An introduction to pseudo-differential operators

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Introduction

The spirit of these notes is to use the famous Weyl law (on the asymptotic distribution of eigenvalues of the Laplace operator on a compact manifold) as a case study to introduce and illustrate one of the many applications of the pseudo-differential calculus. The material presented here corresponds to a 24 hours course taught in Toulouse in 2012 and 2013. We introduce all tools required to give a complete proof of the Weyl law, mainly the semiclassical pseudo-differential calculus, and then of course prove it! The price to pay is that we avoid presenting many classical concepts or results which are not necessary for our purpose (such as Borel summations, principal symbols, invariance by diffeomorphism or the Gårding inequality). More importantly, we neither discuss important subjects such as dynamical aspects (relationship with the geodesic flow, the Egorov Theorem) nor quantum mechanical interpretations. We hope to treat them in a future course.

The prerequisite is some familiarity with basic functional analysis, distributions theory and Fourier transform on the Schwartz space, but we don't assume any knowledge on differentiable manifolds to which the first (short) chapter is devoted. These notes are self contained but we include a bibliography with classical textbooks on microlocal analysis for the interested reader.

Chapter 1

Background on analysis on manifolds

In this section we briefly introduce basic notions and notation about manifolds, integration and differential operators thereon for readers with no background on these topics.

Definition 1.1. *A* **smooth manifold** *of dimension n is a Hausdorff*[1] *topological space M which is σ-compact*[2] *such that, for any $m \in M$,*

1. *there exists an homeomorphism $\kappa : M \supset U \to V \subset \mathbb{R}^n$, between an open neighborhood U of m and an open subset V of $\mathbb{R}^n$.*
2. *If $\kappa_1 : U_1 \to V_1$ and $\kappa_2 : U_2 \to V_2$ are as in item 1, then the map*

$$\kappa_2 \circ \kappa_1^{-1} : \kappa_1(U_1 \cap U_2) \to \kappa_2(U_1 \cap U_2)$$

is smooth.

Note that item 1 is of topological nature hence has a clear sense on M. In item 2, $\kappa_1(U_1 \cap U_2)$ and $\kappa_2(U_1 \cap U_2)$ are open subsets of $\mathbb{R}^n$ so the notion of smoothness is clear. Note also that, in item 2, $\kappa_2 \circ \kappa_1^{-1}$ is automatically a diffeomorphism, since its inverse $\kappa_1 \circ \kappa_2^{-1}$ is smooth as well by definition (swap the roles of κ_1 and κ_2).

An open subset U as in item 1 is called a **coordinate patch** (at m) and the triple (U, V, κ) a **coordinate chart** or a **local coordinates** system (strictly speaking, the coordinates are the n components of the map κ). A collection of charts (U_i, V_i, κ_i) such that $\cup_i U_i = M$ is called an **atlas**.

Exercise 1.2 (The 2-sphere. Part 1)**.** *Consider the 2-sphere $\mathbb{S}^2$,*

$$\mathbb{S}^2 := \{m = (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

[1] in french: "séparé"

[2] ie a countable union of compact sets

1. *Show that* $\kappa_3 : m \in \mathbb{S}^2 \cap \{z > 0\} \mapsto (x, y)$ *is a local chart onto an open subset of* $\mathbb{R}^2$ *to be determined. Compute* κ_3^{-1}

2. *Same question with* $\kappa_2 : m \in \mathbb{S}^2 \cap \{y > 0\} \mapsto (x, z)$.

3. *Check directly that* $\kappa_3 \circ \kappa_2^{-1}$ *is a diffeomorphism between open subset of* $\mathbb{R}^2$.

4. *Let* $V = (0, 2\pi) \times (0, \pi)$. *Determine the range* $U \subset \mathbb{S}^2$ *of*

$$V \ni (\theta, \phi) \mapsto \big(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\big) \in \mathbb{S}^2.$$

Show that $m \in U \mapsto (\theta, \phi) \in V$ *define local coordinates.*

The notion of continuity is well defined on a topological space. In particular, we can consider the space $C^0(M)$ of (complex valued) continuous functions on M. If M is not compact, it is also useful to introduce the space $C^0_0(M)$ of continuous functions vanishing outside a compact set.

The manifold structure allows to define the notion of smoothness.

Definition 1.3. *A function* $\varphi : M \to \mathbb{R}$ *(or* $\mathbb{C}$*) is* **smooth** *if, for all coordinate chart* (U, V, κ)*, the map* $\varphi \circ \kappa^{-1} : V \to \mathbb{R}$ *is smooth.*

Exercise 1.4. *Check that we obtain an equivalent definition if we consider the coordinate charts of an atlas rather than* all *coordinate charts.*

It is straightforward to check that smooth functions on M form a vector space, which we denote by $C^\infty(M)$. One also defines the subspace $C^\infty_0(M)$ of smooth functions vanishing outside a compact set. Note that, if M is compact, $C^\infty(M) = C^\infty_0(M)$.

In the sequel, we shall use the following standard operatorial notation

$$\kappa_*\varphi = \varphi \circ \kappa^{-1}, \qquad \kappa^*\psi = \psi \circ \kappa, \tag{1.1}$$

for functions φ defined on U and functions ψ defined on V. $\kappa^*\psi$ is called the **pullback** of ψ (which is a function on U) and $\kappa_*\varphi$ the **pushforward** of φ. Obviously,

$$\kappa^*\kappa_* = I, \qquad \kappa_*\kappa^* = I,$$

as operators on functions on U and functions on V respectively.

Proposition 1.5 (Partition of unity)**.** *Assume that* M *is a smooth compact manifold and that we are given a finite open cover of* M,

$$M = \bigcup_{i=1}^N W_i, \qquad W_i \text{ open subset of } M.$$

Then there exist $\theta_i \in C_0^\infty(W_i)$, $i = 1, \ldots, N$, such that

$$1 = \sum_{i=1}^{N} \theta_i \qquad \text{on } M.$$

Furthermore, each θ_i can be taken of the form $\theta_i = \varphi_i^2$ for some $\varphi_i \in C_0^\infty(W_i)$.

A useful application of this result is that, when $W_1, \ldots, W_N$ are coordinate patches, each smooth function f on M can be written $\sum_i \theta_i f$, ie as a (finite) sum of functions supported in coordinate patches.

Proof. Since each W_i is open, for any $m \in W_i$ we can choose open subsets U_m^i and $\widetilde{U}_m^i$ contained in a coordinate patch at m such that

$$m \in U_m^i \Subset \widetilde{U}_m^i \subset W_i. \tag{1.2}$$

Then, by compactness, we obtain a finite open cover of M

$$M = \bigcup_{i=1}^{N} \big(U^i_{m^i_1} \cup \cdots \cup U^i_{m^i_{n_i}} \big). \tag{1.3}$$

By pulling back cutoffs on $\mathbb{R}^n$, we can select

$$\phi^i_{m^i_k} \in C_0^\infty\big(\widetilde{U}^i_{m^i_k}\big) \qquad \text{such that} \qquad \phi^i_{m^i_k} = 1 \ \text{ on } U^i_{m^i_k} \qquad \text{and} \qquad \phi^i_{m^i_k} \geq 0 \ \text{ on } M. \tag{1.4}$$

We then introduce

$$\Phi_i = \left(\sum_{k=1}^{n_i} \phi^i_{m^i_k} \right)^2, \qquad \Phi = \sum_{i=1}^{N} \Phi_i.$$

Clearly Φ_i belongs to $C_0^\infty(W_i)$ by (1.2) and (1.4). By (1.4), we also have $\Phi_i \geq 1$ on each $U^i_{m^i_k}$ hence on their union (over k). Therefore (1.3) implies that $\Phi \geq 1$ on M so that we can define

$$\varphi_i = \frac{1}{\sqrt{\Phi}} \sum_{k=1}^{n_i} \phi^i_{m^i_k} \in C_0^\infty(W_i),$$

which obviously satisfies $\sum_{i=1}^{N} \varphi_i^2 = 1$. $\square$

For convenience, we assume in the sequel that $\boxed{M \text{ is compact.}}$

Exercise 1.6. *Show that $C^\infty(M)$ is dense in $C^0(M)$ equipped with the norm $||\varphi||_\infty = \sup_M |\varphi|$.*

Definition 1.7. *A smooth* **volume density** *on M is a non negative Borel measure $d\mu$ such that, for any chart (U, V, κ), there exists a smooth positive function v_κ such that*

$$\int \varphi d\mu = \int_{\kappa(U)} \varphi \circ \kappa^{-1}(x) v_\kappa(x) dx, \tag{1.5}$$

for all $\varphi \in C_0^0(U)$. Here dx stands for the Lebesgue measure on $\kappa(U) \subset \mathbb{R}^n$.

In the sequel, we shall denote

$$\kappa_* d\mu = v_\kappa(x) dx, \tag{1.6}$$

as a short hand for (1.5). Unlike (1.1), this is only a notation since we have not defined the notion of pullback and pushforward for volume densities (and don't wish to do so for the moment).

Example (the 2-sphere. Part 2)**.** According to the notation of Exercise 1.2, 4th question, one defines a volume density on $U \subset \mathbb{S}^2$ by considering

$$\kappa_* d\mu = \sin\phi d\phi d\theta, \qquad \theta \in (0, 2\pi), \ \phi \in (0, \pi), \tag{1.7}$$

with $\kappa(\omega) = (\theta, \phi)$. This is the so called *induced measure*[3] on $\mathbb{S}^2$. We point out that this measure is natural to the extent that it allows to justify

$$\int_{\mathbb{R}^3} f(x, y, z) dxdydz = \int_0^\infty \left(\int_{\mathbb{S}^2} f(r\omega) d\mu(\omega) \right) r^2 dr. \tag{1.8}$$

Indeed, using the diffeomorphism

$$(r, \theta, \phi) \mapsto (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi),$$

which maps $(0, \infty) \times (0, 2\pi) \times (0, \pi)$ onto the complement of a Lebesgue negligible set in $\mathbb{R}^3$ and whose Jacobian is $-r^2 \sin\phi$, one has

$$\int_{\mathbb{R}^3} f(x, y, z) dxdydz = \int_{r=0}^\infty \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} f(r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi) \sin\phi d\phi d\theta r^2 dr,$$

which gives an explicit meaning to (1.8).

Exercise 1.8 (Change of coordinates)**.** *Show that if (U_1, V_1, κ_1) and (U_2, V_2, κ_2) are two charts (on a general manifold M) with non empty $U_1 \cap U_2$, then*

$$v_{\kappa_2} = \left|\det d\big(\kappa_1 \circ \kappa_2^{-1}\big)\right| \, v_{\kappa_1} \circ \big(\kappa_1 \circ \kappa_2^{-1}\big),$$

on $\kappa_2(U_1 \cap U_2)$.

[3]induced by the Lebesgue measure on $\mathbb{R}^3$

Remark. By continuity and positivity of v_κ in (1.5), we have the following useful property: for all $K \Subset U$, there exists $C > 0$ such that

$$C^{-1} \int |\kappa_* \varphi| dx \leq \int |\varphi| d\mu \leq C \int |\kappa_* \varphi| dx, \tag{1.9}$$

for all continuous φ with $\mathrm{supp}(\varphi) \subset K$.

Exercise 1.9. *Show that if $\varphi \in C^0(M)$ and $\int_M |\varphi| d\mu = 0$ then $\varphi \equiv 0$.*

This exercise implies that, for $p \in [1, \infty)$, the map $\varphi \mapsto \left(\int_M |\varphi|^p \right)^{1/p}$ is a norm on $C^0(M)$. This allows to state the following definition.

Definition 1.10. *Fix a smooth volume density $d\mu$. The* **Lebesgue spaces** $L^p(M) := L^p(M, d\mu)$ *are*

$$L^p(M) = \textit{closure of } C^\infty(M) \textit{ for the norm } \left(\int |\varphi|^p d\mu \right)^{1/p},$$

for $1 \leq p < \infty$.

In particular, this will allow us to use the Hilbert space $L^2(M)$ whose inner product will be denoted by $(.,.)_{L^2(M)}$ and satisfies

$$(\varphi, \phi)_{L^2(M)} = \int_M \overline{\varphi} \phi d\mu,$$

at least for $\varphi, \phi \in C^0(M)$. Note the following consequence of (1.9): for all compact subset K of a given coordinate patch U, there exists $C > 0$ such that

$$C^{-1} ||\kappa_* \varphi||_{L^2(\mathbb{R}^n)} \leq ||\varphi||_{L^2(M)} \leq C ||\kappa_* \varphi||_{L^2(\mathbb{R}^n)}, \tag{1.10}$$

for all continuous φ such that $\mathrm{supp}(\varphi) \subset K$.

Exercise 1.11. *Let $u \in L^2(M)$. Let $K \Subset U$ be a compact subset of a coordinate patch U. Show that there exists a unique $u_\kappa \in L^2(\kappa(K), dx)$ such that*

$$(u, \varphi)_{L^2(M)} = \int \overline{u_\kappa(x)} \big(\kappa_* \varphi \big)(x) v_\kappa(x) dx, \qquad \varphi \in C_0^\infty(K).$$

Definition 1.12. *A* **differential operator** *on M is a linear map $P : C^\infty(M) \to C^\infty(M)$ such that*

$$\mathrm{supp}(P\varphi) \subset \mathrm{supp}(\varphi), \qquad \varphi \in C^\infty(M).$$

This is an abstract definition. The following proposition gives the explicit structure of such operators and shows that we recover the usual notion of differential operator on an open subset of $\mathbb{R}^n$. This result is known as the Peetre Theorem.

Proposition 1.13. *Let P be a differential operator on M and (U, V, κ) be a coordinate chart. Then, for all $K \Subset U$ there exist $m \in \mathbb{N}$ and smooth functions $a_\alpha \in C^\infty(\kappa(K))$, $|\alpha| \leq m$, such that*

$$(P\varphi) \circ \kappa^{-1}(x) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha \big(\varphi \circ \kappa^{-1}\big)(x), \tag{1.11}$$

for all $\varphi \in C_0^\infty(K)$.

Note that (1.11) can be written more compactly in operator form as

$$\kappa_* P \kappa^* = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha, \qquad \text{on } C_0^\infty\big(\kappa(K)\big). \tag{1.12}$$

For completeness, we prove Proposition 1.13 in Appendix A, though this proof will play no role in the sequel and might be skipped in the first reading.

Chapter 2

The Weyl law: statement of the problem

In this section, we state the Weyl law for the Laplace Beltrami operator on a compact manifold and also record some related useful results. The detailed proofs will be given later and this part can be seen as a motivation for the semiclassical pseudo-differential calculus which will be investigated in the next sections.

To fully define the Laplace Beltrami operator (or Laplacian) on a manifold, one needs to introduce the definition of a *Riemannian metric* which in turn requires to define objects such as the tangent bundle and tensors. Although they are fundamental in differential geometry, the precise knowledge of these objects is not necessary to state and prove the Weyl law so we will only record the properties and formulas relative to the Laplacian and the Riemannian volume density which we shall need. We refer for instance to [3] for an introduction to Riemannian geometry.

In the sequel, we shall call a **Riemannian manifold** a smooth manifold M of dimension n on which there are a volume density $d\mathrm{vol}_g$ and a differential operator Δ_g with the following properties:

1. Δ_g is **formally selfadjoint** on $C_0^\infty(M)$ **with respect to** $d\mathrm{vol}_g$, ie
$$\int_M \overline{\Delta_g \varphi} \ \phi \ d\mathrm{vol}_g = \int_M \overline{\varphi} \ \Delta_g \phi \ d\mathrm{vol}_g, \qquad \varphi, \phi \in C_0^\infty(M). \tag{2.1}$$

2. Δ_g is **non positive**, ie
$$\int_M \overline{\Delta_g \varphi} \ \varphi \ d\mathrm{vol}_g \leq 0, \qquad \varphi \in C_0^\infty(M). \tag{2.2}$$
Note that, by item 1, the left hand side of (2.2) is real valued.

3. **In local coordinates**: for every coordinate chart (U, V, κ), there exists a $n \times n$ symmetric matrix $(g^{jk}(x))$ with smooth and real valued coefficients on V such that
$$\big(g^{jk}(x)\big)_{1 \leq j,k \leq n} \quad \text{is positive definite at every point } x \in V, \tag{2.3}$$

and, if we set

$$|g(x)| := \det(g^{jk}(x))^{-1/2},$$

then

$$\kappa_* d\mathrm{vol}_g = |g(x)|dx, \tag{2.4}$$

and

$$\begin{aligned}\kappa_* \Delta_g \kappa^* &= \sum_{j,k=1}^{n} |g(x)|^{-1} \frac{\partial}{\partial x_j} \left(|g(x)| g^{jk}(x) \frac{\partial}{\partial x_k} \right) \\ &= \sum_{j,k=1}^{n} g^{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} + |g(x)|^{-1} \frac{\partial}{\partial x_j} \left(|g(x)| g^{jk}(x) \right) \frac{\partial}{\partial x_k}. \end{aligned} \tag{2.5}$$

Exercise 2.1. *Check directly that the operator defined by the right hand side of (2.5) is symmetric with respect to $|g(x)|dx$ on $C_0^\infty(V)$.*

Example (the 2-sphere. Part 3)**.** On the same coordinate patch as in Exercise 1.2 and the example after Definition 1.7, the matrix $(g^{jk}(x))$ is defined by

$$\begin{pmatrix} 1 & 0 \\ 0 & \left(\sin\phi\right)^{-2} \end{pmatrix},$$

hence

$$\begin{aligned}\kappa_* d\mathrm{vol}_g &= \sin\phi d\phi d\theta, \\ \kappa_* \Delta_g \kappa^* &= \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin^2\phi} \frac{\partial^2}{\partial \theta^2} + \frac{\cos\phi}{\sin\phi} \frac{\partial}{\partial \phi}. \end{aligned}$$

Note in particular that $\kappa_* d\mathrm{vol}_g$ is nothing but (1.7).

Theorem 2.2 (Diagonalization of Δ_g)**.** *Let M be a compact Riemannian manifold. There exists an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $L^2(M) = L^2(M, d\mathrm{vol}_g)$ of C^∞ functions such that*

$$-\Delta_g e_j = \lambda_j e_j,$$

with

$$0 \le \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots, \qquad \lim_{j\to\infty} \lambda_j = +\infty.$$

Note that once the existence of smooth eigenfunctions is established, it is clear that the eigenvalues must be nonnegative since, by (2.2),

$$\lambda_j = (-\Delta_g e_j, e_j)_{L^2(M)} \ge 0.$$

Proof of Theorem 2.2. See Section 7.2.

The functions e_j are called the **eigenfunctions** of Δ_g and the real numbers λ_j the **eigenvalues** of $-\Delta_g$. One can then define the **couting function** of eigenvalues by

$$N(\lambda) := \#\{j \in \mathbb{N} \mid \lambda_j \leq \lambda\},$$

whose asymptotic behaviour is the purpose of the Weyl law. Note that $N(\lambda)$ is finite for each $\lambda \in \mathbb{R}$ since λ_j goes to infinity as $j \to \infty$.

Theorem 2.3 (Weyl law)**.** *If M is a compact Riemannian manifold of dimension n, then*

$$N(\lambda) \sim (2\pi)^{-n} \omega_n \mathrm{vol}_g(M) \lambda^{n/2}, \qquad \lambda \to +\infty,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $\mathrm{vol}_g(M) = \int_M d\mathrm{vol}_g$.

In the rest of this section, we introduce the *functional calculus* associated to Δ_g and explain its role in the proof of Theorem 2.3. This will in particular motivate the analysis of functions of semiclassical operators which will be studied later.

Let us denote by $\mathcal{B}(\mathbb{R})$ the algebra of bounded Borel functions on $\mathbb{R}$ (actually, piecewise continuous will be sufficient here). For any $u \in L^2(M)$, which can be uniquely written as

$$u = \sum_{j \in \mathbb{N}} u_j e_j, \qquad u_j = (e_j, u)_{L^2(M)},$$

with convergence in $L^2(M)$, and any $f \in \mathcal{B}(\mathbb{R})$, we set

$$f(-\Delta_g) u := \sum_{j \in \mathbb{N}} f(\lambda_j) u_j e_j. \tag{2.6}$$

Obviously this defines an element in $L^2(M)$ since it is a sum of orthogonal terms such that

$$\sum_j |f(\lambda_j) u_j|^2 \leq \sup_{\mathbb{R}} |f|^2 \sum_j |u_j|^2 = \big(\sup |f|\big)^2 ||u||^2_{L^2(M)} < \infty. \tag{2.7}$$

It is also easy to check that the map $u \mapsto f(-\Delta_g)u$ is linear on $L^2(M)$ and continuous since, by (2.7),

$$||f(-\Delta_g)||_{L^2(M) \to L^2(M)} \leq \sup_{\mathbb{R}} |f|. \tag{2.8}$$

In the following proposition, $\mathcal{L}(L^2(M))$ denotes the algebra of bounded operators on $L^2(M)$.

