PIECEWISE ISOMETRIES OF THE HYPERBOLIC PLANE

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Abstract. We introduce and study first properties of a family of piecewise isometries of the hyperbolic plane. We study an exemple for which all points are periodic and the dynamic on the boundary is conjugate to an irrational rotation.

1. Introduction

2. Basic definitions

2.1. **The hyperbolic plane.** We consider the standard hyperbolic plane $\mathbb{H}$. We will also deal with the Poincarré disk model $\mathbb{D}$ of the hyperbolic plane and move from one model to the other depending on computationnal convenience. Roughly, $\mathbb{H}$ is the upper half complex plane, endowed with the hyperbolic metric.

$$\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$$

It is mapped onto the open unit disk $\mathbb{D} = \{z \in \mathbb{C}, |z| = 1\}$ by the map :

$$z \mapsto i\frac{z+i}{z-i}.$$

The boundary of the hyperbolic plane is the closed real line $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ (and the circle of radius 1 for $\mathbb{D}$).

2.2. **Isometries of the hyperbolic plane.** Isometries of $\mathbb{H}$ form a group isomorphic to $PSL(2, \mathbb{R})$. Isometries of the hyperbolic plane are maps

$$\left\{z \mapsto \frac{az+b}{cz+d};\ a, b, c, d \in \mathbb{R},\ ad - bc = 1\right\}.$$

For $g \in SL(2, \mathbb{R})$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1$, we write $g \cdot z = \frac{az+b}{cz+d}$ its action on $\mathbb{C}$. It acts on $\mathbb{H}$ and the quotient by $-I$ yields the same map. I think we are talking about positive isometries.
This action can obviously be extended to the boundary $\partial\mathbb{H}$.
Forme dans $\mathbb{D}$. Positive isometries can be written in $\mathbb{D}$ as

$$\left\{z \mapsto \frac{az+\bar{c}}{cz+\bar{a}};\ a, c \in \mathbb{C},\ |a|^2 - |c|^2 = 1\right\}.$$

or

$$\left\{z \mapsto \lambda\frac{z-z_0}{\bar{z}_0 z+1};\ \lambda, z_0 \in \mathbb{C},\ |\lambda| = 1\right\}.$$

Note that $|c| = \sinh\frac{1}{2}\rho(0, g0)$ and $|a| = \cosh\frac{1}{2}\rho(0, g0)$ Is it worth specifying the link : given an isometry of $\mathbb{H}$, with parameters a, b, c, d, we send it to an isometry of $\mathbb{D}$ by ...

Les classiques. Among isometries of $\mathbb{H}$, we have $m_\alpha : z \mapsto \alpha z = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} \end{pmatrix} \cdot z$, $\iota : z \mapsto -\frac{1}{z}$, $\kappa : z \mapsto z+1$, and $z \mapsto \frac{\cos(\theta) z + \sin\theta}{-\sin(\theta) z + \cos\theta}$.

Lemma 1. *Let Δ and Δ' be two geodesics. There is an isometry k_0 mapping Δ onto Δ', and, if $\Delta = i\mathbb{R}_+^*$ is the imaginary axis,*

$$\{k \in : k(i\mathbb{R}_+^*) = \Delta'\} = \{k_0 \circ m_\alpha, \alpha \in \mathbb{R}_+^*\} \cup \{k_0 \circ \iota \circ m_\alpha, \alpha \in \mathbb{R}_+^*\}.$$

Among these maps, there is one with minimal trace. That is when the minimal length is minimal (fixed point on the geodesic, or minimal distance)

Proof. This is essentially a known fact. Follows from the fact that an isometry is determined by the images of three points of the boundary, here the basis of the geodesic and another one to determine α. The last assertion has to be specified.

2.2.1. *Classification.* Isometries of $\mathbb{H}$ split in three main categories, elliptic, parabolic and hyperbolic, according to whetether $\text{trace}(M) < 2$, $\text{trace}(M) = 2$ and $\text{trace}(M) > 2$. The fixed points are conjugate with non zero imaginary part, real and unique, real and distinct.
For g elliptic, g is conjugate to a rotation and $\text{trace}^2(g) = 4\cos^2(\theta/2))$. Expression is simple if the fixed point is $i \in \mathbb{H}$ or $0 \in \mathbb{D}$. Otherwise if the center is v and the other fixed point v_1, we have $\frac{g(z)-v}{g(z)-v_1} = e^{i\theta}\frac{z-v}{z-v_1}$.
For g hyperbolic, denote $T = \inf_z \rho(z, gz)$ the translation length of g. We have $\text{trace}^2(g) = 4\cosh^2(T/2))$. We note that if $g(z) = \alpha z$, then $\cosh^2(T/2)) = 1 + \frac{(1-\alpha)^2}{4\alpha}$. T is the translation length along the axis of g, geodesic from one to the other fixed point.

2.2.2. *Subgroups generated by two elements.* Remarque sur les sous groupes engendrés par deux éléments. Discrets ? Action sur le bord.

