the class τ would only result in permutations of the rows of the matrices.
A couple (τ, A) where τ is a combinatorial type and $A \in A(\tau)$ is all we need to determine a nice triangulation of the triangle $[123]$, the vertex 3 is determined by the angles EA. We will use the following notation.

Definition 7 *We will denote by (τ, A) the partition of the triangle $(1, 2, 3)$, the vertices 1 and 2 being fixed and its angles determined by $E.A$. The partition is constructed by the bisection process described by τ.*

4 Types of maps

We want to enumerate the bijective piecewise similarities with a given number of triangles. We suppose the partitions to be nice and we associate to each of them a combinatorial type. A piecewise similarity maps each vertex of each triangle of the first partition to a vertex of a corresponding triangle of the target partition.
From the point of view of combinatorial types, the map is a permutation on the vertices. Not all possible permutations are allowed though, the triangles themselves can be permuted as well as the vertices inside a triangle. But a triple of vertices consisting in a triangle must still correspond to a triangle after the permutation. We will say that a permutation $\Sigma \in S_{3n}$ is *admissible* if there are $(\sigma, s_1, \ldots, s_n) \in S_n \times S_3^n$, such that Σ can be written in the form

$$\Sigma(3i + k) = 3(\sigma(i) - 1) + s_i(k), \quad 1 \leq i \leq n, 1 \leq k \leq 3.$$

We denote $\overline{S}_n \subset S_{3n}$ the subset of admissible permutations on $3n$ elements. If $\Sigma \in \overline{S}_n$, we also denote Σ its $3n \times 3n$ permutation matrix. Notice that $|\overline{S}_n| = n!6^n$.

Given two combinatorial types τ and τ' with $|\tau| = |\tau'|$, two angles vector A and A', and an admissible permutation Σ, we consider the piecewise *affine* map f_Σ mapping the triangles of (τ, A) onto the triangles of (τ', A') in the order prescribed by Σ. Precisely, for $t \in \tau$ and $t' \in \tau'$ we have for all $1 \leq i \leq n$, the image of the triangle $(t(3i)t(3i+1)t(3i+2))$ is the triangle $(t'(\Sigma(3i))t'(\Sigma(3i+1))t'(\Sigma(3i+2))$.

Remark 3 *Notice that the identity permutation in S_{3n} may yield a non trivial map as soon as the types are distinct.*

An affine map is a similarity if and only if it preserves the angles of a non degenerate triangle. Hence the map f_Σ is a P.W.S. if and only if $A' = \Sigma A$. According to section 3, the triangles partitioned by (τ, A) and (τ', A') are the same if and only if $E(\tau)A = E(\tau')A'$. It is then natural to make the following definition.

Definition 8 *For all integer n, all combinatorial types τ, τ' with $|\tau| = |\tau'| = n$ and all admissible permutation $\Sigma \in \overline{S}_n$, we denote $\mathcal{A}(\tau, \tau', \Sigma)$ the set of solutions in $A \in]0, \pi[^{3n}$ of the* *equations*

$$\begin{array}{rcl} E(\tau)A & = & E(\tau')\Sigma A \\ C(n)A & = & C(n)\Sigma A = \pi \\ V(\tau)A & = & V(\tau')\Sigma A = \pi. \end{array} \tag{6}$$

where $E(\tau), V(\tau), C(n)$ are the matrices defined above.

We see that given τ, τ' and Σ, the angle vectors $A \in \mathcal{A}(\tau, \tau', \Sigma)$ are such that A and ΣA describe two partitions with similar triangles. It is then easy to find the piecewise similarity corresponding to this transformation.
Conversely, we can always consider any given piecewise similarity f to be defined on nice triangulations, cf. proposition 1. Both partitions, possibly up to a similarity, can be described by two combinatorial types τ and τ' and two angle vectors. As discussed above, the map corresponds to some permutations Σ of the angles. Then certainly f is included in the set of P.W.S. given by the solutions $\mathcal{A}(\tau, \tau', \Sigma)$. We shall also use the same notation $\mathcal{A}(\tau, \tau', \Sigma)$ to denote the set of the corresponding maps. We summarize this remark in the following proposition.

Proposition 2 *Up to conjugacy, we have:*

$$\mathcal{S} = \bigcup_{\tau, \tau', \Sigma} \mathcal{A}(\tau, \tau', \Sigma)$$

Where τ and τ' are two combinatorial types with the same number of triangles and $\Sigma \in \overline{S}_{|\tau|}$.

It can happen that two different pairs of types lead to the same map (possibly up to a similarity). For instance, the top and bottom pairs of combinatorial types pictured in figure 12 are "equivalent" and will give the same maps up to a similarity.
We will say that two pairs (τ_1, τ_2) and (τ_1', τ_2') are *equivalent* if there exists a similarity S such that for all partitions (P_1, P_2) and (P_1', P_2') following these types we have

$$(P_1, P_2) = (SP_1', SP_2').$$

These partitions being non-degenerate, i.e. all their atoms have non-empty interiors. This equivalence can be expressed combinatorially in terms of permutations of the vertices of the initial triangles [123]:

Definition 9 *Two pairs (τ_1, τ_2) and (τ_1', τ_2') are* equivalent *if there exist $\sigma \in S_3$ such that $\tau_1' = \tau_1 \circ \tilde{\sigma}$ and $\tau_2' = \tau_2 \circ \tilde{\sigma}$, where $\tilde{\sigma}(i) = \sigma(i)$ for $i \leq 3$ and $\tilde{\sigma}(i) = i$ for $i > 3$.*

We must notice that the bijectivity of the P.W.S. implies that there is a symmetry between the pair (τ_1, τ_2) and the pair (τ_2, τ_1). The inverse map of a similarity is a similarity. More formally: $\mathcal{A}(\tau_1, \tau_2, \Sigma) = \mathcal{A}(\tau_2, \tau_1, \Sigma^{-1})$.

We are now ready to enumerate the piecewise similarities. For all integer n, we can enumerate the combinatorial types of partitions with n triangles using section 3. Then for each pair of such combinatorial types (τ, τ'), and all admissible permutation Σ we can solve the linear system (6) to determine the angle vectors in $\mathcal{A}(\tau, \tau', \Sigma)$. We call this set a *solution*. A solution

is then an intersection of two n-dimensional simplices. The intersection may be empty, leading to no solution at all or it is again a simplex. If it is non empty, then (τ, τ', Σ) may be called the (combinatorial) type of the corresponding P.W.S. For n triangles, the number of possible types of P.W.S. is bounded by $n!6^n(3^n(n-1)!)^2 = 54^n n!(n-1)!^2$.
This bound is a bit crude, but a better estimation would involve tedious computations unnecessary for our purpose.

Let us now mention a few relevant properties of these solutions.

Definition 10 *For short, the dimension of a solution will be the dimension of the simplex* $\mathcal{A}(\tau, \tau', \Sigma)$.

The dimension zero solutions are constituted of only one point and will often be called *fixed* in the following.
For such a fixed solution $a = (\alpha_1, \ldots, \alpha_{3n})$ with n triangles, all the angles will clearly be rational multiples of π. Moreover, they have the same denominator up to possible simplifications with the numerator. In other words, all the denominators of the angles must be divisors of the same integer, there exists an integer q such that $a \in (\frac{1}{q}\mathbb{Z}\pi)^{3n}$.
As we only deal with maps of the euclidian plane, we can use the complex numbers to express the vertices of the triangles and the maps themselves. All these quantities can be computed only performing operations in the number field $\mathbb{Q}(e^{i\alpha_1}, \ldots, e^{i\alpha_{3n}})$. As there exists a q such that $\alpha_i = p_i\pi/q$ for every i, the number field is finitely generated, its dimension over $\mathbb{Q}$ is the degree of the cyclotomic polynomial of order $2q$, that is $\phi(2q)$, ϕ being the Euler function.

Definition 11 *We will call the* degree *of a fixed solution* $(\alpha_1, \ldots, \alpha_{3n})$ *the dimension of the cyclotomic number field* $\mathbb{Q}(e^{i\alpha_1}, \ldots, e^{i\alpha_{3n}})$.

