PERSISTENCE OF WANDERING INTERVALS IN SELF-SIMILAR AFFINE INTERVAL EXCHANGE TRANSFORMATIONS

XAVIER BRESSAUD, PASCAL HUBERT, AND ALEJANDRO MAASS

ABSTRACT. In this article we prove that given a self-similar interval exchange transformation $T_{(\lambda,\pi)}$, whose associated matrix verifies a quite general algebraic condition, there exists an affine interval exchange transformation with wandering intervals that is semi-conjugated to it. That is, in this context the existence of Denjoy counterexamples occurs very often, generalizing the result of M. Cobo in [C].

1. INTRODUCTION

Since the work of Denjoy [D] it is known that every C^1-diffeomorphism of the circle such that the logarithm of its derivative is a function of bounded variation has no wandering intervals. There is no analogous result for interval exchange transformations. Levitt in [L] found an example of a non-uniquely ergodic affine interval exchange transformation with wandering intervals. Latter, Camelier and Gutierrez [CG], using Rauzy induction technique exhibited a uniquely ergodic affine interval exchange transformation with wandering intervals. Moreover, this example is semi-conjugated to a self-similar interval exchange transformation. In geometric language, it means that this interval exchange transformation is induced by a pseudo-Anosov diffeomorphism. In combinatorial terms, the symbolic system is generated by a substitution

An interval exchange transformation (IET) is defined by the length of the intervals $\lambda = (\lambda_1, \dots, \lambda_r)$ and a permutation π. It is denoted by $T_{(\lambda,\pi)}$. To define an affine interval exchange transformation (AIET) one additional information is needed; the slope of the map on each interval. This is a vector $(w_1, \dots, w_r)$ with $w_i > 0$ for $i = 1, \dots, r$. Camelier and Gutierrez remarked that a necessary condition for an AIET to be semi-conjugated to the interval exchange transformation $T_{(\lambda,\pi)}$ is that the vector $\log(w) = (\log(w_1), \dots, \log(w_r))$ is orthogonal to λ.

The conjugacy of an affine interval exchange transformation with an interval exchange transformation was studied in details by Cobo [C]. He proved that the regularity of the conjugacy depends on the position of the vector $\log(w)$ in the flag of the lyapunov exponents of the Rauzy-Veech-Zorich induction. In particular, assume that $T_{(\lambda,\pi)}$ is self-similar, which means that λ is an eigenvector of a positive $r \times r$ matrix R obtained by applying Rauzy induction a finite number of times. Cobo proves that if $\log(w)$ belongs to the contracting space of ${}^tR$ then f is C^1 conjugated to $T_{(\lambda,\pi)}$. If $\log(w)$ is orthogonal to λ and is not in the contracting

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space of ${}^tR$ then any conjugacy between f and $T_{(\lambda,\pi)}$ is not an absolutely continuous function. Moreover, Camelier and Guttierez example shows that conjugacy between f and $T_{(\lambda,\pi)}$ does not always exist.
In this paper, we prove the following result:

Theorem 1. *Let $T_{(\lambda,\pi)}$ be a self-similar interval exchange transformation and R the associated matrix obtained by Rauzy induction. Let θ_1 be the Perron-Frobenius eigenvalue of R. Assume that R has an eigenvalue θ_2 such that*

(1) *θ_2 is a conjugate of θ_1,*
(2) *θ_2 is a real number,*
(3) $1 < \theta_2 (< \theta_1)$.

Then there exists an affine interval exchange transformation f with wandering intervals that is semi-conjugated to $T_{(\lambda,\pi)}$.

This result means that Denjoy counterexamples occur very often (see section 5).

1.1. **Reader's guide.** Camelier-Gutierrez [CG] and Cobo [C] developed an strategy to prove the existence of a wandering interval in an affine interval exchange transformation f which is semi-conjugated with a given IET. We explain it in section 4. This strategy allowed them to achieve a first concrete example. Here we explore the limits of this method in order to consider a large (and in some sense abstract) family of IET.
Let $T_{(\lambda,\pi)}$ be a self-similar interval exchange transformation with associated matrix R. A main property of such kind of system is that the symbolic coding of its orbits with respect to the partition defined by the intervals of the IET is given by a substitution or equivalently a substitutive system on the alphabet $\{1,\dots,r\}$. Our approach is to translate our problem into properties of such symbolic dynamical system.
Roughly speaking, to create a wandering interval from the interval exchange transformation $T_{(\lambda,\pi)}$, one blows up an orbit of $T_{(\lambda,\pi)}$. Let $\gamma = (\gamma_1,\dots,\gamma_r)$ be the vector of the logarithm of the slopes of the affine interval exchange transformation f constructed from the IET $T_{(\lambda,\pi)}$. Remark that if I is an interval contained in subinterval j defining the AIET then the length $|f(I)|$ is equal to $e^{\gamma_j}|I|$. So, the difficulty is to insure that the total length of a wandering interval remains finite. More precisely, if the symbolic coding of an orbit of the $T_{(\lambda,\pi)}$ is $x = (x_n)_{n\in\mathbb{Z}}$ and we put $\gamma(x_n) = \gamma_{x_n}$ for any $n \in \mathbb{Z}$, we have to check that the series

$$\sum_{n\geq 1} e^{-\gamma(x_0)-\dots-\gamma(x_{n-1})} \text{ and } \sum_{n\geq 1} e^{\gamma(x_{-n})+\dots+\gamma(x_{-1})} \tag{1.1}$$

converge.
This is certainly not true for a generic point x of the symbolic system associated to $T_{(\lambda,\pi)}$. Let $\ell(x)$ be the broken line with vertices $(n, \gamma(x_0)+\dots+\gamma(x_{n-1}))_{n\in\mathbb{N}}$ and $(-n, -\gamma(x_{-n})-\dots-\gamma(x_{-1}))_{n\geq 1}$. Since γ is orthogonal to λ, for a generic point x, the line $\ell(x)$ oscillates around 0 as predicted by Hálasz's Theorem ([Ha]). If the vector γ is not in the contracting space of ${}^tR$ the amplitude of the oscillations tends to infinity with speed

$$n^{\log(\theta_1)/\log(\theta_2)}.$$

It is hoped that the series (1.1) converge if the y-coordinate of the broken line $\ell(x)$ is positive and tends to infinity fast enough as n tends to $\pm\infty$. Points x such that

the broken line $\ell(x)$ is positive are called *minimal points*. Those are the main tool of the paper.
This analysis applies to a very large class of substitutions and not only to substitutions arising from interval exchange transformations. Section 3 gives an algorithm to construct minimal points. We prove that the prefix-suffix decomposition of any minimal point is ultimately periodic. From this analysis, we deduce that for any minimal point x one has

$$\liminf_{n\to\infty} \frac{\gamma(x_0)+\ldots+\gamma(x_n)}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} > 0 \text{ and } \liminf_{n\to\infty} \frac{-\gamma(x_{-n})-\ldots-\gamma(x_{-1})}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} > 0 \tag{1.2}$$

Formulas in (1.2) imply immediately the convergence of the series in (1.1). Moreover, formulas in (1.2) have its own interest. It is a strengthening of a result by Adamczewski [Ad] about discrepancy of substitutive systems.
Even if the fractal curves studied by Dumont and Thomas in [DT1], [DT2] are not considered explicitly in the article, they were a source of inspiration for the authors. These curves correspond to the renormalization of the broken lines $\ell(x)$ and appear in subsection 3.3 in another language.
In section 5, we discuss the hypothesis of the main result in a geometric language. We exhibit many examples where our hypothesis on the matrix R are fulfilled.

2. Preliminaries

2.1. **Words and sequences.** Let A be a finite set. One calls it an *alphabet* and its elements *symbols*. A *word* is a finite sequence of symbols in A, $w = w_0 \ldots w_{\ell-1}$. The length of w is denoted $|w| = \ell$. One also defines the empty word ε. The set of words in the alphabet A is denoted A^* and $A^+ = A^* \setminus \{\varepsilon\}$. We will need to consider words indexed by integer numbers, that is, $w = w_{-m} \ldots w_{-1}.w_0 \ldots w_\ell$ where $\ell, m \in \mathbb{N}$ and the dot separates negative and non-negative coordinates. If necessary we call them *dotted words*.
The set of one-sided infinite sequences $x = (x_i)_{i\in\mathbb{N}}$ in A is denoted by $A^{\mathbb{N}}$. Analogously, $A^{\mathbb{Z}}$ is the set of two-sided infinite sequences $x = (x_i)_{i\in\mathbb{Z}}$. A point $x \in A^{\mathbb{N}}$ is ultimately periodic if it can be written as $x = uvvvvv\ldots$ with $u, v \in A^*$. The word v is called a *generating word* of x, its length $|v|$ a *period* of x and $|u|$ a *preperiod*. If u is the empty word the point is said to be periodic. Its is clear that generating words and periods are not unique.
Given a sequence x in A^+, $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$ one denotes $x[i,j]$ the sub-word of x appearing between indexes i and j. Similarly one defines $x(-\infty, i]$ and $x[i, \infty)$. Let $w = w_{-m} \ldots w_{-1}.w_0 \ldots w_\ell$ be a (dotted) word in A. One defines the cylinder set $[w]$ as $\{x \in A^{\mathbb{Z}} : x[-m,\ell] = w\}$.
The shift map $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ (or $T : A^{\mathbb{N}} \to A^{\mathbb{N}}$) is given by $T(x) = (x_{i+1})_{i\in\mathbb{Z}}$ for $x = (x_i)_{i\in\mathbb{Z}}$ (resp. $T(x) = (x_{i+1})_{i\in\mathbb{N}}$ for $x = (x_i)_{i\in\mathbb{N}}$). A subshift is any shift invariant and closed (for the product topology) subset of $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$. A subshift is minimal if all of its orbits by the shift are dense.
In what follows we will use the shift map in several contexts, in particular restricted to a subshift. To simplify notations we keep the name T all the time.

2.2. **Substitutions and minimal points.** We refer to [Qu] and [F] and references therein for the general theory of substitutions.