2.3. **Piecewise isometries of the hyperbolic plane.** Let Δ be a geodesic of $\mathbb{H}$. Denote Δ_+ and Δ_- the two part of $\mathbb{H}$ delimited by Δ (one open, one closed). Let g and h be two isometries of $\mathbb{H}$. We consider the map $f = f_{\Delta,g,h}$ defined on $\mathbb{H}$ by :

$$f(z) = \begin{cases} g(z) & \text{if } z \in \Delta_- \\ h(z) & \text{if } z \in \Delta_+ \end{cases}$$

We denote $\mathcal{H}$ the set of all such maps. A few remarks:

- We can choose $\Delta = i\mathbb{R}_+^*$. Up to a conjugacy.
- The map f is bijective if and only if $g(0) = h(0)$ and $g(\infty) = h(\infty)$, so that $g(\Delta) = h(\Delta)$, globally. We denote $\mathcal{H}^*$ the set of bijective PWI of $\mathbb{H}$.
- The dynamic is not very fun if $g(\Delta) \cap \Delta = \emptyset$. Indeed, $g(\Delta_+) \subset \Delta_+$ and all $z \in \Delta_-$ ends up in Δ_+. Well this is the bad case, but there could be some ping-pong. Indeed if $g(\Delta_+) \subset \Delta_-$, then we can iterate untill either something like $g(\Delta_+) \subset \Delta_+$ or $g(\Delta) \cap \Delta \neq \emptyset$
- About the dynamics on the boundary : a piecewise homography of the real line. (or of the circle ?). We could forget about what happens in the disk and study this map. It is bijective. Bijectivity on the boundary tells us where 0 and ∞ are mapped and hence ensures bijectivity on the whole disk... Not so clear if more pieces.

Definition 2. *We say that the map f is* ***discrete*** *if the subgroup generated by g and h is discrete.*

Coding. We define *periodic islands.* We code orbits using the natural partition. That is for $z \in \mathbb{H}$, we set $i(z) = \pm 1$ if $z \in \Delta_{\pm}$ and we associate to $z \in \mathbb{H}$ the coding of its orbit i.e. the sequence $(i(f^n z))_{n\in\mathbb{Z}}$. The set of points having a given coding may have different forms. If it contains an open set we say it is an island. If the coding is periodic we say the island is periodic. Usually (lemma ?), periodic island contain a fixed point. A word in $\{-1, +1\}^{\mathbb{Z}}$ is admissible if there is a point having it as coding.

Definition 3. *A subset A of $\mathbb{H}$ is called a* periodic island *or a* cell *if all points in it have the same coding.*

We distinguish the *trivial* cells, corresponding to codes $\underline{0}$ and $\underline{1}$. Is there a simple condition for the trivial cells to be bounded ?
More general HPWI. Consider a (finite) partition of $\mathbb{H}$ and associate to each atom an isometry. The map taking every point $z \in P_i$ to $h_i(z)$ is a HPWI. If it is bijective (i.e. if $(h(P_i))$ form a partition ?) then it is a BHPWI.
Induction. Induction makes sense as for any other map. The induced map may be a HPWI. It is if and only if the returns are finite, so the induced partition is finite. Questions of boundaries.

Proposition 4. *If the subgroup is discrete, then all islands are polygons. Their union tiles the plane.*

3. Classification

3.1. **Parameterization.** A priori, a map f in $\mathcal{H}$ depends on Δ, f and g that is on 8 parameters. Obvious conjugacy of isometries reduces the interesting space to 6 parameters. Is it clear ? Let us see. We say that $f \sim f'$ if there is a map ψ (it must me an isometry) such that $f' \circ \psi = \psi \circ f$. On $\Delta_+ \cap \psi^{-1}(\Delta'_+)$, $h' \circ \psi = \psi \circ h$ shows that ψ must be isometric. Hence ψ is a BH4PWI. [Starting with f and a ψ that would be piecewise, construct f' in $\mathcal{H}$]. So we must define a stronger equivalence relation telling that the maps are conjugate by an isometry. Same thing asking ψ to be an isometry. Then $f' = \psi \circ f \circ \psi^{-1}$. So f' writes :

$$f'(z) = \begin{cases} \psi(g(\psi^{-1}(z))) & \text{if } z \in \psi(\Delta_-) \\ \psi(h(\psi^{-1}(z))) & \text{if } z \in \psi(\Delta_+) \end{cases}$$

or shortly, $f' = f_{\psi(\Delta),\psi\circ g\circ\psi^{-1},\psi\circ h\circ\psi^{-1}}$. A choice of ψ changes Δ and moves the isometries in conjugacy classes. For instance it means that we can decide that one of the isomtries is taken to be the simplest in its conjugacy class.
Then bijectivity reduces 2 parameters, since $g(0) = h(0)$ and $g(\infty) = h(\infty)$. We are left with 3 parameters.

3.1.1. *The model family.* Choose $r, s, t \in \mathbb{R}^3_+$. We consider the map

$$f(z) = \begin{cases} g \cdot z & \text{if } \Re(z) < 0 \\ h \cdot z & \text{if } \Re(z) \geq 0, \end{cases}$$

where k is an isometry with $\text{trace}^2(k) = r$, $g(i\mathbb{R}^*_+) = h(i\mathbb{R}^*_+) = k(i\mathbb{R}^*_+)$ and $\text{trace}^2(g) = s$, $\text{trace}^2(h) = t$. If $r < 4$, we ask further $k(i) = i$. If $r > 4$ we ask $k(1) = 1$. Otherwise, $k(0) = 0$. On a encore un probleme d'orientation, non ? If $r, s, t < 4$, we set $4\cos^2(\theta/2) = r$ and the same for φ, ψ.