A solution with dimension greater than zero will also be called a simplicial solution, they have a somehow similar expression. The functions of the parameters $f_i(x_1, \ldots, x_p)$ have the following form:

$$f_i(x_1, \ldots, x_p) = d\pi + \sum_{j=1}^{p} a_j x_j \quad (a_i, d) \in \left(\frac{1}{q}\mathbb{Z}\right)^2$$

for $i \in 1, \ldots, 3n$, p the dimension of the solution and a unique integer q. The parameters x_i range in intervals whose bounds are of the same form as above. At least one of them has fixed bounds in $\frac{1}{q}\mathbb{Z}\pi$.

The following definition is also natural.

Definition 12 *We say that a solution is* direct *if all the associated transformations have positive determinant. We say that a solution is* reverse *if all the associated transformations have negative determinant and finally that it is* mixed *otherwise.*

As the triangles in our lists are oriented, it is clear that if the permutation of the vertices of a given triangle is even, then the resulting similarity will preserve orientation. Thus, if we limit the investigation to direct piecewise similarities only, we have $n!3^n$ allowed permutations instead of $n!6^n$, which can save us a significant amount of computational time.

Given an angle vector in a solution and the permutation attached to it, we know all the angles of both partitions. Once we constructed them, as we know by the permutations which triangle in the first partition is supposed to be mapped onto which one of the second; we can compute the transformations. They are similarities as they preserve the angles and can possibly be isometries. A solution is a P.W.I. if and only if all the associated transformations on all its atoms have their determinants equal to 1 or -1. We cannot say that a simplicial solution has a definite type, in general it can contain both piecewise similarities and piecewise isometries, cf. figure 11. Notice that we still do not have a nice algorithmic way to discriminate P.W.I. in a simplicial solution containing P.W.S.

To avoid some redundancy in the solutions, we will introduce the following definition:

Definition 13 *We will say that a piecewise similarity is* irreducible*, if given any pair of twin triangles, each of them bears a different similarity.*

Simply stated, this means that the map has the minimum number of atoms. It cannot be reduced, by gluing two or more elements to a piecewise similarity on a nice triangulation with fewer triangles.

To find all the piecewise similarities on n triangles, we will first determine all possible combinatorial types on n atoms. There are finitely many of them, indeed fewer than $(n-1)!3^{n-1}$, cf. section 3. Then we pick all the pairs of combinatorial types and list all the solutions for all admissible permutations.

Clearly, the amount of computations grows dramatically fast with the number of triangles. In the following sections we will perform an exhaustive enumeration for the cases with two triangles. We will also have a look at some three-triangles cases.

5 First computations

5.1 Two triangles

We start with the simplest case, that is, when we only have two atoms. By the construction described above, given our reference triangle $(1, 2, 3)$ we have three possible bisections. For each of them, the new vertex 4 will land on a different side. As we must specify a pair of partitions, we are led to 9 possible pairs, many of them being equivalent. For instance, the pairs of figure 4 are clearly equivalent. In fact, each pair can be "rotated" three times and finally we have only three cases to consider, cf. figure 5. In the following, we will make an extensive exploration of them.

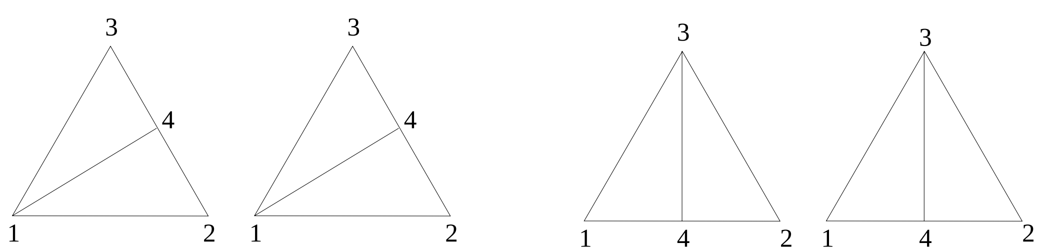

Figure 4: The two above pairs are equivalents up to a cyclic permutation of the vertices.

5.1.1 The "tower-case"

The first case we will investigate has a structure already encountered in other references. A 2-triangles case involving this bisection scheme has been extensively studied in [9] where the author shows one of the first self-similar dynamics encountered in the field of piecewise isometries. Generalizations to more than two triangles of this structure in the form of "towers of triangles" have been considered subsequently in [10]. Such towers, originally found "by hand" can be retrieved by systematic computations.

In this section and the following, the phase space will be a triangle labelled $(1, 2, 3)$ and 1 being the lower-left corner. All the triangles will be oriented counter-clockwise. The bisections of the triangles correspond to the lists $((1, 2, 4)(1, 4, 3))$ and $((1, 2, 4)(4, 2, 3))$ respectively, as shown in figure 5 (top triangle).

For this model, the computer checked all admissible permutations of the vertices between the

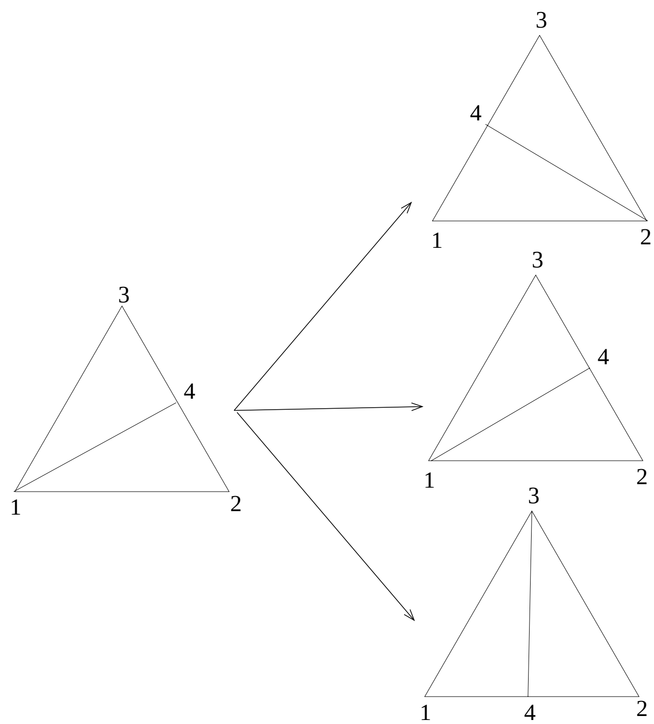

Figure 5: The three cases to consider.

two triangles. The solutions are in the 6-dimensional open cube $I =]0, \pi[^6$, one dimension for each of the six angles. Their order is based on the triangles list, i.e. for the list $((1, 2, 4), (1, 4, 3))$ each component $(a_1, a_2, a_3, a_4, a_5, a_6) \in I$ corresponds to the angles

$$(\widehat{124}), (\widehat{412}), (\widehat{241}), (\widehat{143}), (\widehat{314}), (\widehat{431})$$

respectively. We are thus led to solve the following equations under the constraints of staying in I and for all allowed permutation matrix Σ.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} . \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \\ \pi \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} .\Sigma. \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \\ \pi \end{pmatrix}$$

and moreover, to ensure that the exterior angles coincides:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} .\Sigma. \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} .$$

There is a total of 8 solutions, 2 of them are fixed, direct and correspond to two piecewise isometries. The first one is of degree 4 and the second one of degree 6. In the following we will only list the *irreducible* cases. The fixed solutions are listed in figure 6 while the simplicial solutions are listed in figure 8.

Number	Permutation	Angles
1	(231645)	$(2\pi/5,\ \pi/5,\ 2\pi/5,\ 3\pi/5,\ \pi/5,\ \pi/5)$
2	(312645)	$(3\pi/7,\ 2\pi/7,\ 2\pi/7,\ 5\pi/7,\ \pi/7,\ \pi/7)$

Figure 6: Fixed solutions table for section 5.1.1 case.

The first solution listed in table 6 is well-known, it is the case extensively studied in [9]. The second and only other fixed solution has also been investigated by the same author and his collaborator in [11]. These solutions have proven very interesting and have intricate behaviour. About the other solutions, we conducted an automated investigation and their behaviors appear to be near-trivial. All are constituted by a reflection on one of the atoms and a rotation on the other one, leading to periodic or quasi-periodic motions. Thus, the only cases of interest in that configuration are the fixed solutions listed above. Let us recall that a first idea of the dynamics of such a map is given by the *mosaic* of the map, i.e. the union of the backward and forward images of the discontinuity lines (for more details about standard tools for the study of the dynamics of P.W.I., see among others [20]).