A *substitution* is a map $\sigma : A \to A^+$. It naturally extends to A^+, $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$; for $x = (x_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}}$ the extension is given by

$$\sigma(x) = \ldots \sigma(x_{-2})\sigma(x_{-1}).\sigma(x_0)\sigma(x_1)\ldots$$

where the central dot separates negative and non-negative coordinates of $\sigma(x)$. A further natural convention is that the image of the empty word ε is ε.
Let M be the matrix with indices in A such that M_{ab} is the number of times letter b appears in $\sigma(a)$ for any $a, b \in A$. We observe that there is no a classical convention about the construction of the matrix associated to a substitution. In [Qu], ${}^tM$ is considered instead of M. The substitution is primitive if there is $N > 0$ such that for any $a \in A$, $\sigma^N(a)$ contains any other letter of A (here σ^N means N consecutive iterations of σ). Under primitivity one can assume without loss of generality that $M > 0$. For $w \in A^*$ let $L(w) = (L(w)_a : a \in A)^t$ be the vector such that $L(w)_a$ is the number of times a appears in w. A simple computation yields to $L(\sigma(w)) = {}^tML(w)$.
Let $X_\sigma \subseteq A^{\mathbb{Z}}$ be the subshift defined from σ. That is, $x \in X_\sigma$ if and only if any subword of x is a subword of $\sigma^N(a)$ for some $N \in \mathbb{N}$ and $a \in A$.
Assume σ is primitive. Given a point $x \in X_\sigma$ there exists a unique sequence $(p_i, c_i, s_i)_{i\in\mathbb{N}} \in (A^* \times A \times A^*)^{\mathbb{N}}$ such that for each $i \in \mathbb{N}$: $\sigma(c_{i+1}) = p_ic_is_i$ and

$$\ldots \sigma^3(p_3)\sigma^2(p_2)\sigma^1(p_1)p_0.c_0s_0\sigma^1(s_1)\sigma^2(s_2)\sigma^3(s_3)\ldots$$

is the central part of x, where the dot separates negative and non-negative coordinates. This sequence is called the prefix-suffix decomposition of x (see for instance [CS]).
If only finitely many suffixes s_i are nonempty, then there exists $a \in A$ and non-negative integers ℓ and q such that

$$x[0,\infty) = c_0s_0\sigma^1(s_1)\ldots\sigma^\ell(s_\ell)\lim_{n\to\infty}\sigma^{nq}(a)$$

Analogously, if only finitely many p_i are nonempty, then

$$x(-\infty,-1] = \lim_{n\to\infty}\sigma^{np}(b)\sigma^m(p_m)\ldots\sigma^1(p_1)p_0$$

for some $b \in A$ and non-negative integers p and m.
Let θ_1 be the Perron-Frobenius eigenvalue of M. Let $\lambda = (\lambda(a) : a \in A)^t$ be a strictly positive right eigenvector of M associated to θ_1. We will also assume the following algebraic property that we call (AH): M has an eigenvalue θ_2 which is a conjugate of θ_1. Notice that this property coincides with hypothesis (1) of Theorem 1.
The following lemma are important consequences of the algebraic property (AH).

Lemma 2. *Let $\eta : \mathbb{Q}[\theta_1] \to \mathbb{Q}[\theta_2]$ be the field homomorphism that sends θ_1 to θ_2. The vector $\gamma = \eta(\lambda) = (\eta(\lambda(a)) : a \in A)^t$ is an eigenvector of M associated to θ_2.*

Proof. The field homomorphism η naturally extends to $\mathbb{Q}[\theta_1]^A$. Since λ belongs to $\mathbb{Q}[\theta_1]^A$ (up to normalization), then one deduces that $M\eta(\lambda) = \theta_2\eta(\lambda)$. Thus, $\eta(\lambda)$ is an eigenvector of M associated to θ_2. □

Lemma 3. *Let γ be the eigenvector of M associated to θ_2 as in Lemma 2. Then for any A-tuple of non-negative integers $(n_a : a \in A)$, $\sum_{a\in A} n_a\gamma(a) = 0$ implies $n_a = 0$ for any $a \in A$.*

Proof. Assume $\sum_{a\in A} n_a\gamma(a) = 0$. Since $\gamma = \eta(\lambda)$, applying η^{-1} one gets that $\sum_{a\in A} n_a\lambda(a) = 0$. This equality implies that $n_a = 0$ for every $a \in A$ because the coordinates of λ are positive. □

The following concepts are crucial in the sequel. Let $\gamma = \eta(\lambda)$ as in Lemma 2:

(1) for $w = w_0 \dots w_{l-1} \in A^+$, define $\gamma(w) = \gamma(w_0) + \dots + \gamma(w_{l-1})$.
(2) for $x \in X_\sigma$, define $\gamma_0(x) = 0$, $\gamma_n(x) = \sum_{i=0}^{n-1} \gamma(x_i)$ for $n > 0$ and $\gamma_n(x) = -\sum_{i=n}^{-1} \gamma(x_i)$ for $n < 0$. Put $\Gamma(x) = \{\gamma_n(x) : n \in \mathbb{Z}\}$.
(3) Given a (dotted) word $w = w_{-m} \dots w_{-1}.w_0 \dots w_{l-1}$, define $\gamma_0(w) = 0$, $\gamma_n(w) = \sum_{i=0}^{n-1} \gamma(w_i)$ for $0 < n \le l$, $\gamma_n(w) = -\sum_{i=n}^{-1} \gamma(w_i)$ for $-m \le n < 0$ and $\Gamma(w) = \{\gamma_n(x) : -m \le n \le l\}$.

Definition 4. *The best occurrence of $a \in A$ in the (dotted) word $w = w_{-m} \dots w_{-1}.w_0 \dots w_{l-1}$ is the symbol w_i in w such that $w_i = a$ and $\gamma_{i+1}(w) = \min\{\gamma_{j+1}(w) : -m \le j < l, w_j = a\}$. By Lemma 3, under hypotheses (AH) this symbol is well defined and unique.*

Definition 5. *One says that $x \in X_\sigma$ is minimal for γ if $\gamma_n(x) \ge 0$ for any $n \in \mathbb{Z}$. The set of minimal points for γ is denoted by $\mathcal{M}_\sigma(\gamma)$. It is important to mention that if x is a minimal point of a substitution satisfying hypothesis (AH) then, by Lemma 3, $\gamma_n(x) > 0$ whenever $n \neq 0$.*

2.3. **Affine interval exchange transformations.** Let $0 = a_0 < a_1 < \dots < a_{r-1} < a_r = 1$ and $A = \{1, \dots, r\}$.

An *affine interval exchange transformation* (AIET) is a bijective map $f : [0,1) \to [0,1)$ of the form $f(t) = w_i t + v_i$ if $t \in [a_{i-1}, a_i)$ for $i \in A$. The vector $w = (w_1, \dots, w_r)$ is called the slope of f. We assume furthermore the slope is strictly positive.

An *interval exchange transformation* (IET) is an AIET with slope $w = (1, \dots, 1)$. Commonly an IET is given by a vector $\lambda = (\lambda_1, \dots, \lambda_r)$ such that $\lambda_i = |a_i - a_{i-1}|$ for $i \in A$ and a permutation π of A which indicates the way intervals $[a_{i-1}, a_i)$'s are rearranged by the IET. Clearly, $a_i = \sum_{j=1}^{i} \lambda_j$. We use $T_{(\lambda,\pi)}$ to refer to the IET associated to λ and π.

One says the AIET f is semi-conjugated with the IET $T_{(\lambda,\pi)}$ if there is a monotonic, surjective and continuous map $h : [0,1) \to [0,1)$ such that $h \circ f = T_{(\lambda,\pi)} \circ h$.

Let $T_{(\lambda,\pi)}$ be an interval exchange transformation. There is a natural symbolic coding of the orbit of any point $t \in [0,1)$ by $T_{(\lambda,\pi)}$. Consider the partition $\alpha = \{[0, a_1), \dots, [a_{i-1}, a_i), \dots, [a_{r-1}, 1)\}$ and define $\phi(t) = (t_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}}$ by $t_i = j$ if and only if $T^i_{(\lambda,\pi)}(t) \in [a_{j-1}, a_j)$. The set $\phi([0,1))$ is invariant for the shift but it is not necessarily closed, then one considers its closure $X = \overline{\phi([0,1))}$. This procedure produces a semi-conjugacy (factor map) $\varphi : (X, T) \to ([0,1), T_{(\lambda,\pi)})$. If t is not in the orbit of the extreme points $0, a_1, \dots, 1$, then it has a unique preimage by φ. If not, it has at most two preimages corresponding to the coding of $(\lim_{s\to t^-} T^i_{(\lambda,\pi)}(s))_{i\in\mathbb{Z}}$.

We use freely concepts related to Rauzy-Zorich-Veech induction. Rauzy induction was defined in [Ra], extended to zippered rectangles by Veech [Ve], and accelerated by Zorich [Zo]. For a complete description about the Rauzy-Veech-Zorich induction see also the expository papers by Zorich [Zo2] and Yoccoz [Yo].

An IET $T_{(\lambda,\pi)}$ is *self-similar* if it can be recovered from itself after finitely many steps of Rauzy inductions (up to normalization). More precisely, there exists a loop in the Rauzy diagram and an associated *Perron-Frobenius* matrix R such that

$$\theta_1 \lambda = {}^t R \lambda$$

with θ_1 the dominant eigenvalue of ${}^t R$.
For a self-similar IET $T_{(\lambda,\pi)}$ there is a direct relation between the subshift X and the matrix R associated to $T_{(\lambda,\pi)}$. Indeed, there exists a substitution $\sigma : A \to A^+$ with associated matrix $M = {}^t R$ such that $X_\sigma = X$ (see [CG] and references therein). If the IET $T_{(\lambda,\pi)}$ is minimal then the subshift X_σ is minimal too. In the sequel, we will use the fact that the substitution σ is primitive which implies that X_σ is minimal. Nevertheless, no specific property of substitutions obtained from $T_{(\lambda,\pi)}$ will be needed for our purpose.
The relation between self-similar IET and pseudo-Anosov diffeomorphisms is explained in [Ve].