Proposition 2.4. *The map*

$$\mathcal{B}(\mathbb{R}) \ni f \mapsto f(-\Delta_g) \in \mathcal{L}(L^2(M)) \tag{2.9}$$

is a continuous morphism of algebras. In particular

$$f_1(-\Delta_g) f_2(-\Delta_g) = (f_1 f_2)(-\Delta_g),$$

for all $f_1, f_2 \in \mathcal{B}(\mathbb{R})$. *Furthermore,*

$$f(-\Delta_g)^* = \overline{f}(-\Delta_g),$$

for all $f \in \mathcal{B}(\mathbb{R})$. *In particular, if* f *is real valued then* $f(-\Delta_g)$ *is selfadjoint. Finally*

$$f \geq 0 \qquad \Longrightarrow \qquad f(-\Delta_g) \geq 0. \tag{2.10}$$

Proof. Left to the reader as an exercise. □

The morphism (2.9) is usually called the **functional calculus** of the Laplacian. Of course, it has nothing to do with the particular structure of the operator Δ_g and only uses that it can be diagonalized in a orthonormal basis with real eigenvalues.

The interest of the functional calculus in proof of the Weyl law is the following. Let $\mathbb{1}_{[0,1]}$ be the characteristic function of $[0,1]$. Then, for all $\lambda > 0$, $\mathbb{1}_{[0,1]}(-\Delta_g/\lambda)$ is a finite rank (selfadjoint) operator. According to 2.6, it is given by

$$\mathbb{1}_{[0,1]}(-\Delta_g/\lambda)u = \sum_{\lambda_j \in [0,\lambda]} (e_j, u)_{L^2(M)} e_j.$$

The key observation is that

$$N(\lambda) = \mathrm{tr}\big(\mathbb{1}_{[0,1]}(-\Delta_g/\lambda)\big), \tag{2.11}$$

which follows from

$$\mathrm{tr}\big(\mathbb{1}_{[0,1]}(-\Delta_g/\lambda)\big) = \sum_{j \in \mathbb{N}} \mathbb{1}_{[0,1]}(\lambda_j/\lambda) = \sum_{\lambda_j \in [0,\lambda]} 1. \tag{2.12}$$

Here tr denotes the **trace** of the operator which, in the present situation where the operator has finite rank and is selfadjoint, is the well defined sum of its (non zero) eigenvalues. We shall recall the notion of trace for operators in infinite dimension in Section 4.2 and see that the present formula coincides with the general definition (see Definition 4.11).

By (2.11), the proof of the Weyl law will be reduced to the computation of trace asymptotics of functions of Δ_g. Analyzing the orthogonal projection $\mathbb{1}_{[0,1]}(-\Delta_g/\lambda)$ is a difficult question but, as we shall see, studying *smooth* functions of Δ_g will be sufficient. We shall prove the following result.

Theorem 2.5. *Let* $f \in C_0^\infty(\mathbb{R})$. *Then*

$$\lim_{\lambda \to +\infty} \lambda^{-n/2} \mathrm{tr}\big(f(-\Delta_g/\lambda)\big) = (2\pi)^{-n} \mathrm{vol}_g(M) \int_{\mathbb{R}^n} f(|\eta|^2) d\eta.$$

Proof. See Section 7.3.

Using this theorem, it is now easy to prove Theorem 2.3 by a classical monotonicity argument.

Proof of Theorem 2.3. Fix $f_-, f_+ \in C_0^\infty(\mathbb{R})$ such that

$$f_- \leq \mathbb{1}_{[0,1]} \leq f_+. \tag{2.13}$$

By (2.10) and (2.11), we have

$$\operatorname{tr}\big(f_-(-\Delta_g/\lambda)\big) \leq N(\lambda) \leq \operatorname{tr}\big(f_+(-\Delta_g/\lambda)\big). \tag{2.14}$$

This can also be checked directly by remarking that the eigenvalues of the operators $\big(f_+ - \mathbb{1}_{[0,1]}\big)(-\Delta_g/\lambda)$ and $\big(\mathbb{1}_{[0,1]} - f_-\big)(-\Delta_g/\lambda)$ are non negative. Multiplying (2.14) by $\lambda^{-n/2}$ and then taking the liminf in the first inequality and the limsup in the second one, Theorem 2.5 implies that

$$C_M \int f_-(|\eta|^2)d\eta \leq \liminf_{\lambda\to\infty} \lambda^{-n/2}N(\lambda) \leq \limsup_{\lambda\to\infty} \lambda^{-n/2}N(\lambda) \leq C_M \int f_+(|\eta|^2)d\eta, \tag{2.15}$$

where $C_M = (2\pi)^{-n}\mathrm{vol}_g(M)$. By a standard approximation procedure whose details are left to the reader, we can select sequences $f_{-,k}, f_{+,k} \in C_0^\infty(\mathbb{R})$ vanishing outside a fixed compact neighborhood of $[0,1]$ (*e.g.* $[-1,2]$), uniformly bounded, satisfying (2.13) and such that

$$\lim_{k\to\infty} f_{-,k} = \mathbb{1}_{(0,1)}, \qquad \lim_{k\to\infty} f_{+,k} = \mathbb{1}_{[0,1]}.$$

Since the sphere $\{|\eta| = 1\}$ has zero Lebesgue measure, we note that both limits of $f_{-,k}(|\eta|^2)$ and $f_{+,k}(|\eta|^2)$ coincide with the characteristic function of the unit ball almost everywhere. By using (2.15) for $f_{-,k}, f_{+,k}$ and then by letting k go to infinity, we obtain

$$\liminf_{\lambda\to\infty} \lambda^{-n/2}N(\lambda) = \limsup_{\lambda\to\infty} \lambda^{-n/2}N(\lambda) = C_M \int_{|\eta|<1} d\eta,$$

which completes the proof. $\square$

Chapter 3

Pseudodifferential calculus

3.1 The Fourier transform

In this short section, we briefly review basic facts about the Fourier transform. We fix in particular some notation and conventions (*e.g.* on the definition of the Fourier transform) but also introduce the notion of *seminorms* which will be of constant use in this chapter.

In the sequel, we will extensively use the **japanese bracket** $\langle x \rangle$ defined by

$$\langle x \rangle = (1 + |x|^2)^{1/2},$$

for $x \in \mathbb{R}^n$.

Definition 3.1. *The* **Schwartz space** $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ *is the set of smooth functions* $\mathbb{R}^n \to \mathbb{C}$ *such that, for all* $\alpha \in \mathbb{N}^n$ *and all* $N \geq 0$,

$$|\partial_x^\alpha u(x)| \leq C_{\alpha N} \langle x \rangle^{-N}, \qquad x \in \mathbb{R}^n.$$

Exercise 3.2. *Prove that* $u \in \mathcal{S}$ *if and only if, for all* $\alpha, \beta \in \mathbb{N}^n$,

$$|x^\beta \partial_x^\alpha u(x)| \leq C_{\alpha\beta}.$$

It is convenient to introduce the following so called **seminorms**

$$\mathcal{N}_N^{\mathcal{S}}(u) := \max_{|\alpha| \leq N} \sup_{\mathbb{R}^n} \langle x \rangle^N |\partial_x^\alpha u(x)|,$$

for $N \geq 0$ and $u \in \mathcal{S}$. They are actually norms on $\mathcal{S}$, however the natural topology of $\mathcal{S}$ is not given by a single norm but by the whole family of (semi)norms $\big(\mathcal{N}_N^{\mathcal{S}}\big)_N$. It will not be necessary to describe in detail this topology (so called *Fréchet space* topology) but we will need many estimates involving such seminorms which is the reason why we introduce them.

Seminorms allow to define **convergent sequences** in $\mathcal{S}$ as follows: given $u \in \mathcal{S}$ and (u_j) a sequence in $\mathcal{S}$,

$$u_j \to u \ \text{ in } \mathcal{S} \qquad \overset{\text{def}}{\Longleftrightarrow} \qquad \text{for all } N \geq 0, \ \ \mathcal{N}_N^{\mathcal{S}}\big(u_j - u\big) \to 0.$$

Definition 3.3. *For $u \in \mathcal{S}$, the* **Fourier transform** *of u is*

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi} u(y) dy, \qquad \xi \in \mathbb{R}^n.$$

In the following proposition, we give a first illustration of the use of seminorms in continuity issues on the Schwartz space.

Proposition 3.4. *The map $u \mapsto \hat{u}$ is linear from $\mathcal{S}$ to $\mathcal{S}$ and continuous in the following sense: for all $N \geq 0$ there exists $C > 0$ and $M > 0$ such that*

$$\mathcal{N}_N^{\mathcal{S}}(\hat{u}) \leq C \mathcal{N}_M^{\mathcal{S}}(u), \qquad u \in \mathcal{S}.$$

Proof. The smoothness of $\hat{u}$ follows by standard differentiation under the integral sign. Furthermore

$$\begin{aligned} \xi^\alpha \partial_\xi^\beta \hat{u}(\xi) &= i^{|\alpha|-|\beta|} \int_{\mathbb{R}^d} \partial_y^\alpha \big(e^{-iy\cdot\xi}\big) y^\beta u(y) dy \\ &= (-i)^{|\alpha|+|\beta|} \int_{\mathbb{R}^d} e^{-iy\cdot\xi} \partial_y^\alpha \big(y^\beta u(y)\big) dy \end{aligned}$$

the second line being obtained by integrations by part. Using that, for some M and C depending on α and β,

$$\langle y \rangle^{d+1} |\partial_y^\alpha \big(y^\beta u(y)\big)| \leq C \mathcal{N}_M^{\mathcal{S}}(u),$$

we obtain

$$|\xi^\alpha| \big|\partial_\xi^\beta \hat{u}(\xi)\big| \leq C \mathcal{N}_M^{\mathcal{S}}(u), \qquad u \in \mathcal{S}.$$

By remarking that, for each integer $N \geq 0$,

$$\langle \xi \rangle^N \leq C_N \sum_{|\alpha| \leq N} |\xi^\alpha|, \qquad \xi \in \mathbb{R}^d,$$

the result follows. $\square$

Exercise 3.5. *Using Proposition 3.4, check that if $u_j \to u$ in $\mathcal{S}$ then $\hat{u}_j \to \hat{u}$ in $\mathcal{S}$.*

We recall without proof the following **Fourier inversion formula**

$$u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \hat{u}(\xi) d\xi, \tag{3.1}$$

for all $u \in \mathcal{S}$ and $x \in \mathbb{R}^n$. Using this formula, we obtain the following identities which will motivate the definition of pseudo-differential operators in the next section. By differentiation under the integral sign, we have

$$\partial_j u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} i\xi_j \hat{u}(\xi) d\xi, \tag{3.2}$$

meaning that, up the factor i, differentiation with respect to x_j corresponds to multiplication by ξ_j on the Fourier side. To deal more easily with the i factor, one introduces

$$D_j = \frac{1}{i}\partial_j, \qquad D = (D_1, \dots, D_n), \tag{3.3}$$

so that a more general form of (3.2) is

$$D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \xi^\alpha \hat{u}(\xi) d\xi. \tag{3.4}$$

3.2 Definition of pseudo-differential operators

To motivate the definition of pseudo-differential operators, we consider first a differential operator

$$P = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \tag{3.5}$$

which we write in term of D^α (recall (3.3)) rather than ∂^α. If we set

$$p(x,\xi) := \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha, \tag{3.6}$$

which is a polynomial in ξ with x dependent coefficients, then (3.4) implies that

$$\begin{aligned} Pu(x) &= \sum_{|\alpha| \le m} a_\alpha(x) (2\pi)^{-n} \int e^{ix\cdot\xi} \xi^\alpha \hat{u}(\xi) d\xi, \\ &= \sum_{|\alpha| \le m} (2\pi)^{-n} \int e^{ix\cdot\xi} a_\alpha(x) \xi^\alpha \hat{u}(\xi) d\xi, \end{aligned}$$

that is

$$Pu(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} p(x,\xi) \hat{u}(\xi) d\xi. \tag{3.7}$$

The function p is called the **symbol** of the operator P. Pseudo-differential operators are a generalization of differential operators in that they are defined by symbols which are non necessarily polynomials with respect to ξ. Let us introduce the symbols we shall consider.

Definition 3.6. *For $m, \mu \in \mathbb{R}$, the space $S^{\mu,m} = S^{\mu,m}(\mathbb{R}^n \times \mathbb{R}^n)$ is the set of smooth functions $a : \mathbb{R}^{2n} \to \mathbb{C}$ such that, for all $\alpha, \beta \in \mathbb{N}^n$,*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle x \rangle^\mu \langle \xi \rangle^{m-|\beta|}, \qquad x, \xi \in \mathbb{R}^n.$$

When $\mu = 0$, we use the standard notation

$$S^m := S^{0,m}.$$

Exercise 3.7. *1. Show that, in (3.6), $p \in S^m$ iff the functions a_α are bounded and all their derivatives are bounded.*

2. Let f be a smooth function non vanishing on $\mathbb{R}^d$. Show that, for $\gamma \neq 0$,

$$\partial^\gamma \left(\frac{1}{f} \right) = \text{ linear combination of } \frac{\partial^{\gamma_1} f \cdots \partial^{\gamma_j} f}{f^{1+j}},$$

with $1 \leq j \leq |\gamma|$, $\gamma_1 + \cdots + \gamma_j = \gamma$ and $\gamma_1, \ldots, \gamma_j \neq 0$.

3. Show that if $a \in S^m$ (with $m \in \mathbb{R}$) satisfies $|a(x,\xi)| \geq c\langle \xi \rangle^m$ for some $c > 0$ (one says that a is **elliptic***) then $1/a \in S^{-m}$.*

We can now define pseudo-differential operators.

Definition 3.8. *Given $u \in \mathcal{S}$ and $a \in S^{\mu,m}$, we set*

$$a(x,D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi) \hat{u}(\xi) d\xi.$$

The operator $a(x,D)$ is called the **pseudo-differential operator** *of symbol a and will also be denoted by*

$$Op(a) := a(x,D).$$

Examples. 1) By (3.7), differential operators are pseudo-differential ones with symbols which are polynomial with respect to ξ.
2) The Fourier inversion formula reads

$$Op(1) = I. \tag{3.8}$$

3) If $z \in \mathbb{C} \setminus [0,+\infty)$ and if we set $a_z(\xi) = \dfrac{1}{|\xi|^2 - z}$, then $a_z \in S^{-2}$ (see Exercise 3.7) and, by (3.7),

$$(-\Delta - z) Op(a_z) = I,$$

which means (at least formally) that the inverse of $-\Delta - z$ is a pseudo-differential operator.

The third example is an important motivation for the introduction of pseudo-differential operators for it shows that they are convenient tools to analyze (and construct) the inverse of an elliptic operator (ie an operator with an elliptic symbol, in the sense of item 3 of Exercise 3.7). However, this example is of very special nature since it deals with constant coefficients operators (ie x independent). In this case, finding an inverse is an easy task. We shall see in Chapter 6 how pseudo-differential operators allow to analyze the inverse of general (x dependent) elliptic operators, including operators on manifolds.

To give a first flavour of what happens for operators with variable coefficients, we suggest the reader to work out the following exercise.

Exercise 3.9. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be smooth and such that $\partial^\alpha V$ is bounded for all α. Set*

$$p(x,\xi) = |\xi|^2 + V(x) + i, \qquad a(x,\xi) = \big(|\xi|^2 + V(x) + i\big)^{-1}.$$

1. *Show that $p \in S^2$, $a \in S^{-2}$.*
2. *Check that $p(x,D) = -\Delta + V(x) + i$.*
3. *Show that there exists $r \in S^{-1}$ such that, for all $u \in \mathcal{S}$,*

$$p(x,D)a(x,D)u = u + r(x,D)u.$$

The last question shows that, in general, $Op(p)Op(1/p)$ is not the identity, which only happens for constant coefficients operators. There is a corrective term $Op(r)$. We shall see in the sequel how the pseudo-differential calculus allows to deal with such remainders and is a more robust tool for PDEs than the pure Fourier analysis.

Before entering the core of the subject in the next section, it remains to define **semiclassical** pseudo-differential operators.

Definition 3.10 (Semiclassical operators)**.** *For $a \in S^{\mu,m}$ and $h \in (0,1]$, we set*

$$Op_h(a) = a(x,hD) := Op(a_h)$$

with $a_h(x,\xi) := a(x,h\xi)$.

The interest of this definition, compared to Definition 3.8, will be clearer below when we shall control important estimates with respect to h. However, we can already motivate its introduction as follows. Consider a general differential operator P as in (3.5) and split its symbol (3.6) into homogeneous pieces

$$p = \sum_{j=0}^{m} p_{m-j}, \qquad p_{m-j}(x,\xi) = \sum_{|\alpha|=m-j} a_\alpha(x)\xi^\alpha.$$

Obviously, we have $p_{m-j}(x,h\xi) = h^{m-j}p_{m-j}(x,\xi)$ hence

$$h^m p(x,\xi) = \sum_{j=0}^{m} h^j p_{m-j}(x,h\xi),$$

which implies that

$$h^m P = \sum_{j=0}^{m} h^j Op_h(p_{m-j}). \tag{3.9}$$

Now recall that one of our main goals is to prove Theorem 2.5 which involves the operator Δ_g/λ where λ is large. By setting $h^2 = 1/\lambda$, we see from (3.9) (and the expression (2.5) of Δ_g in local coordinates) that

$$\Delta_g/\lambda = h^2\Delta_g,$$

is, in local coordinates, a sum of semiclassical operators.

3.3 Symbolic calculus

In this section, we will see that pseudo-differential operators form a class which is stable under composition and adjunction. We work in the semiclassical setting for two reasons: it will be important for the final application to the Weyl law and it covers the 'standard' calculus corresponding to $h = 1$.

As we did for the Schwartz space in Section 3.1, it is important to introduce suitable seminorms on $S^{\mu,m}$. We will use

$$\mathcal{N}_N^{S^{\mu,m}}(a) = \max_{|\alpha+\beta|\leq N} \sup_{\mathbb{R}^{2n}} \langle x\rangle^{-\mu}\langle\xi\rangle^{-m+|\beta|}\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|.$$

When there is no confusion, we shall drop the dependence on $S^{m,\mu}$ from the notation and write $\mathcal{N}_N$ for $\mathcal{N}_N^{S^{m,\mu}}$.

Exercise 3.11. *Let m_1, m_2, μ_1, μ_2 be real numbers such that $m_1 \leq m_2$ and $\mu_1 \leq \mu_2$.*

1. *Show that $S^{\mu_1,m_1} \subset S^{\mu_2,m_2}$.*
2. *Show that this embedding is continuous ie that, for all $N_2 \in \mathbb{N}$, there exist C and N_1 such that*

$$\mathcal{N}_{N_2}^{S^{\mu_2,m_2}}(a) \leq C\mathcal{N}_{N_1}^{S^{\mu_1,m_1}}(a),$$

for all $a \in S^{\mu_1,m_1}$.

Exercise 3.12. *Let m_1, m_2, μ_1, μ_2 be real numbers.*

1. *Show that if $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$ then $ab \in S^{\mu_1+\mu_2,m_1+m_2}$.*
2. *Show that the map $(a,b) \mapsto ab$ is continuous ie that, for all $N \in \mathbb{N}$, there exist C and N_1, N_2 such that*

$$\mathcal{N}_N^{S^{\mu_1+\mu_2,m_1+m_2}}(ab) \leq C\mathcal{N}_{N_1}^{S^{\mu_1,m_1}}(a)\mathcal{N}_{N_2}^{S^{\mu_2,m_2}}(b),$$

for all $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$.

3. *Show similarly that, for fixed $\alpha, \beta \in \mathbb{N}^n$, the map $a \mapsto \partial_x^\alpha\partial_\xi^\beta a$ is continuous from S^{μ_1,m_1} to $S^{\mu_1,m_1-|\beta|}$.*

The following proposition mainly states that pseudo-differential operators are (continuous) linear maps on $\mathcal{S}$.