Lemma 5. *This is well defined. We give expressions.*

As for an expression : case $r < 4$. Then k must be rotation of angle θ about i. We check that we can choose α and β so that $g(z) = k(\alpha z)$ and $h(z) = k(\beta z)$. Then we relate α to φ and β to ψ. It is as well of interest to specify where lie the fixed elements of g and h, roughly on the bissector of $i\mathbb{R}_+^*$ and $k(i\mathbb{R}_+^*)$. Let us give the expression, firstly in $\mathbb{H}$. Note that $k(z) = \frac{\cos(\theta)z+\sin(\theta)}{-\sin(\theta)z+\cos(\theta)}$. We must check that

$$\alpha = \frac{\cos\varphi/2 - \sqrt{\cos^2(\varphi/2) - \cos^2(\theta/2)}}{\cos\varphi/2 + \sqrt{\cos^2(\varphi/2) - \cos^2(\theta/2)}}$$

This holds if $\theta < \varphi$; otherwise, what ? In terms of traces, this becomes :

$$\alpha = \frac{\sqrt{s} - \sqrt{s-r}}{\sqrt{s} + \sqrt{s-r}} = \frac{(\sqrt{s} - \sqrt{s-r})^2}{s-s+r} = \frac{1}{r}(2s-r-2\sqrt{s(s-r)}) = -1+2\frac{s}{r}(1-\sqrt{1-\frac{r}{s}})$$

Note that

$$\frac{\alpha-1}{\alpha+1} = \frac{\sqrt{s} - \sqrt{s-r} - (\sqrt{s} + \sqrt{s-r})}{\sqrt{s} - \sqrt{s-r} + \sqrt{s} + \sqrt{s-r}} = -\frac{\sqrt{s-r}}{\sqrt{s}} = -\sqrt{1-\frac{r}{s}}$$

So that

$$\left(\frac{\alpha-1}{\alpha+1}\right)^2 = 1 - \frac{r}{s}$$

$$\frac{s}{r} = \frac{1}{1-\left(\frac{\alpha-1}{\alpha+1}\right)^2} = \frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha-1)^2} = \frac{(\alpha+1)^2}{\alpha^2+2\alpha+1-\alpha^2+2\alpha+1} = \frac{(\alpha+1)^2}{4\alpha}$$

$$s = \frac{r}{4}\left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right)^2$$

$$s = \frac{r}{4}(\alpha + \frac{1}{\alpha} + 2)$$

$$r\alpha^2 + 2(r-2s)\alpha + r = 0$$

$$\alpha^2 + 2(1-2\frac{s}{r})\alpha + 1 = 0$$

The solution looks like :

$$\alpha = 1-2\frac{s}{r}\pm\sqrt{(1-2\frac{s}{r})^2-1} = 1-2\frac{s}{r}\pm\sqrt{1-4\frac{s}{r}+4\frac{s^2}{r^2}-1} = 1-2\frac{s}{r}\pm 2\sqrt{\frac{s}{r}}\sqrt{-1+\frac{s}{r}}.$$

Note that this means

$$\mathrm{trace}^2(g) = \mathrm{trace}^2(k)\mathrm{trace}^2(m_\alpha)/4 = \mathrm{trace}^2(k)\cosh^2(T/2).$$

In this context the map g is :

$$g(z) = \frac{\cos(\theta)\alpha z + \sin(\theta)}{-\sin(\theta)\alpha z + \cos(\theta)}$$

$$g(z) = \frac{\cos(\theta)(\cos\varphi/2 - \sqrt{\cos^2(\varphi/2) - \cos^2(\theta/2)})z + \sin(\theta)(\cos\varphi/2 + \sqrt{\cos^2(\varphi/2) - \cos^2(\theta/2)})}{-\sin(\theta)(\cos\varphi/2 - \sqrt{\cos^2(\varphi/2) - \cos^2(\theta/2)})z + \cos(\theta)(\cos\varphi/2 + \sqrt{\cos^2(\varphi/2) - \cos^2(\theta/2)})}$$

Well, let do it in the disk $\mathbb{D}$ now. Here $k(z) = e^{i\theta}z$ while

$$g(z) = e^{i\theta}\frac{z + i\frac{\alpha-1}{\alpha+1}}{-i\frac{\alpha-1}{\alpha+1}z - 1} = e^{i\theta}\frac{(\alpha+1)z + i(\alpha-1)}{-i(\alpha-1)z - (\alpha+1)}$$

$$g(z) = e^{i\theta}\frac{z - i\sqrt{1-\frac{r}{s}}}{i\sqrt{1-\frac{r}{s}}z - 1} = e^{i\theta}\frac{\sqrt{s}z - i\sqrt{s-r}}{i\sqrt{s-r}z - s} = \frac{\sqrt{s}e^{i\theta/2}z - i\sqrt{s-r}e^{i\theta/2}}{i\sqrt{s-r}e^{-i\theta/2}z - \sqrt{s}e^{-i\frac{\theta}{2}}}$$

It is almost in standard form with $a' = \sqrt{s}e^{i\theta/2}$ and $c' = i\sqrt{s-r}e^{i\theta/2}$. We have $|a'|^2 - |c'|^2 = s - s - r = r$. So for standard form we should divide by $\sqrt{r}$:

$$g(z) = \frac{\sqrt{\frac{s}{r}}e^{i\theta/2}z - i\sqrt{\frac{s}{r}-1}e^{i\theta/2}}{i\sqrt{\frac{s}{r}-1}e^{-i\theta/2}z - \sqrt{\frac{s}{r}}e^{-i\frac{\theta}{2}}}$$