5.1.2 The "Symmetric" Case

In our attempt to list all the possible piecewise isometries, and even all possible piecewise similarities, we mentioned three different pairs of combinatorial types. This section deals with the "symmetric pair" $((1,2,4)(1,4,3))$ gives $((1,2,4)(1,4,3))$. Following the same process as before, we will compute all the possible solutions given by all the admissible permutations of the vertices. This time, there is a total of 27 solutions, 5 of them are direct. We have 6 fixed

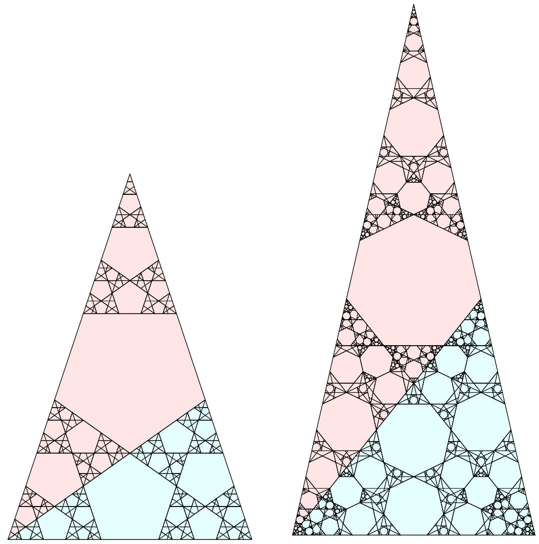

Figure 7: Left pictures: mosaic for solution number 1, Right: mosaic for solution number 2.

Solution Number 4	
Permutation Angles Constraints	(213564) $(\frac{\pi}{3}-\frac{x_3}{3},\ x_3, \frac{2\pi}{3}-\frac{2x_3}{3},\ \frac{\pi}{3}+\frac{2x_3}{3},\ \frac{\pi}{3}-\frac{4x_3}{3},\ \frac{\pi}{3}+\frac{2x_3}{3})$ $0 < x_3 < \pi/4$
Solution Number 5	
Permutation Angles Constraints	(213645) $(\frac{\pi}{3}+\frac{x_3}{3},\ x_3, \frac{2\pi}{3}-\frac{4x_3}{3},\ \frac{\pi}{3}+\frac{4x_3}{3},\ \frac{\pi}{3}-\frac{2x_3}{3},\ \frac{\pi}{3}-\frac{2x_3}{3})$ $0 < x_3 < \pi/2$
Solution Number 6	
Permutation Angles Constraints	(231645) $(\frac{\pi}{2}-\frac{x_3}{2},\ x_3, \frac{\pi}{2}-\frac{x_3}{2},\ \frac{\pi}{2}+\frac{x_3}{2},\ \frac{\pi}{2}-\frac{3x_3}{2},\ x_3)$ $0 < x_3 < \pi/3$
Solution Number 7	
Permutation Angles Constraints	(312546) $(\pi-2x_3,\ x_3,\ x_3,\ \pi-x_3,\ \pi-3x_3,\ -\pi+4x_3)$ $\pi/4 < x_3 < \pi/3$

Figure 8: Simplicial solutions table for section 5.1.1 case.

Number	Permutation	Angles
3	(564231)	$(2\pi/5,\ \pi/5,\ 2\pi/5,\ \pi/5,\ 3\pi/5,\ \pi/5)$
4	(645312)	$(3\pi/5,\ \pi/5,\ \pi/5,\ \pi/5,\ 2\pi/5,\ 2\pi/5)$

Figure 9: Fixed irreducible solutions table for section 5.1.2 case.

solutions, all of them are piecewise isometries, the first four are of degree 2 and leave their atoms invariant, thus only the remaining solutions, of degree 4, will be listed in figure 9.

Solutions number 3 and 4 do have a non-trivial dynamics but they are conjugated to the well known dynamics of case [9] already encountered in the previous section, cf. figure 10 for their mosaics of cells.

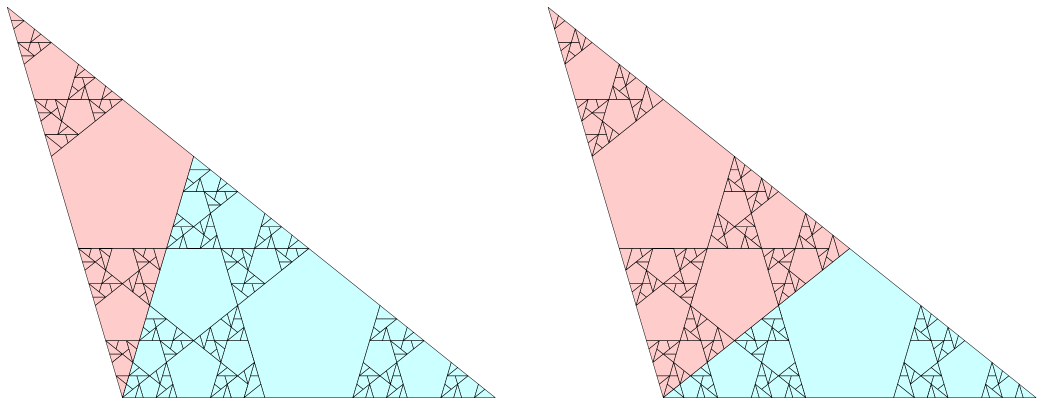

Figure 10: Left picture: mosaic for solution number 3, right: mosaic for solution number 4. Both are conjugated to solution number 1 of section 5.1.1.

There are many simplicial solutions, but most of them, solutions 7 to 21, leave their atoms invariant. Also, solution number 27 is not irreducible. Among the remaining solutions, the numbers 23 to 26 are constituted by a rotation and a reflection, leading to only periodic or pseudo-periodic orbits. Here we call *pseudo-periodic orbits* the orbits which densely fill a circle or a finite number of circles. The last case to consider, number 22, is a family of piecewise similarities, its partitions for some value of the parameter are shown in figure 11. We see that this family contains mostly piecewise similarities, in fact everytimes except for $x = \pi/4$. However, the dynamics of this solution appears to be trivial as well, it consists merely on the exchange of the two atoms.

To conclude, this pair of combinatorial type does not bring anything new as the only non-trivial solutions are conjugated to some of the solutions from the previous section.

5.1.3 Third Case

The third case to consider, as shown in figure 5 and given by the lists of triangles $((1,2,4)(1,4,3))$ and $((1,4,3)(4,2,3))$ can be viewed as the inverse of the "tower case", investigated above. Up to a rotation, this is clear as shown in fig. 12.

As all the solutions must be essentially bijective, we expect the solutions in this section to be the inverses of the solutions of section 5.1.1. This case thus requires no further investigations as it brings nothing new in terms of dynamics.
From all these computations, we can say that the only non-trivial piecewise isometric dynamics on 2 triangles are cases displayed in figure 7, both of them are already known in the literature.

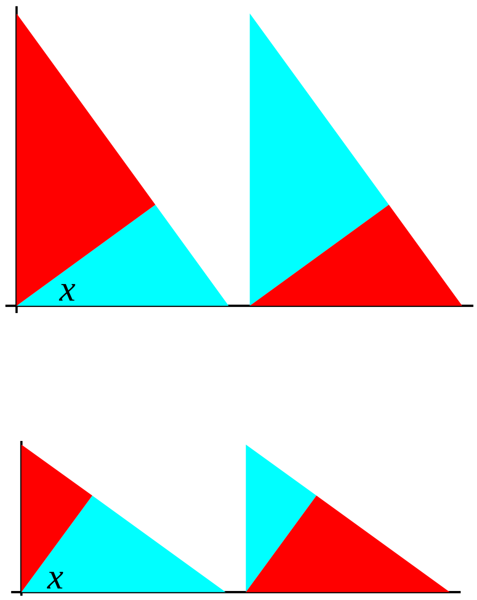

Figure 11: The only solution giving piecewise similarities encountered so far, the angles are $(\pi/2 - x, x, x, \pi/2, \pi/2, \pi/2 - x, x)$, for $0 < x < \pi/2$. The plots show 4 examples in the family with 4 increasing values of x. We note that the upper-right figure displays a piecewise isometry, thus illustrating the fact that a simplicial solution can contain both piecewise similarities and piecewise isometries.