Proposition 3.13. *For all $h \in (0,1]$, all $a \in S^{\mu,m}$, $Op_h(a)$ maps $\mathcal{S}$ into itself. Actually, for fixed h, the bilinear map*

$$(a,u) \mapsto Op_h(a)u$$

is continuous from $S^{\mu,m} \times \mathcal{S}$ to $\mathcal{S}$.

Proof. See Proposition 3.22.

This proposition is useful for it shows for instance that one can compose two pseudo-differential operators, as endomorphisms on the Schwartz space. The continuity property will be used to define the action of pseudo-differential operators on temperate distributions by duality (see Definition 3.18). However, we already point out that the most important mapping properties of pseudo-differential operators for the applications concern L^2 or Sobolev spaces rather than the Schwartz space. This will be investigated in Section 5.1.

We now state the main results of this chapter, which will be refered to as **symbolic calculus**, and which describe the composition and the adjoint of pseudo-differential operators.

Theorem 3.14 (composition)**.** *If $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$, then*

$$Op_h(a)Op_h(b) = Op_h\big((a\#b)(h)\big)$$

where, if we set

$$(a\#b)_j = \sum_{|\alpha|=j} \frac{1}{\alpha!}\partial_\xi^\alpha a D_x^\alpha b,$$

we have for all J,

$$(a\#b)(h) = \sum_{j<J} h^j (a\#b)_j + h^J r_J^\#(a,b,h),$$

for some

$$r_J^\#(a,b,h) \in S^{\mu_1+\mu_2,m_1+m_2-J}.$$

More precisely, the map $(a,b) \mapsto r_J^\#(a,b,h)$ is bilinear and equicontinuous in the following sense: for all seminorm $\mathcal{N}_N$ in $S^{\mu_1+\mu_2,m_1+m_2-J}$, there exist $C > 0$ and seminorms $\mathcal{N}_{N_1}^{S^{\mu_1,m_1}}$ and $\mathcal{N}_{N_2}^{S^{\mu_2,m_2}}$ such that

$$\mathcal{N}_N(r_J^\#(a,b,h)) \leq C\mathcal{N}_{N_1}^{S^{\mu_1,m_1}}(a)\mathcal{N}_{N_2}^{S^{\mu_2,m_2}}(b),$$

for all $a \in S^{\mu_1,m_1}$, all $b \in S^{\mu_2,m_2}$ and $h \in (0,1]$.

Proof. Consequence of Proposition 3.30.

The equicontinuity is the fact that the constant C and the seminorms are independent of h. We also note that it is straightforward to check the continuity of the bilinear map

$$(a,b) \mapsto (a\#b)_j \in S^{\mu_1+\mu_2,m_1+m_2-j}$$

on $S^{\mu_1,m_1} \times S^{\mu_2,m_2}$, by using Exercise 3.12. In particular, the map

$$(a,b) \mapsto (a\#b)(h)$$

is equicontinuous.

Exercise 3.15. *1. Check that if a and b are polynomial in ξ, this theorem follows directly from the Leibniz rule.*

2. Show that if a and b do not depend on x, then $Op_h(a)Op_h(b) = Op_h(ab)$.

We next consider the adjoint. To this end, we denote

$$(u, v) := \int_{\mathbb{R}^n} \overline{u(x)} v(x) dx, \qquad u, v \in \mathcal{S}, \tag{3.10}$$

which is the inner product of $L^2(\mathbb{R}^n)$, but restricted to Schwartz functions.

Theorem 3.16 (adjoint). *For all $a \in S^{\mu,m}$ and $u, v \in \mathcal{S}$, we can write*

$$\big(u, Op_h(a)v\big) = \big(Op_h(a^*(h))u, v\big)$$

where, if we set

$$a_j^* := \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{a},$$

we have for all J,

$$a^*(h) = \sum_{j<J} h^j a_j^* + h^J r_J^*(a, h)$$

for some

$$r_J^*(a, h) \in S^{\mu, m-J}.$$

More precisely, the map $a \mapsto r_J^*(a, h)$ is antilinear and equicontinuous in the sense that, for all seminorm $\mathcal{N}_N$ in $S^{\mu,m-J}$, there exist $C > 0$ and a seminorm $\mathcal{N}_{N_1}^{S^{\mu,m}}$ such that

$$\mathcal{N}_N(r_J^*(a, h)) \leq C \mathcal{N}_{N_1}^{S^{\mu,m}}(a), \tag{3.11}$$

for all $a \in S^{\mu,m}$ and $h \in (0, 1]$.

Proof. Consequence of Proposition 3.25.

Similarly to Theorem 3.14, here again the symbols of the expansion depend continuously on a: for all j, the map

$$a \mapsto a_j^* \in S^{\mu, m-j}$$

is antilinear and continuous on $S^{\mu,m}$. This follows from Exercise 3.12.

In a formal way, Theorem 3.16 means that

$$Op_h(a)^* = Op_h(a^*(h)). \tag{3.12}$$

This is formal since the adjoint on the left hand side has not been properly defined. Nevertheless, we will use freely this notation in the sequel (to be completely rigorous one may consider (3.12) as a definition of $Op_h(a)^*$ and then Theorem 3.16 states that this definition is compatible with the usual definition of an adjoint).

Exercise 3.17. *1. If a is polynomial in ξ, check that Theorem 3.16 is a direct consequence of the Leibniz formula.*

2. If a does not depend on x, show that

$$\big(u, Op_h(a)v\big) = \big(Op_h(\overline{a})u, v\big), \qquad u, v \in \mathcal{S}.$$

Remark. Both Theorem 3.14 and 3.16 have been stated in the semi-classical framework, but they hold in particular for $h = 1$.

One useful application of Proposition 3.13 and Theorem 3.16 is to allow to define the action of a pseudo-differential operator on a temperate distribution.

Let us denote by $\langle ., . \rangle_{\mathcal{S}', \mathcal{S}}$ the bilinear pairing between $\mathcal{S}'$ and $\mathcal{S}$. For a temperate distribution $u \in \mathcal{S}'$ and a Schwartz function $\mathcal{S}$, we then set

$$(u, \varphi) := \overline{\langle u, \overline{\varphi} \rangle}_{\mathcal{S}', \mathcal{S}}. \tag{3.13}$$

This extends the definition of (3.10) to the case when u is a distribution (recall that if u is a function, then $\langle u, v \rangle_{\mathcal{S}', \mathcal{S}} = \int uv$).

Definition 3.18. *If $u \in \mathcal{S}'$ and $a \in S^{\mu,m}$, one defines $Op_h(a)u$ to be the temperate distribution given by*

$$\big(Op_h(a)u, \varphi\big) := \big(u, Op_h(a^*(h))\varphi\big), \qquad \varphi \in \mathcal{S}.$$

We note that this definition makes sense, ie indeed defines a temperate distribution, since

$$\begin{aligned} \big|\big(Op_h(a)u, \varphi\big)\big| \;=\; \big|\big(u, Op_h(a^*(h))\varphi\big)\big| \;&\leq\; C_u \mathcal{N}^{\mathcal{S}}_{N_u}\left(Op_h(a^*(h))\varphi\right) \\ &\leq\; C'_{u,h} \mathcal{N}^{\mathcal{S}}_{N'_{u,h}}(\varphi) \end{aligned}$$

where the first inequality follows from the fact that $u \in \mathcal{S}'$ and the second one from Proposition 3.13.

Exercise 3.19. *Check that if $u \in \mathcal{S}$ then Definition 3.18 coincides with the previous definition of a pseudo-differential operator acting on $\mathcal{S}$.*

3.4 Proofs

We shall derive the results of Section 3.3 from a more general framework described in this section whose main results are Propositions 3.25 and 3.30. The interest of these propositions is to give sufficiently explicit formulas for the remainders in the symbolic calculus theorems. They allow not only to prove easily Theorems 3.14 and 3.16, but also to extend those theorems to other classes of symbols.

Definition 3.20. *Let $\nu \in \mathbb{R}$. The space $\mathcal{A}^\nu = \mathcal{A}^\nu(\mathbb{R}^{2n})$ is the space of smooth functions $\mathbb{R}^{2n} \to \mathbb{C}$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}(1+|x|+|\xi|)^\nu,$$

or equivalently such that the following (semi)norms are finite for all N

$$\mathcal{N}_N^{\mathcal{A}^\nu}(a) := \max_{|\alpha|+|\beta|\le N} \sup_{\mathbb{R}^{2n}} (1+|x|+|\xi|)^{-\nu} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|.$$

As before, the seminorms allow to define the notion of convergent sequence by

$$a_j \to a \ \text{ in } \mathcal{A}^\nu \qquad \overset{\text{def}}{\Longleftrightarrow} \qquad \text{for all } N \ge 0, \ \ \mathcal{N}_N^{\mathcal{A}^\nu}\big(a_j - a\big) \to 0.$$

The classes $\mathcal{A}^\nu$ contain the classes $S^{\mu,m}$ in the following sense.

Proposition 3.21. *Fix $m, \mu \in \mathbb{R}$. Then*

$$S^{\mu,m} \subset \mathcal{A}^{|\mu|+|m|},$$

continuously in the following sense: for all $N \ge 0$, there exist $C > 0$ and $M > 0$ such that

$$\mathcal{N}_N^{\mathcal{A}^{|\mu|+|m|}}(a) \le C \mathcal{N}_M^{S^{\mu,m}}(a),$$

for all $a \in S^{\mu,m}$.

Proof. It simply follows from the fact that

$$\langle x\rangle^\mu \langle \xi\rangle^m \le C(1+|x|+|\xi|)^{|\mu|+|m|}$$

and thus

$$(1+|x|+|\xi|)^{-|\mu|-|m|}|a(x,\xi)| \le C\langle x\rangle^{-\mu}\langle\xi\rangle^{-m}|a(x,\xi)|,$$

which implies the estimate with $N = M = 0$. Higher order seminorms are treated similarly using that $\langle\xi\rangle^{m-|\beta|} \le \langle\xi\rangle^m$. ☐

Conformally to Definition 3.8, we still denote

$$Op(a)u = a(x,D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi,$$

for $u \in \mathcal{S}$ and $a \in \mathcal{A}^\nu$.

Proposition 3.22. *The map $(a,u) \mapsto Op(a)u$ is bilinear from $\mathcal{A}^\nu \times \mathcal{S}$ to $\mathcal{S}$ and continuous in the sense that, for all $N \ge 0$, there exist $C > 0$ and $M, M' \ge 0$ such that*

$$\mathcal{N}_N^{\mathcal{S}}(a(x,D)u) \le C\mathcal{N}_M^{\mathcal{A}^\nu}(a)\mathcal{N}_{M'}^{\mathcal{S}}(u),$$

for all $u \in \mathcal{S}$ and all $a \in \mathcal{A}^\nu$.

Proof. The smoothness of $a(x,D)u$ follows by differentiation under the integral sign. By integration by part and the Leibniz rule, we have

$$\begin{aligned} x^\alpha \partial_x^\beta a(x,D)u &= \sum_{\gamma\le\beta} C_\beta^\gamma i^{|\gamma|-|\alpha|} \int_{\mathbb{R}^n} \partial_\xi^\alpha \big(e^{ix\cdot\xi}\big)(\partial_x^{\beta-\gamma}a)(x,\xi)\xi^\gamma \hat{u}(\xi)d\xi \\ &= \sum_{\gamma\le\beta} C_\beta^\gamma i^{|\gamma|+|\alpha|} \sum_{\delta\le\alpha} C_\alpha^\delta \int_{\mathbb{R}^n} e^{ix\cdot\xi}(\partial_x^{\beta-\gamma}\partial_\xi^{\alpha-\delta}a)(x,\xi)\partial_\xi^\delta\big(\xi^\gamma\hat{u}(\xi)\big)d\xi. \end{aligned}$$

Now, using the rough estimate

$$(1+|x|+|\xi|)^\nu \le C\langle x\rangle^{|\nu|}\langle\xi\rangle^{|\nu|},$$

we observe that for some M and C depending on α,β, we have

$$\big|\partial_x^{\beta-\gamma}\partial_\xi^{\alpha-\delta}a(x,\xi)\big| \le C\langle x\rangle^{|\nu|}\langle\xi\rangle^{|\nu|}\mathcal{N}_M^{\mathcal{A}^\nu}(a), \tag{3.14}$$

for all $a\in\mathcal{A}^\nu$ and all $\gamma\le\beta$, $\delta\le\alpha$. On the other hand, by Proposition 3.4, there exist C and M' such that

$$|\partial_\xi^\delta\big(\xi^\gamma\hat{u}(\xi)\big)| \le C\langle\xi\rangle^{-n-1-|\nu|}\mathcal{N}_{M'}^{\mathcal{S}}(u), \tag{3.15}$$

for all $u\in\mathcal{S}$ and all $\gamma\le\beta$, $\delta\le\alpha$. Using (3.14) and (3.15), we obtain

$$|x^\alpha|\big|\partial_x^\beta a(x,D)u\big| \le C\langle x\rangle^{|\nu|}\mathcal{N}_M^{\mathcal{A}^\nu}(a)\mathcal{N}_{M'}^{\mathcal{S}}(u),$$

which, as in end of the proof of Proposition 3.4, implies that, for each integer N and each multiindex β, there are C,M,M' such that

$$\langle x\rangle^{N-|\nu|}\big|\partial_x^\beta a(x,D)u\big| \le C\mathcal{N}_M^{\mathcal{A}^\nu}(a)\mathcal{N}_{M'}^{\mathcal{S}}(u).$$

Since this is true for all N, the same estimate holds with $\langle x\rangle^N$ instead of $\langle x\rangle^{N-|\nu|}$ in the left hand side, after the possible replacement of C, M and M' by larger values. This completes the proof. □

Exercise 3.23. *Check, using Proposition 3.22, that if $a_j\to a$ in $\mathcal{A}^\nu$ and $u_j\to u$ in $\mathcal{S}$, then $Op(a_j)u_j\to Op(a)u$ in $\mathcal{S}$.*

We now state a very useful lemma saying that any symbol in $\mathcal{A}^\nu$ can be approached by a sequence in $C_0^\infty(\mathbb{R}^{2n})$ which is bounded in $\mathcal{A}^\nu$ and converges to a in $\mathcal{A}^{\nu'}$ for all $\nu'>\nu$.

Lemma 3.24. *Let $\chi\in C_0^\infty(\mathbb{R}^{2n})$ such that $\chi\equiv 1$ near 0. For $a\in\mathcal{A}^\nu$, define*

$$a_j(x,\xi) := a(x,\xi)\chi(x/j,\xi/j), \qquad j\ge 1.$$

Then,

1. *for all seminorm $\mathcal{N}_N$ of $\mathcal{A}^\nu$, there exists C such that*

$$\mathcal{N}_N(a_j) \leq C\mathcal{N}_N(a), \qquad j \geq 1, \ a \in \mathcal{A}^\nu.$$

2. *For all $\nu' > \nu$, $a_j \to a$ in $\mathcal{A}^{\nu'}$ as $j \to \infty$.*

Proof. By the Leibniz rule, $\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)$ is equal to

$$\chi(x/j,\xi/j)\partial_x^\alpha \partial_\xi^\beta a(x,\xi) + \sum_{\substack{\gamma\leq\alpha, \ \delta\leq\beta \\ \gamma+\delta\neq 0}} j^{-|\gamma|-|\delta|} C_\alpha^\gamma C_\beta^\delta (\partial_x^\gamma \partial_\xi^\delta \chi)(x/j,\xi/j)\partial_x^{\alpha-\gamma}\partial_\xi^{\beta-\delta} a(x,\xi).$$

In particular, using the boundedness of χ and its derivatives, we get for each fixed N and $|\alpha+\beta| \leq N$,

$$|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)| \leq C(1+|x|+|\xi|)^\nu \mathcal{N}_N^{\mathcal{A}^\nu}(a),$$

from which the first item follows. To prove the second one, we observe that the previous computation yields

$$\begin{aligned} \mathcal{N}_N^{\nu'}(a-a_j) &\leq \max_{|\alpha+\beta|\leq N} \sup_{\mathbb{R}^{2d}} |1-\chi(x/j,\xi/j)|(1+|x|+|\xi|)^{-\nu'} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| + Cj^{-1}\mathcal{N}_N^{\mathcal{A}^{\nu'}}(a) \\ &\lesssim j^{\nu-\nu'} + j^{-1}, \end{aligned}$$

using that $1-\chi(x/j,\xi/j)$ is supported in $|x|+|\xi| \gtrsim j$ and the fact that

$$(1+|x|+|\xi|)^{-\nu'}\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq (1+|x|+|\xi|)^{\nu-\nu'}\mathcal{N}_N^{\mathcal{A}^\nu}(a).$$

The proof is complete. □

Proposition 3.25. *Fix $\nu \in \mathbb{R}$. For all $a \in \mathcal{A}^\nu$ and all $u, v \in \mathcal{S}$, one has*

$$\big(v, Op(a)u\big)_{L^2} = \big(Op(a^*)v, u\big)_{L^2},$$

where, for all $K \geq 0$,

$$a^* = \sum_{|\alpha|\leq K} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha \overline{a} + r_K, \tag{3.16}$$

and the antilinear map $a \mapsto r_K$ is continuous from $\mathcal{A}^\nu$ to $\mathcal{A}^\nu$. The structure of r_K is as follows: for all integers M_1, M_2 such that

$$2M_1 > n + |\nu|, \qquad 2M_2 > n + |\nu|, \tag{3.17}$$

there exists a family of functions $(b_{\alpha,\gamma_1,\gamma_2})$ indexed by

$$|\alpha| = K+1, \qquad |\gamma_1| \leq 2M_1, \qquad |\gamma_2| \leq 2M_2, \tag{3.18}$$

such that

$$|b_{\alpha,\gamma_1,\gamma_2}(t,z,\zeta)| \leq C\langle z\rangle^{-2M_1}\langle\zeta\rangle^{-2M_2}, \qquad t\in(0,1),\ z\in\mathbb{R}^n,\ \zeta\in\mathbb{R}^n, \tag{3.19}$$

and such that for all $a \in \mathcal{A}^\nu$

$$r_K(x,\xi) = \sum_{\alpha,\gamma_1,\gamma_2} \int_0^1 \iint_{\mathbb{R}^{2n}} e^{-iz\cdot\zeta}\big(\partial_x^{\alpha+\gamma_1}\partial_\xi^{\alpha+\gamma_2}\overline{a}\big)(x+tz,\xi+\zeta)b_{\alpha,\gamma_1,\gamma_2}(t,z,\zeta)dzd\zeta dt. \tag{3.20}$$

We split the proof of this proposition into several lemmas.

Lemma 3.26 (Peetre's inequality)**.** *Fix a dimension* $d \geq 1$. *For all* $\nu \in \mathbb{R}$, *there exists* $C > 0$ *such that*

$$\langle X+Y\rangle^\nu \leq C\langle X\rangle^\nu\langle Y\rangle^{|\nu|}, \qquad X,Y\in\mathbb{R}^d.$$

In particular, if $d = n$,

$$(1+|x+z|+|\xi+\zeta|)^\nu \leq C(1+|x|+|\xi|)^\nu\langle z\rangle^{|\nu|}\langle\xi\rangle^{|\nu|}, \qquad x,\xi,z,\zeta\in\mathbb{R}^n. \tag{3.21}$$

Proof. We may replace $\langle X\rangle$ by $1+|X|$ everywhere. Then

$$(1+|X+Y|) \leq (1+|X|+|Y|) \leq (1+|X|)(1+|Y|),$$

and by raising this equality to the power ν we get the result when $\nu \geq 0$. If $\nu < 0$, the result for $-\nu$ yields

$$(1+|X|)^{-\nu} \leq (1+|X+Y|)^{-\nu}(1+|-Y|)^{|\nu|}$$

and we obtain the result for ν by multiplying this inequality by $(1+|X|)^\nu(1+|X+Y|)^\nu$. To prove (3.21), we simply observe that

$$(1+|x+z|+|\xi+\zeta|)^\nu \leq C(1+|x|+|\xi|)^\nu(1+|z|+|\zeta|)^{|\nu|}$$

and that $(1+|z|+|\zeta|)^{|\nu|} \leq (1+|z|)^{|\nu|}(1+|\zeta|)^{|\nu|}$. The proof is complete. □

Lemma 3.27. *Fix* $K \geq 0$. *For all* $\nu\in\mathbb{R}$, *the map* $a \mapsto r_K$ *is continuous from* $\mathcal{A}^\nu$ *to* $\mathcal{A}^\nu$.