Hence if $a = \sqrt{\frac{s}{r}}e^{i\theta/2}$ and $c = i\sqrt{\frac{s}{r}-1}e^{-i\theta/2}$, then $|a|^2 - |c|^2 = 1$. Have a look at the trace: $\mathrm{trace}(g) = 2\Re(a) = 2\sqrt{\frac{s}{r}}\cos\theta/2 = \sqrt{s}$ since remember that $r = 4\cos^2\theta/2$. In fact we could write this all in terms of r and s : $e^{i\theta/2} = (\sqrt{r} + i\sqrt{4-r})/2$, so,

$$a = \sqrt{\frac{s}{r}}(\sqrt{r} + i\sqrt{4-r})/2 = (\sqrt{s} + i\sqrt{\frac{4s}{r} - s})/2$$

while

$$c = i\sqrt{\frac{s}{r} - 1}(\sqrt{r} + i\sqrt{4-r})/2 = (i\sqrt{s-r} - \sqrt{\frac{4s}{r} - s - 4 + r})/2$$

Essayons en θ, φ :

$$a = \sqrt{\frac{\cos^2(\varphi/2)}{\cos^2(\theta/2)}}e^{i\theta/2} = \frac{\cos(\varphi/2)}{\cos(\theta/2)}e^{i\theta/2}$$

$$c = \frac{\sqrt{\cos^2(\varphi/2) - \cos^2(\theta/2)}}{\cos(\theta/2)}e^{i(\theta+\pi)/2}$$

Remark 6. It seems that not all values of the parameters are allowed.

Lemma 7. *Every BH2PWI ($\neq I$) is conjugate to some $f_{r,s,t}$.*

Remark 8. More precisely, s and t give the trace of g and h. r is the trace of another map mapping the separatrix onto its image with constraints on its fixed points. We could say, if intersection, that it is the angle between them. Is it possible to choose k in such way that if no intersection, r gives the distance between separatrix and its image.

Proof. The argument is straightforward. We set $s = \mathrm{trace}^2(g)$and $t = \mathrm{trace}^2(h)$. Distinguish according to whether $q = \Delta \cap \Delta'$ is (i) empty, (ii) reduced to one point and (iii) a geodesic. In all cases, we consider isometries mapping the Δ to $i\mathbb{R}_+^*$.
(i) Firstly assume that $\bar{\Delta} \cap \bar{\Delta}' = \emptyset$. Choose K such that K maps Δ onto $i\mathbb{R}_+^*$ and $K(\Delta'_\infty) = 1$
(ii) Choose K such that K maps Δ onto $i\mathbb{R}_+^*$ and $K(\Delta \cap \Delta') = i$. We denote k the unique (elliptic) isometry mapping Δ onto $\Delta'' = K(\Delta')$ and fixing i. We set $r = \mathrm{trace}^2(k)$. We claim that $K \circ f \circ K^{-1} = f_{r,s,t}$. We observe that $r < s, t$. Indeed, $g' = K \circ g \circ K^{-1}$ and $h' = K \circ h \circ K^{-1}$ map $i\mathbb{R}_+^*$ onto Δ'' ; so we can write $g' = k \circ m_\alpha$ where α is chosen $g'(z_0) = k(\alpha z_0)$; we also have $k \circ m_\beta$. It follows from Lemma **??** that f and g write $k \circ m_\alpha$ and $k \circ m_\beta$.
(iii) This case is not very exciting. It means that $g(z) = \pm 1/(kz)$, up to an isometry mapping the separatrix onto Δ. In any case the dynamics of the PWI is not very interesting. Anyhow for completeness, we note that this is coherent.

4. The Fibonacci example

Let $\alpha =$.

$$\alpha = \frac{\cos\varphi/2 - \sqrt{\cos^2(\varphi/2) - \cos^2(\theta/2)}}{\cos\varphi/2 + \sqrt{\cos^2(\varphi/2) - \cos^2(\theta/2)}}$$

$$\alpha = \frac{\cos\pi/5 - \sqrt{\cos^2(\pi/5) - \cos^2(\pi/4)}}{\cos\pi/5 + \sqrt{\cos^2(\pi/5) - \cos^2(\pi/4)}}$$

$$\alpha = \frac{\cos\pi/5 - \sqrt{\cos^2(\pi/5) - 1/2}}{\cos\pi/5 + \sqrt{\cos^2(\pi/5) - 1/2}} = \frac{\cos\pi/5 - \sqrt{\cos(2\pi/5)/2}}{\cos\pi/5 + \sqrt{\cos(2\pi/5)/2}}$$

On se rappelle que : $\cos^2(\pi/5) = (1 + \frac{1+\sqrt{5}}{2})/4 = (1+\phi)/4$, avec $\phi^2 = \phi + 1$. Donc $\cos^2(\pi/5) = \phi/4$. Il vient :

$$\alpha = \frac{\phi/2 - \sqrt{\phi^2/4 - 1/2}}{\phi/2 + \sqrt{\phi^2/4 - 1/2}}$$

$$\alpha = \frac{\phi - \sqrt{\phi - 1}}{\phi + \sqrt{\phi - 1}}$$

$$\alpha = \phi(1 - \sqrt{\phi - 1}) = \phi - \sqrt{\phi}$$

We are going to study more into details the map defined by

$$f(z) = \begin{cases} \frac{1+z}{1-z} & \text{if } \Re(z) < 0 \\ \frac{1+\alpha z}{1-\alpha z} & \text{if } \Re(z) \geq 0 \end{cases}$$