3. Construction of minimal points

Let $\sigma : A \to A^+$ be a primitive substitution with associated matrix $M > 0$. Let θ_1, θ_2, λ and γ be as in subsection 2.2. In addition, assume θ_2 verifies the hypotheses of Theorem 1. By Perron-Frobenius theorem, γ has negative and positive coordinates. The main objective of the section is to give a combinatorial construction of minimal points in this case.

3.1. Existence of minimal points.

The following two lemma follow directly from the equality $M\gamma = \theta_2 \gamma$. Their simple proofs are left to the reader.

Lemma 6. *Let $m \in \mathbb{N}$ and $w \in A^+$. Then $\gamma(\sigma^m(w)) = \theta_2^m \gamma(w)$.*

Lemma 7. *Let $w = w_0 \dots w_{l-1} \in A^+$ and $m \in \mathbb{N}$. Write*

$$\sigma^m(w) = \sigma^m(w_0) \dots \sigma^m(w_{l-1})$$

The minimum of $\Gamma(\sigma^m(w))$ is attained in a coordinate corresponding to some $\sigma^m(w_i)$, where w_i is the best occurrence of this symbol in w.

Lemma 8. *Let $a \in A$ such that $\gamma(a) > 0$ and $m \in \mathbb{N}$. Write $\sigma^m(a) = p_m s_m$ where the minimum of $\Gamma(\sigma^m(a))$ is attained at $\gamma_i(\sigma^m(a))$ and $i = |p_m|$. Then $\gamma(s_m) \geq \theta_2^m \gamma(a)$. In particular, $|s_m|$ grows exponentially fast with m.*

Proof. Observe that from Lemma 6 $\gamma(p_m) + \gamma(s_m) = \theta_2^m \gamma(a)$. Since $\gamma(p_m) \leq 0$ the result follows. □

Proposition 9. $\mathcal{M}_\sigma(\gamma) \neq \emptyset$

Proof. Since γ has positive and negative coordinates and X_σ is minimal, then there exist $b, c \in A$ such that bc is a subword of a point in X_σ and $\gamma(b) < 0, \gamma(c) > 0$ holds.
Let $n \geq 0$ and define the dotted word $u_n = \sigma^n(b).\sigma^n(c)$. The sequence $\Gamma(u_n)$ attains its minimum at some $N_n \in \{-|\sigma^n(b)|, \dots, -1, 0, \dots, |\sigma^n(c)|\}$. Define the dotted word $v_n = u_n[-|\sigma^n(b)|, N_n - 1].u_n[N_n, |\sigma^n(c)| - 1] = v_n^-.v_n^+$. The minimum of $\Gamma(v_n)$ is attained at coordinate 0, and is equal to 0.
By Lemma 8 there is a subsequence $(n_i)_{i \in \mathbb{N}}$ such that

$$\lim_{i\to\infty} |v_{n_i}^-| = \lim_{i\to\infty} |v_{n_i}^+| = \infty$$

By compactness and eventually taking once again a subsequence there exists $x \in X_\sigma$ such that for any $m \in \mathbb{N}$ there is $i \in \mathbb{N}$ with $|v_{n_i}^-| \geq m$, $|v_{n_i}^+| \geq m$ and $x \in [v_{n_i}^-.v_{n_i}^+]$.

Thus $\Gamma(x[-m,m]) \subseteq \mathbb{R}^+$ and its minimum is zero at zero coordinate. This implies that $x \in \mathcal{M}_\sigma(\gamma)$. $\square$

3.2. **The** *best strategy* **algorithm.** In what follows we develop a procedure to construct *minimal* points for γ that will become useful in next subsections.

3.2.1. *The basic procedure.* The following procedure will allow to construct the prefix-suffix decomposition of a minimal point for γ.

Step 0: For each $a \in A$ write $\sigma(a) = p_0^{a,0} c_0^{a,0} s_0^{a,0}$ where $\Gamma(\sigma(a))$ attains its minimum at $\gamma_{|p_0^{a,0}|}(\sigma(a))$.

Step 1: Let $a \in A$. By Lemma 7, the minimum of $\Gamma(\sigma^2(a))$ comes from $\sigma(b)$ for some $b \in A$ in its best occurrence in $\sigma(a)$. Write $\sigma(a) = p_1^{a,1} c_1^{a,1} s_1^{a,1}$ where $c_1^{a,1} = b$ is the best occurrence of b in $\sigma(a)$. Put $w_1(a) = \sigma(p_1^{a,1}) p_0^{b,0} . c_0^{b,0} s_0^{b,0} \sigma(s_1^{a,1})$, where the dot separates negative and non-negative coordinates. Let $p_0^{a,1} = p_0^{b,0}$, $c_0^{a,1} = c_0^{b,0}$ and $s_0^{a,1} = s_0^{b,0}$. The sequence $(p_i^{a,1}, c_i^{a,1}, s_i^{a,1})_{i=0}^{1}$ is called the best strategy for symbol a at step 1. By construction $\Gamma(w_1(a)) \subseteq \mathbb{R}^+$ and the minimum is equal to zero at coordinate zero.

Step n+1: assume that in previous step we have constructed for each symbol $a \in A$ the best strategy $(p_i^{a,n}, c_i^{a,n}, s_i^{a,n})_{i=0}^{n}$. This sequence verifies:

(i) for $0 \leq i \leq n$, $\sigma(c_{i+1}^{a,n}) = p_i^{a,n} c_i^{a,n} s_i^{a,n}$ (here $c_{n+1}^{a,n} = a$). Moreover, each $c_i^{a,n}$ is the best occurrence of this symbol in $\sigma(c_{i+1}^{a,n})$.

(ii) $\Gamma(w_n(a)) \subseteq \mathbb{R}^+$ and its minimum is zero at zero coordinate, where

$$w_n(a) = \sigma^n(p_n^{a,n}) \dots \sigma(p_1^{a,n}) p_0^{a,n} . c_0^{a,n} s_0^{a,n} \sigma(s_1^{a,n}) \dots \sigma^n(s_n^{a,n})$$

Observe that as a non-dotted word $w_n(a)$ is equal to $\sigma^{n+1}(a)$.

Now we proceed as in step 1. Consider $a \in A$. By Lemma 7, the minimum of $\Gamma(\sigma^{n+2}(a))$ comes from $\sigma^{n+1}(b)$ for some $b \in A$ in its best occurrence in $\sigma(a)$. Write $\sigma(a) = p_{n+1}^{a,n+1} c_{n+1}^{a,n+1} s_{n+1}^{a,n+1}$ where $c_{n+1}^{a,n+1} = b$ is the best occurrence of b in $\sigma(a)$. The finite sequence $(p_i^{a,n+1}, c_i^{a,n+1}, s_i^{a,n+1})_{i=0}^{n+1}$ where $(p_i^{a,n+1}, c_i^{a,n+1}, s_i^{a,n+1}) = (p_i^{b,n}, c_i^{b,n}, s_i^{b,n})$ for $0 \leq i \leq n$ is a best strategy for a at step $n+1$ and verifies conditions (i) and (ii) by construction.

3.2.2. *Finitely many minimal points for γ.* For each $a \in A$ and $n \in \mathbb{N}$ consider the cylinder set $C^{a,n} = [w_n(a)]$, where $w_n(a)$ is the dotted word defined in previous subsection. It is clear from the basic procedure that for any $a \in A$ and $n \in \mathbb{N}$ there exists a unique $b \in A$ such that $C^{a,n+1} \subseteq C^{b,n}$. Thus, by compactness, there exist at most $|A|$ infinite decreasing sequences of the form $(C^{a_n,n})_{n\in\mathbb{N}}$. Let $C_1, \dots, C_\ell$ with $\ell \leq |A|$ be the collection of intersections of such sequences. Remark that such sets are finite.

Given $x \in X$ a minimal point for γ with prefix-suffix decomposition $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ and $n \in \mathbb{N}$, there is $a_n \in A$ such that $(p_i, c_i, s_i) = (p_i^{a_n,n}, c_i^{a_n,n}, s_i^{a_n,n})$ for $0 \leq i \leq n$. Therefore, $x \in C_i = \bigcap_{n\in\mathbb{N}} C^{a_n,n}$ for some $1 \leq i \leq \ell$.

The following proposition is plain.

Proposition 10. *There are finitely many minimal points for γ.*

We will see later that minimal points for γ have ultimately periodic prefix-suffixe decom- position. This fact yields to an alternative proof of previous proposition.

3.3. **Serie associated to a minimal point.** Define $\overline{\mathcal{S}} = \{(p_i, c_i, s_i)_{i\in\mathbb{N}} : \forall\ i > 0,\ \sigma(c_i) = p_{i-1}c_{i-1}s_{i-1}\}$ and $\underline{\mathcal{S}} = \{(p_i, c_i, s_i)_{i\in\mathbb{N}} : \forall\ i \geq 0,\ \sigma(c_i) = p_{i+1}c_{i+1}s_{i+1}\}$. Observe that finite sequences taken from sequences in $\overline{\mathcal{S}}$ and $\underline{\mathcal{S}}$ coincide once reversed.

Let $a \in A$ and $n \geq 1$. Then $\sigma^n(a)$ can be decomposed as

$$\sigma^n(a) = \sigma^{n-1}(p_1)\dots\sigma(p_{n-1})p_nc_ns_n\sigma(s_{n-1})\dots\sigma^{n-1}(s_1)$$

where for all $1 \leq i \leq n$, $\sigma(c_{i-1}) = p_ic_is_i$ (we have considered $c_0 = a$). This decomposition is not unique. To a and the finite sequence $(p_i, c_i, s_i)_{i=1}^n$ one associates the finite sum:

$$v(a; (p_i, c_i, s_i)_{i=1}^n) = \sum_{i=1}^{n} \theta_2^{-i}\gamma(p_i)$$

Clearly, given $\mathbf{x} = (p_i^{\mathbf{x}}, c_i^{\mathbf{x}}, s_i^{\mathbf{x}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ with $c_0^{\mathbf{x}} = a$, the series

$$v(a; \mathbf{x}) = \lim_{n\to\infty} v(a; (p_i^{\mathbf{x}}, c_i^{\mathbf{x}}, s_i^{\mathbf{x}})_{i=1}^n) = \sum_{i\geq 1} \theta_2^{-i}\gamma(p_i^{\mathbf{x}})$$

exists.