Proof. Fix M_1, M_2 satisfying (3.17) and multi-indices γ,β. Then there exist $C, N \geq 0$ such that, for all α,γ_1,γ_2 satisfying (3.18), we have

$$\begin{aligned}\big|\partial_x^\gamma\partial_\xi^\beta\big(\partial_x^{\alpha+\gamma_1}\partial_\xi^{\alpha+\gamma_2}\overline{a}\big)(x+tz,\xi+\zeta)\big| &\leq \mathcal{N}_N^{\mathcal{A}^\nu}(a)(1+|x+tz|+|\xi+\zeta|)^\nu,\\ &\leq C\mathcal{N}_N^{\mathcal{A}^\nu}(a)(1+|x|+|\xi|)^\nu\langle z\rangle^{|\nu|}\langle\zeta\rangle^{|\nu|},\end{aligned}$$

using (3.21) in the second line and the fact that $|t|\leq 1$. Using (3.19), it follows that r_K is a smooth function that can be differentiated under the integral sign and satisfies

$$\big|\partial_x^\gamma\partial_\xi^\beta r_K(x,\xi)\big| \leq C\mathcal{N}_N^{\mathcal{A}^\nu}(a)(1+|x|+|\xi|)^\nu\int\int_{\mathbb{R}^{2n}}\langle z\rangle^{|\nu|-2M_1}\langle\zeta\rangle^{|\nu|-2M_1}dzd\zeta,$$

where the integral is finite by (3.17). This means exactly that $a\mapsto r_K$ is continuous on $\mathcal{A}^\nu$. □

Lemma 3.28. *If (3.16) holds for all $a \in \mathcal{S}(\mathbb{R}^{2n})$ then it holds for all ν and all $a \in \mathcal{A}^\nu$.*

Proof. Fix $\nu \in \mathbb{R}$ and M_1, M_2 satisfying (3.17). Fix $a \in \mathcal{A}^\nu$. We can then choose $\nu' > \nu$ sufficiently close to ν so that

$$2M_1 > n + |\nu'|, \qquad 2M_2 > n + |\nu'|.$$

Define a_j as in Lemma 3.24 and denote by r_K^j the symbol obtained by replacing a by a_j in (3.20). Since we assume that (3.16) holds for symbols in the Schwartz space, we have

$$\big(v, Op(a_j)u\big)_{L^2} = \sum_{|\alpha| \le K} \frac{1}{\alpha!} \big(Op(D_x^\alpha \partial_\xi^\alpha \overline{a}_j)v, u\big)_{L^2} + \big(Op(r_K^j)v, u\big)_{L^2}, \tag{3.22}$$

for all $u, v \in \mathcal{S}$ and $j \ge 1$. As $j \to \infty$, we have $a_j \to a$ in $\mathcal{A}^{\nu'}$ by Lemma 3.24 and thus

$$D_x^\alpha \partial_\xi^\alpha \overline{a}_j \to D_x^\alpha \partial_\xi^\alpha \overline{a}, \qquad r_K^j \to r_K \qquad \text{in } \mathcal{A}^{\nu'},$$

using Lemma 3.27 for r_K^j. Now if $u \in \mathcal{S}(\mathbb{R}^n)$, Proposition 3.22 shows that we have the following convergences in $\mathcal{S}(\mathbb{R}^n)$

$$Op(a_j)u \to Op(a)u, \qquad Op(D_x^\alpha \partial_\xi^\alpha \overline{a}_j)v \to Op(D_x^\alpha \partial_\xi^\alpha \overline{a})v, \qquad Op(r_K^j)v \to Op(r_K)v.$$

This allows to let $j \to \infty$ in (3.22) which yields the result. □

Lemma 3.29. *For all $a \in \mathcal{S}(\mathbb{R}^{2n})$, define*

$$a^*(x,\xi) = (2\pi)^{-n} \int\!\!\int_{\mathbb{R}^{2n}} e^{-iz\cdot\zeta} \overline{a}(x+z, \xi+\zeta) d\zeta dz. \tag{3.23}$$

Then $a^* \in \mathcal{S}(\mathbb{R}^{2n})$ and

$$\big(v, Op(a)u\big)_{L^2} = \big(Op(a^*)v, u\big)_{L^2}, \qquad u, v \in \mathcal{S}. \tag{3.24}$$

Proof. To show that a^* belongs to $\mathcal{S}(\mathbb{R}^{2n})$ it suffices to check that, for all $\alpha, \beta, \gamma, \delta$ there exists C such that

$$\big| x^\gamma \xi^\delta \partial_x^\alpha \partial_\xi^\beta a^*(x,\xi) \big| \le C, \qquad x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^n. \tag{3.25}$$

Since $\overline{a}(x+z, \xi+\zeta)$ and its derivatives decay fast with respect to z, ζ, locally uniformly with respect to x, ξ, a^* is well defined and, by differentiation under the integral sign, is smooth and

$$\partial_x^\alpha \partial_\xi^\beta a^*(x,\xi) = (2\pi)^{-n} \int\!\!\int_{\mathbb{R}^{2n}} e^{-iz\cdot\zeta} \big(\partial_x^\alpha \partial_\xi^\beta \overline{a}\big)(x+z, \xi+\zeta) dz d\zeta.$$

By writing

$$x^\gamma = (x+z-z)^\gamma = \sum_{\gamma_1 \le \gamma} C_\gamma^{\gamma^1} (x+z)^{\gamma-\gamma_1} (-z)^{\gamma_1}$$

and similarly $\xi^\delta = (\xi + \zeta - \zeta)^\delta$, we obtain that $x^\gamma \xi^\delta \partial_x^\alpha \partial_\xi^\beta a^*(x,\xi)$ is a linear combination of integrals of the form

$$\int\int_{\mathbb{R}^{2n}} z^{\gamma_1} \zeta^{\gamma_2} e^{-iz\cdot\zeta} a_{\alpha,\beta,\gamma_1,\gamma_2}(x+z, \xi+\zeta) dz d\zeta, \tag{3.26}$$

where $a_{\alpha,\beta,\gamma_1,\gamma_2} \in \mathcal{S}(\mathbb{R}^{2n})$ is given by

$$a_{\alpha,\beta,\gamma_1,\gamma_2}(y,\eta) = y^{\gamma-\gamma_1} \eta^{\delta-\gamma_2} (\partial_x^\alpha \partial_\xi^\beta \overline{a})(y,\eta).$$

By integration by part with respect to ζ, (3.26) reads

$$(-i)^{|\gamma_1|} \int\int_{\mathbb{R}^{2n}} e^{-iz\cdot\zeta} \partial_\zeta^{\gamma_1} \big(\zeta^{\gamma_2} a_{\alpha,\beta,\gamma_1,\gamma_2}(x+z,\xi+\zeta)\big) dz d\zeta,$$

which, by integration by part in z to handle the powers of ζ remaining after the expansion of the derivative according to the Leibnitz rule, is a linear combination of integrals of the form

$$\int\int_{\mathbb{R}^{2n}} e^{-iz\cdot\zeta} \partial_z^{\gamma_2'} \partial_\zeta^{\gamma_1'} a_{\alpha,\beta,\gamma_1,\gamma_2}(x+z,\xi+\zeta) dz d\zeta,$$

with $\gamma_1' \leq \gamma_1$ and $\gamma_2' \leq \gamma_2$. Since $\partial_\zeta^{\gamma_1'} \partial_z^{\gamma_2'} a_{\alpha,\beta,\gamma_1,\gamma_2}$ belongs to $\mathcal{S}(\mathbb{R}^{2n})$ the integral above is bounded in $\mathbb{C}$ as x, ξ vary in $\mathbb{R}^n$ so we get (3.25). Let us now prove (3.24). For all $u, v \in \mathcal{S}$, we have

$$\begin{aligned}
\big(v, a(x,D)u\big)_{L^2} &= \int \overline{v(x)} a(x,D) u(x) dx \\
&= (2\pi)^{-n} \int\int e^{ix\cdot\xi} \overline{v(x)} a(x,\xi) \hat{u}(\xi) d\xi dx \\
&= (2\pi)^{-n} \int\int\int e^{i(x-y)\cdot\xi} \overline{v(x)} a(x,\xi) u(y) dy d\xi dx \\
&= (2\pi)^{-n} \int\int\int e^{i(x-y)\cdot\xi} \overline{\overline{a}(x,\xi) v(x)} u(y) dy d\xi dx.
\end{aligned}$$

From this calculation, we see that

$$\begin{aligned}
\big(v, Op(a)u\big)_{L^2} &= \int\int \overline{v(x)} A(x,y) u(y) dy dx \\
&= \int\int \overline{B(x,y) v(y)} u(x) dx dy
\end{aligned}$$

where

$$A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,\xi) d\xi, \tag{3.27}$$

is the kernel of $Op(a)$ and

$$B(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \overline{a}(y,\xi) d\xi.$$

To prove (3.24) it suffices to see that $B(x,y)$ is the kernel of $Op(a^*)$ namely

$$B(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a^*(x,\xi) d\xi. \tag{3.28}$$

Denoting by $\hat{\cdot}$ the Fourier transform with respect to ξ, this means that $\widehat{\overline{a}}(y, y-x) = \widehat{a^*}(x, y-x)$, ie

$$\widehat{\overline{a}}(x+z, z) = \widehat{a^*}(x,z), \qquad x, z \in \mathbb{R}^n,$$

or equivalently, by taking the inverse Fourier transform with respect to z, that

$$\begin{aligned} a^*(x,\xi) &= (2\pi)^{-n} \int e^{iz\cdot\xi} \widehat{\overline{a}}(x+z, z) dz \\ &= (2\pi)^{-n} \int\int e^{iz\cdot(\xi-\eta)} \widehat{\overline{a}}(x+z,\eta) d\eta dz. \end{aligned}$$

The change of variable $\eta - \xi = \zeta$ in the last integral shows that the last equality is precisely (3.23). Hence (3.28) holds true and this completes the proof. □

Proof of Proposition 3.25. By Lemma 3.28, we may assume that $a \in \mathcal{S}(\mathbb{R}^{2n})$.
Step 1: the expansion. The starting point is to write a^* given by (3.23) as an iterate integral

$$a^*(x,\xi) = (2\pi)^{-n} \int \left(\int_{\mathbb{R}^{2n}} e^{-iz\cdot\zeta} \overline{a}(x+z, \xi+\zeta) d\zeta \right) dz. \tag{3.29}$$

Expanding $\overline{a}(x+z,\xi+\zeta)$ by the Taylor formula in z, we get

$$\overline{a}(x+z,\xi+\zeta) = \sum_{|\alpha|\le K} \frac{1}{\alpha!} (\partial_x^\alpha \overline{a})(x,\xi+\zeta) z^\alpha + \sum_{|\alpha|=K+1} \frac{K+1}{\alpha!} R_{K,\alpha}(x,z,\xi+\zeta) z^\alpha, \tag{3.30}$$

where

$$R_{K,\alpha}(x,z,\eta) = \int_0^1 (1-t)^K (\partial_x^\alpha \overline{a})(x+tz,\eta) dt. \tag{3.31}$$

Inserting each term of the expansion $\sum_{|\alpha|\le K} \cdots$ of (3.30) into the integral in ζ of (3.29), which makes sense for the integrand decays fast with respect to ζ, we obtain

$$\begin{aligned} \int z^\alpha e^{-iz\cdot\zeta} (\partial_x^\alpha \overline{a})(x,\xi+\zeta) d\zeta &= \int (i\partial_\zeta)^\alpha e^{-iz\cdot\zeta} (\partial_x^\alpha \overline{a})(x,\xi+\zeta) d\zeta, \\ &= (-i)^{|\alpha|} \int e^{-iz\cdot\zeta} (\partial_x^\alpha \partial_\xi^\alpha \overline{a})(x,\xi+\zeta) d\zeta \\ &= (-i)^{|\alpha|} \widehat{\partial_x^\alpha \partial_\xi^\alpha \overline{a}}(x,z) e^{iz\cdot\xi}. \end{aligned} \tag{3.32}$$

Integrating now (3.32) with respect to z according to (3.29) and using the Fourier inversion formula, we get

$$(2\pi)^{-n} \int \left(\int z^\alpha e^{-iz\cdot\zeta} (\partial_x^\alpha \overline{a})(x,\xi+\zeta) d\zeta \right) dz = D_x^\alpha \partial_\xi^\alpha \overline{a}(x,\xi).$$

This shows that (3.16) holds with

$$r_K(x,\xi) = (2\pi)^{-n} \int \left(\sum_{|\alpha|=K+1} \frac{K+1}{\alpha!} \int z^\alpha e^{-iz\cdot\zeta} R_{K,\alpha}(x,z,\xi+\zeta) d\zeta \right) dz. \tag{3.33}$$

Note that we are not allowed at the moment to swap the sum over α and the integral with respect to z for this is ony a semi-convergent integral (ie iterate ones). We now proceed to the analysis of such terms.

Step 2: the remainder. By the same integrations by part as those leading to (3.32), we have

$$\int z^\alpha e^{-iz\cdot\zeta} R_{K,\alpha}(x,z,\xi+\zeta) d\zeta = \int e^{-iz\cdot\zeta} D_\xi^\alpha R_{K,\alpha}(x,z,\xi+\zeta) d\zeta. \tag{3.34}$$

Note that these integrals make clearly sense since (3.31) decays fast with respecto to η. To get some decay with respect to z, we next integrate by part thanks to

$$\langle z\rangle^{-2M_1}(1-\Delta_\zeta)^{M_1} e^{-iz\cdot\zeta} = e^{-iz\cdot\zeta},$$

so that we can write the right hand side of (3.34) as

$$\langle z\rangle^{-2M_1} \int e^{-iz\cdot\zeta}(1-\Delta_\xi)^{M_1} D_\xi^\alpha R_{K,\alpha}(x,z,\xi+\zeta) d\zeta.$$

Using now that, for fixed ξ, the amplitude of the above integral decays fast with respect to ζ and using Peetre's inequality we have

$$\left|(1-\Delta_\xi)^{M_1} D_\xi^\alpha R_{K,\alpha}(x,z,\xi+\zeta)\langle z\rangle^{-2M_1}\right| \leq C_{x,\xi}\langle z\rangle^{|\mu|-2M_1}\langle\zeta\rangle^{-n-1},$$

which is integrable with respect to z and ζ if (3.17) holds. We can then use the Fubini Theorem in the right hand side of (3.33) to get

$$r_K(x,\xi) = \sum_{|\alpha|=K+1} \frac{K+1}{\alpha!} \int \left(\int e^{-iz\cdot\zeta}(1-\Delta_\xi)^{M_1} D_\xi^\alpha R_{n,\alpha}(x,z,\xi+\zeta)\langle z\rangle^{-2M_1} dz \right) d\zeta.$$

We finally integrate by part in the above integral with respect to z thanks to

$$\langle\zeta\rangle^{-2M_2}(1-\Delta_z)^{M_2} e^{-iz\cdot\zeta} = e^{-iz\cdot\zeta},$$

with M_2 as in (3.17). Expanding the corresponding derivatives according to the Leibniz rule and taking (3.31) into account, we obtain a linear combination of integrals of the form (3.20). $\square$

Proof of Theorem 3.16. We apply Proposition 3.25 to $a_h(x,\xi) = a(x, h\xi)$. According to (3.16) with $K = J-1$, we have

$$\begin{aligned} Op(a_h^*) &= \sum_{|\alpha|<J} \frac{1}{\alpha!} Op(\partial_\xi^\alpha D_x^\alpha \overline{a}_h) + Op(r_{J-1}) \\ &= \sum_{|\alpha|<J} \frac{1}{\alpha!} h^{|\alpha|} Op_h(\partial_\xi^\alpha D_x^\alpha \overline{a}) + Op(r_{J-1}) \end{aligned}$$

since $\partial_\xi^\alpha D_x^\alpha \overline{a}_h(x,\xi) = h^{|\alpha|}\big(\partial_\xi^\alpha D_x^\alpha \overline{a}\big)(x, h\xi)$. By (3.20), we also have

$$Op(r_{J-1}) = h^J Op_h(r_J^*(a,h))$$

with $r_J^*(a,h)$ which is the function of (x,ξ) given by

$$\sum_{\alpha,\gamma_1,\gamma_2} h^{|\gamma_2|} \int_0^1 \int\!\!\int_{\mathbb{R}^{2n}} e^{-iz\cdot\zeta}\big(\partial_x^{\alpha+\gamma_1}\partial_\xi^{\alpha+\gamma_2}\overline{a}\big)(x+tz, \xi+h\zeta) b_{\alpha,\gamma_1,\gamma_2}(t,z,\zeta)dzd\zeta dt. \quad (3.35)$$

Our remaing task is to prove (3.11). Since a belongs to $S^{\mu,m}$ and $|\alpha+\gamma_2| \geq J$, we have

$$\begin{aligned} \big|\partial_x^{\alpha+\gamma_1}\partial_\xi^{\alpha+\gamma_2}\overline{a}(x+tz,\xi+h\zeta)\big| &\leq \langle x+tz\rangle^\mu \langle \xi+h\zeta\rangle^{m-J} \mathcal{N}_{N_1}^{S^{m,\mu}}(a) \\ &\leq C\langle x\rangle^\mu\langle\xi\rangle^{m-J}\langle z\rangle^{|\mu|}\langle\zeta\rangle^{|m-J|}\mathcal{N}_{N_1}^{S^{m,\mu}}(a) \end{aligned}$$

using the Peetre inequality and the fact that $|t|, |h| \leq 1$, and where we can take

$$N_1 = 2(M_1 + M_2 + J),$$

by (3.18). If M_1 and M_2 have been chosen[1] so that $|\mu| - 2M_1 < -n$ and $|m-J| - 2M_2 < -n$, we obtain from (3.19) that

$$\int_{[0,1]\times\mathbb{R}^{2n}} \big|\big(\partial_x^{\alpha+\gamma_1}\partial_\xi^{\alpha+\gamma_2}\overline{a}\big)(x+tz,\xi+h\zeta)b_{\alpha,\gamma_1,\gamma_2}(t,z,\zeta)\big|dtdzd\zeta \leq C\langle x\rangle^\mu\langle\xi\rangle^{m-J}\mathcal{N}_{N_1}^{S^{m,\mu}}(a),$$

hence that

$$\mathcal{N}_0^{S^{\mu,m-J}}(r_J^*(a,h)) \leq C\mathcal{N}_{N_1}^{S^{m,\mu}}(a),$$

which is precisely (3.11) for $N=0$. The general case (ie $N \geq 0$) is obtained similarly after differentiation of $r_J^*(a,h)$ under the integral sign in (3.35). $\square$

In the next proposition, we consider the composition of pseudo-differential operators.

Proposition 3.30. *Let $\nu, \kappa \in \mathbb{R}$. For all $a \in \mathcal{A}^\nu$ and $b \in \mathcal{A}^\kappa$, one has*

$$Op(a)Op(b) = Op(c),$$

[1] note that the choice will depend on J

with $c \in \mathcal{A}^{\nu+\kappa}$ such that, for all integer $K \geq 0$, there exists a bilinear map $(a,b) \mapsto r_K$ from $\mathcal{A}^\nu \times \mathcal{A}^\kappa$ to $\mathcal{A}^{\nu+\kappa}$ such that

$$c(x,\xi) = \sum_{|\alpha| \leq K} c_\alpha(x,\xi) + r_K(x,\xi), \qquad c_\alpha(x,\xi) = \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi),$$

and with the following structure: for all integers $M_1, M_2 \geq 0$ such that

$$2M_1 > n + |\nu|, \qquad 2M_2 > n + |\kappa|,$$

there exists a family of functions $(d_{\alpha,\gamma_1,\gamma_2})$ indexed by

$$|\alpha| = K+1, \qquad |\gamma_1| \leq 2M_1, \qquad |\gamma_2| \leq 2M_2,$$

such that

$$|d_{\alpha,\gamma_1,\gamma_2}(t,\zeta,z)| \leq C\langle\zeta\rangle^{-2M_1}\langle z\rangle^{-2M_2}, \qquad t \in (0,1),\ z \in \mathbb{R}^n,\ \zeta \in \mathbb{R}^n,$$

and such that, for all $a \in \mathcal{A}^\nu$ and all $b \in \mathcal{A}^\kappa$,

$$r_K(x,\xi) = \sum_{\alpha,\gamma_1,\gamma_2} \int_0^1 \iint_{\mathbb{R}^{2n}} e^{-iz\cdot\zeta}\big(\partial_\xi^{\alpha+\gamma_1} a\big)(x,\xi+\zeta)\big(\partial_x^{\alpha+\gamma_2} b\big)(x+tz,\xi) d_{\alpha,\gamma_1,\gamma_2}(t,\zeta,z)\, dz d\zeta dt.$$

The proof is very similar to the proof of Proposition 3.25 so we shall only sketch the main steps. We hope that this more synthetic exposition will be a pedagogic alternative to the detailed proof of Proposition 3.25.