That is on the left hand side, there is a rotation by $\pi/2$ about i, while on the right hand side, it is a rotation by $2\pi/5$ about a center $z_0 = a(1+i)$, with $a =$. Anotherway to say that is that we do a rotation by $\pi/2$ about i, but if we are on the right side, we first apply the hyperbolic translation with ratio α and axis $i\mathbb{R}_+^*$. The center i is stable by g, while, for h it is of period 5. We consider the polygon $i, h(i), h^2(i), h^3(i), h^4(i)$. The convex hull is a polygon P. Since $h(P)$ is included in the right hand half plane, f is periodic on P. The polygon P is the periodic island with coding $\underline{1}$.
We consider the subgroup generated by g and h. It is discrete. We can divide P into five triangles, each of them forming a fundamental domain of the subgroup. This should be checked. Another point is that tha tiling is bounded by the $i\mathbb{R}_+^*$. Is this enough to check that all island are copies of P. We can introduce the graph of these cells with edges between adjacent cells. This is the Cayley graph of some group (?) dual or something like that. Is it clear that m_α is in the group ? En tous cas, on ne l'utilise jamais : il serait interessant de se demander les elements du groupe qu'on atteint quand on suit des orbites. Qd meme il faut eviter d'ecrire des conneries.

4.1. **Results.** The dynamical partition is the pentagonal tiling. All points are periodic. The dynamics on the boundary is a piecewise homography. It is topologically (?) conjugated to the rotation by angle φ. More precisely, the coding is the Fibonacci subshift. We should say something about the codings of the periodic islands.

4.2. **Induction.** The basic lemma is about induction. We define the domain Λ_1 to be the south east quarter of $\mathbb{D}$, $\Lambda_1 = \{z \in \mathbb{D}, \Re(z) \geq 0, \Im(z) \leq 0\}$. We consider the map $\phi = m_\alpha$ and denote $\Lambda_0 = \phi^{-1}(\Lambda_1)$ and $\Lambda_n = \phi^n(\Lambda_0)$. Consider the induced map on the domain Λ_0. It is a isometry whose partition has two pieces. This map is self similar in the following sense :

$$f_{\Lambda_1} = \phi f_{\Lambda_0} \phi^{-1}.$$

$$f_{\Lambda_{n+1}} = \phi f_{\Lambda_n} \phi^{-1}.$$

Lemma 9.

Let us be more synthetic.

$$\begin{array}{llllll} f_{\Lambda_0}: & P_1 \stackrel{h}{\to} & P_2 & & & \\ & Q_1 \stackrel{h}{\to} & Q_2 \stackrel{h}{\to} & Q_3 \stackrel{g}{\to} & Q_4 \stackrel{g}{\to} & Q_5 \end{array}$$

or equivalently,

$$f_{\Lambda_0}(z) = \left\{ \begin{array}{ll} h(z) & \text{if } z \in P_1 \\ g^2 \circ h^2(z) & \text{if } z \in Q_1 \end{array} \right.$$

Wenaturally set $h_0 = h$ and $g_0 = g^2 \circ h^2$. For the induced map on Λ_1 :

$$\begin{array}{lllll} f_{\Lambda_1}: & P_1 \stackrel{h_0}{\to} & P_2 \stackrel{h_0}{\to} & P_3 \stackrel{g_0}{\to} & P_4 \\ & Q_1 \stackrel{h_0}{\to} & Q_2 \stackrel{g_0}{\to} & Q_3 & \end{array}$$

$$f_{\Lambda_1}(z) = \left\{ \begin{array}{ll} g_0(h_0^2(z)) & \text{if } z \in P_1' \\ g_0(h_0(z)) & \text{if } z \in Q_1' \end{array} \right.$$

So that more generally, $h_{n+1} = g_n \circ h_n^2$ and $g_{n+1} = g_n \circ h_n$.

4.3. **Symbolic dynamics of the periodic islands.** ¿From this business, we can deduce the symbolic dynamics of the periodic islands in this particular case. Firstly we notice that if at level $n+1$, the coding in terms of the partition (P, Q) is given by a word u on the alphabet $\{p, q\}^{\mathbb{N}}$, then at step n, it is given by $\sigma(u)$, where

$$\left\{ \begin{array}{lll} \sigma(p) & = & ppq \\ \sigma(q) & = & pq \end{array} \right.$$

The fixed point of σ is given by $u = ppqppqpqppqppqpqppqpq\cdots$. write it in more usual form $aabaababaabaababaabab$. The same argument at level 0

$$\left\{ \begin{array}{lll} s(p) & = & + \\ s(q) & = & ++-- \end{array} \right.$$