Let $v(a) = \min\{v(a; \mathbf{x}) : \mathbf{x} \in \underline{\mathcal{S}} \text{ with } c_0^{\mathbf{x}} = a\}$. A sequence $\mathbf{x} \in \underline{\mathcal{S}}$ with $c_0^{\mathbf{x}} = a$ such that $v(a; \mathbf{x}) = v(a)$ is said to be *minimal for* a.

The best strategy for symbol a at step $n \geq 1$ given by the algorithm produces a finite sequence $(p_i^{a,n}, c_i^{a,n}, s_i^{a,n})_{i=0}^n$. Set $v_n(a) = \sum_{i=0}^n \theta^{-n+i-1}\gamma(p_i^{a,n})$. It follows that $v_n(a) = v(a; (p_{n-i}^{a,n}, c_{n-i}^{a,n}, s_{n-i}^{a,n})_{i=0}^n)$.

Lemma 11. *For every $a \in A$ and $n \geq 1$, $v_n(a)$ is minimal among the $v(a; (p_i, c_i, s_i)_{i=1}^{n+1})$ and $v(a) = \lim_{n\to\infty} v_n(a)$.*

Proof. The first fact is analogous to say that $(p_i^{a,n}, c_i^{a,n}, s_i^{a,n})_{i=0}^n$ is the best strategy. Moreover, $|v_n(a) - v(a)| \leq K\theta_2^{-n}$ for some constant $K > 0$. This implies the desired result. □

Lemma 12. *Let $a \in A$ and $l \in \mathbb{N}$. Assume there is a finite sequence $(\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=1}^l$ such that for infinitely many $n \in \mathbb{N}$, $(p_{n-j+1}^{a,n}, c_{n-j+1}^{a,n}, s_{n-j+1}^{a,n})_{j=1}^l = (\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=1}^l$ (when $l = 0$ this hypothesis is void). Then, there exists $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ such that $(\mathbf{y}_j)_{j=1}^l = (\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=1}^l$, $c_0^{\mathbf{y}} = a$ and $v(a) = v(c_0^{\mathbf{y}}; \mathbf{y})$.*

Proof. For any $n \in \mathbb{N}$ where the property of the lemma holds consider the sequence

$$\mathbf{y}^{(n)} = \mathbf{y}_0^{(n)} \dots \mathbf{y}_{n+1}^{(n)} = (p, a, s)(p_n^{a,n}, c_n^{a,n}, s_n^{a,n})\dots(p_0^{a,n}, c_0^{a,n}, s_0^{a,n})$$

where $\sigma(b) = pas$ for some $b \in A$.

Let $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}}$ be the limit of a subsequence $(\mathbf{y}^{(n_i)})_{i\in\mathbb{N}}$. It follows by construction that $(\mathbf{y}_j)_{j=1}^l = (\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=1}^l$, $c_0^{\mathbf{y}} = a$ and $\sigma(c_i^{\mathbf{y}}) = p_{i+1}^{\mathbf{y}}c_{i+1}^{\mathbf{y}}s_{i+1}^{\mathbf{y}}$ for any $i \geq 0$. Also, $c_{i+1}^{\mathbf{y}}$ is the best occurrence of this symbol in $\sigma(c_i^{\mathbf{y}})$.

Let $i, L \in \mathbb{N}$ such that $(p_j^{\mathbf{y}}, c_j^{\mathbf{y}}, s_j^{\mathbf{y}}) = (p_{n_i-j+1}, c_{n_i-j+1}, s_{n_i-j+1})$ for $1 \leq j \leq L$. Since $\gamma(p_k^{a,n})$ and $\gamma(p_k^{\mathbf{y}})$ are bounded independently of n, k and $\mathbf{y}$, there is $C > 0$ such that

$$|v(a) - \sum_{k\geq 1} \theta_2^{-k}\gamma(p_k^{\mathbf{y}})| \leq |v(a) - v_{n_i}(a)| + C\theta_2^{-L}$$

Let $\epsilon > 0$. By Lemma 11, considering i and L large enough one deduces

$$|v(a) - \sum_{k\geq 1} \theta_2^{-k}\gamma(p_k^{\mathbf{y}})| \leq |v(a) - v_{n_i}(a)| + C\theta_2^{-L} \leq \epsilon$$

Since ϵ is arbitrary one concludes $v(c_0^{\mathbf{y}}) = v(a) = \sum_{k\geq 1} \theta_2^{-k}\gamma(p_k^{\mathbf{y}})$. $\square$

One says that a point $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ verifies the *continuation property* if $v(c_i^{\mathbf{y}}) = v(c_i^{\mathbf{y}}; T^i(\mathbf{y}))$ for all $i \geq 0$, where T is the shift map. It is clear that $T^i(\mathbf{y})$ has the continuation property too, for any $i \in \mathbb{N}$. In fact, as we prove in the next lemma, to satisfy the continuation property it is enough to be minimal for $c_0^{\mathbf{y}}$.

Lemma 13. *If $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ is minimal for $c_0^{\mathbf{y}}$ (that is, $v(c_0^{\mathbf{y}}) = v(c_0^{\mathbf{y}}; \mathbf{y})$) then $\mathbf{y}$ verifies the continuation property.*

Proof. Let $a = c_0^{\mathbf{y}}$, $b = c_1^{\mathbf{y}}$ and $\mathbf{z} = (p_i^{\mathbf{z}}, c_i^{\mathbf{z}}, s_i^{\mathbf{z}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ with $c_0^{\mathbf{z}} = b$ and $v(b; \mathbf{z}) = v(b)$ given by Lemma 12 (considering $l = 0$). The sequence $\mathbf{w} = \mathbf{y}_0\mathbf{y}_1T(\mathbf{z})$ belongs to $\underline{\mathcal{S}}$ and verifies $v(a; \mathbf{w}) = \theta_2^{-1}\gamma(p_1^{\mathbf{y}}) + \theta_2^{-1}v(b)$. Thus, if $v(b; T(\mathbf{y})) > v(b)$, from $v(a) = v(a; \mathbf{y}) = \theta_2^{-1}\gamma(p_1^{\mathbf{y}}) + \theta_2^{-1}v(b; T(\mathbf{y}))$, one deduces that $v(a; \mathbf{w}) < v(a)$ which is a contradiction. $\square$

In particular, this lemma proves that sequences $\mathbf{y}$ constructed in Lemma 12 verifies the continuation property.

3.4. **Minimal points have ultimately periodic prefix-suffix decomposition.** In this section we prove that any minimal point $x \in X_\sigma$ has ultimately periodic prefix-suffix decomposition. That is, if $\bar{x} = (p_i, c_i, s_i)_{i\in\mathbb{N}}$ is the prefix-suffix decomposition of x, then $T^{p+q}(\bar{x}) = T^q\bar{x}$ for some $p > q \geq 0$. If $q = 0$ one says x is a periodic minimal point.

Lemma 14. *For every $a \in A$ there exists a ultimately periodic point $\mathbf{x}(a) = (p_i^{\mathbf{x}(a)}, c_i^{\mathbf{x}(a)}, s_i^{\mathbf{x}(a)})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ with $c_0^{\mathbf{x}(a)} = a$ and $v(a; \mathbf{x}(a)) = v(a)$ (so, $\mathbf{x}(a)$ has the continuation property).*

Proof. Let $a \in A$ and $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ with $c_0^{\mathbf{y}} = a$ and $v(a; \mathbf{y}) = v(a)$ given by Lemma 12 (considering $l = 0$). We are going to construct another one with ultimately periodic decomposition.
Let $0 < q < p$ be such that $\mathbf{y}_q = \mathbf{y}_p$ and $c_{q-1}^{\mathbf{y}} = c_{p-1}^{\mathbf{y}} = b$. The ultimately periodic sequence $\mathbf{x}(a) = \mathbf{y}_0 \ldots \mathbf{y}_{q-1}\mathbf{y}_q \ldots \mathbf{y}_{p-1}\mathbf{y}_q \ldots \mathbf{y}_{p-1} \ldots \in \underline{\mathcal{S}}$ since $\sigma(c_{p-1}^{\mathbf{y}}) = p_q^{\mathbf{y}}c_q^{\mathbf{y}}s_q^{\mathbf{y}}$ by hypothesis. We are going to prove that $v(a; \mathbf{x}(a)) = v(a)$.
Observe that, by Lemma 12 and Lemma 13,

$$v(b) = \sum_{i\geq q} \theta_2^{-(i-q+1)}\gamma(p_i^{\mathbf{y}}) \text{ and } v(b) = \sum_{i\geq p} \theta_2^{-(i-p+1)}\gamma(p_i^{\mathbf{y}}) \ .$$

Thus, $v(b) = \sum_{i=q}^{p-1} \theta_2^{-(i-q+1)}\gamma(p_i^{\mathbf{y}}) + \sum_{i\geq p} \theta_2^{-(i-q+1)}\gamma(p_i^{\mathbf{y}}) = \sum_{i=q}^{p-1} \theta_2^{-(i-q+1)}\gamma(p_i^{\mathbf{y}}) + \theta_2^{-(p-q)}v(b)$. If we denote $B = \sum_{i=q}^{p-1} \theta_2^{-(i-q+1)}\gamma(p_i^{\mathbf{y}})$, then $v(b) = B\sum_{i\geq 0} \theta_2^{-(p-q)i}$. Consequently,

$$v(a) = \sum_{i=1}^{q-1} \theta_2^{-i}\gamma(p_i^{\mathbf{y}}) + \theta_2^{-(q-1)}B\sum_{i\geq 0} \theta_2^{-(p-q)i}$$