Proof. **Step 1: r_K depends continuously on (a,b).** By Peetre's inequality, for each term in the sum defining r_K we have

$$|\big(\partial_\xi^{\alpha+\gamma_1} a\big)(x,\xi+\zeta)\big(\partial_x^{\alpha+\gamma_2} b\big)(x+tz,\xi)| \lesssim \mathcal{N}^{\mathcal{A}^\nu}_{|\alpha+\gamma_1|}(a)\mathcal{N}^{\mathcal{A}^\kappa}_{|\alpha+\gamma_2|}(b)(1+|x|+|\xi|)^{\nu+\kappa}\langle\zeta\rangle^{|\nu|}\langle z\rangle^{|\kappa|}$$

for all $x,\xi,z,\zeta \in \mathbb{R}^n$, $t \in (0,1)$ and $(a,b) \in \mathcal{A}^\nu \times \mathcal{A}^\kappa$. The decay of $d_{\alpha,\gamma_1,\gamma_2}$ and the conditions on M_1, M_2 show that the integrals (with respect to (z,ζ)) are convergent and with modulus of order

$$\mathcal{N}^{\mathcal{A}^\nu}_{|\alpha+\gamma_1|}(a)\mathcal{N}^{\mathcal{A}^\kappa}_{|\alpha+\gamma_2|}(b)(1+|x|+|\xi|)^{\nu+\kappa}.$$

Similar estimates holds for derivatives with respect to x,ξ (note that we can differentiate under the integral sign) which shows that $(a,b) \mapsto r_K$ takes values in $\mathcal{A}^{\nu+\kappa}$ and depends continuously on (a,b).

Step 2: We can assume symbols belongs to $\mathcal{S}(\mathbb{R}^{2n})$. Indeed assume that the result holds for symbols in $\mathcal{S}(\mathbb{R}^{2n})$ and let $a \in \mathcal{A}^\nu, b \in \mathcal{A}^\kappa$. Fix $\nu' > \nu$ and $\kappa' > \kappa$. We can then pick $a_j, b_j \in \mathcal{S}(\mathbb{R}^{2n})$ such that

$$a_j \to a \ \text{ in } \ \mathcal{A}^{\nu'}, \qquad b_j \to b \ \text{ in } \ \mathcal{A}'_\kappa,$$

as $j \to \infty$. Denote c_α^j, r_K^j the expressions of c_α, r_K related to a_j and b_j. Then, clearly,

$$c_\alpha^j = \frac{1}{\alpha!} \partial_\xi^\alpha a_j D_x^\alpha b_j \to \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b = c_\alpha \qquad \text{in } \mathcal{A}^{\nu'+\kappa'}$$

and, by Step 1,

$$r_K^j \to r_K \qquad \text{in } \mathcal{A}^{\nu'+\kappa'}.$$

In particular, for all $u \in \mathcal{S}$, we have

$$\sum_{|\alpha| \leq K} Op(c_\alpha^j)u + Op(r_K^j)u \to \sum_{|\alpha| \leq K} Op(c_\alpha)u + Op(r_K)u, \qquad \text{in } \mathcal{S}. \tag{3.36}$$

Therefore, if Proposition 3.30 holds for all Schwartz symbols, namely if the left hand side of (3.36) equals $Op(a_j)Op(b_j)u$, then by letting $j \to \infty$ and using Proposition 3.22 (or Exercise 3.23) we get

$$Op(a)Op(b)u = \sum_{|\alpha| \leq K} Op(c_\alpha)u + Op(r_K)u,$$

for all $u \in \mathcal{S}$.

Step 3: Computation of the kernel. By Step 2, we can consider Schwartz symbols. This allows to consider

$$A(x,w) = (2\pi)^{-n} \int e^{i(x-w)\cdot\theta} a(x,\theta) d\theta, \qquad B(w,y) = (2\pi)^{-n} \int e^{i(w-y)\cdot\xi} b(w,\xi) d\xi$$

which are the kernels of $Op(a)$ and $Op(b)$, ie

$$Op(a)u(x) = \int A(x,w)u(w)dw, \qquad Op(b)u(w) = \int B(w,y)u(y)dy.$$

See also (3.27). The kernel of $Op(a)Op(b)$ is then

$$\begin{aligned} C(x,y) &= \int A(x,w)B(w,y)dw \\ &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \left((2\pi)^{-n} \int \int e^{i(x-w)\cdot(\theta-\xi)} a(x,\theta) b(w,\xi) dw d\theta \right) d\xi \end{aligned}$$

which, after the change of variables $z = w - x$, $\zeta = \theta - \xi$, is precisely the kernel of $Op(c)$ with

$$c(x,\xi) = (2\pi)^{-n} \int \int e^{-iz\cdot\zeta} a(x, \xi+\zeta) b(x+z, \xi) dz d\zeta. \tag{3.37}$$

Step 4: Taylor expansion and integration by part. We expand $a(x,\xi+\zeta)b(x+z,\xi)$ by the Taylor formula with respect to z and then integrate with respect to ζ. After the same integration by part as in the proof of Proposition 3.25 we see that

$$\int e^{-iz\cdot\zeta} a(x,\xi+\zeta)b(x+z,\xi)d\zeta$$

is the sum of

$$\sum_{|\alpha|\leq K} \frac{1}{\alpha!}\int e^{-iz\cdot\zeta}(\partial_\xi^\alpha a)(x,\xi+\zeta)(D_x^\alpha b)(x,\xi)d\zeta = e^{iz\cdot\xi}\sum_{|\alpha|\leq K}\frac{1}{\alpha!}\widehat{\partial_\xi^\alpha a}(x,z)(D_x^\alpha b)(x,\xi) \quad (3.38)$$

(where $\widehat{\ }$ is the Fourier transform with respect to ζ) and of

$$\sum_{|\alpha|=K+1}\frac{K+1}{\alpha!}\int\int_0^1 e^{-iz\cdot\zeta}(\partial_\xi^\alpha a)(x,\xi+\zeta)(D_x^\alpha b)(x+tz,\xi)(1-t)^K dtd\zeta. \quad (3.39)$$

The integration of (3.38) with respect to z yields the expected $\sum_{|\alpha|\leq K} c_\alpha(x,\xi)$. In (3.39), we integrate by part thanks to

$$\langle z\rangle^{-2M_2}(1-\Delta_\zeta)^{M_2}e^{-iz\cdot\zeta} = e^{-iz\cdot\zeta},$$

then integrate with respect to z and then swap the integrations with respect to z and ζ, for we have sufficient decay in z and ζ. We finally integrate by part using

$$\langle\zeta\rangle^{-2M_1}(1-\Delta_z)^{M_1}e^{-iz\cdot\zeta} = e^{-iz\cdot\zeta},$$

and we obtain the expect form of r_K. □

Proof of Theorem 3.14. Similar to the proof of Theorem 3.16 and left to the reader as an exercise. □

Chapter 4

Some tools of spectral theory

In this section $\mathcal{H}$ and $\mathcal{K}$ are two separable Hilbert spaces (over $\mathbb{C}$). Everywhere $\mathcal{L}(\mathcal{H},\mathcal{K})$ is the (Banach) space of continuous (or bounded) linear maps from $\mathcal{H}$ to $\mathcal{K}$. We denote the norm of $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ by $||A||_{\mathcal{H}\to\mathcal{K}}$. If $\mathcal{H}=\mathcal{K}$, we set as usual $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H},\mathcal{H})$.

4.1 Hilbert-Schmidt operators

Lemma 4.1. *Let $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$. If $(e_j)_{j\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ are orthonormal bases of $\mathcal{H}$ and $\mathcal{K}$ respectively, then*

$$\sum_j ||Ae_j||_{\mathcal{K}}^2 = \sum_k ||A^* f_k||_{\mathcal{H}}^2.$$

Proof. For each j, we have

$$||Ae_j||_{\mathcal{K}}^2 = \sum_k \left|(f_k, Ae_j)_{\mathcal{K}}\right|^2 = \sum_k \left|(A^* f_k, e_j)_{\mathcal{K}}\right|^2.$$

Summing over j and swapping the summations with respect to j and k, we get

$$\sum_j ||Ae_j||_{\mathcal{K}}^2 = \sum_k \sum_j \left|(A^* f_k, e_j)_{\mathcal{K}}\right|^2 = \sum_k ||A^* f_k||_{\mathcal{H}}^2$$

which is precisely the result. □

Definition 4.2. *An operator $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is* **Hilbert-Schmidt** *if*

$$||A||_{\mathrm{HS}} := \left(\sum_j ||Ae_j||_{\mathcal{K}}^2 \right)^{1/2} < \infty,$$

for some orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $\mathcal{H}$. We denote by $S_2(\mathcal{H},\mathcal{K})$ the set of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H}=\mathcal{K}$, we denote it by $S_2(\mathcal{H})$.

Note that $||A||_{\mathrm{HS}}$ is independent of the choice of the orthonormal basis by Lemma 4.1.

Proposition 4.3. *1. If $A \in S_2(\mathcal{H}, \mathcal{K})$, then $A^* \in S_2(\mathcal{K}, \mathcal{H})$ and*

$$||A||_{\mathrm{HS}} = ||A^*||_{\mathrm{HS}}.$$

2. For all $A \in S_2(\mathcal{H}, \mathcal{K})$, we have

$$||A||_{\mathcal{H}\to\mathcal{K}} \leq ||A||_{\mathrm{HS}}.$$

3. $S_2(\mathcal{H}, \mathcal{K})$ is a vector space and $||\cdot||_{\mathrm{HS}}$ is a norm thereon.

4. $S_2(\mathcal{H}, \mathcal{K})$ is complete.

5. If $\mathcal{H}_1$ and $\mathcal{K}_1$ are separable Hilbert spaces and

$$A \in S_2(\mathcal{H}, \mathcal{K}), \qquad B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}), \qquad C \in \mathcal{L}(\mathcal{K}, \mathcal{K}_1)$$

then $CAB \in S_2(\mathcal{H}_1, \mathcal{K}_1)$ and

$$||CAB||_{\mathrm{HS}} \leq ||C||_{\mathcal{K}\to\mathcal{K}_1} ||A||_{\mathrm{HS}} ||B||_{\mathcal{H}_1\to\mathcal{H}}.$$

Proof. Item 1 follows directly from Lemma 4.1. To prove item 2, we fix $u \in \mathcal{H}$ and write

$$u = \lim_{N\to\infty} u_N, \qquad u_N = \sum_{j\leq N} (e_j, u)_{\mathcal{H}} e_j.$$

Then, by the triangle inequality and the Cauchy-Schwartz inequality,

$$\begin{aligned} ||Au_N||_{\mathcal{K}} &\leq \sum_{j\leq N} |(e_j, u)_{\mathcal{H}}| ||Ae_j||_{\mathcal{K}} \\ &\leq \left(\sum_{j\leq N} |(e_j, u)_{\mathcal{H}}|^2 \right)^{1/2} \left(\sum_{j\leq N} ||Ae_j||_{\mathcal{K}}^2 \right)^{1/2} \\ &\leq ||A||_{\mathrm{HS}} ||u||_{\mathcal{H}}. \end{aligned}$$

Letting N go to infinity and using the continuity of A, we obtain $||Au||_{\mathcal{K}} \leq ||A||_{\mathrm{HS}} ||u||_{\mathcal{H}}$ which yields the result. The proof of item 3 is a routine which we omit; we only point out that $||A||_{\mathrm{HS}} = 0$ only if $A = 0$ by item 2. Let us now prove item 4. Let (A_j) be a Cauchy sequence in $S_2(\mathcal{H}, \mathcal{K})$. By item 2, it is a Cauchy sequence in $\mathcal{L}(\mathcal{H}, \mathcal{K})$ hence converges in operator norm to a bounded operator A. It remains to show that A is Hilbert-Schmidt and that $||A - A_j||_{\mathrm{HS}} \to 0$. Fix $\epsilon > 0$. Then for $J > 0$ large enough

$$||A_k - A_j||_{\mathrm{HS}} \leq \epsilon, \qquad j, k \geq J.$$

This implies in particular that,

$$\sum_{m\leq N} ||(A_k - A_j)e_m||_{\mathcal{K}}^2 \leq \epsilon^2, \qquad N \geq 0,\ j,k \geq J.$$

Thus, by letting k go to infinity for fixed N and j and then N to infinity, we see that A is Hilbert-Schmidt and that $||A - A_j||_{\mathrm{HS}} \leq \epsilon$ for $j \geq J$. Therefore $||A - A_j||_{\mathrm{HS}} \to 0$. We finally prove item 5. We first observe that

$$||CAe_j||_{\mathcal{K}_1}^2 \leq ||C||_{\mathcal{K}\to\mathcal{K}_1}^2 ||Ae_j||_{\mathcal{K}}^2$$

hence by summing over j, we see that $CA \in S_2(\mathcal{H}, \mathcal{K}_1)$ and that $||CA||_{\mathrm{HS}} \leq ||C||_{\mathcal{K}\to\mathcal{K}_1}||A||_{\mathrm{HS}}$. To handle the case when $B \neq I_{\mathcal{H}}$, we observe that

$$CAB = \big(B^*(CA)^*\big)^*$$

which shows that CAB is Hilbert-Schmidt and that

$$||CAB||_{\mathrm{HS}} \leq ||B^*||_{\mathcal{H}\to\mathcal{H}_1}||CA||_{\mathrm{HS}} \leq ||C||_{\mathcal{K}\to\mathcal{K}_1}||A||_{\mathrm{HS}}||B||_{\mathcal{H}_1\to\mathcal{H}},$$

using item 1 and the fact that the operator norms of an operator and its adjoint coincide. $\square$

Proposition 4.4. *Let $A \in S_2(\mathcal{H}, \mathcal{K})$ and $(f_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{K}$. Define*

$$\Pi_N = \text{ \textit{orthogonal projection on} span } \{f_k \mid k \leq N\}.$$

Then

$$||\Pi_N A - A||_{\mathrm{HS}} \to 0, \qquad N \to \infty.$$

Proof. By item 1 of Proposition 4.3, it is equivalent to show that $||A^*\Pi_N - A^*||_{\mathrm{HS}} \to 0$. Writing this Hilbert-Schmidt norm in term of the orthonormal basis (f_k), we obtain

$$||A^*(\Pi_N - 1)||_{\mathrm{HS}}^2 = \sum_{k>N} ||A^* f_k||_{\mathcal{H}}^2$$

which clearly goes to zero as $N \to \infty$. $\square$

Remark. Of course if we consider a projection P_N onto the N first vectors of an orthonormal basis of $\mathcal{H}$, we also have $AP_N \to A$ in the Hilbert-Schmidt class. This follows directly from Proposition 4.4 by taking the adjoint.

Using that Π_N is a finite rank operator and the property 2 in Proposition 4.3, we derive automatically the following corollary.

Corollary 4.5. *Hilbert-Schmidt operators are compact.*

We conclude this section with an important example of Hilbert-Schmidt operator. To any $K \in L^2(\mathbb{R}^{2n})$, we can associate the sesquilinear form

$$Q(v,u) = \int\int \overline{v(x)} K(x,y) u(y) dy dx, \qquad u, v \in L^2(\mathbb{R}^n).$$

By the Cauchy-Schwarz inequality, Q is obviously continuous on $L^2(\mathbb{R}^n)^2$ hence there exists a unique bounded operator $A_K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that

$$Q(v,u) = \big(v, A_K u\big)_{L^2(\mathbb{R}^n)}.$$

Definition 4.6. *A_K is the operator with L^2* **kernel** *K.*

Proposition 4.7. *The operator A_K belongs to $S_2\big(L^2(\mathbb{R}^n)\big)$ and*

$$||A_K||_{\mathrm{HS}} = ||K||_{L^2(\mathbb{R}^{2n})}.$$

Proof. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$. Then the countable family

$$\big(e_k \otimes \overline{e_j}\big)_{(j,k)\in\mathbb{N}^2}, \qquad e_k \otimes \overline{e_j}(x,y) := e_k(x)\overline{e_j(y)}$$

is an orthonormal basis of $L^2(\mathbb{R}^{2n})$: that this is an orthonormal system is a simple calculation which we omit and proving that finite linear combinations are dense follows from the density of $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^{2n})$ and the density of finite linear combinations of (e_j) in $L^2(\mathbb{R}^n)$. Then

$$\begin{aligned} ||A_K||_{\mathrm{HS}}^2 = \sum_j ||A_K e_j||^2_{L^2(\mathbb{R}^n)} &= \sum_j \sum_k \big|\big(e_k, A_K e_j\big)_{L^2(\mathbb{R}^n)}\big|^2 \\ &= \sum_j \sum_k \big|Q(e_k, e_j)\big|^2 \\ &= \sum_{(j,k)} \big|\big(e_k \otimes \overline{e_j}, K\big)_{L^2(\mathbb{R}^{2n})}\big|^2 = ||K||^2_{L^2(\mathbb{R}^{2n})}, \end{aligned}$$

completes the proof. □

4.2 Trace class operators

Definition 4.8. *A linear operator $T : \mathcal{H} \to \mathcal{H}$ is* **trace class** *if there exist*

1. *a positive integer N,*
2. *separable Hilbert spaces $\mathcal{K}_1, \ldots, \mathcal{K}_N$,*
3. *Hilbert Schmidt operators $A_1, \ldots, A_N$ and $B_1, \ldots, B_N$ with $A_j, B_j \in S_2(\mathcal{H}, \mathcal{K}_j)$,*

such that

$$T = \sum_{j=1}^{N} A_j^* B_j. \tag{4.1}$$

We denote by $S_1(\mathcal{H})$ *the set of trace class operators on* $\mathcal{H}$ *and define*

$$||T||_{\mathrm{tr}} := \inf \left\{ \sum_{j=1}^{N} ||A_j||_{\mathrm{HS}} ||B_j||_{\mathrm{HS}} \right\},$$

the infimum being taken over all N, $(\mathcal{K}_j)$, (A_j), (B_j) *as in 1,2,3 such that (4.1) holds.*

Note that this definition implies that trace class operators are bounded and more precisely Hilbert-Schmidt.

Proposition 4.9. *1. For all* $T \in S_1(\mathcal{H})$,

$$||T||_{\mathcal{H}\to\mathcal{H}} \leq ||T||_{\mathrm{tr}}.$$

2. $S_1(\mathcal{H})$ *is a vector space and* $||\cdot||_{\mathrm{tr}}$ *is a norm thereon.*
3. *If* T *is trace class, then* T^* *is trace class and*

$$||T^*||_{\mathrm{tr}} \leq ||T||_{\mathrm{tr}}.$$

4. *If* $\mathcal{K}$ *is another separable Hilbert space and*

$$A, B \in \mathcal{L}(\mathcal{K}, \mathcal{H}), \qquad T \in S_1(\mathcal{H}),$$

then $A^*TB \in S_1(\mathcal{K})$ *and*

$$||A^*TB||_{\mathrm{tr}} \leq ||A||_{\mathcal{K}\to\mathcal{H}} ||T||_{\mathrm{tr}} ||B||_{\mathcal{K}\to\mathcal{H}}.$$

Proof. 1. For all $\epsilon > 0$, we can find

$$N^\epsilon, \qquad \mathcal{K}_1^\epsilon, \ldots, \mathcal{K}_{N^\epsilon}^\epsilon, \qquad A_1^\epsilon, \ldots, A_{N^\epsilon}^\epsilon, \qquad B_1^\epsilon, \ldots, B_{N^\epsilon}^\epsilon$$

such that

$$T = \sum_{j \leq N^\epsilon} (A_j^\epsilon)^* B_j^\epsilon, \tag{4.2}$$

and

$$||T||_{\mathrm{tr}} \ \leq \ \sum_{j=1}^{N^\epsilon} ||A_j^\epsilon||_{\mathrm{HS}} ||B_j^\epsilon||_{\mathrm{HS}} \ < \ ||T||_{\mathrm{tr}} + \epsilon. \tag{4.3}$$

Using

$$\begin{aligned} ||T||_{\mathcal{H}\to\mathcal{H}} &\leq ||\sum_{j\leq N^\epsilon}(A_j^\epsilon)^* B_j^\epsilon||_{\mathcal{H}\to\mathcal{H}} \\ &\leq \sum_{j\leq N^\epsilon}||A_j^\epsilon||_{\mathcal{H}\to\mathcal{H}}||B_j^\epsilon||_{\mathcal{H}\to\mathcal{H}} \\ &\leq \sum_{j\leq N^\epsilon}||A_j^\epsilon||_{\mathrm{HS}}||B_j^\epsilon||_{\mathrm{HS}}, \end{aligned}$$

we obtain that $||T||_{\mathcal{H}\to\mathcal{H}} < ||T||_{\mathrm{tr}} + \epsilon$. Since this true for all ϵ, we obtain the desired inequality.