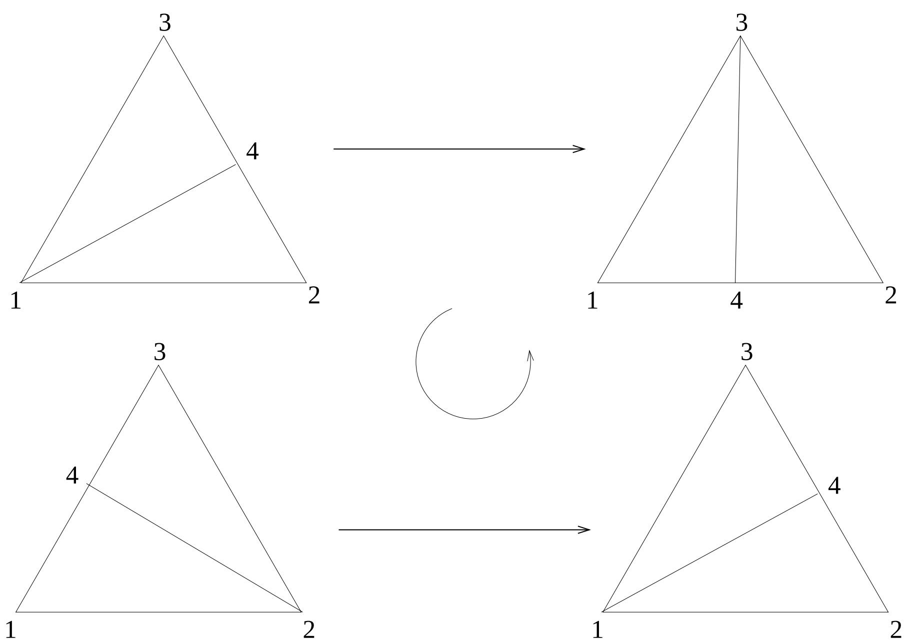

Figure 12: The top plot shows the pair of type to consider. The bottom plot shows the same type but up to a counter-clockwise rotation of $\pi/3$ applied on both triangles. We recognize the swapped pair of section 5.1.1.

5.2 Three triangles

If the number of cases to deal with and the number of solutions in the two-triangles investigation remained relatively low, this will be no longer the case with three triangles. First, we have 15 different combinatorial types with three triangles, they are shown in figure 13. Then there are $15^2 = 225$ pairs. We can as above put an equivalence relation on the pairs to avoid the cases which can be deduced from others by a rotation or a flip, for instance, we have:

$$(1,2) \sim (6,9) \text{ or } (4,3) \sim (7,10) \text{ etc.}$$

We are then led to 75 cases. Moreover we ignore the swapping of combinatorial types inside a pair. Indeed, for example, the pair $(1,2)$ will lead to maps that are the inverses of the maps found for $(2,1)$. This way, we save 20 cases and only 55 cases are left to consider, they are listed below:

((1, 1), (1, 6), (1, 7), (1, 8), (1, 9), (1, 10), (1, 11), (1, 12), (1, 13), (1, 14), (1, 15), (2,2),
(2, 6), (2, 7), (2, 8), (2, 9), (2, 10), (2, 11), (2, 12), (2, 13), (2, 14), (2, 15), (3, 3), (3, 6), (3, 7),
(3, 8), (3, 9), (3, 10), (3, 11), (3, 12), (3, 13), (3, 14), (3, 15), (4, 4), (4, 6), (4, 7), (4, 8),
(4, 9), (4, 10), (4, 11), (4, 12), (4, 13), (4, 14), (4, 15), (5, 5), (5, 6), (5, 7), (5, 8), (5, 9),
(5, 10), (5, 11), (5, 12), (5, 13), (5, 14), (5, 15)).

The 3089 solutions took about twenty minutes to compute on a 3 Ghz desktop PC. The number of solutions for a given pair can vary greatly, from 9 up to 297. We have 449 direct solutions and 810 fixed solutions. The following little table gives the number of fixed solutions by degree.

Degree	2	4	6	8	10	12
Number of Solutions	130	328	258	6	56	32

Among these solutions we can spot some known cases. For example, the case studied in [10] belongs (up to a rotation) to the set of solutions given by the pair $(3,15)$. Similarly, the cases (in fact their inverses) studied in [1] belong to the set of solutions of the pair $(2,8)$. As it would be impossible to review systematically such a vast number of cases in the present paper, we will only focus on two pairs giving a fair amount of interesting solutions. The complete listing of all the solutions with a quick preliminary analysis is available in the electronic supplement [8].

Although the choice we made may seem arbitrary we hope the great number of solutions it provides as well as the variety of them make it suitable for our illustrative purpose.

5.2.1 Tower Case Again

In this section, we will be interested in the pair $(3,15)$. We have been working with the lists $((1,5,4)(5,2,4)(1,4,3))$ and $((4,2,5)(4,5,3)(3,1,4))$. Among its 45 fixed and 86 simplices of solutions, we find, up to a rotation, the case studied in [10], displayed in figure 14.

As we said, there are 45 fixed solutions and some of them have highly non-trivial behaviors, the table below gives their distribution by degree.

Degree	2	4	6	8	10	12
Number of Solutions	2	26	15	0	1	1

Only 27 of them are piecewise isometries, the rest being piecewise similarities. These solutions display a fair variety of behaviors, some seem to have a near-trivial behaviour, that is, their mosaics of n-cells stabilize at a certain level to a finite partition. Many of them have a highly non-trivial behaviour though. A detailed study of these cases would be impossible here for

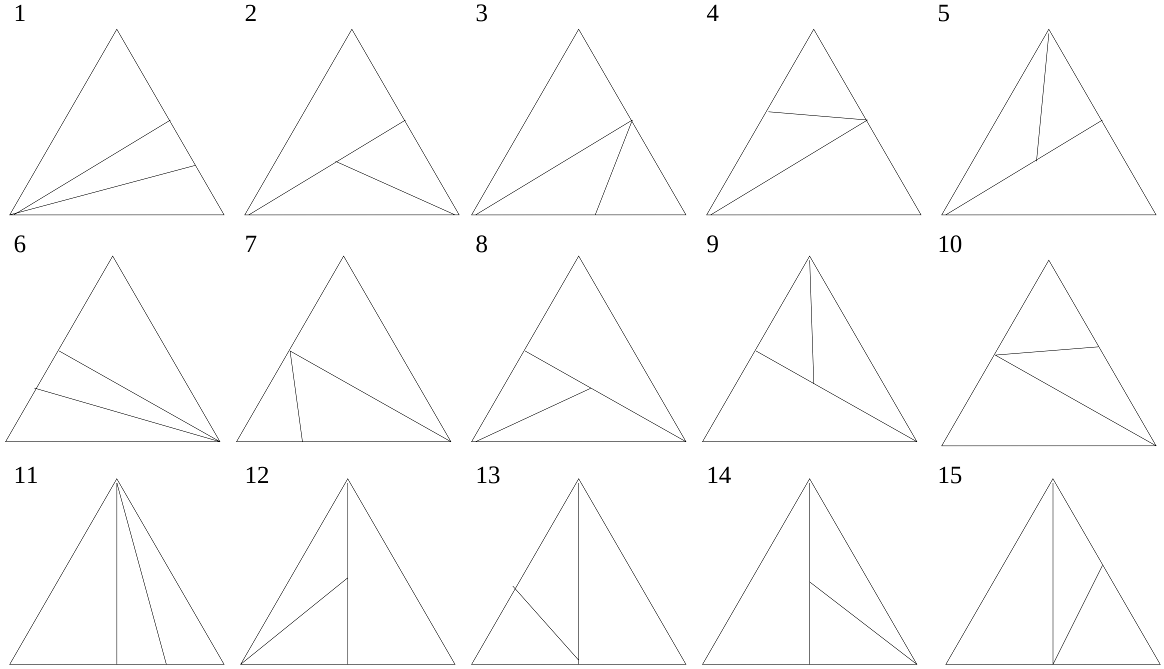

Figure 13: The 15 different combinatorial types with three triangles.