On the other hand, a direct computation yields to

$$v(a; \mathbf{x}(a)) = \sum_{i=1}^{q-1} \theta_2^{-i}\gamma(p_i^{\mathbf{y}}) + \theta_2^{-(q-1)}(\sum_{i\geq 0} \theta_2^{-(p-q)i}B) \ ,$$

which implies, $v(a; \mathbf{x}(a)) = v(a)$. $\square$

To each ultimately periodic sequence $\mathbf{x}(a)$ constructed in previous lemma one can associate a point x in the symbolic space X_σ with periodic prefix-suffix decomposition with generating word

$$(p_0, c_0, s_0), \ldots, (p_{p-q}, c_{p-q}, s_{p-q}) = (p_{p-1}^{\mathbf{x}(a)}, c_{p-1}^{\mathbf{x}(a)}, s_{p-1}^{\mathbf{x}(a)}), \ldots, (p_q^{\mathbf{x}(a)}, c_q^{\mathbf{x}(a)}, s_q^{\mathbf{x}(a)}) \ .$$

Even if, by construction, this point is associated to the minimal value $v(b)$, there is no reason for it to be a minimal point.
Without loss of generality we will do the following simplification. By iterating σ enough times one can assume that all ultimately periodic sequences constructed in Lemma 14 are of period 1 and of preperiod 1. That is, for each letter $a \in A$, $c_0^{\mathbf{x}(a)} = a$ and $\mathbf{x}_i(a) = (p^{(a)}, \hat{a}, s^{(a)})$ for all $i \geq 1$. The letter $a \in A$ is *periodic* if $\hat{a} = a$ and one denotes $\hat{A}$ the subset of periodic letters. Since, the construction of Lemma 14 implies that $v(c_i^{\mathbf{x}(a)}) = v(c_i^{\mathbf{x}(a)}; T^i(\mathbf{x}(a)))$ for $0 \leq i \leq p-1$, then under this simplification $v(\hat{a}) = v(\hat{a}; T(\mathbf{x}(a)))$.

Lemma 15. *Let $\mathbf{y} \in \underline{\mathcal{S}}$ verifying the continuation property. Then, for any $i \geq 1$ the point $\mathbf{y}^{(i)} = \mathbf{y}_0 \ldots \mathbf{y}_i T(\mathbf{x}(c_i^{\mathbf{y}}))$ has the continuation property too.*

Proof. Let $i \geq 1$ and $1 \leq j \leq i$. From the continuation property one deduces that $v(c_j^{\mathbf{y}}) = \sum_{k=1}^{i-j} \theta_2^{-k} \gamma(p_{k+j}^{\mathbf{y}}) + \theta_2^{-(i-j)} v(c_i^{\mathbf{y}})$. But, $v(c_i^{\mathbf{y}}) = v(c_i^{\mathbf{y}}; \mathbf{x}(c_i^{\mathbf{y}}))$ and $v(\hat{c}_i^{\mathbf{y}}) = v(\hat{c}_i^{\mathbf{y}}; T(\mathbf{x}(c_i^{\mathbf{y}})))$, then $\mathbf{y}^{(i)} = \mathbf{y}_0 \ldots \mathbf{y}_{i-1} T(\mathbf{x}(c_i^{\mathbf{y}}))$ has the continuation property too. □

Lemma 16. *Let $\mathbf{x}, \mathbf{y} \in \underline{\mathcal{S}}$ such that $(\mathbf{x}_i)_{i \geq l+1} = (\mathbf{y}_i)_{i \geq l+1}$ and $c_0^{\mathbf{x}} = c_0^{\mathbf{y}} = a$. If $v(a; \mathbf{x}) = v(a; \mathbf{y})$ then $(\mathbf{x}_i)_{i \geq 1} = (\mathbf{y}_i)_{i \geq 1}$.*

Proof. Let $\mathbf{x} = (p_i^{\mathbf{x}}, c_i^{\mathbf{x}}, s_i^{\mathbf{x}})_{i \in \mathbb{N}}$ and $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i \in \mathbb{N}}$. From the hypothesis one deduces that

$$\sum_{i=1}^{l} \theta_2^{-i} \gamma(p_i^{\mathbf{x}}) = \sum_{i=1}^{l} \theta_2^{-i} \gamma(p_i^{\mathbf{y}})$$

and consequently

$$\gamma(\sigma^{l-1}(p_1^{\mathbf{x}}) \ldots p_l^{\mathbf{x}}) = \gamma(\sigma^{l-1}(p_1^{\mathbf{y}}) \ldots p_l^{\mathbf{y}}) \ .$$

But words $\sigma^{l-1}(p_1^{\mathbf{x}}) \ldots p_l^{\mathbf{x}}$ and $\sigma^{l-1}(p_1^{\mathbf{y}}) \ldots p_l^{\mathbf{y}}$ are both prefixes of $\sigma^l(a)$. Then, without loss of generality one can suppose that $\sigma^{l-1}(p_1^{\mathbf{x}}) \ldots p_l^{\mathbf{x}}$ is a prefix of $\sigma^{l-1}(p_1^{\mathbf{y}}) \ldots p_l^{\mathbf{y}}$. This implies that

$$\gamma(\sigma^{l-1}(p_1^{\mathbf{y}}) \ldots p_l^{\mathbf{y}}) - \gamma(\sigma^{l-1}(p_1^{\mathbf{x}}) \ldots p_l^{\mathbf{x}}) = \sum_{a \in A} n_a \gamma(a) = 0$$

with $n_a \geq 0$ for any $a \in A$. Therefore, by the algebraic condition (Lemma 3) they $\gamma(\sigma^{l-1}(p_1^{\mathbf{y}}) \ldots p_l^{\mathbf{y}}) = \gamma(\sigma^{l-1}(p_1^{\mathbf{x}}) \ldots p_l^{\mathbf{x}})$. This implies $(p_i^{\mathbf{x}}, c_i^{\mathbf{x}}, s_i^{\mathbf{x}}) = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})$ for $1 \leq i \leq l$. □

Theorem 17. *The prefix-suffix decomposition of any minimal point is ultimately periodic.*

Proof. Let $x \in X_\sigma$ be a minimal point with prefix-suffixe decomposition $(p_i, c_i, s_i)_{i \in \mathbb{N}}$. There exists a finite sequence $(\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=0}^{l}$ such that $(\bar{p}_0, \bar{c}_0, \bar{s}_0) = (\bar{p}_l, \bar{c}_l, \bar{s}_l)$ and for infinitely many $i \in \mathbb{N}$, $(p_{i-j}, c_{i-j}, s_{i-j})_{j=0}^{l} = (\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=0}^{l}$.
Let $a = \bar{c}_0 = \bar{c}_l$. By Lemma 12, there is a point $\mathbf{y} \in \underline{\mathcal{S}}$ verifying the continuation property such that $(\mathbf{y}_j)_{j=0}^{l} = (\bar{p}_j, \bar{c}_j, \bar{s}_j)_{j=0}^{l}$, $v(a; \mathbf{y}) = v(a)$ and $v(a; T^l(\mathbf{y})) = v(a)$.

Since, $v(a) = v(a; \mathbf{x}(a))$, then by Lemma 15 the sequence $\mathbf{z} = \mathbf{y}_0 \dots \mathbf{y}_l T(\mathbf{x}(a))$ has the continuation property and $v(a) = v(a; \mathbf{z})$ holds. Therefore, by Lemma 16, one concludes that $(\mathbf{x}(a)_i)_{i\geq 1} = (\mathbf{z}_i)_{i\geq 1}$.
We have proved that $a \in \hat{A}$, that is $a = \hat{a}$, and that the word $(p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})$ appears infinitely many times in the prefix-suffixe decomposition of x. Now we prove that $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ is ultimately periodic with generating word $(p^{(a)}, a, s^{(a)})$. Assume this result does not hold. Then there is $b \neq a$ in A such that

$$(p_i, c_i, s_i)(p_{i-1}, c_{i-1}, s_{i-1})(p_{i-2}, c_{i-2}, s_{i-2}) = (p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})(p, b, s)$$

for infinitely many $i \in \mathbb{N}$.
By Lemma 12, there is a point $\mathbf{w} \in \underline{\mathcal{S}}$ verifying the continuation property and such that $\mathbf{w}_0\mathbf{w}_1\mathbf{w}_2 = (p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})(p, b, s)$. Since $v(b) = v(b; T^2(\mathbf{w}))$ and $v(b) = v(b; \mathbf{x}(b))$, by Lemma 15, the points $\mathbf{u} = \mathbf{w}_0\mathbf{w}_1\mathbf{w}_2 T(\mathbf{x}(b))$ and $\mathbf{v} = \mathbf{x}(a)_0\mathbf{x}(b)$ have the continuation property. Since $\mathbf{u}$ and $\mathbf{v}$ are ultimately equal, then, by Lemma 16, one concludes $a = b$ which is a contradiction. This proves the theorem. □

We stress the fact that it is possible to construct examples with minimal points having ultimately periodic but not periodic prefix-suffix decomposition.

3.5. **Convergence of series associated to minimal points.** We will need the following general property of prefix-suffix decomposition.