Let us now prove item 2. Proving that $S_1(\mathcal{H})$ is a vector space (ie a subspace of $\mathcal{L}(\mathcal{H})$) is a routine which uses that the sum of two sums as in the right hand side of (4.1) is still of this form. Let us check that $||\cdot||_{\mathrm{tr}}$ is a norm. Observe first that $||T||_{\mathrm{tr}} = 0$ if and only if $T = 0$ by item 1 and the obvious fact that $||0||_{\mathrm{tr}} = 0$. Let next $T, \widetilde{T}$ be in $S_1(\mathcal{H})$ and write

$$T = \sum_{j=1}^{N} A_j^* B_j, \qquad \widetilde{T} = \sum_{j=1}^{\widetilde{N}} \widetilde{A}_j^* \widetilde{B}_j,$$

according to Definition 4.8. Then

$$||T + \widetilde{T}||_{\mathrm{tr}} \leq \sum_{j=1}^{N} ||A_j||_{\mathrm{HS}}||B_j||_{\mathrm{HS}} + \sum_{j=1}^{\widetilde{N}} ||\widetilde{A}_j||_{\mathrm{HS}}||\widetilde{B}_j||_{\mathrm{HS}}.$$

Taking first the infimum over the set of $N, (\mathcal{K}_j)$, (A_j) and (B_j) as in Definition 4.8, we obtain

$$||T + \widetilde{T}||_{\mathrm{tr}} \leq ||T||_{\mathrm{tr}} + \sum_{j=1}^{\widetilde{N}} ||\widetilde{A}_j||_{\mathrm{HS}}||\widetilde{B}_j||_{\mathrm{HS}}.$$

Taking next the infimum over the set of $\widetilde{N}, (\widetilde{\mathcal{K}}_j)$, $(\widetilde{A}_j)$ and $(\widetilde{B}_j)$, we conclude that

$$||T + \widetilde{T}||_{\mathrm{tr}} \leq ||T||_{\mathrm{tr}} + ||\widetilde{T}||_{\mathrm{tr}}.$$

Let us now fix $\lambda \in \mathbb{C}$ and show that $||\lambda T||_{\mathrm{tr}} = |\lambda| ||T||_{\mathrm{tr}}$. Since we know that $||0||_{\mathrm{tr}} = 0$ we may assume that $\lambda \neq 0$. Using (4.1), we have $\lambda T = \sum_{j=1}^{N} \lambda A_j^* B_j$, hence

$$||\lambda T||_{\mathrm{tr}} \leq |\lambda| \sum_{j\leq N} ||A_j||_{\mathrm{HS}}||B_j||_{\mathrm{HS}},$$

and by taking the same infimum as in Definition 4.8,

$$|\lambda|^{-1}||\lambda T||_{\mathrm{tr}} \leq ||T||_{\mathrm{tr}},$$

that is $||\lambda T||_{\mathrm{tr}} \leq |\lambda| ||T||_{\mathrm{tr}}$. Writing $T = \lambda^{-1}\lambda T$, we also have $||T||_{\mathrm{tr}} \leq |\lambda|^{-1}||\lambda T||_{\mathrm{tr}}$ and the result follows.

We now prove item 3. Using (4.1), we have $T^* = \sum_{j\leq N} B_j^* A_j$ hence T^* is trace class and

$$||T^*|| \leq \sum_{j\leq N} ||B_j^*||_{\mathrm{HS}} ||A_j||_{\mathrm{HS}} = \sum_{j\leq N} ||A_j||_{\mathrm{HS}} ||B_j||_{\mathrm{HS}}.$$

By taking the same infimum as in Definition 4.8, we conclude that $||T^*||_{\mathrm{tr}} \leq ||T||_{\mathrm{tr}}$.

Let us finally prove item 4. Using (4.1) and item 5 of Proposition 4.3 we see that

$$A^*TB = \sum_{j=1} (A_jA)^*(B_jB)$$

is trace class and that

$$||A^*TB|| \leq ||A||_{\mathcal{K}\to\mathcal{H}} \left(\sum_{j=1}^{N} ||A_j||_{\mathrm{HS}} ||B_j||_{\mathrm{HS}} \right) ||B||_{\mathcal{K}\to\mathcal{H}},$$

which, by taking the infimimum over N, $(\mathcal{K}_j)$, (A_j), (B_j), yields the result. □

Lemma 4.10. *Let $T \in S_1(\mathcal{H})$ and $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then the sum*

$$\sum_{j\in\mathbb{N}} \big(e_j, Te_j\big)_{\mathcal{H}}$$

is (absolutely) convergent and does not depend on the choice of the orthonormal basis.

Proof. It suffices to prove the result when $T = A^*B$ with $A, B : \mathcal{H} \to \mathcal{K}$ Hilbert-Schmidt operators, for some separable Hilbert space $\mathcal{K}$. Pick an orthonormal basis (f_k) of $\mathcal{K}$. Then, for each j,

$$\big(e_j, Te_j\big)_{\mathcal{H}} = \big(Ae_j, Be_j\big)_{\mathcal{K}} = \sum_k \overline{\big(f_k, Ae_j\big)}_{\mathcal{K}} \big(f_k, Be_j\big)_{\mathcal{K}}. \tag{4.4}$$

On the other hand, using that A is Hilbert-Schmidt,

$$\sum_j \sum_k \big|\big(f_k, Ae_j\big)_{\mathcal{K}}\big|^2 = \sum_j ||Ae_j||_{\mathcal{K}}^2 < \infty,$$

and similarly for B. Therefore, seen as families indexed by (k, j), $\big(f_k, Ae_j\big)_{\mathcal{K}}$ and $\big(f_k, Be_j\big)_{\mathcal{K}}$ are l^2, so when we sum (4.4) over j we are allowed to swap the sums with respect to j and k so that

$$\begin{aligned}
\sum_j \big(e_j, Te_j\big)_{\mathcal{H}} &= \sum_k \sum_j \overline{\big(f_k, Ae_j\big)}_{\mathcal{K}} \big(f_k, Be_j\big)_{\mathcal{K}} \\
&= \sum_k \sum_n \big(e_j, A^* f_k\big)_{\mathcal{H}} \overline{\big(e_j, B^* f_k\big)}_{\mathcal{H}} \\
&= \sum_k \big(B^* f_k, A^* f_k\big)_{\mathcal{H}} \qquad (4.5)
\end{aligned}$$

which is independent of $(e_j)_{j\in\mathbb{N}}$. $\square$

This lemma allows to state the following definition.

Definition 4.11. *The* **trace** *of $T \in S_1(\mathcal{H})$ is the complex number*

$$\mathrm{tr}(T) := \sum_{j\in\mathbb{N}} \big(e_j, Te_j\big)_{\mathcal{H}},$$

for some (hence all) orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $\mathcal{H}$. If we wish to specify the Hilbert space on which T is defined, we shall sometimes use the notation $\mathrm{tr}_{\mathcal{H}}(T)$ for $\mathrm{tr}(T)$.

Exercise 4.12. *Assuming that Theorem 2.2 is proved, check that (2.12) holds with* tr *defined according to Definition 4.11.*

Proposition 4.13. *1. The trace is linear on $S^1(\mathcal{H})$ and for all $T \in S_1(\mathcal{H})$,*

$$\big|\mathrm{tr}(T)\big| \leq ||T||_{\mathrm{tr}}. \tag{4.6}$$

In particular, the trace is continuous on $S_1(\mathcal{H})$.

2. We have the identities

$$\begin{aligned} \mathrm{tr}_{\mathcal{H}}(A^*B) &= \mathrm{tr}_{\mathcal{K}}(BA^*), \qquad A, B \in S_2(\mathcal{H},\mathcal{K}), & (4.7)\\ \mathrm{tr}_{\mathcal{H}}(A^*TB) &= \mathrm{tr}_{\mathcal{K}}(TBA^*), \qquad T \in S_1(\mathcal{K}),\ A, B \in \mathcal{L}(\mathcal{H},\mathcal{K}). & (4.8)\end{aligned}$$

The second property is often called **cyclicity** of the trace.

Proof. 1. The linearity is obvious. Let us prove (4.6). Using (4.1), we have

$$\begin{aligned} |\mathrm{tr}(T)| \leq \sum_{j\leq N} |\mathrm{tr}(A_j^*B_j)| &\leq \sum_{j\leq N}\left(\sum_{m\in\mathbb{N}} |\big(A_je_m, B_je_m\big)_{\mathcal{K}_j}|\right)\\ &\leq \sum_{j\leq N}\left(\sum_{m\in\mathbb{N}} ||A_je_m||_{\mathcal{K}_j}||B_je_m||_{\mathcal{K}_j}\right)\\ &\leq \sum_{j\leq N} ||A_j||_{\mathrm{HS}}||B_j||_{\mathrm{HS}}, \end{aligned}$$

so the estimate follows by taking the infimum over $N, (\mathcal{K}_j), (A_j), (B_j)$.

Let us now prove item 2. We observe first that (4.7) follows directly from (4.5). To prove (4.8), we may assume by linearity that $T = A_1^*B_1$ with $A_1, B_1 \in S_2(\mathcal{H},\mathcal{K}_1)$. Let then Π_N be the orthogonal projection onto (the span of) the N first vectors of an orthonormal basis of $\mathcal{H}$, as in Lemma 4.4. We observe that

$$\mathrm{tr}_{\mathcal{H}}(A^*TB) = \lim_{N\to\infty} \mathrm{tr}_{\mathcal{H}}(\Pi_N A^*TB) \tag{4.9}$$

by continuity of the trace and the fact that $\Pi_N A^* TB = \big(\Pi_N A^* A_1^*\big) B_1 B$ with $\big(\Pi_N A^* A_1^*\big) \to A^* A_1^*$ in the Hilbert-Schmidt class by Proposition 4.4. For fixed N, $\Pi_N A^*$ is Hilbert-Schmidt hence, by (4.7), we have

$$\operatorname{tr}_{\mathcal{H}}(\Pi_N A^* TB) = \operatorname{tr}_{\mathcal{K}}(TB\Pi_N A^*). \tag{4.10}$$

Since $TB\Pi_N A^*$ reads $A_1^*(B_1 B \Pi_N A^*)$ where the bracket converges to $B_1 B A^*$ in the Hilbert-Schmidt class by Proposition 4.4, we obtain

$$\operatorname{tr}_{\mathcal{K}}(TBA^*) = \lim_{N\to\infty} \operatorname{tr}_{\mathcal{K}}(TB\Pi_N A^*). \tag{4.11}$$

By combining (4.9), (4.10) and (4.11), we get the result. $\square$

We conclude this section with a useful example of calculation of a trace. We consider Hilbert-Schmidt operators with L^2 kernels, as in Definition 4.6.

Proposition 4.14. *Let $A = A_{K_1}$ and $B = A_{K_2}$ be operators on $L^2(\mathbb{R}^n)$ with L^2 kernels $K_1, K_2 \in L^2(\mathbb{R}^{2n})$. Then*

$$\operatorname{tr}_{L^2(\mathbb{R}^n)}\big(AB\big) = \iint K_1(x,y)K_2(y,x)dxdy.$$

Proof. Let us observe first that A^* is the operator with L^2 kernel $K_1^*(x,y) := \overline{K_1(y,x)}$. Observe also that

$$\big(K_1^*, K_2\big)_{L^2(\mathbb{R}^{2n})} = \iint K_1(x,y)K_2(y,x)dxdy.$$

The conclusion follows then from the following calculation. Given an orthonormal basis (e_j) of $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}
\operatorname{tr}(AB) = \sum_j \big(e_j, ABe_j\big)_{L^2(\mathbb{R}^n)} &= \sum_j \big(A^*e_j, Be_j\big)_{L^2(\mathbb{R}^n)} \\
&= \sum_j \sum_k \overline{\big(e_k, A^*e_j\big)}_{L^2(\mathbb{R}^n)} \big(e_k, Be_j\big)_{L^2(\mathbb{R}^n)} \\
&= \sum_j \sum_k \overline{\big(e_k \otimes \overline{e_j}, K_1^*\big)}_{L^2(\mathbb{R}^{2n})} \big(e_k \otimes \overline{e_j}, K_2\big)_{L^2(\mathbb{R}^{2n})} \\
&= \big(K_1^*, K_2\big)_{L^2(\mathbb{R}^{2n})},
\end{aligned}$$

using in the fourth line that $(e_k \otimes \overline{e_j})_{(j,k)}$ is an orthonormal basis of $L^2(\mathbb{R}^{2n})$ (see Proposition 4.7). $\square$

4.3 Functional calculus via the Helffer-Sjöstrand formula

In this paragraph, for a function defined on $\mathbb{C}$ or $\mathbb{R}^2$, we will use the operator

$$\overline{\partial} = \partial_x + i\partial_y.$$

We recall that if a function $g = g(x+iy)$ is holomorphic on some open subset of $\mathbb{C}$, then $\overline{\partial}g = 0$.

Definition 4.15. *Let $f \in C_0^\infty(\mathbb{R})$. An almost analytic extension of f is a function $\widetilde{f} \in C_0^\infty(\mathbb{R}^2)$ such that*

1. *for all $N \geq 0$, $\overline{\partial}\widetilde{f}(x,y) = \mathcal{O}(|y|^N)$,*
2. *$\widetilde{f}(x,0) = f(x)$.*

We comment that the first condition is equivalent to the fact that $\overline{\partial}\widetilde{f}$ vanishes at infinite order on $\{y = 0\}$, ie that

$$\partial_y^k \overline{\partial}\widetilde{f}(x,0) = 0, \qquad k \geq 0, \ x \in \mathbb{R}. \tag{4.12}$$

If one identifies $\mathbb{R}^2$ and $\mathbb{C}$, this means that $\overline{\partial}\widetilde{f}$ vanishes at infinite order on the real line. Using this identification, the second condition states that the restriction of $\widetilde{f}$ on the real line coincides with f.

We also record that, since $\widetilde{f}$ is compactly supported, the first condition has no influence on the growth of $\overline{\partial}\widetilde{f}$ at infinity. More precisely, by choosing $\chi \in C_0^\infty(\mathbb{R}^2)$ which is equal to 1 near the support of $\widetilde{f}$ and non negative, we have

$$\left|\overline{\partial}\widetilde{f}(x,y)\right| \leq C_N |y|^N \chi(x,y), \qquad (x,y) \in \mathbb{R}^2.$$

This estimate is easily obtained by writing that $\overline{\partial}\widetilde{f}$ coincides with the remainder of its Taylor expansion with respect to y to order N multiplied by χ.

The next proposition gives a simple explicit way to construct almost analytic extensions. It can be found on Chapter 8 of [2].

Proposition 4.16. *Let $f \in C_0^\infty(\mathbb{R})$. Let $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R})$ such that*

$$\chi_1 \equiv 1 \ \text{ near the support of } f, \qquad \chi_2 \equiv 1 \ \text{ near } 0.$$

Then

$$\widetilde{f}(x,y) := \chi_1(x)\chi_2(y)\frac{1}{2\pi}\int e^{i(x+iy)\xi}\chi_2(y\xi)\widehat{f}(\xi)d\xi$$

is an almost analytic extension of f.

Proof. Obviously, we have $\widetilde{f}(x,0) = \chi_1(x)f(x) = f(x)$ by the Fourier inversion formula. Since $e^{i(x+iy)\xi}$ is holomorphic, $\overline{\partial}\widetilde{f}$ is, up to the constant $1/2\pi$, the sum of the following three terms,

$$\chi_1'(x)\chi_2(y)\int e^{i(x+iy)\xi}\chi_2(y\xi)\widehat{f}(\xi)d\xi \tag{4.13}$$

$$i\chi_1(x)\chi_2'(y)\int e^{i(x+iy)\xi}\chi_2(y\xi)\widehat{f}(\xi)d\xi \tag{4.14}$$

$$i\chi_1(x)\chi_2(y)\int e^{i(x+iy)\xi}\chi_2'(y\xi)\xi\widehat{f}(\xi)d\xi. \tag{4.15}$$

In (4.13), we expand $e^{-y\xi}\chi_2(y\xi)$ by the Taylor formula which gives a remainder of the form

$$\chi_1'(x)\chi_2(y)\int e^{ix\xi}\mathcal{O}((y\xi)^N)\widehat{f}(\xi)d\xi = \mathcal{O}(y^N)$$

and a linear combination of terms of the form

$$\chi_1'(x)\chi_2(y)\int e^{ix\xi}(y\xi)^k\widehat{f}(\xi)d\xi = 0,$$

since, by the Fourier inversion formula, the integral equals $y^k f^{(k)}(x)$ up to a multiplicative constant and since χ_1' vanishes on $\mathrm{supp}(f)$. Obviously, (4.14) vanishes near $y = 0$. In (4.15), using that χ_2' vanishes near 0, the integral can be written

$$\int e^{i(x+iy)\xi}\frac{\chi_2'(y\xi)}{(y\xi)^N}y^N\xi^{N+1}\widehat{f}(\xi)d\xi = \mathcal{O}(y^N),$$

and this completes the proof. □

In the sequel, for a continuous function $B(x,y)$ defined on $\mathbb{R}^2\setminus\{y=0\}$, or equivalently on $\mathbb{C}\setminus\mathbb{R}$, with values in a Banach space, we shall denote

$$\int_{|\mathrm{Im}z|\geq\epsilon}\overline{\partial}\widetilde{f}(z)B(z)L(dz) := \int_{|y|\geq\epsilon}\left(\int_{\mathbb{R}}\overline{\partial}\widetilde{f}(x,y)B(x,y)dx\right)dy, \qquad \epsilon>0,$$

and

$$\int_{\mathbb{C}}\overline{\partial}\widetilde{f}(z)B(z)L(dz) := \lim_{\epsilon\to 0}\int_{|\mathrm{Im}z|\geq\epsilon}\overline{\partial}\widetilde{f}(z)B(z)L(dz), \tag{4.16}$$

when the limit exists.

The following proposition will be of constant use: it justifies the existence of integrals involving almost analytic extensions and, in the applications, it will allow to estimate certain remainder terms.

Proposition 4.17. *Fix $f \in C_0^\infty(\mathbb{R})$ and $\widetilde{f}$ an almost analytic extension of f supported in $[a,b] + i[c,d]$. For all continuous function*

$$B : [a,b] + i[c,d] \setminus \mathbb{R} \to \mathcal{B}$$

with values in a Banach space $\mathcal{B}$ and such that, for some $C, M \geq 0$,

$$||B(z)||_{\mathcal{B}} \leq C |\mathrm{Im}(z)|^{-M}, \qquad z \in [a,b] + i[c,d] \setminus \mathbb{R},$$

the following hold:

1. *the integral*
$$\int_{\mathbb{C}} \overline{\partial} \widetilde{f}(z) B(z) L(dz)$$
is well defined in the sense of (4.16),

2. *we have the bound*
$$\left| \left| \int_{\mathbb{C}} \overline{\partial} \widetilde{f}(z) B(z) L(dz) \right| \right|_{\mathcal{B}} \leq C \sup_{[a,b]+i[c,d] \setminus \mathbb{R}} \left| \left| \mathrm{Im}(z)^M B(z) \right| \right|_{\mathcal{B}}.$$

Proof. By standard results, the map $y \mapsto \int_{\mathbb{R}} \overline{\partial} \widetilde{f}(x,y) B(x+iy) dx$ is continuous on $[c,d] \setminus \{0\}$ and satisfies

$$\left| \left| \int_{\mathbb{R}} \overline{\partial} \widetilde{f}(x,y) B(x+iy) dx \right| \right|_{\mathcal{B}} \leq (b-a) \sup_{x \in [a,b]} \left| \left| y^M B(x+iy) \right| \right|_{\mathcal{B}} \sup_{x \in [a,b]} \left| y^{-M} \overline{\partial} \widetilde{f}(x,y) \right|,$$

for all $y \in [c,d] \setminus \{0\}$. The result follows easily after integration with respect to y. $\square$

The main interest of almost analytic extensions in these notes is the following Cauchy type formula.