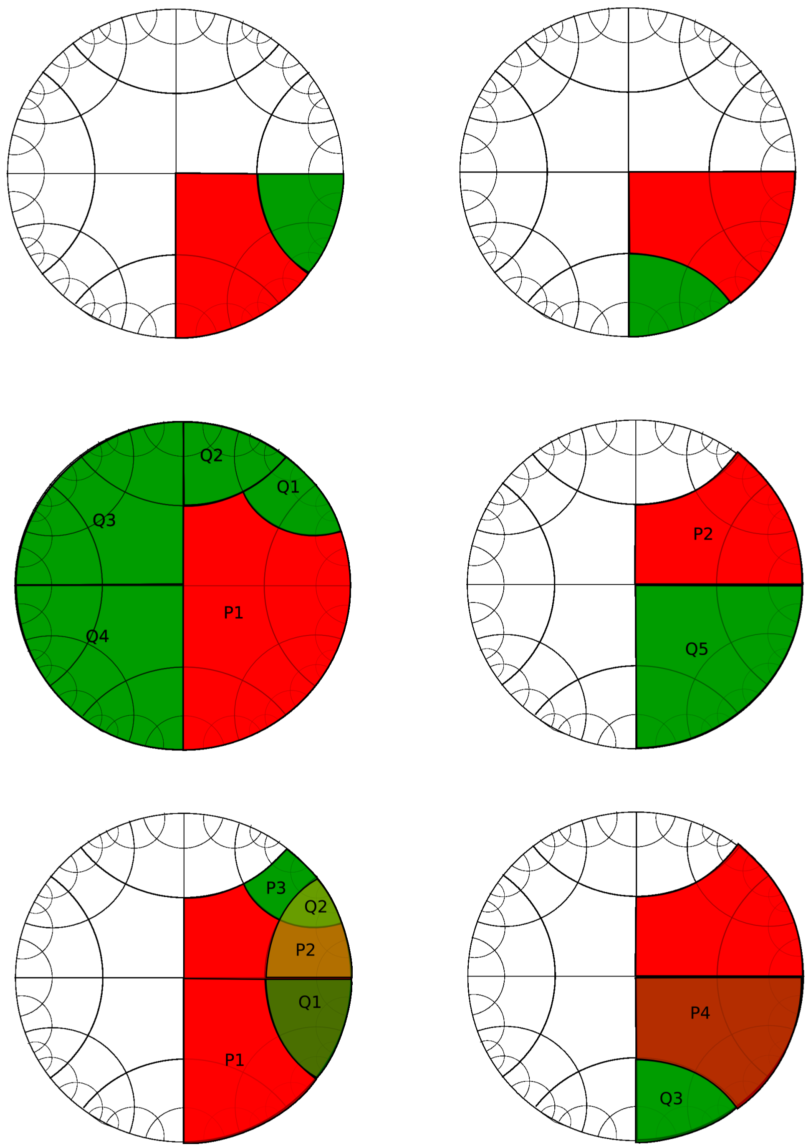
This seems to say that the orbit of 0 is coded by :

$$u_\infty = ++++--++++--+++--++++--++++--+++--++++--+++--$$

It is not completely clear to me if it is the best coding we can hope for. It sounds like if the language was Fibonacci at some level, but shifted (i.e; the fixed point is not the usual) all this is standard but tedious.

In any case, we must conclude something about periodic orbits. The deal is that when inducing, we leave a pentagon which has a periodic orbit. Formally, if $\Lambda_n = P_1 \cup Q_1$, then $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 = \Lambda_{n-1} \setminus P^*$. It is obvious that in Λ_{n-1}, this guy, P^* is invariant (rotated by $2\pi/5$; tut tuuut, maybe not exactly : could be

Figure 1: Induction



Explications : We induce on $\Lambda_0 = P_1 \cup Q_1$. The piece P_1 is mapped onto $P_2 \subset \Lambda_0$ (right hand side), while Q_1 is mapped onto Q_2, then Q_3, Q_4, before to come back to $Q_5 \subset \Lambda_0$.

Then, we show that when we induce this map on Λ_1, we obtain the same map up to conjugacy.

$6\pi/5$ and some recusion... Check it). Anyhow, since it is in the P_1 of level $n-1$, we can recover its coding by substitution. It is given by $s \circ \sigma^{n-1}(p)$. We obtain :

$$+$$

$$++++--$$

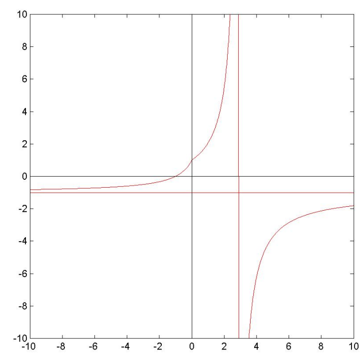
$$++++--++++--+++--$$

$$++++--++++--+++--++++--++++--+++--++++--+++--$$

and so on. In terms of the length of these orbits, the recursion is $p_{n+1} = 2p_n + q_n$ and $q_{n+1} = p_n + q_n$, while $p_1 = 1$ and $q_1 = 0$. The effective number of + is $p_n + 2q_n$ and the number of - is $2q_n$. The length is $p_n + 4q_n$. This writes in termsof the Fibonacci sequence.

4.4. **Map on the boundary.** Denote $\tilde{f}$ the extension of f to the boundary $\partial\mathbb{H}$. It is a piecewise bijective homography of the real line defined by the same formula as f.

The map $\tilde{f}$.