Lemma 18. *Let $x \in X_\sigma$ with prefix-suffix decomposition $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ and consider $n \geq 0$ an integer. Let $N \geq 1$ be the smallest possible integer such that $x_1 \dots x_{n+1}$ is a prefix of $s_0\sigma(s_1)\dots\sigma^{N-1}(s_{N-1})$. Let $(p_i^{(n)}, c_i^{(n)}, s_i^{(n)})_{i\in\mathbb{N}}$ be the prefix-suffix decomposition of $T^{n+1}(x)$ (recall T is the shift map). Then, $(p_i^{(n)}, c_i^{(n)}, s_i^{(n)})_{i\geq N} = (p_i, c_i, s_i)_{i\geq N}$ and*

$$\sigma^{N-1}(p_{N-1})\dots p_0c_0x_1\dots x_n = \sigma^{N-1}(p_{N-1}^{(n)})\dots p_0^{(n)}$$

Proof. By hypothesis, $\sigma(c_N) = p_{N-1}c_{N-1}s_{N-1} = p_{N-1}^{(n)}c_{N-1}^{(n)}s_{N-1}^{(n)}$ where $p_{N-1}c_{N-1}$ is a prefix of $p_{N-1}^{(n)}$. This implies that $(p_i^{(n)}, c_i^{(n)}, s_i^{(n)})_{i\geq N} = (p_i, c_i, s_i)_{i\geq N}$. Also, from the same equality one deduces that

$$\sigma^{N-1}(p_{N-1})\dots p_0c_0s_0\dots\sigma^{N-1}(s_{N-1}) = \sigma^{N-1}(p_{N-1}^{(n)})\dots p_0^{(n)}c_0^{(n)}s_0^{(n)}\dots\sigma^{N-1}(s_{N-1}^{(n)})$$

But, from construction, x_{n+1} corresponds to $c_0^{(n)}$ and thus $x_1\dots x_n$ is a suffix of $\sigma^{N-1}(p_{N-1}^{(n)})\dots p_0^{(n)}$. One concludes,

$$\sigma^{N-1}(p_{N-1})\dots p_0c_0x_1\dots x_n = \sigma^{N-1}(p_{N-1}^{(n)})\dots p_0^{(n)}$$

as desired. □

Lemma 19. *Let $\mathbf{y} \in \underline{\mathcal{S}}$ such that $c_0^{\mathbf{y}} = a \in \hat{A}$ and $v(a; \mathbf{y}) = v(a)$. Then, $\mathbf{y}_1 = (p^{(a)}, a, s^{(a)})$.*

Proof. Put $c_0^{\mathbf{y}} = a$. First we prove that $v(c_1^{\mathbf{y}}) = v(c_1^{\mathbf{y}}; T(\mathbf{y}))$. Let $\mathbf{z} = \mathbf{y}_0\mathbf{y}_1 T(\mathbf{x}(c_1^{\mathbf{y}})) \in \underline{\mathcal{S}}$. If the assertion is not true then

$$v(a) = \theta_2^{-1}(\gamma(p_1^{\mathbf{y}}) + v(c_1^{\mathbf{y}}; T(\mathbf{y}))) > \theta_2^{-1}(\gamma(p_1^{\mathbf{y}}) + v(c_1^{\mathbf{y}})) = v(a; \mathbf{z}) \geq v(a)$$

which is a contradiction. Thus, $v(c_1^{\mathbf{y}}) = v(c_1^{\mathbf{y}}; T(\mathbf{y}))$ and furthermore $v(a) = v(a; \mathbf{z})$. Then, the point $\mathbf{w} = (p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})T(\mathbf{z})$ verifies $v(a) = v(a; \mathbf{w})$. But $\mathbf{w}$ and $\mathbf{x}(a)$ are ultimately equal, then by Lemma 16, $\mathbf{y}_1 = (p^{(a)}, a, s^{(a)})$. □

Lemma 20. *Let $x \in X_\sigma$ be a minimal point. Then,*

$$\liminf_{n\to\infty} \frac{\gamma(x_0 \dots x_n)}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} > 0 \text{ and } \liminf_{n\to\infty} \frac{-\gamma(x_{-n} \dots x_{-1})}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} > 0$$

Proof. We only prove the first inequality, the other one can be shown analogously. Assume the result does not hold. Then, for a subsequence $(n_i)_{i\in\mathbb{N}}$,

$$\lim_{i\to\infty} \frac{\gamma(x_0 \dots x_{n_i})}{n_i^{\frac{\log(\theta_2)}{\log(\theta_1)}}} = 0$$

Let $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ be the prefix-suffix decomposition of x and let $a \in \hat{A}$ such that $(p^{(a)}, a, s^{(a)})$ is its generating word (recall it is ultimately periodic).
(1) First we assume $s^{(a)}$ is different from the empty word. For i enough large, let $N_i \geq 1$ be the minimal integer such that $x_1 \dots x_{n_i+1}$ is the prefix of $s_0 \dots \sigma^{N_i-1}(s^{(a)})$ (recall $s_n = s^{(a)}$ for large enough n).
Consider the prefix-suffix decomposition $(p_j^{(n_i)}, c_j^{(n_i)}, s_j^{(n_i)})_{j\in\mathbb{N}}$ of $T^{n_i+1}(x)$. By Lemma 18,

$$\sigma^{N_i-1}(p_{N_i-1}^{(n_i)}) \dots \sigma(p_1^{(n_i)}) p_0^{(n_i)} = \sigma^{N_i-1}(p_{N_i-1}) \dots \sigma(p_1) p_0 x_0 \dots x_{n_i}$$

Then,

$$\sum_{j=N_i-1}^{0} \theta_2^j \gamma(p_j^{(n_i)}) = \sum_{j=N_i-1}^{0} \theta_2^j \gamma(p_j) + \gamma(x_0 \dots x_{n_i})$$

Dividing by $\theta_2^{N_i}$ one gets,

$$\sum_{j=1}^{N_i} \theta_2^{-j} \gamma(p_{N_i-j}^{(n_i)}) = \sum_{j=1}^{N_i} \theta_2^{-j} \gamma(p_{N_i-j}) + \theta_2^{-N_i} \gamma(x_0 \dots x_{n_i}) \tag{3.1}$$

Observe that n_i behaves like $\theta_1^{N_i}$, so $\lim_{i\to\infty} \theta_2^{-N_i} \gamma(x_0 \dots x_{n_i}) = 0$. Also, since $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ is ultimately periodic with generating word $(p^{(a)}, a, s^{(a)})$, then

$$\lim_{i\to\infty} \sum_{j=1}^{N_i} \theta_2^{-j} \gamma(p_{N_i-j}) = \sum_{j\geq 1} \theta_2^{-j} \gamma(p^{(a)}) = v(a; \mathbf{x}(a)) = v(a)$$

Therefore, taking the limit when $i \to \infty$ in (3.1) one gets

$$\lim_{i\to\infty} \sum_{j=1}^{N_i} \theta_2^{-j} \gamma(p_{N_i-j}^{(n_i)}) = v(a) \tag{3.2}$$

Now, for i enough large consider the finite sequences:

$$\mathbf{y}^{(n_i)} = (p^{(a)}, a, s^{(a)})(p_{N_i-1}^{(n_i)}, c_{N_i-1}^{(n_i)}, s_{N_i-1}^{(n_i)}) \dots (p_0^{(n_i)}, c_0^{(n_i)}, s_0^{(n_i)})$$

By construction, $p^{(a)}a$ is a strict prefix of $p_{N_i-1}^{(n_i)} c_{N_i-1}^{(n_i)}$ unless $n_i + 1 = 0$ which is not the case. As in the proof of Lemma 12, by taking a subsequence one constructs from the finite sequence $\mathbf{y}^{(n_i)}$ a limit point $\mathbf{y} = (p_i^{\mathbf{y}}, c_i^{\mathbf{y}}, s_i^{\mathbf{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$.

Let $L \geq 1$. Then, for i enough large,

$$\begin{aligned} v(a;\mathbf{y}) &= \sum_{j\geq 1} \theta_2^{-j}\gamma(p_j^{\mathbf{y}}) = \sum_{j=1}^{L} \theta_2^{-j}\gamma(p_j^{\mathbf{y}}) + \sum_{j>L} \theta_2^{-j}\gamma(p_j^{\mathbf{y}}) \\ &= \sum_{j=1}^{L} \theta_2^{-j}\gamma(p_{N_i-j}^{(n_i)}) + \sum_{j>L} \theta_2^{-j}\gamma(p_j^{\mathbf{y}}) \\ &= \sum_{j=1}^{N_i} \theta_2^{-j}\gamma(p_{N_i-j}^{(n_i)}) - \sum_{j=L+1}^{N_i} \theta_2^{-j}\gamma(p_{N_i-j}^{(n_i)}) + \sum_{j>L} \theta_2^{-j}\gamma(p_j^{\mathbf{y}}) \end{aligned}$$

Let $\epsilon > 0$ and consider L such that $\sum_{j>L} \theta_2^{-j} \leq \epsilon/C$ where $C = \max\{|\gamma(p)| : p \text{ is a prefix of } \sigma(b) \text{ for some } b \in A\}$. One obtains for i big enough that

$$|v(a;\mathbf{y}) - v(a)| \leq |\sum_{j=1}^{N_i} \theta_2^{-j}\gamma(p_{N_i-j}^{(n_i)}) - v(a)| + 2\epsilon$$