Proposition 4.18. *Let $f \in C_0^\infty(\mathbb{R})$ and $\widetilde{f} \in C_0^\infty(\mathbb{R}^2)$ be an almost analytic extension of f. Then*

$$\frac{1}{2\pi} \int_{\mathbb{C}} \overline{\partial} \widetilde{f}(z) (\lambda - z)^{-1-j} L(dz) = \frac{(-1)^j}{j!} f^{(j)}(\lambda), \tag{4.17}$$

for all integer $j \geq 0$ and $\lambda \in \mathbb{R}$.

Proof. Observe on one hand that $\partial_x^j \widetilde{f}$ is an almost analytic extension of $f^{(j)}$ (use for instance (4.12)) and on the other hand that, by integrations by part in x,

$$\int_{|\mathrm{Im} z| \geq \epsilon} \overline{\partial} \widetilde{f}(z) (\lambda - z)^{-1-j} L(dz) = \frac{(-1)^j}{j!} \int_{|\mathrm{Im} z| \geq \epsilon} \overline{\partial} \partial_x^j \widetilde{f}(z) (\lambda - z)^{-1} L(dz).$$

Therefore, it suffices to prove the result when $j = 0$ which we do now. For fixed ϵ, we integrate by part with $\overline{\partial}$ and use that $\overline{\partial}(\lambda - z)^{-1} = 0$ to get

$$\int_{|\mathrm{Im} z| \geq \epsilon} \overline{\partial}\widetilde{f}(z)(\lambda - z)^{-1}L(dz) = i\int_{\mathbb{R}} \widetilde{f}(x, -\epsilon)\frac{1}{\lambda - x + i\epsilon} - \widetilde{f}(x, \epsilon)\frac{1}{\lambda - x - i\epsilon}dx. \tag{4.18}$$

Then, using

$$\widetilde{f}(x, \pm\epsilon) = f(x) \pm \epsilon\partial_y \widetilde{f}(x, 0) + \mathcal{O}(\epsilon^2\langle x\rangle^{-2})$$

and the fact that

$$\frac{1}{|\lambda - x \pm i\epsilon|} \leq \frac{1}{\epsilon},$$

the right hand side of (4.18) can be written

$$2\epsilon\int_{\mathbb{R}} f(x)\frac{dx}{(x-\lambda)^2 + \epsilon^2} + i\epsilon\int_{\mathbb{R}} \partial_y\widetilde{f}(x, 0)\frac{2(x-\lambda)}{(x-\lambda)^2+\epsilon^2}dx + \mathcal{O}(\epsilon)$$

that is,

$$2\int_{\mathbb{R}} f(\lambda + \epsilon t)\frac{dt}{1+t^2} - i\epsilon\int_{\mathbb{R}} \partial_x\partial_y\widetilde{f}(\lambda + s, 0)\ln(s^2 + \epsilon^2)ds + \mathcal{O}(\epsilon). \tag{4.19}$$

By dominated convergence, using the bound $\left|\ln(s^2+\epsilon^2)\right| \leq C\max\left(|\ln\epsilon|, \langle s\rangle\right)$, (4.19) converges to $2\pi f(\lambda)$ as ϵ goes to zero. $\square$

Exercise 4.19. *Check that if $\mathcal{B} = \mathcal{L}(\mathcal{H}, \mathcal{K})$ and if $u \in \mathcal{H}$ then*

$$\left(\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)B(z)L(dz)\right)u = \int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)B(z)uL(dz).$$

Proposition 4.22 below is the main result of this section. Its interest is to give expressions of functions of Δ_g in term of its resolvent. We note that the definition of the resolvent given below uses implicitly the result of Theorem 2.2; of course the proof of Theorem 2.2 (given in the next chapter) will not use Proposition 4.22.

Definition 4.20 (Resolvent of the Laplacian)**.** *For $z \in \mathbb{C} \setminus [0, +\infty)$ we define*

$$R(z) = (-\Delta_g - z)^{-1} := f_z(-\Delta_g),$$

with $f_z(\lambda) = (\lambda - z)^{-1}$, using (2.6).

Notice that, by (2.8), one has in particular

$$\left|\left|(-\Delta_g - z)^{-1}\right|\right|_{L^2(M)\to L^2(M)} \leq \frac{1}{|\mathrm{Im}(z)|}, \qquad z \in \mathbb{C}\setminus\mathbb{R}. \tag{4.20}$$

Exercise 4.21. *Show that the map $z \mapsto R(z)$ is continuous from $\mathbb{C}\backslash[0,+\infty)$ to $\mathcal{L}(L^2(M))$. Hint. Use (and prove) that $R(z) - R(\zeta) = -(z-\zeta)R(z)R(\zeta)$.*

Proposition 4.22 (Helffer-Sjöstrand formula)**.** *Let $f \in C_0^\infty(\mathbb{R})$ and $\widetilde{f}$ be an almost analytic extension of f. Then*

$$f(-\Delta_g) = \frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(-\Delta_g - z)^{-1}L(dz). \tag{4.21}$$

Proof. Both sides of the identity are well defined bounded operators (by Proposition 4.17 for the right hand side) hence it suffices to check that they coincide on a dense subspace. We consider the subspace a finite linear combinations of eigenfunctions of Δ_g, ie of vectors of the form $u = \sum_{j\leq J}\left(e_j,u\right)_{L^2(M)} e_j$. The right hand side of (4.21) applied to such a vector reads

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(-\Delta_g - z)^{-1}L(dz)u &= \sum_{j\leq J}\left(e_j,u\right)_{L^2(M)}\frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(-\Delta_g - z)^{-1}e_jL(dz) \\
&= \sum_{j\leq J}\left(e_j,u\right)_{L^2(M)}\frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(\lambda_j - z)^{-1}e_jL(dz) \\
&= \sum_{j\leq J}\left(e_j,u\right)_{L^2(M)} f(\lambda_j)e_j, \\
&= f(-\Delta_g)u,
\end{aligned}$$

using Proposition 4.18 to go from the second to the third line. This completes the proof. □

Chapter 5

L^2 bounds for pseudo-differential operators

5.1 L^2 estimates

We recall the following notation which will use extensively in this chapter,

$$S^m = S^{0,m}, \qquad (u,v) = \int_{\mathbb{R}^n} \overline{u}v, \qquad ||u||^2 = (u,u).$$

Recall that S^m was introduced in Definition 3.6 and that $(.,.)$ and $||\cdot||$ are respectively the inner product and the norm of $L^2(\mathbb{R}^n)$.

The main result of this section is the following.

Theorem 5.1. *Fix the dimension n. Then there exist $C > 0$ and a seminorm $\mathcal{N}_N^{S^0}$ of S^0 such that*

$$||Op(a)u|| \leq C\mathcal{N}_N^{S^0}(a)||u||, \tag{5.1}$$

for all $a \in S^0$ and all $u \in \mathcal{S}$.

This result means that pseudo-differential operators with symbols in S^0 are bounded on L^2. We point out that this L^2 boundedness remains true for a larger class of symbols (*e.g.* symbols in $\mathcal{A}^0$, see Definition 3.20) by the so called *Calderón-Vaillancourt theorem* whose proof is more technical. Moreover, Theorem 5.1 is sufficient for many interesting applications, in particular for those considered in these notes.

Here is the semiclassical version of Theorem 5.1.

Corollary 5.2. *There exist $C > 0$ and $N \geq 0$ (depending on n) such that*

$$||Op_h(a)u|| \leq C\mathcal{N}_N^{S^0}(a)||u||,$$

for all $a \in S^0$, all $u \in \mathcal{S}$ and all $h \in (0,1]$.

Notice that the right hand side does not depend on h. This corollary is a straightforward consequence of Theorem 5.1 and the following exercise.

Exercise 5.3. *For $a \in S^0$, denote $a_h(x,\xi) = a(x, h\xi)$. Show that for all $N \geq 0$ there exists $C > 0$ such that*

$$\mathcal{N}_N^{S^0}(a_h) \leq C\mathcal{N}_N^{S^0}(a),$$

for all $a \in S^0$ and $h \in (0,1]$.
Hint. Use that $h^{|\beta|}\langle\xi\rangle^{\beta} \leq \langle h\xi\rangle^{|\beta|}$.

Before proving Theorem 5.1, we record a few comments and consequences.

Since $\mathcal{S}$ is dense in $L^2(\mathbb{R}^n)$, Theorem 5.1 implies that $Op(a)$ has a unique continuous extension as a linear map on $L^2(\mathbb{R}^n)$ (also called its *closure*) which we shall denote by $\overline{Op(a)}$, or even simply by $Op(a)$ when there is no possible confusion (of course, a similar convention will be used for $Op_h(a)$ in the semiclassical case).

Exercise 5.4. *Let $u \in L^2(\mathbb{R}^n)$ and $a \in S^0$. Considering L^2 functions as temperate distributions, show that $\overline{Op(a)}u$, defined as above, coincides with $Op(a)u$ in the sense of Definition 3.18*[1]*.*

From the L^2 boundedness of pseudo-differential operators, we will easily derive the continuity of pseudo-differential operators on the Sobolev spaces $H^s(\mathbb{R}^n)$. We recall that, for $s \in \mathbb{R}$, $H^s(\mathbb{R}^n)$ is the space of temperate distributions u whose Fourier transform $\hat{u}$ belongs to L^2_{loc} and such that

$$||u||^2_{H^s} := (2\pi)^{-n} \int \langle\xi\rangle^{2s} \left|\hat{u}(\xi)\right|^2 d\xi < \infty.$$

Equivalently, $H^s(\mathbb{R}^n)$ is the closure of the Schwartz space for the norm $||\cdot||_{H^s}$ defined above. In the case when $s \in \mathbb{N}$, these definitions are equivalent to require that $\partial^\alpha u$ (in the distributions sense) belongs to $L^2(\mathbb{R}^n)$ for all $|\alpha| \leq s$.

Exercise 5.5. *Let u be a temperate distribution (or $u \in L^2(\mathbb{R}^n)$) and $s \in \mathbb{R}$. Denote*

$$\langle D\rangle^s = Op(\langle\xi\rangle^s).$$

1. *Show that $u \in H^s(\mathbb{R}^n)$ iff $\langle D\rangle^s u \in L^2(\mathbb{R}^n)$, and that*

$$||u||_{H^s} = ||\langle D\rangle^s u||_{L^2}.$$

2. *Show that $\langle D\rangle^s$ is a bijective isometry from $H^s(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with inverse $\langle D\rangle^{-s}$.*

Corollary 5.6 (Action on Sobolev spaces)**.** *Let $m, s \in \mathbb{R}$. Let $a \in S^m$. Then $Op(a)$ maps continuously $H^s(\mathbb{R}^n)$ on $H^{s-m}(\mathbb{R}^n)$.*

[1] here we consider the case when $h = 1$ since the dependence on h is irrelevant for this question.

Proof of Corollary 5.6. By symbolic calculus, we can write

$$\langle D\rangle^{s-m} Op(a)\langle D\rangle^{-s} = Op(b),$$

for some $b \in S^0$. Therefore, we have

$$Op(a) = \langle D\rangle^{m-s} Op(b)\langle D\rangle^{s},$$

where $\langle D\rangle^s$ maps H^s in L^2 by Exercise 5.5, $Op(b)$ maps L^2 in L^2 by Theorem 5.1 and $\langle D\rangle^{m-s}$ maps L^2 in H^{s-m} by Exercise 5.5. The result follows. □

The rest of the section is devoted to the proof of Theorem 5.1 which rests on the following lemma.

Lemma 5.7 (The Schur test)**.** *Let K be a continuous function on $\mathbb{R}^{2n}$ such that*

$$\sup_{x\in\mathbb{R}^n} \int |K(x,y)|dy \leq M, \qquad \sup_{y\in\mathbb{R}^n} \int |K(x,y)|dx \leq M.$$

Set

$$\mathcal{K}u(x) = \int K(x,y)u(y)dy.$$

Then

$$||\mathcal{K}u|| \leq M||u||,$$

for all $u \in \mathcal{S}$.

Proof. We leave as an exercise (Exercise 5.8 below) the proof that $\mathcal{K}u$ is a well defined continuous function. Then

$$\begin{aligned} |\mathcal{K}u(x)| &\leq \int |K(x,y)u(y)|dy = \int |K(x,y)|^{1/2}|K(x,y)|^{1/2}|u(y)|dy \\ &\leq \left(\int |K(x,y)|dy\right)^{1/2} \left(\int |K(x,y)||u(y)|^2 dy\right)^{1/2} \end{aligned} \tag{5.2}$$

by the Cauchy-Schwartz inequality. Therefore, squaring this inequality and integrating with respect to x, we get

$$\int |\mathcal{K}u(x)|^2 dx \leq M \int\int |K(x,y)||u(y)|^2 dydx \leq M^2||u||^2$$

using the Fubini Theorem. This completes the proof. □

Exercise 5.8. *Check that $\mathcal{K}u$ is a well defined continuous function for every $u \in \mathcal{S}$.*

We shall see that the L^2 boundedness of $Op(a)$ is a fairly direct consequence of the Schur test if $a \in S^{-\epsilon}$ for some $\epsilon > 0$. To deal with the case $\epsilon = 0$, we will use the following lemma.

Lemma 5.9. *Fix $N \geq 0$. For all $a \in S^0$ denote $M(a) := \mathcal{N}_N^{S^0}(a)$. Then*

$$b := (2M(a)^2 - |a|^2)^{1/2}$$

belongs to S^0 and there exists $C > 0$ such that

$$\mathcal{N}_N^{S^0}(b) \leq CM(a), \tag{5.3}$$

for all $a \in S^0$.

Proof. If $M(a) = 0$, then $a = b = 0$ and the result is trivial. Otherwise, after division by $M(a)$, we may assume that $M(a) = 1$. Then b is smooth and

$$|b|^2 = 2 - |a|^2 \leq 2 = 2 \sup_{\mathbb{R}^{2n}} |a|^2 \leq 2M(a)^2.$$

If $|\alpha| + |\beta| > 0$, an induction shows that $\partial_x^\alpha \partial_\xi^\beta b$ is a (universal) linear combination of

$$\big(2 - |a|^2\big)^{-k/2} \partial_x^{\alpha_1} \partial_\xi^{\beta_1} a_1 \cdots \partial_x^{\alpha_j} \partial_\xi^{\beta_j} a_j, \qquad a_1, \ldots, a_j = a \text{ or } \overline{a}, \tag{5.4}$$

with

$$\alpha_1 + \cdots + \alpha_j = \alpha, \qquad \beta_1 + \cdots + \beta_j = \beta, \qquad 1 \leq j \leq 2N, \qquad 1 \leq k \leq 2N.$$

Thus, if $|\alpha + \beta| \leq N$

$$\langle \xi \rangle^{|\beta|} |\partial_x^\alpha \partial_\xi^\beta b(x, \xi)| \lesssim 1,$$

for all $(x, \xi) \in \mathbb{R}^{2n}$ and all a such that $M(a) = 1$, which proves (5.3). For $|\alpha + \beta| > N$, the form of $\partial_x^\alpha \partial_\xi^\beta b$ given by (5.4) remains of course valid and allows to check that $|\partial_x^\alpha \partial_\xi^\beta b(x, \xi)| \lesssim C_{a,\alpha,\beta} \langle \xi \rangle^{-|\beta|}$, ie that $b \in S^0$ (but $C_{a,\alpha,\beta}$ is not any longer uniform with respect to a). $\square$

Proof of Theorem 5.1. Let us fix $m > n$ and consider first the case when $a \in S^{-m}$. The kernel of $Op(a)$ is

$$K(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, \xi) d\xi.$$

(See (3.27) where this formula was obtained). Note that the assumption on m guarantees the convergence of the integral and its boundedness with respect to (x, y). By integration by part, we have

$$\big(1 + (x_1 - y_1)^2\big) K(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \big(1 - \partial_{\xi_1}^2\big) a(x, \xi) d\xi,$$

whose (modulus of the) right hand side is bounded by

$$\left((2\pi)^{-n} \int \langle \xi \rangle^{-m} d\xi \right) \sup_{\mathbb{R}^{2n}} \big| (1 - \partial_{\xi_1}^2) a(x, \xi) \big| \langle \xi \rangle^m.$$

Repeating the integrations by part with respect to the other variables, we obtain

$$|K(x,y)| \leq C\langle x_1 - y_1\rangle^{-2} \cdots \langle x_n - y_n\rangle^{-2} \mathcal{N}_{N_0}^{S^{-m}}(a),$$

for some $N_0 > 0$ (here $N_0 = 2n$ is sufficient). By the Schur test, this implies that

$$||Op(a)u|| \leq C\mathcal{N}_{N_0}^{S^{-m}}(a)||u||. \tag{5.5}$$

Now assume only that $a \in S^{-m/2}$. By symbolic calculus (Theorems 3.14 and 3.16 with $h = 1$), one can write

$$Op(a)^* Op(a) = Op(\tilde{a}), \tag{5.6}$$

for some $\tilde{a} \in S^{-m}$ depending continuously on a. Therefore, we have

$$\begin{aligned} ||Op(a)u||^2 = \big(u, Op(a)^* Op(a)u\big) = \big(u, Op(\tilde{a})u\big) &\leq ||u||||Op(\tilde{a})u|| \\ &\leq C\mathcal{N}_{N_0}^{S^{-m}}(\tilde{a})||u||^2, \end{aligned} \tag{5.7}$$

using (5.5) in the last step. The continuous dependence of $\tilde{a}$ on a shows that, for some N_1,

$$\mathcal{N}_{N_0}^{S^{-m}}(\tilde{a}) \leq C\mathcal{N}_{N_1}^{S^{-m/2}}(a)^2.$$

Using (5.7), this shows that (5.5) holds with m replaced by $m/2$ and N_0 replaced by N_1. Iterating this procedure, we obtain for each $k \in \mathbb{N}$ the existence of C_k and N_k such that

$$||Op(a)u|| \leq C_k \mathcal{N}_{N_k}^{S^{-m/2^k}}(a)||u||. \tag{5.8}$$

At this point, since k can be chosen as large as we wish, we obtain the L^2 boundedness of $Op(a)$ whenever $a \in S^{-\epsilon}$ for some $\epsilon > 0$. To complete the proof for $a \in S^0$, we proceed as follows. Let us fix k such that $m/2^k \leq 1$. By symbolic calculus, the symbol $\tilde{a}$ in (5.6) satifies $\tilde{a} - \overline{a}a \in S^{-1}$, therefore

$$\begin{aligned} \big|\big|Op(a)u\big|\big|^2 &= \big(u, Op(a^*)Op(a)u\big) \\ &= \big(u, Op(|a|^2)u\big) + \big(u, Op(r_a)u\big), \end{aligned} \tag{5.9}$$

for some $r_a \in S^{-1}$ depending continuously on $a \in S^0$. Using the notation of Lemma 5.9 with some N to be chosen latter, we have similarly

$$\begin{aligned} \big|\big|Op(b)u\big|\big|^2 &= \big(u, Op(|b|^2)u\big) + \big(u, Op(r_b)u\big) \\ &= 2M_a^2||u||_{L^2}^2 - \big(u, Op(|a|^2)u\big) + \big(u, Op(r_b)u\big), \end{aligned} \tag{5.10}$$

with $r_b \in S^{-1}$ depending continuously on $b \in S^0$. Using that $\big|\big|Op(b)u\big|\big|^2$ is nonnegative, (5.9) and (5.10) yield

$$\big|\big|Op(a)u\big|\big|^2 \leq 2M(a)^2||u||^2 + \big(u, Op(r_a)u\big) + \big(u, Op(r_b)u\big).$$

Using (5.8) for r_a and r_b, we obtain

$$\big|\big|Op(a)u\big|\big|^2 \leq \left(2M(a)^2 + C_k\mathcal{N}_{N_k}^{S^{-m/2^k}}(r_a) + C_k\mathcal{N}_{N_k}^{S^{-m/2^k}}(r_b)\right)||u||^2. \quad (5.11)$$

We now observe that $r_a = r_{-1}^{\#}(a^*, a)$ (see Theorems 3.14 and 3.16 for the notation) so that, by continuity of $a \mapsto a^*$ in S^0 and continuity of $r_{-1}^{\#}(a^*, a) \in S^{-1} \subset S^{-m/2^k}$ with respect to a^* and a in S^0, there exists M_k such that

$$\mathcal{N}_{N_k}^{S^{-m/2^k}}(r_a) \lesssim \mathcal{N}_{M_k}^{S^0}(a)^2, \quad (5.12)$$

for all $a \in S^0$. Using this bound for b and choosing $N = M_k$ in Lemma 5.9, we obtain

$$\mathcal{N}_{N_k}^{S^{-m/2^k}}(r_b) \lesssim \mathcal{N}_{M_k}^{S^0}(b)^2 \lesssim \mathcal{N}_{M_k}^{S^0}(a)^2, \quad (5.13)$$

Summing up, (5.11), (5.12) and (5.13) show that

$$||Op(a)u||_{L^2}^2 \lesssim \mathcal{N}_N^{S^0}(a)^2||u||^2, \qquad a \in S^0, \ u \in \mathcal{S},$$

which is precisely the result. □

5.2 Hilbert-Schmidt estimates

The main result of this section is the following.