Proposition 10. *The map $\tilde{f}$ is (topologically) conjugated to the rotation of angle $\tilde{\theta} =$ on the unit circle.*

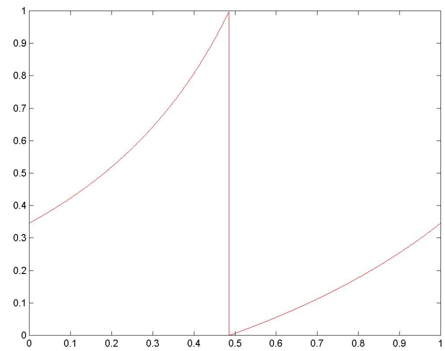
This result immediatly yields unique ergodicity of the system and hence existence and uniqueness of an invariant measure. As we may see, this measure is absolutely contnuous with respect to Lebesgue but does not have a density as smooth as one could have expected (naively). Well, the density is Holder, but not given by pieces of homography or such formulae. We will detail further.

Proof. The induction scheme proposed to study the map f still provides an induction scheme on the boundary. Putting a $\tilde{.}$ on parts of $\mathbb{H}$ do denote the trace of their adherence on $\partial\mathbb{H}$, we describe it.
We shall firstly concentrate on $\tilde{f}_{\tilde{\Lambda}}$ which is self-induced. As seen on the interval $[0, 1]$ of $\mathbb{R} = \partial\mathbb{H}$, it writes:

$$\tilde{f}_{\tilde{\Lambda}_1}(z) = \begin{cases} \alpha\frac{1+z}{1-z} & \text{if } 0 \leq z < \frac{1-\alpha}{1+\alpha} \;\; (z \in \tilde{P}_1) \\ \alpha\frac{(\alpha+1)z+\alpha-1}{(\alpha-1)z+\alpha+1} & \text{if } \frac{1-\alpha}{1+\alpha} < z \leq 1 \;\; (z \in \tilde{Q}_1) \end{cases}$$

and a its graph is shown on Figure **??**. Self-induction readily follows from self-induction of the map f_{Λ_1}. The difference is now that points in $\partial\mathbb{H}$ have "coding" in the adherence (i.e. we are not restricted to peiodic orbits). We build a map from $\widetilde{\Lambda}_1$ to a symbolic space and then to the interval on which lives the rotation with same coding. This is standard.
To do things properly, we consider the map R_α defined on $[0, 1]$ by $R_\alpha(x) = x + \alpha$ (mod 1). It is classical to induce on $[0, \alpha]$. One step induction provides a rotation

The induced map $\tilde{f}_{\tilde{\Lambda}_1}$.



of angle α^2 so that, after renormalisation, we get the same map. More specifically, $R^{(\alpha)}_{[0,\alpha[}(x) = x + \alpha^2 \pmod{\alpha}$. So if ϕ_α denote the multiplication by α,

$$R^{(\alpha)}_{[0,\alpha[} \circ \phi_\alpha = \phi_\alpha \circ R^{(\alpha)}.$$

or

$$R^{(\alpha)}_{[0,\alpha[}(\alpha x) = \alpha R^{(\alpha)}(x)$$

Now consider the renormalization map $x \mapsto \frac{1}{\alpha} x \bmod (1)$. Let $i_n(x)$ be the dynamical partition. Observe that, for all integer n, if $i_m(x) = 2$ for all $m < n$ while $i_n(x) < 2$, then $i_m(R(x)) = 0$ for all $m < n$ while $i_n(R(x)) = i_n(x) + 1$ and $i_m(R(x)) = i_m(x)$, for all $m > n$.

We observe that the combinatorics of this renormalization is the same as that of $\tilde{f}_{\tilde{\Lambda}_1}$. Indeed, we must have a look at the "renormalisation map" on $\partial\mathbb{H}$. It is the piecewise homography defined by

$$\mathcal{R}(z) = \begin{cases} \frac{z}{\alpha} & \text{if } 0 \le x < \alpha \\ \frac{1}{\alpha} \frac{1+z}{1-z} & \text{if } \alpha \le x < \alpha \frac{1+\alpha}{1-\alpha} \\ \frac{1}{\alpha} \frac{(1-\alpha)x - \alpha(\alpha+1)}{(\alpha+1)x + \alpha(\alpha-1)} & \text{if } \alpha \frac{1+\alpha}{1-\alpha} \le x < 1 \end{cases}$$

It is the piecewise homography defined by

$$\mathcal{R}(z) = \begin{cases} \frac{z}{\alpha} & \text{if } 0 \le x < \alpha \\ \frac{1}{\alpha} h^{-1}(z) & \text{if } \alpha \le x < \alpha \frac{1+\alpha}{1-\alpha} \\ \frac{1}{\alpha} h^{-2}(z) & \text{if } \alpha \frac{1+\alpha}{1-\alpha} \le x < 1 \end{cases}$$

Hence it is clear that if $i_0(x) < 2$, then $i_0(h(x)) = i_0(x) + 1$ and $\mathcal{R}(h(x)) = \mathcal{R}(x)$. More generally, if $i_m(x) = 2$ for all $m < n$ while $i_n(x) < 2$, then : $i_n(h(x)) = i_0(\mathcal{R}^n(h(x))) = i_0(h(\mathcal{R}^n(x))) = i_0(\mathcal{R}^n(x)) = i_n(x)$ and for $m < n$, $i_m(h(x)) = i_0(\mathcal{R}^m(h(x))) = i_0(h(\mathcal{R}^m(x))) = 0$ since,when $i_0(x) = 2$, $i_0(h(x)) = 0$.

Remark 11. It is completely general. Here we have a SFT which tells us the possible codings.