Finally, from (3.2) one concludes $|v(a;\mathbf{y}) - v(a)| \leq 2\epsilon$ and thus $v(a;\mathbf{y}) = v(a)$. By Lemma 19, $\mathbf{y}_1 = (p^{(a)}, a, s^{(a)})$. This implies $(p^{(a)}, a, s^{(a)}) = (p_{N_i-1}^{(n_i)}, c_{N_i-1}^{(n_i)}, s_{N_i-1}^{(n_i)})$ for some large i, which is a contradiction.
(2) Now suppose $s^{(a)}$ is the empty word. Then, considering a power of σ if necessary we can assume that $(x_n)_{n\geq N} = \lim_{m\to\infty} \sigma^m(b)$ for some $N \in \mathbb{N}$ and $b \in A$. If we write $\sigma(b) = bs$ one obtains $x_N x_{N+1} \ldots = bs\sigma(s)\sigma^2(s)\ldots$.
Let $\mathbf{x} = (\varepsilon, b, s)_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$. We have $v(b;\mathbf{x}) = 0$. We claim $v(b) = 0$. Suppose this is not true. Then $v(b) < 0$ and for $k \in \mathbb{N}$ large enough one has $\sum_{i=1}^{k} \theta_2^{k-i}\gamma(p_i^{\mathbf{x}(b)}) \leq K\theta_2^k$ with $K < 0$. That is, γ applied to the prefix

$$\sigma^k(p_1^{\mathbf{x}(b)})\sigma^{k-1}(p_2^{\mathbf{x}(b)})\ldots p_k^{\mathbf{x}(b)}$$

of $\sigma^{k+1}(b)$ can be as negative as we want if k increases. This implies that $\gamma_n(x) < 0$ for some $n \in \mathbb{N}$, which is imposible since x is a minimal point. Then $v(b) = 0$. On the other hand, $v(b) = \sum_{j\geq 1} \theta_2^{-j}\gamma(p^{(b)})$ which implies $\gamma(p^{(b)}) = 0$. But $\gamma(p^{(b)}) = \sum_{c\in A} n_c\gamma(c)$, where n_c is the number of times the symbol c appears in $p^{(b)}$. Then by Lemma 3 $p^{(b)}$ is the empty word ε and $\mathbf{x}(b) = \mathbf{x}$.
Finally we prove that $\gamma(x_N \ldots x_{N+i}) > 0$ for all $i \geq 1$. Suppose that $\gamma(x_N \ldots x_{N+i}) < 0$ for some $i \geq 1$. Let $l \geq 1$ such that $x_N \ldots x_{N+i} = b\sigma(s)\ldots\sigma^{l-1}(s)s^-$ with s^- a prefix of $\sigma^l(s)$. By Lemma 6 $\gamma(\sigma(b)) = \theta_2\gamma(b) = \gamma(b)+\gamma(s)$, then $\gamma(s) = (\theta_2-1)\gamma(b)$. We deduce,

$$\gamma(x_N \ldots x_{N+i}) = \gamma(b) + \sum_{j=1}^{l-1} \theta_2^j\gamma(s) + \gamma(s^-) = \theta_2^l\gamma(b) + \gamma(s^-)$$

Let $k \geq 0$ be an integer and write $\sigma^l(s) = s^-s^+$. Then $\sigma^{l+k}(s) = \sigma^k(s^-s^+) = \sigma^k(s^-)\sigma^k(s^+)$ and $b\sigma(s)\ldots\sigma^{l+k-1}(s)\sigma^k(s^-)$ is a prefix of $b\sigma(s)\ldots\sigma^{l+k}(s)$. We have,

$$\gamma(b\sigma(s)\ldots\sigma^{l+k-1}(s)\sigma^k(s^-)) = \gamma(b) + \sum_{j=1}^{l+k-1} \theta_2^j\gamma(s) + \theta_2^k\gamma(s^-) = \theta_2^k(\theta_2^l\gamma(b) + \gamma(s^-))$$

But $\theta_2^k(\theta_2^l\gamma(b) + \gamma(s^-))$ can be as negative as we want, so for some $M \in \mathbb{N}$ $\gamma(x_0 \dots x_M) < 0$ which contradicts the fact that x is a minimal point. Therefore $\gamma(x_N \dots x_{N+i}) > 0$ for all $i \geq 1$.
To conclude one uses the proof of part (1) with b instead of a to deduce that

$$\liminf_{n\to\infty} \frac{\gamma(x_N \dots x_{N+n})}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} > 0$$

and then

$$\liminf_{n\to\infty} \frac{\gamma(x_0 \dots x_n)}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} = \liminf_{n\to\infty} \frac{\gamma(x_0 \dots x_{N-1}) + \gamma(x_N \dots x_{N+n})}{n^{\frac{\log(\theta_2)}{\log(\theta_1)}}} > 0$$

□

The following proposition is plain.

Proposition 21. *Let $x \in X_\sigma$ be a minimal point. Then,*

$$\sum_{n\geq 1} e^{-\gamma(x_0 \dots x_{n-1})} < \infty \text{ and } \sum_{n\geq 1} e^{\gamma(x_{-n} \dots x_{-1})} < \infty$$

4. Proof of the Main Theorem

The arguments of this section follows the strategy developed in the works of [CG] and [C].
Let $T_{(\lambda,\pi)}$ be a self-similar interval exchange transformation and R its associated matrix. Assume R verifies hypotheses of Theorem 1.
Let X_σ be the substitutive system associated to $T_{(\lambda,\pi)}$ and let $M = {}^tR$ be the associated matrix. Consider a minimal point $x \in X_\sigma$. By Proposition 21,

$$K = \sum_{n\geq 1} e^{\gamma(x_{-n} \dots x_{-1})} + 1 + \sum_{n\geq 1} e^{-\gamma(x_0 \dots x_{n-1})} < \infty$$

Let $t = \varphi(x)$. That is, x is the coding of t or x is the coding of $(\lim_{s\to t^-} T^i(s))_{i\in\mathbb{Z}}$ in the case t is in the orbit of one of the a_i's. To simplify notations we assume the first case holds, the other one is analogous.
Define the probability measure μ_t on $[0,1)$ by

$$\mu_t = \frac{1}{K}\left(\sum_{n\geq 1} e^{\gamma(x_{-n} \dots x_{-1})} \delta_{T^{-n}_{(\lambda,\pi)} t} + \delta_t + \sum_{n\geq 1} e^{-\gamma(x_0 \dots x_{n-1})} \delta_{T^{n}_{(\lambda,\pi)} t} \right)$$

Lemma 22. *For every Borel set $I \subseteq [0,1)$*

$$\mu_t(T_{(\lambda,\pi)}(I)) = \sum_{i=1}^{r} e^{-\gamma_i} \mu_t(I \cap [a_{i-1}, a_i))$$

Proof. It is enough to consider $I = [a_{i-1}, a_i)$ for $i \in A$. One has,

$$
\begin{aligned}
&\mu_t(T_{(\lambda,\pi)}(I))\\
&= \frac{1}{K}\left(\sum_{n\geq 1} e^{\gamma(x_{-n}\ldots x_{-1})}\delta_{T^{-n}_{(\lambda,\pi)}t} + \delta_t + \sum_{n\geq 1} e^{-\gamma(x_0\ldots x_{n-1})}\delta_{T^{n}_{(\lambda,\pi)}t}\right)(T_{(\lambda,\pi)}(I))\\
&= \frac{1}{K}\left(\sum_{n\geq 1} e^{\gamma(x_{-n}\ldots x_{-1})}\delta_{T^{-n-1}_{(\lambda,\pi)}t} + \delta_{T^{-1}_{(\lambda,\pi)}t} + \sum_{n\geq 1} e^{-\gamma(x_0\ldots x_{n-1})}\delta_{T^{n-1}_{(\lambda,\pi)}t}\right)(I)\\
&= \frac{1}{K}\left(\sum_{n\geq 1} e^{-\gamma(x_{-n})}e^{\gamma(x_{-n}\ldots x_{-1})}\delta_{T^{-n}_{(\lambda,\pi)}t} + e^{-\gamma(x_0)}\delta_t + \sum_{n\geq 1} e^{-\gamma(x_n)}e^{-\gamma(x_0\ldots x_{n-1})}\delta_{T^{n}_{(\lambda,\pi)}t}\right)(I)\\
&= e^{-\gamma_i}\mu_t(I)
\end{aligned}
$$

where in the last equality we use the fact that $T^n_{(\lambda,\pi)}(t)\in I$ if and only if $\gamma(x_n)=\gamma_i$. □

Define $g:[0,1)\to[0,1)$ by $g(s)=\mu_t([0,s])$. This function is nondecreasing, right continuous and has left limits. Let $i\in A$. Denote $a'_i=T(a_i)$ and define $b_i=\lim_{a\to a_i^-} g(a)$ and $b'_i=\lim_{a'\to(a'_i)^-} g(a')$. Then at interval $[b_{i-1},b_i)$ define linearly the AIET f with image $[b'_{i-1},b'_i)$. The slope vector of f is $w=(e^{-\gamma_1},\ldots,e^{-\gamma_r})$. Indeed,

$$
\frac{b'_i-b'_{i-1}}{b_i-b_{i-1}}=\frac{\mu_t([a'_{i-1},a'_i))}{\mu_t([a_{i-1},a_i))}=e^{-\gamma_i}
$$

where the last equality follows from Lemma 22.
Let $h:[0,1)\to[0,1)$ be the map defined by: $h(v)=u$ if $g(u)=v$ and $h(v)=u$ if $\lim_{w\to u^-} g(w)\leq v\leq g(u)$. Clearly h is surjective, continuous and non decreasing. Since μ_t has atoms, then h is not injective
The following lemma allows to conclude Theorem 1.

Lemma 23. *The map h defines a semi-conjugacy between the AIET f and $T_{(\lambda,\pi)}$. Moreover, f has wandering intervals.*

Proof. The semi-conjugacy follows from construction. The interval

$$
I=(\lim_{s\to t^-} g(s),g(t)]
$$

is a wandering interval for h. □

5. Pseudo-Anosov diffeomorphisms and eigenvalues of matrices obtained by Rauzy induction

In this section, we discuss the hypothesis of Theorem 1 in a geometric language. Our hypothesis is that the Perron-Frobenius eigenvalue θ_1 of the matrix R has a real conjugate $\theta_2>1$.
We recall that every interval exchange transformation $T_{(\lambda,\pi)}$ is realized as the first return map of a flow on a translation surface $\mathcal{S}$ which genus $g(\pi)$ only depends on the permutation π (and not on λ). This translation surface is not unique. If $T_{(\lambda,\pi)}$ is a periodic point of the Rauzy induction, one can choose $\mathcal{S}$ fixed by a pseudo-Anosov diffeomorphism ϕ (see [Th] for an enlightening discussion on pseudo-Anosov diffeomorphisms). The eigenvalue θ_1 is the dominant eigenvalue of the action of ϕ

on the absolute homology of $\mathcal{S}$. Therefore θ_1 is an algebraic number of degree at most $2g(\pi)$ over $\mathbb{Q}$.
Heuristically, after the work of Avila and Viana [AV], it is reasonable to believe that a "generic" pseudo-Anosov satisfies our hypothesis. Nevertheless, it seems extremely difficult to understand the eigenvalues of *all* pseudo-Anosov diffeomorphisms. In this section, we want to explain that our hypothesis are often satisfied. They are not always satisfied: for instance, the conjugates of the Arnoux-Yoccoz pseudo-Anosov are not real. Situations much worse do exist.