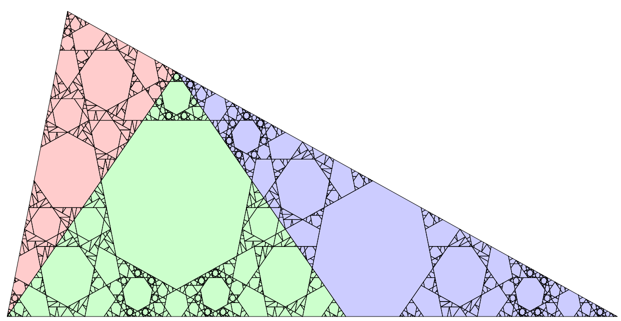

Figure 14: The cases from [10], found among the solutions given the pairs $(3, 15)$.

obvious space reasons, besides, it would be out the scope of this paper. Instead, we will merely list the non-trivial cases and display some of them, among the most interesting.

We split the list according to the degree of the solutions. We have two degree 2 solutions, both of them have finite mosaics. Among the 26 degree 4 solutions, 10 have finite mosaics. Among the P.W.S. solutions of degree 4, 6 have a finite number of attracting points as the examples of figure 19 and yield uninteresting dynamics, the table in figure 15 lists the remaining. All these solutions have non-trivial behaviors, two of them are closely similar to the case studied in [9]. In fact, a simple induction leads to the same map. Looking at figure 15 we notice that an angle vector can appear several times, attached to a different permutation and thus leading to several different dynamics. An interesting example of this phenomenon is given by solutions 27 and 43, whose mosaics are both displayed in figure 16. Solution 43 is especially interesting as it displays unusual features, indeed, its mosaics seems to be dense and, at least according to the initial conditions we tried, the dynamics seems to be minimal. Such amazing properties would more than justify further investigations in forthcoming works.

Nb	Angles	Permutation	Type	Remark
9	$(3\pi/5, \pi/5, \pi/5, \pi/5, 2\pi/5, 2\pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(231798546)	$P.W.I.$	conj.[9]
10	$(3\pi/5, \pi/5, \pi/5, \pi/5, 2\pi/5, 2\pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(231798645)	$P.W.I.$	1 refl.
21	$(\pi/3, \pi/3, \pi/3, \pi/6, 2\pi/3, \pi/6, \pi/2, \pi/12, 5\pi/12)$	(645213987)	$P.W.I.$	2 refl.
24	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(645312789)	$P.W.I.$	conj.[9]
26	$(\pi/3, \pi/3, \pi/3, \pi/6, 2\pi/3, \pi/6, \pi/2, \pi/12, 5\pi/12)$	(645321987)	$P.W.I.$	2 refl.
27	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5)$	(645798123)	$P.W.I.$	1 refl.
28	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(645798321)	$P.W.I.$	conj.[9]
30	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(645978321)	$P.W.I.$	1 refl.
31	$(2\pi/5, \pi/10, \pi/2, \pi/5, 3\pi/5, \pi/5, 3\pi/10, 3\pi/10, 2\pi/5)$	(645987123)	$P.W.S.$	1 att.pt.
37	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(879564321)	$P.W.I.$	2 refl.
43	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(897564321)	$P.W.I.$	1 refl.

Figure 15: Table of degree 4 solutions. Abbreviations: "refl." stands for reflection, "att.pt." stands for attracting point, "conj." stands for conjugacy with the map encountered in [9].

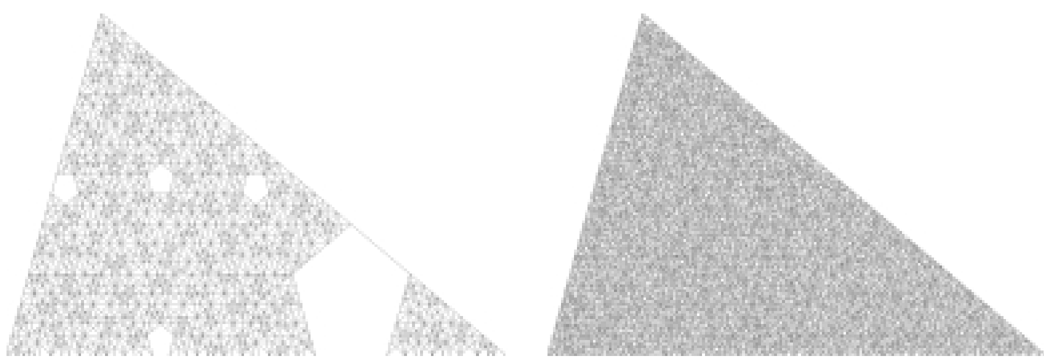

Figure 16: The mosaics for solutions number 27 and 43. Though the angles are the same, different permutations yield different dynamics.

We have 15 degree 6 solutions, all of them are P.W.S., 7 of them lead to simple dynamics. Among the non-trivial case, solution 22 is (up to a rotation) precisely the case studied in [10]. Solution number 20 is interesting because it is a case of a "fake" P.W.S., indeed, as we shall see in the next section, inducing on a well-chosen set yield a piecewise isometry, which is enough to describe the whole dynamics. Figure 21 shows 3 non-trivial cases of piecewise similarities.

We only have 2 solutions of higher degree. Solution 23 and 25 have degree 10 and 12 respectively, they are presented in figure 18. Both of them displays a highly non-trivial dynamics, cf. figure 20. We will not present any simplicial solution here, instead, after a remark on the behaviour of a particular piecewise similarity solution in the next section, we will present a

Nb	Angles	Permutation	Type	Remark
14	$(3\pi/7, 3\pi/7, \pi/7, \pi/7, 4\pi/7, 2\pi/7, 4\pi/7, \pi/7, 2\pi/7)$	(312798546)	$P.W.S.$	
17	$(4\pi/7, \pi/7, 2\pi/7, 2\pi/7, 3\pi/7, 2\pi/7, 3\pi/7, \pi/7, 3\pi/7)$	(321978465)	$P.W.S.$	
20	$(4\pi/7, \pi/7, 2\pi/7, 3\pi/7, 3\pi/7, \pi/7, 4\pi/7, \pi/7, 2\pi/7)$	(564798321)	$P.W.S.$	conj. P.W.I.
22	$(2\pi/7, 2\pi/7, 3\pi/7, \pi/7, 5\pi/7, \pi/7, 3\pi/7, \pi/7, 3\pi/7)$	(645231789)	$P.W.I.$	cf. [10]
38	$(4\pi/7, \pi/7, 2\pi/7, \pi/7, 3\pi/7, 3\pi/7, 2\pi/7, \pi/7, 4\pi/7)$	(879645321)	$P.W.S.$	
41	$(3\pi/7, 3\pi/7, \pi/7, \pi/7, 4\pi/7, 2\pi/7, 4\pi/7, \pi/7, 2\pi/7)$	(897213546)	$P.W.S.$	
44	$(7\pi/9, \pi/9, \pi/9, \pi/3, 2\pi/9, 4\pi/9, 4\pi/9, 2\pi/9, \pi/3)$	(978123456)	$P.W.S.$	
45	$(3\pi/7, \pi/7, 3\pi/7, 2\pi/7, 4\pi/7, \pi/7, 3\pi/7, 2\pi/7, 2\pi/7)$	(987546123)	$P.W.S.$	

Figure 17: Table of degree 6 solutions.

simplicial family belonging to a different pair of types and whose dynamics will be strongly reminiscent of another well-known dynamical family, namely the maps from [1].

Nb	Angles	Permutation	Type
23	$(2\pi/11, 2\pi/11, 7\pi/11, \pi/11, 9\pi/11, \pi/11, 3\pi/11, 3\pi/11, 5\pi/11)$	(645231978)	$P.W.I.$
25	$(6\pi/13, \pi/13, 6\pi/13, 3\pi/13, 7\pi/13, 3\pi/13, 4\pi/13, 4\pi/13, 5\pi/13)$	(645312978)	$P.W.I.$

Figure 18: Table of degree 10 and 12 solutions.