We define a map from $[0, 1]$ to itself by taking the coding ω_x of x and map x onto the point with coding ω_x of R_α. Those guys live in a SFT that has not been specified but which must be the same. Why ? What is the acertive argument ?

Properties of the conjugacy.

Lemma 12. *The derivative of the iterated map $\mathcal{R}^n$ satisfies :*

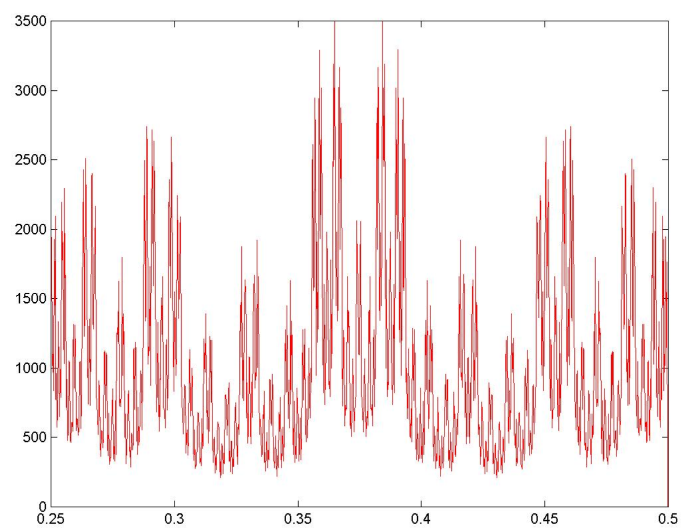
$$1/C \leq \alpha^n (\mathcal{R}^n)'(x) \leq C.$$

Convergence to 0 is enough to guarantee continuity of our map. Indeed, the map is increasing (topological conjugacy to the SFT respects the lexicographical ordering). It remains to check that two points x and y are not mapped two far. But if they are close, they are (in general) contained in a common cylinder which is itself mapped onto a cylinder of size α^n. So a control of the size of the cylinders is enough.
More precisely : let x be a point in the interval $[0, 1]$ and let $\epsilon > 0$. Either x is always in smaller and smaller cylinders, either it is on a "boundary point". In one case we use one cylinder, in the other one there are always two cylinder covering a neighbourhood. Choose n so that $\alpha^n < \epsilon$. Consider the measure of the cylinder of order n around x (or of the min of the two "adjacent" cylinders) This measure is positive. Choose δ smaller than this value. Any point δ close to x has its image ϵ close to that of x. If the lemma is true then the control is Lipshitz. Indeed, the measure of the cylinder writes

$$\int_0^1 dt = \int_{C_n(x)} (\mathcal{R}^n)'(u) du.$$

Attention. Pour avoir exactement la mesure il faut tenir compte des coupures liees au fait qu'on a pas un full shift mais un SFT. Ca corse notoirement les choses. Je ne sais pas s'il faut aller jusque la...
To conclude we have to put this back onto the whole $\partial\mathbb{H}$. This is easy, but of course changes the angle. The density will go through as well. □

Invariant measure



View of the frequencies for $\tilde{f}_{\tilde{\Lambda}_1}$ as seen on a part of $\partial\mathbb{D}$.

Remark 13. We do the explicit computation of the invariant density on $\partial\mathbb{H}$, in terms of the density ρ on $[0,1]$. Note that $h^{-1}(z) = \frac{1}{\alpha}\frac{z-1}{z+1}$ and $(h^{-1})'(z) = \frac{1}{\alpha}\frac{2}{(1+z)^2}$. Let $\rho_1(z) = \rho(h^{-1}(z))(h^{-1})'(z)$, $\rho_2(z) = \rho_1(h^{-1}(z))(h^{-1})'(z)$ and $\rho_3(z) = \rho_2(h^{-1}(z))(h^{-1})'(z)$. Let as well $\rho_{-1}(z) = \rho(g(z))g'(z)$ and $\rho_{-2}(z) = \rho_1(g(z))g'(z)$. We finally set

$$\rho(z) = \begin{cases} \rho_{-2}(z) & \text{if } x < -1 \\ \rho_{-1}(z) & \text{if } {}_1 \leq x < 0 \\ \rho(z) & \text{if } 0 \leq x < 1 \\ \rho_1(z) & \text{if } 1 \leq x < \frac{1+\alpha}{1-\alpha} \\ \rho_2(z) & \text{if } \frac{1+\alpha}{1-\alpha} \leq x < \frac{1+\alpha^2}{1-2\alpha-\alpha^2} \\ \rho_3(z) & \text{if } x > \frac{1+\alpha^2}{1-2\alpha-\alpha^2} \end{cases}$$

Nota : on peut ecrire mieux les bornes.

The measure $\rho(z)dz$ is invariant. We just have to check that

$$\rho(\tilde{f}(z))\tilde{f}'(z) = \rho(z).$$

Remark 14. In a first stage I naively thought

The map is conjugate to a rotation by angle $\tilde{\phi}$. Indeed, if we let $\psi(z) = \int_{-\infty}^{z} \rho(x)dx$, map $\mathbb{R}$ onto $[0,1]$, we have

$$\tilde{f} = \psi^{-1} \circ R_{\tilde{\phi}} \circ \psi.$$

This can be checked by direct computation but may also be seen as a consequence of the induction scheme.

5. Other examples

Acknowledgments.

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