5.1. **Existence of a conjugate θ_2 with $|\theta_2| \geq 1$.** A pseudo-Anosov diffeomorphism preserves the symplectic form induced by the intersection form. Thus if z is an eigenvalue of the automorphism ϕ_* of $H_1(\mathcal{S}, \mathbb{Z})$, its inverse z^{-1} is also an eigenvalue of ϕ_*. Consequently, $\frac{1}{\theta_1}$ is an eigenvalue of ϕ_*. If it is the only Galois conjugate of θ_1, it means that θ_1 is an algebraic number of degree 2. It is classical (see [KS] for instance) that the surface $\mathcal{S}$ is then a covering of a torus (a square tiled surface up to normalization). Therefore hypothesis (1) is satisfied if and only if the surface $\mathcal{S}$ is not a square tiled surface. Thus, this hypothesis is very natural and simple to check.

5.2. **Real conjugates.** The second hypothesis is more subtle to analyze.
A pseudo-Anosov diffeomorphism is obtained by Thurston's construction if it is the product of two affine Dehn twists T_h and T_v along two multi-curves filling a surface (see [Th]).
After normalization, the derivatives of the Dehn twists in the natural parameters of the translation surface are

$$T_h = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},\ T_v = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

where a and b are positive real numbers and ab is an algebraic number.
An element f of the group generated by T_h and T_v is a pseudo-Anosov diffeomorphism if the absolute value of the trace $t(f)$ of the corresponding matrix is larger than 2. For every pseudo-Anosov diffeomorphism obtained by Thurston's construction, the conjugates of $t(f)$ are real numbers (see [HL]). The dominant root of the action of f on the homology is the real number $\theta_1 > 1$ with $\theta_1 + \theta_1^{-1} = t(f)$. The number θ_1 (or one of its power) is the Perron-Frobenius eigenvalue of the matrix obtained by Rauzy induction considered in the present paper (see [Ve]). Let θ' be a conjugate of θ_1 and $t'(f) = \theta' + \theta'^{-1}$ a (*real*) conjugate of $t(f)$. θ' is a real number with $\theta' > 1$ if $|t'(f)| > 2$. It is a complex number of modulus one if $|t'(f)| < 2$. This directly comes from the fact that $\theta' + \theta'^{-1} = t'(f)$.
For instance, the diffeomorphisms $f_{n,m} = T_h^n T_v^m$ are pseudo-Anosov diffeomorphisms if n, m are positive integers. In fact the absolute value of the trace of $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}^m$ is larger than 2 because $nmab > 0$.
Thus $\theta' > 1$ if $|t'(f)| = |2 + nm(ab)'| > 2$ (where $(ab)'$ is a real number). This is satisfied for all couples (n, m) except for a finite number of exceptions. Using more sophisticated arguments, $|t'(f)| > 2$ if n and m are positive integers.

6. Examples

In this section, we give an infinite family of examples that satisfy the hypothesis of Theorem 1. These examples are constructed in the simplest non trivial Rauzy

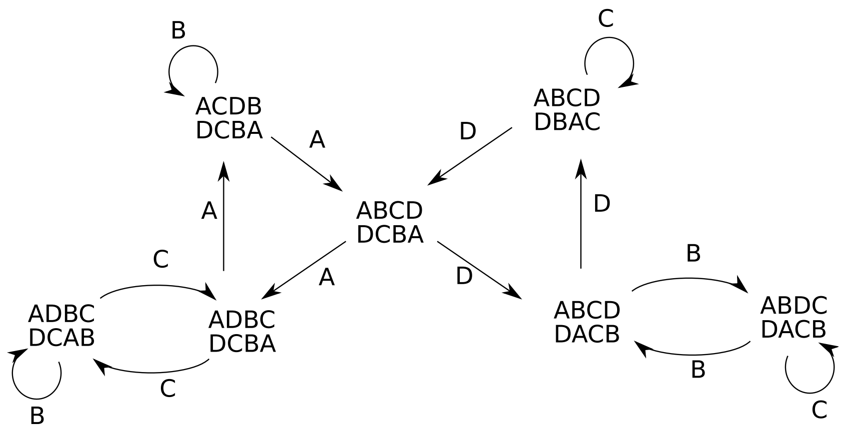

FIGURE 1. Rauzy diagram.

diagram. This Rauzy diagram is the combinatorial data of interval exchange transformations on 4 intervals. The diagram is drawn in figure 1. It is an oriented graph. The vertices are pairs of permutations. Every vertex possesses two outgoing arrows towards the pairs of permutations reachable after one step of Rauzy induction. The labels are letters defined by this choice.

We recall that a Rauzy class is the closure of a given pair of permutations under Rauzy induction and that self similar interval exchange transformations correspond to loops in the Rauzy diagram. Marmi-Moussa-Yoccoz gave an if and only if condition ensuring that a loop is realized by a self similar interval exchange transformation. In our situation, the Marmi-Moussa-Yoccoz's criterion says that a loop is admissible if all the labels A, B, C, D appear along the loop. This condition will be fulfilled by the family of examples studied here.

Every arrow of the diagram induces a substitution. Given an interval exchange transformation T on an interval I, let T' its image under Rauzy induction acting on I'. The substitution is defined by following the images of elements of the partition I' in the partition I under T until they come back to I'. The product of these substitutions gives the substitution corresponding to the loop or to the self similar interval exchange transformation associated to the loop. This is the substitution that we study in the paper.

Our family of examples is the following. The origin of the loop is the symmetric permutation

$$\begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$$

Then, we follow the labels $DBC^nBDCDAAA$ where C^n means that the loop labelled by C is done n times. The substitution on the alphabet $\{1, 2, 3, 4\}$ is

$$\sigma_n = \begin{cases} 1 & \mapsto & 14 \\ 2 & \mapsto & 1423^n24 \\ 3 & \mapsto & 1423^{n+1}23^{n+1}24 \\ 4 & \mapsto & 1423^{n+1}24 \end{cases}$$

The corresponding matrix is

$$M_n = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 2 \\ 0 & n & 2(n+1) & n+1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$

Proposition 24. *For $n \geq 2$, the Matrix M_n has two eigenvalues larger than 1 ($\theta_1^{(n)} > \theta_2^{(n)} > 1$), moreover $\theta_2^{(n)}$ is a conjugate of the Perron-Frobenius eigenvalue $\theta_1^{(n)}$.*

The characteristic polynomial of M_n is $P_n = X^4 - (2n+7)X^3 + (5n+11)X^2 - (2n+7)X + 1$. In the sequel the dependence in n of the eigenvalues is omitted to simplify the notations. As the polynomial is reciprocal and the matrix is primitive, $(\theta_1, \ldots \theta_4)$ satisfy the following relations: $\theta_4 = 1/\theta_1$, $\theta_3 = 1/\theta_2$,

$$\theta_1 > 1 \geq |\theta_2| \geq |1/\theta_2| \geq 1 > 1/\theta_1.$$

Let $t = \theta_1 + 1/\theta_1$ and $s = \theta_2 + 1/\theta_2$. ($s$ and t are real numbers).

Lemma 25. *If $n \geq 2$, we have $\theta_2 > 1$.*

Proof. As P is a polynomial with real coefficients, if θ_2 is not real, then $1/\theta_2 = \overline{\theta_2}$ and the modulus of θ_2 is equal to one. Let be $s = \theta_2 + 1/\theta_2$. If the modulus of θ_2 is one then $|s| \leq 2$. Moreover, if θ_2 is real, the sign of s is the same as the sign of θ_2. To prove that θ_2 is real and larger than 1, it is enough to prove that $s > 2$.
t and s satisfy $t+s = \theta_1 + 1/\theta_1 + \theta_2 + 1/\theta_2 = 2n+7$, $st = \theta_1\theta_2 + 1/(\theta_1\theta_2) + \theta_1/\theta_2 + \theta_2/\theta_1 = n+9-2 = n+7$. Consequently s and t are roots of the polynomial

$$X^2 - (5+n)X + n + 7. \tag{6.1}$$

We also remark that $t \leq s$ because $\theta_1 > |\theta_2| \leq 1$ and the map $x \mapsto x + 1/x$ is increasing on $(1, \infty($. This yieds to

$$s = 2n + 7 - \sqrt{4n^2 + 8n + 13}, n \geq 1. \tag{6.2}$$

If $n \leq 2$, we check that $s > 2$. Consequently, $\theta_2 > 1$. □

Lemma 26. *The eigenvalue θ_2 is a Galois conjugate of θ_1.*

Proof. If θ_2 is not a conjugate of θ_1 then θ_2 is an algebraic number of degree at most 2. Consequently, $s = \theta_2 + 1/\theta_2$ is a rational number. Because of (6.1), s is an algebraic integer, thus s is an integer. By equation (6.2), there exist an integer a such that $4n^2+8n+13 = a^2$. This leads to $(2n+2)^2+9 = a^2$. Let b equals $2(n+1)$, we have $(a-b)(a+b) = 3^2$. Thus $(a,b) = (3,0)$ or $(a,b) = (5,4)$. The first solution is meaningless because $b \geq 4$ and the second solution corresponds to $n = 1$. Thus for $n \geq 2$, s is not an integer, thus θ_2 is not a quadratic surd. Therefore θ_2 is a conjugate of θ_1. □

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Institut de Mathématiques de Luminy, 163 avenue de Luminy, Case 907, 13288 Marseille Cedex 9, France.
E-mail address: `bressaud@iml.univ-mrs.fr`

Laboratoire Analyse, Topologie et Probabilités, Case cour A, Faculté des Sciences de Saint-Jerôme, Avenue Escadrille Normandie-Niemen, 13397 Marseille Cedex 20, France.
E-mail address: `hubert@cmi.univ-mrs.fr`

Centro de Modelamiento Matemático and Departamento de Ingeniería Matemática, Universidad de Chile, Av. Blanco Encalada 2120, Santiago, Chile.
E-mail address: `amaass@dim.uchile.cl`