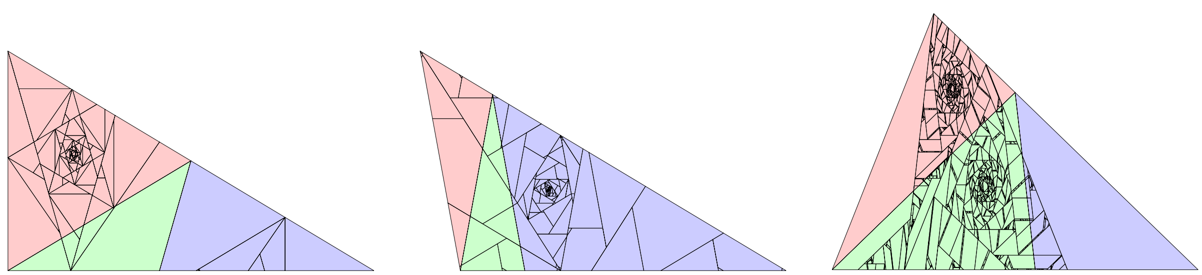

Figure 19: The mosaics for solutions number 11, 13 and 34. We clearly see the attracting points on solutions 11 and 13. Solution 34 has two attracting points and thus a slightly more complicated dynamics.

5.2.2 An Example of a "False" Piecewise Similarity

Each of the fixed solutions mentioned above would require a long and detailed study, we haven't done them, they would go far beyond the scope of this paper. However, in this chapter, we are going to present a preliminary study of a piecewise similarity, namely case number 11, which is displayed in figure 22. We call it T, defined on the atoms P_1, P_2, P_3.

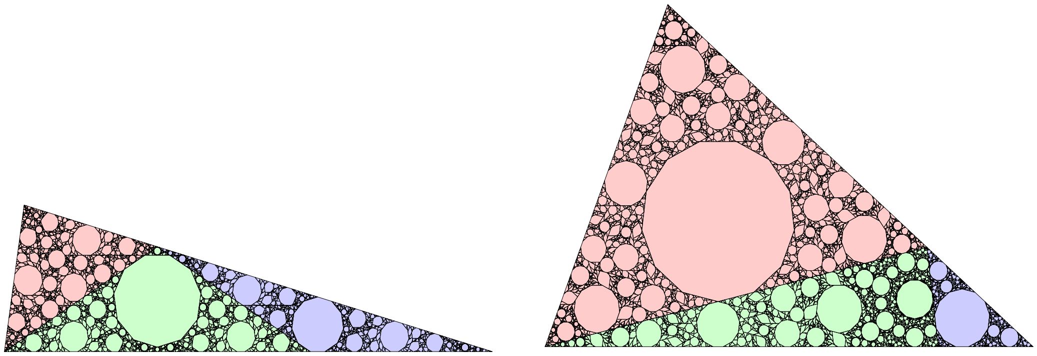

Figure 20: Mosaic of solutions number 23 and 25, of order 10 and 12 respectively.

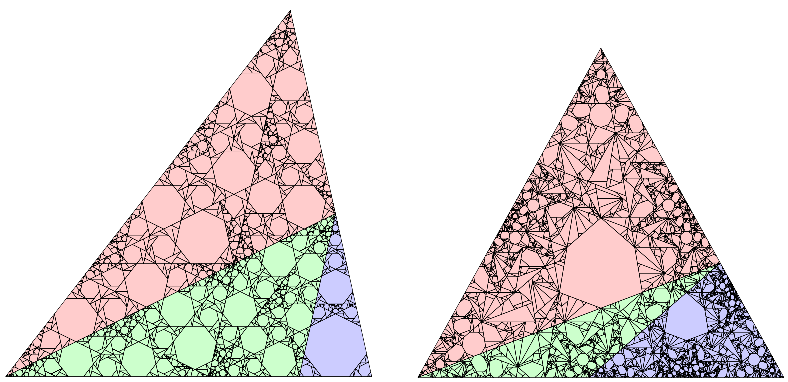

Figure 21: Two non-trivial piecewise similarities corresponding to the cases number 20, 44.

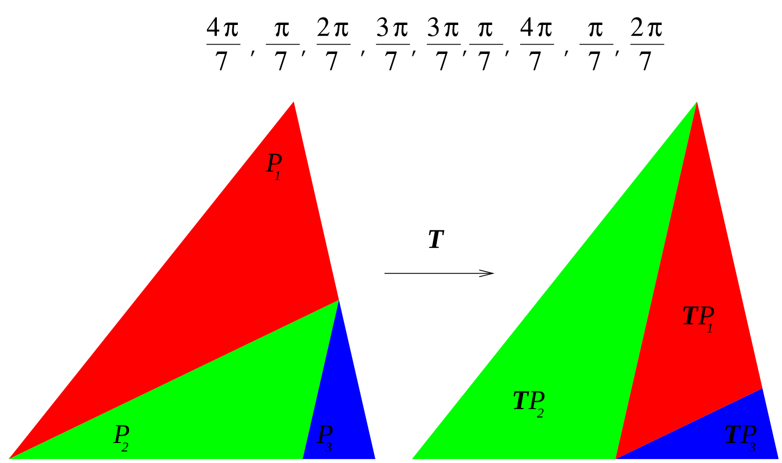

Figure 22: Solution number 11, we clearly see that this is not a piecewise isometry.

The first interesting feature of this map is that in fact, though clearly non-isometric on two of its three atoms, it can be completely described in terms of a standard piecewise isometry.

Indeed, ρ_{P_3}, the first return map in the blue atom P_3 is a piecewise isometry, displayed in figure 23 (left). Moreover, the orbits of all the atoms of the return map cover the whole phase space, except for a finite number of periodic cells as displayed in figure 23 (right). Thus, to describe the dynamics of the whole map, it is enough to describe the dynamics of the piecewise isometry ρ_{P_3}.

We already encountered such a feature in the example displayed in figure 11, but the map was too simple to serve any illustrative purpose, indeed the return map was merely the identity map. Though it might exist, we haven't been able to establish a similar property for other piecewise similarities.

Also based on angles multiple of $\pi/7$, this map exhibits some properties which remind us of the case from [10]. It displays non-uniformly bounded return time. Inducing on the lower right corner triangular set enlightened in green on figure 24, we are led to the map displayed on the top of the same figure. This map is reminiscent of the unbounded return map encountered in [10].

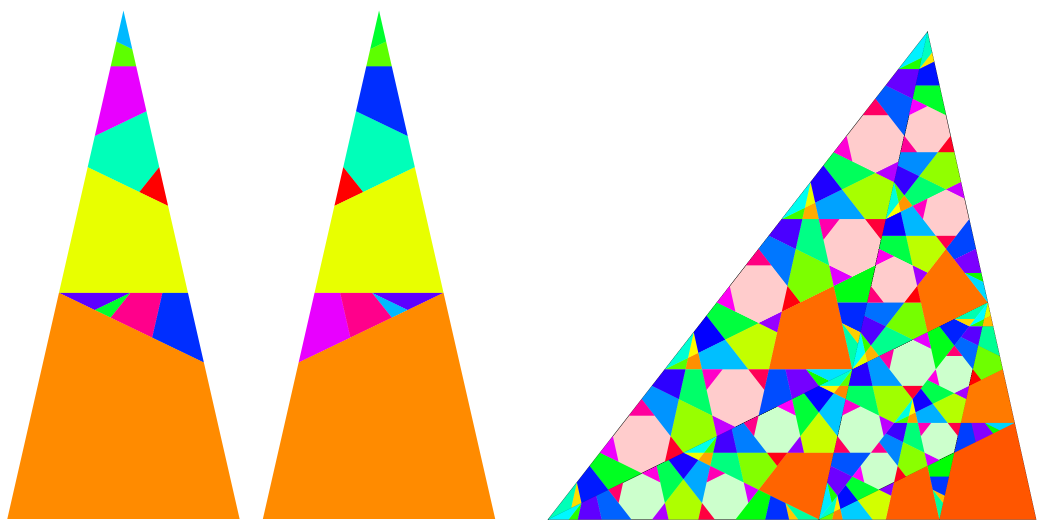

Figure 23: On the left, the first return map in the blue atom of solution 11. On the right, the orbits of its atoms, we can see that the phase-space is completely tiled, up to a finite number of periodic cells.

5.2.3 A.K.T.-like maps

One of the first (and one of the few) examples of piecewise isometries that have been rigorously studied and whose dynamics is fully understood is found in the work by R. Adler, B. Kitchens and C. Tresser, [1] (and subsequently by B. Kahng in [12]). They describe a continuous one-parameter family of maps consisting of a rotation on a "tilted" two-dimensional torus, which is equivalent to a piecewise isometry on a rhombus, cf. figure 25 (left). Its dynamics is described when the rotation angle is equal to $\pi/4$, $\pi/5$ and $2\pi/5$. The case $\pi/4$ in particular, is extensively studied. This map, and sometimes the whole family is ever since frequently referred to as the "A.K.T." map (or, shuffling the letters, the "K.A.T"-map, by analogy with the Arnold

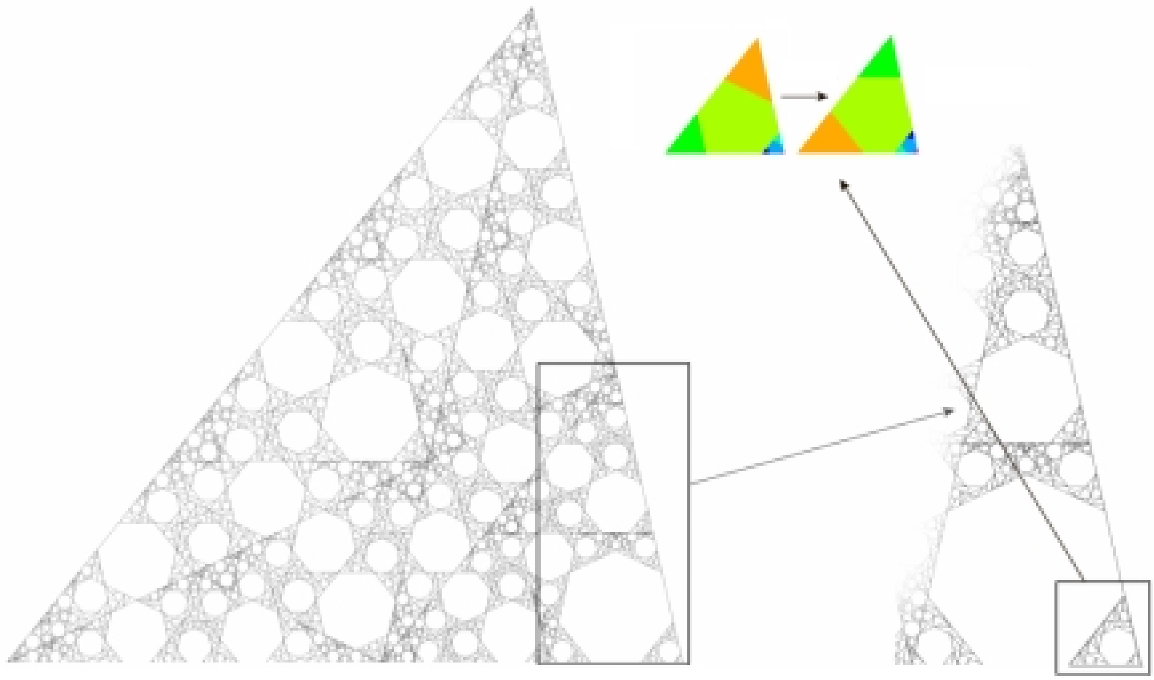

Figure 24: The mosaic of solution number 20, with a blowup of the region on which the first return is made. On the lower-right triangle, an induction gives a piecewise isometry with a self-similar partition, which has been already encountered in [10].

"cat"-map).
Regardless of the angle, the dynamics is always symmetric with respect to the center of the rhombus. Thus, using this symmetry we define a new map on a dart-shaped figure (cf. figure 25 (right)). This shape can be constructed with two bisections of a big triangle. The combinatorial type associated with the dart shape will have three triangles, type number 12 in the table 13 is suitable. The type of the image partition will then be number 5 of the table 13.

Equations for the pair $(12, 5)$ give a total of 96 solutions, 16 are fixed and 80 are simplicial. We have been working with the lists $((1,5,3)(1,4,5)(4,2,3))$ and $((1,5,3)(5,4,3)(1,2,4))$. Among the fixed solutions, 6 are P.W.I. Among the simplicial ones, solution number 35 will particularly focus our interest:

$$\left(\frac{5\pi}{6}-\frac{2x}{3}, \frac{\pi}{6}-\frac{x}{3}, x, \frac{2\pi}{3}-\frac{4x}{3}, \frac{\pi}{6}+\frac{2x}{3}, \frac{\pi}{6}+\frac{2x}{3}, \frac{\pi}{3}-\frac{2x}{3}, \frac{\pi}{3}+\frac{4x}{3}, \frac{\pi}{3}-\frac{2x}{3}\right), \quad 0<x<\frac{\pi}{2}$$

attached to the angle permutation: (132456978). When $x=\pi/8$, the map given by the angles is a P.W.I. and it is precisely the case extensively studied in [1] up to a flip, it is displayed in figure 26.

Changing the angle of the rhombus gives birth to a continuous family of piecewise isometries, some of which have been studied in this context [1, 12, 13, 14], or for maps closely related [16, 17]. This degree of freedom corresponds to the parameter x. Figure 27 shows 12 mosaics for 12 values of x, ranging from $\pi/10$ to $\pi/3$. These plots are quite familiar, among them we can in particular recognize, at the second position of the second row, the case for which the

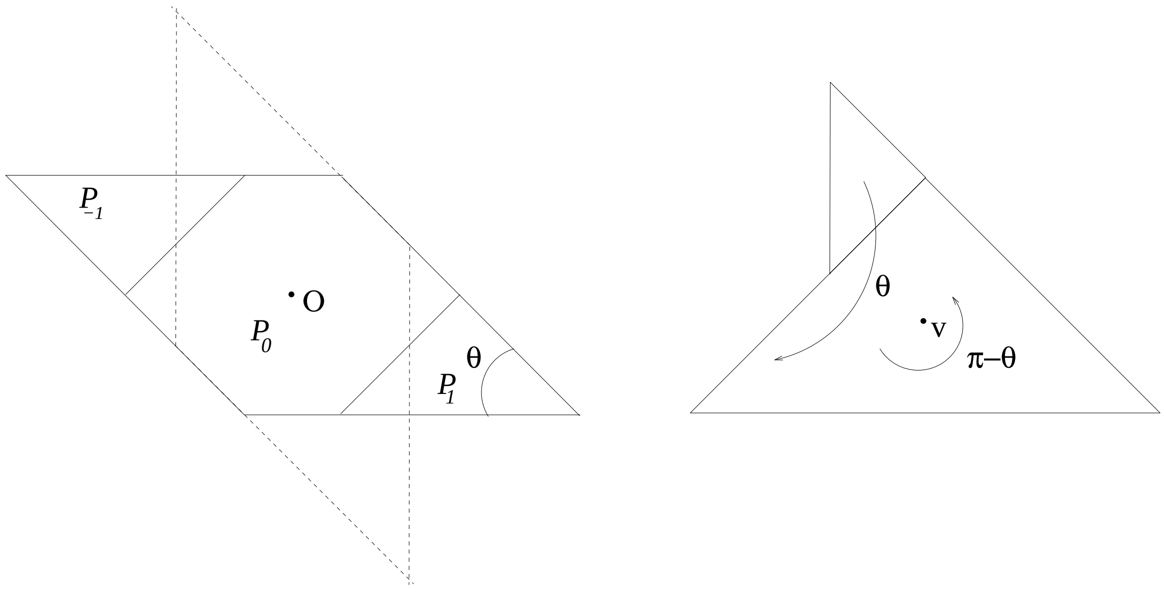

Figure 25: On the left, the rhombus has its sides identified to turn it into a torus, the map is then the rotation around the origin with angle $-\theta$. It translates into a piecewise isometry on atoms P_{-1}, P_0 and P_1. On the right, use is made of the symmetry of the dynamics to draw the so-called dart figure on which, essentially, takes place all the dynamics. The small triangle is rotated by $-\theta$ around the origin of the rhombus, while the big one is rotated around its own center by the angle $\pi - \theta$.

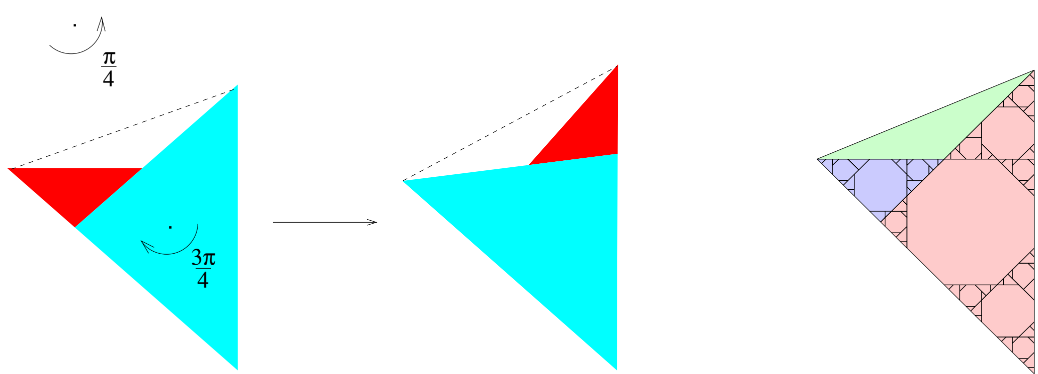

Figure 26: The map illustrated on the left is composed of two rotations, the top one by $\pi/4$ has for fixed point the center of the rhombus as it is described in the text. The bottom one has angle $3\pi/4$ and fixed point the center of the bottom triangle. The remaining triangle, below the dashed line has a mere reflection defined upon it. On the right, we recognize the well-known mosaic of cells, which has been one of the first to be proven exactly self-similar.

rotation angle is based on $\pi/5$ and which has also been described in [1] (and subsequently in [13]). The mosaic is not exactly the same though because of the extra-triangle (the green one) needed in our context to ensure the consistency of the bisection process.

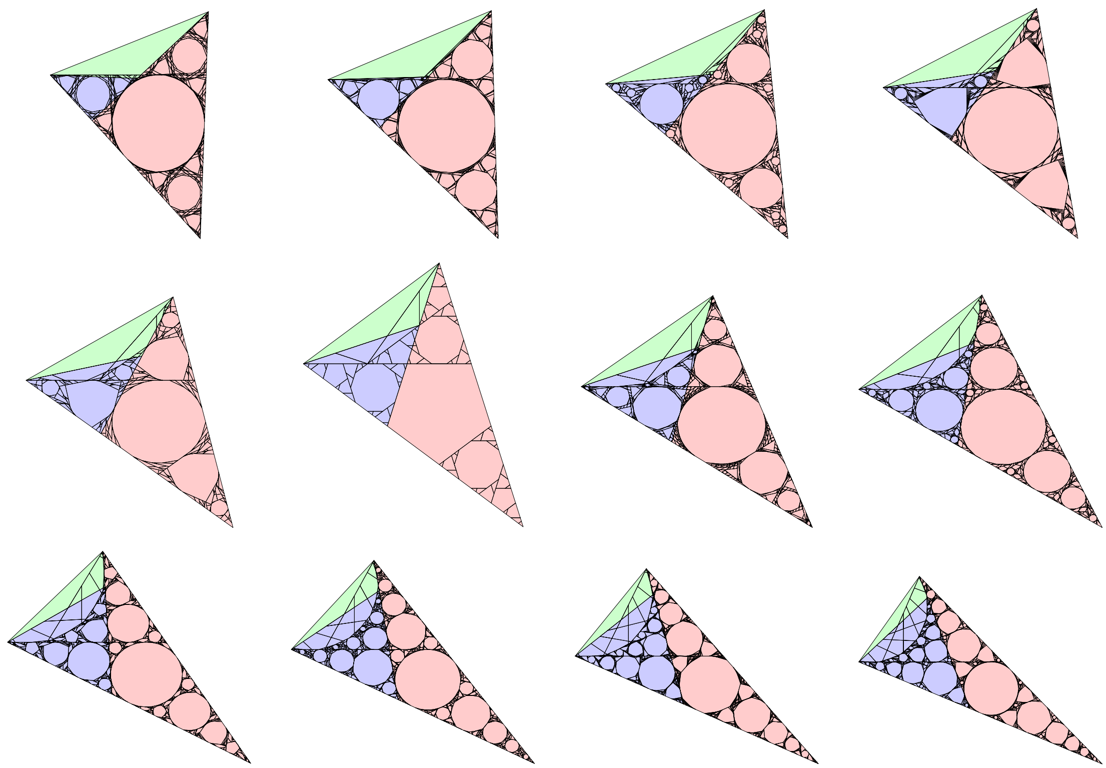

Figure 27: From top to bottom and from left to right, 12 mosaics of cells for 12 increasing values of x, their shapes are familiar, they match closely those of [1] when we change the angles of the rhombus.

6 Perspectives

It remains a lot of work to have a better understanding of the underlying geometry of the set of P.W.S. Notice that this set, for n triangles, can be viewed as the union of "self-intersections" of a $n+1$-dimensional complex embedded in $\mathbb{R}^{3n}$. By self-intersection we mean intersections of distinct faces of the complex which is, roughly speaking, the union of the simplices corresponding to the types up to allowed permutations. Self-intersections correspond to nice partitions that can be re-arranged to form another nice partition (possibly of a different triangle). Notice that if we take the closure of the faces then this object is connected. Moreover, the complex should have a somehow recursive structure as its shape for $n+1$ triangles is based upon the shape for n triangles. However, up to now we were not able to extract useful information out of these considerations, yet the concept is nice and could yield new ideas.

An important point is to understand the "boundaries" of the simplices. In terms of partition types, a boundary corresponds to some angles being zero or π. In particular, the number of triangles in the type is decreasing when some parameters attain a boundary. If the maps in the simplex are all piecewise isometries, we should still have piecewise isometries on the boundary. This is false when the maps are strictly piecewise similarities, it is possible to construct simple examples where it is not the case. For instance, when the ratio of a similarity goes to infinity so that in the limit a triangle vanishes while its image remains.

It might be interesting as well to work with the lengths of the sides of the triangles instead of their angles. The difficulty is that the equations linking the parameters would then have polynomial form and hence, would be trickier to deal with. The interest though, would be to discriminate directly the P.W.I. among the P.W.S., especially in the case when the solution is a simplex.

It seems possible to generalize this construction to more than two dimensions. In the tetrahedron we can define a notion corresponding to a partition by bisection. Given a nice triangulation $(P_i)_{1\leq i\leq n}$ of one face opposite to a vertex indexed by k of the tetrahedron, consider the partition of the tetrahedron with tetrahedra of base P_i and vertices k. Since the elements of this partition are tetrahedra, it makes sense to iterate the process as it was done above.

Another direction for generalizations suggested by our point of view is P.W.I. in non euclidian spaces. The hyperbolic plane as well as the sphere have nice isometry groups and the notion of partitions by bisection makes sense. Some work in this direction is in progress and seems promising.

Our formalism provides also a reasonable context to ask whether or not a given behaviour is typical for bijective P.W.I. For instance we have in mind the question raised by Buzzi and Hubert (during the conference Porq'roll 2002) about genericity of periodic islands among P.W.I. Notice that from this point of view, solution 43 of section 5.2.1 is striking. If it is true that it has no periodic island, as suggested by the mosaic of figure 16, it would be highly interesting to understand why.

We did not say anything about the behaviour of the boundaries of the triangles of the partitions, although they carry an important part of the dynamics (according to the literature). In particular, it is not even clear if essentially bijective maps can be extended to the boundaries in a bijective way.

A real breakthrough would be to use this description to induce in a systematic way. Notice that it is by no mean clear that it is possible. Indeed, in general, if the partition of T is nice, the partition of T^2 is not. However we think there is some hope to find classes of maps for which induction would behave well. If this idea was to be fruitful, the parametrization of the space of P.W.S. would give dynamical information on the maps themselves.

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