Theorem 5.10. *Let $\rho > n/2$. Then, for all $h \in (0,1]$ and $a \in S^{-\rho,-\rho}$*

1. *$Op_h(a)$ is Hilbert-Schmidt on $L^2(\mathbb{R}^n)$,*
2. *its Hilbert-Schmidt norm reads*

$$||Op_h(a)||_{\mathrm{HS}} = (2\pi h)^{-n/2}||a||_{L^2(\mathbb{R}^{2n})}.$$

Before giving the proof, we record the following useful result which is a simple analogue of Lemma 3.24 and whose proof is left as an exercise to the reader.

Exercise 5.11. *Let $m, \mu \in \mathbb{R}$ and $a \in S^{\mu,m}$. Show that, if we set*

$$a_j(x,\xi) = \chi(x/j, \xi/j)a(x,\xi),$$

with $\chi \in C_0^\infty$ which is equal to 1 near $(0,0)$, then for all $\mu' > \mu$ and $m' > m$,

$$a_j \to a \qquad \text{in } S^{\mu',m'}.$$

Proof of Theorem 5.10. Assume first that a belongs to the Schwartz class. Then, using Definition 4.6 (see also the calculation leading to (3.27)), we see that

$$Op_h(a) = A_K, \qquad K(x,y) = (2\pi h)^{-n}\widehat{a}\left(x, \frac{y-x}{h}\right) \tag{5.14}$$

where $\widehat{a}$ is the Fourier transform of $a(x,\xi)$ with respect to ξ. Since

$$\iint \left|\widehat{a}\left(x, \frac{y-x}{h}\right)\right|^2 dxdy = h^n \iint |\widehat{a}(x,z)|^2 dxdz = (2\pi h)^n \iint |a(x,\xi)|^2 dxd\xi,$$

the result follows from Proposition 4.7. In the general case, Exercise 5.11 allows to pick $a_j \in C_0^\infty(\mathbb{R}^{2n})$ which converges to a in $S^{-\rho',-\rho'}$ with $\rho > \rho' > n/2$. Then

$$||Op_h(a_j)||_{\mathrm{HS}} = (2\pi h)^{-n/2}||a_j||_{L^2(\mathbb{R}^{2n})}. \tag{5.15}$$

Since $a_j \to a$ in $L^2(\mathbb{R}^{2n})$, $Op_h(a_j)$ is a Cauchy sequence in $S_2(L^2)$ hence in the space of bounded operators. On the other hand, $a_j \to a$ in $S^{0,0}$ hence $Op_h(a_j) \to Op_h(a)$ pointwise on the Schwartz space. This implies that the limit of $Op_h(a_j)$ in $S_2(L^2)$ is necessarily the L^2 bounded operator $Op_h(a)$ which is thus an Hilbert-Schmidt operator. By letting $j \to \infty$ in (5.15), the result follows. □

A simple and useful consequence of Theorem 5.10 is the following one.

Corollary 5.12. *If $a \in S^{-\epsilon,-\epsilon}$, for some $\epsilon > 0$, then $Op_h(a)$ is a compact operator on $L^2(\mathbb{R}^n)$.*

Proof. By Exercise 5.11, we may choose $a_j \in C_0^\infty(\mathbb{R}^{2n})$ such that $a_j \to a$ in $S^{0,0}$. By Theorem 5.1, this implies the following convergence in operator norm on $L^2(\mathbb{R}^n)$,

$$Op_h(a_j) \to Op_h(a), \qquad j \to \infty.$$

Since each $Op_h(a_j)$ is Hilbert-Schmidt by Theorem 5.10, hence is compact by Corollary 4.5, this implies that $Op_h(a)$ is compact. □

5.3 Trace class estimates

In this section, we give a simple criterion for a pseudo-differential operator to be trace class and give a simple formula for its trace. This is the purpose of the following theorem.

Theorem 5.13. *Fix $\rho > n$.*

1. *For all $a \in S^{-\rho,-\rho}$ and $h \in (0,1]$, $Op_h(a)$ is trace class on $L^2(\mathbb{R}^n)$.*
2. *There exist C and a seminorm $\mathcal{N}_N^{S^{-\rho,-\rho}}$ such that for all $a \in S^{-\rho,-\rho}$ and all $h \in (0,1]$,*

$$||Op_h(a)||_{\mathrm{tr}} \leq Ch^{-n}\mathcal{N}_N^{S^{-\rho,-\rho}}(a).$$

3. For all $a \in S^{-\rho,-\rho}$ and all $h \in (0,1]$,

$$\mathrm{tr}\big(Op_h(a)\big) = (2\pi h)^{-n} \iint a(x,\xi) dx d\xi. \tag{5.16}$$

We will need a general lemma.

Lemma 5.14. *Let $a_1, a_2 \in \mathcal{S}^{\mu,m}$ with $m < -n$. If $Op(a_1) = Op(a_2)$ then $a_1 = a_2$.*

Let us remark that the condition $m < -n$ can be removed, but the proof is more straightforward with this extra condition and sufficient for the present purpose.

Proof. If $u \in \mathcal{S}(\mathbb{R}^n)$ and x is fixed, the condition $m < -n$ allows to write

$$\begin{aligned} Op(a_1)u(x) &= (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a_1(x,\xi) u(y) dy d\xi, \\ &= (2\pi)^{-n} \int \widehat{a}_1(x, y-x) u(y) dy, \end{aligned}$$

where $\widehat{\cdot}$ is the Fourier transform with respect to ξ. Thus the assumption $Op(a_1) = Op(a_2)$ implies that, if we set $b_x(y) = \widehat{a}_1(x, y-x) - \widehat{a}_2(x, y-x)$, we have

$$\int b_x(y) u(y) dy = 0, \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

This implies that, for each fixed x, $b_x(\cdot) \equiv 0$. By taking the inverse Fourier transform, we see that $a_1 = a_2$. ☐

Proof of Theorem 5.13. We will prove item 1 and item 2 simultaneously. Set $w(x,\xi) = \langle x \rangle^{\rho/2} \langle \xi \rangle^{\rho/2}$, which belongs to $S^{\rho/2,\rho/2}$, and

$$W_h := \langle x \rangle^{\rho/2} \langle hD \rangle^{\rho/2} = Op_h(w).$$

On the other hand, introduce

$$B_h := \langle hD \rangle^{-\rho/2} \langle x \rangle^{-\rho/2} = Op_h\left(\langle x \rangle^{-\rho/2} \langle \xi \rangle^{-\rho/2} \right)^*,$$

which is Hilbert-Schmidt by Proposition 5.10 with $||B_h||_{\mathrm{HS}} \le C h^{-n/2}$. Since $B_h W_h = I$ on the Schwartz space, we have

$$Op_h(a) = B_h\big(W_h Op_h(a)\big), \tag{5.17}$$

and therefore

$$\big|\big|Op_h(a)\big|\big|_{\mathrm{tr}} \le ||B_h||_{\mathrm{HS}} \big|\big|W_h Op_h(a)\big|\big|_{\mathrm{HS}}.$$

By symbolic calculus and the fact that $w \in S^{\rho/2,\rho/2}$, we have

$$W_h Op_h(a) = Op_h(c_h), \tag{5.18}$$

for some bounded family $(c_h)_{h\in(0,1]}$ of $S^{-\rho/2,-\rho/2}$ depending continuously on $a \in S^{-\rho,-\rho}$. This implies that for some N and C,

$$||c_h||_{L^2(\mathbb{R}^{2n})} \leq C\mathcal{N}_N^{S^{-\rho,-\rho}}(a), \qquad a \in S^{-\rho,-\rho}, \ h \in (0,1].$$

The conclusion follows then from Proposition 5.10. To prove item 3, it suffices to prove the result when $h = 1$. We start with the following computation. If b, c are Schwartz functions on $\mathbb{R}^{2n}$, then $Op(b)$ and $Op(c)$ are Hilbert-Schmidt so using Proposition 4.14 and (5.14), we have

$$\operatorname{tr}\left(Op(b)Op(c)\right) = (2\pi)^{-2n} \iint \widehat{b}\,(x, y-x)\,\widehat{c}\,(y, x-y)\,dxdy. \tag{5.19}$$

On the other hand, we recall from the symbolic calculus (see (3.37)) that one can write $Op(b)Op(c) = Op(a)$ with

$$a(x,\xi) = (2\pi)^{-n} \iint e^{-iz\cdot\zeta} b(x,\xi+\zeta)c(x+z,\xi)dzd\zeta.$$

Thus, using the change of variables $z = y - x$ and $\zeta = \eta - \xi$, we have

$$\begin{aligned}(2\pi)^{-n} \iint a(x,\xi)dxd\xi &= (2\pi)^{-n} \iiiint e^{-i(y-x)\cdot(\eta-\xi)} b(x,\eta)c(y,\xi)dyd\eta dxd\xi \\ &= (2\pi)^{-n} \iint \widehat{b}\,(x, y-x)\,\widehat{c}\,(y, x-y)\,dydx\end{aligned}$$

which shows together with (5.19) that (5.16) holds when $Op(a) = Op(b)Op(c)$ and b, c are Schwartz functions, ie

$$\operatorname{tr}\big(Op(b)Op(c)\big) = (2\pi)^{-n} \int\int (b\#c)(x,\xi)dxd\xi. \tag{5.20}$$

We next check that this remains true if $b, c \in S^{-\rho/2,-\rho/2}$. Indeed, according to Exercise 5.11, we may approximate b, c by Schwartz functions b_j, c_j for the topology of $S^{-\rho'/2,-\rho'/2}$ with $\rho > \rho' > n$. Then

$$Op(b_j)Op(c_j) = Op(b_j\#c_j),$$

with $b_j\#c_j \to b\#c$ in $S^{-\rho',-\rho'}$ as $j \to \infty$ by Theorem 3.14, and we also have

$$Op(b_j)Op(c_j) \to Op(b)Op(c), \qquad Op(b_j\#c_j) \to Op(b\#c) \qquad \text{in trace class norm,}$$

by Proposition 5.10 in the first case and items 1 and 2 of the present proposition in the second case. On the other hand, we have

$$b_j\#c_j \to b\#c \qquad \text{in } L^1(\mathbb{R}^{2n}),$$

using the embedding $S^{-\rho',-\rho'} \subset L^1(\mathbb{R}^{2n})$. Using (5.20) for b_j and c_j, and then letting j go to infinity, we obtain that (5.20) remains true if $b, c \in S^{-\rho/2,-\rho/2}$. To complete the proof, it suffices to note that, for any $a \in S^{-\rho,-\rho}$, one can write

$$Op(a) = Op(b)Op(c), \tag{5.21}$$

for some b and c in $S^{-\rho/2,-\rho/2}$, using (5.17) and (5.18). Thus

$$\operatorname{tr}\bigl(Op(a)\bigr) = (2\pi)^{-n} \iint (b\#c)(x,\xi)dxd\xi,$$

by the previous step. Since $Op(b)Op(c) = Op(b\#c)$, Lemma 5.14 and (5.21) imply that $a = b\#c$ and the result follows. $\square$

Exercise 5.15. *Show that if $\mathcal{K}$ is an operator with kernel $K \in \mathcal{S}(\mathbb{R}^{2n})$, then $\mathcal{K}$ is trace class and*

$$\operatorname{tr}_{L^2(\mathbb{R}^{2n})}(\mathcal{K}) = \int_{\mathbb{R}^n} K(x,x)dx.$$

Hint. Write $\mathcal{K}$ as $Op(a)$ for some suitable a (use (3.27) to find a).

Chapter 6

Elliptic parametrix and applications

In this chapter, we construct a parametrix for a semiclassical elliptic operator, that is an approximate inverse of $h^2P - z$ if P is a second order elliptic differential operator. Here h is the semiclassical parameter and z is a spectral parameter which, in this chapter, will belong to $\mathbb{C} \setminus [0, +\infty)$. This analysis can be generalized in many directions but we focus on this example which will be sufficient for our applications. In Section 6.1, we construct the parametrix for an operator which is globally elliptic on $\mathbb{R}^n$, to ignore the problems of localization and local charts on a manifold. In Section 6.2, we explain how to use the result on $\mathbb{R}^n$ to obtain fairly directly a parametrix on any relatively compact subset of an open set and also derive the local elliptic regularity theorem.

6.1 Parametrix on $\mathbb{R}^n$

Let P be a differential operator on $\mathbb{R}^n$ of the form

$$P = p_2(x, D) + p_1(x, D) + p_0(x), \tag{6.1}$$

with

$$p_2(x, \xi) = \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k, \qquad p_1(x, \xi) = \sum_{j=1}^{n} b_j(x)\xi_j,$$

such that $a_{jk} = a_{kj}$ for all j, k. We assume that

$$p_{2-j} \in S^{2-j}, \qquad j = 0, 1, 2. \tag{6.2}$$

We refer to Definition 3.6 for S^{2-j} and recall that, according to Exercise 3.7, this condition is equivalent to the fact that a_{jk}, b_j, p_0 are smooth functions which are bounded on $\mathbb{R}^n$ together with all their derivatives. Throughout this section, p_2 will be called the **principal symbol** of P. We assume that p_2 is a real valued and such that, for some $c > 0$,

$$p_2(x, \xi) \geq c|\xi|^2, \qquad x, \xi \in \mathbb{R}^n, \tag{6.3}$$

which is our ellipticity assumption. We consider the semiclassical operator

$$P(h) := h^2 P = \sum_{j=0}^{2} h^j p_{2-j}(x, hD),$$

and look for an approximate inverse for $P(h) - z$ in terms of pseudo-differential operators. Since

$$P(h) - z = Op_h(p_2 - z) + hOp_h(p_1) + h^2 Op_h(p_0),$$

and using the intuition suggested by Exercise 3.9, it is natural to consider the function $(p_2 - z)^{-1}$ which is well defined provided that

$$z \in \mathbb{C} \setminus [0, +\infty), \tag{6.4}$$

since p_2 takes its values in $[0, +\infty)$. We record the following useful result.

Lemma 6.1. *There exists $C > 0$ such that*

$$\left|(p_2(x,\xi) - z)^{-1}\right| \leq C \frac{\langle z \rangle}{d(z, \mathbb{R}^+)} \langle \xi \rangle^{-2},$$

for all $x, \xi \in \mathbb{R}^n$ and all z satisfying (6.4).

Proof. We write

$$\frac{1}{p_2 - z} = \frac{p_2 + 1}{p_2 - z} \frac{1}{p_2 + 1},$$

where, by (6.3), we have

$$0 \leq \frac{1}{p_2(x,\xi) + 1} \leq C \langle \xi \rangle^{-2}$$

and, by the inequalities $|p_2 - z| \geq d(z, \mathbb{R}^+)$ and $d(z, \mathbb{R}^+) \leq |z|$,

$$\left| \frac{p_2 + 1}{p_2 - z} \right| = \left| 1 + \frac{z + 1}{p_2 - z} \right| \leq 1 + \frac{|z| + 1}{d(z, \mathbb{R}^+)} \leq C \frac{\langle z \rangle}{d(z, \mathbb{R}^+)},$$

which completes the proof. $\square$

Proposition 6.2. *For all $z \in \mathbb{C} \setminus [0, +\infty)$, $(p_2 - z)^{-1}$ belongs to S^{-2}. More precisely, for all $\alpha, \beta \in \mathbb{N}^n$, there exists $C_{\alpha\beta}$ and such that*

$$\left| \partial_x^\alpha \partial_\xi^\beta (p_2(x,\xi) - z)^{-1} \right| \leq C_{\alpha\beta} \left(\frac{\langle z \rangle}{d(z, \mathbb{R}^+)} \right)^{|\alpha| + |\beta| + 1} \langle \xi \rangle^{-2-\beta}, \tag{6.5}$$

for all $x, \xi \in \mathbb{R}^n$ and all z satisfying (6.4). Furthermore, the map $z \mapsto (p_2 - z)^{-1}$ is continuous from $\mathbb{C} \setminus [0, +\infty)$ to S^{-2}.

Proof. We start by proving (6.5). The case $\alpha = \beta = 0$ follows from Lemma 6.1. Otherwise, by item 2 of Exercise 3.7, we have to consider terms of the form

$$\begin{aligned}\frac{\partial_x^{\alpha_1}\partial_\xi^{\beta_1}p_2\cdots\partial_x^{\alpha_j}\partial_\xi^{\beta_j}p_2}{(p_2-z)^{1+j}} &= \mathcal{O}\left(\langle\xi\rangle^{2j-|\beta_1|-\cdots-|\beta_j|}\right)\mathcal{O}\left(\langle\xi\rangle^{-2-2j}\frac{\langle z\rangle^{1+j}}{d(z,\mathbb{R}^+)^{1+j}}\right)\\ &= \mathcal{O}\left(\langle\xi\rangle^{-2-|\beta|}\frac{\langle z\rangle^{1+j}}{d(z,\mathbb{R}^+)^{1+j}}\right),\end{aligned}$$

using Lemma 6.1 and the fact that $\beta_1 + \cdots + \beta_j = \beta$. Since $\langle z\rangle/d(z,\mathbb{R}^+) \geq 1$ and $j \leq |\alpha| + |\beta|$, we can replace $1 + j$ by $1 + |\alpha| + |\beta|$ in the last line, and we obtain the expected estimates. This implies in particular that $(p_2 - z)^{-1} \in S^{-2}$. Let us prove the continuous dependence on z. Fix z_0. Then

$$(p_2 - z)^{-1} - (p_2 - z_0)^{-1} = (z - z_0)(p_2 - z)^{-1}(p_2 - z_0)^{-1}.$$

By considering $(p_2-z_0)^{-1}$ as a symbol in S^0 and using the continuity of the map $(a, b) \mapsto ab$ from $S^{-2} \times S^0$ to S^{-2} (see the Exercise 3.12), (6.5) implies that for any seminorm $\mathcal{N}_N$ of S^{-2}, there exists $M \geq 0$ such that

$$\mathcal{N}_N\big((p_2 - z)^{-1} - (p_2 - z_0)^{-1}\big) \leq C|z - z_0|\left(\frac{\langle z\rangle}{d(z,\mathbb{R}^+)}\right)^M.$$

Since the right hand side goes to zero as $z \to z_0$, we get the result. □

To construct our approximate inverse, we try to find a sequence of symbols

$$q_{z,-2-k} \in S^{-2-k}, \qquad k \geq 0,$$

such that, for each N,

$$(P(h) - z)\left(\sum_{k=0}^{N-1} h^k Op_h\big(q_{z,-2-k}\big)\right) = I + h^N Op_h\big(r_{z,-N}(h)\big), \tag{6.6}$$

with $(r_{z,-N}(h))_{h\in(0,1]}$ bounded in S^{-N}, and all these symbols satisfying nice bounds in term of z. To determine the conditions to be satisfied by the symbols $q_{z,-2-k}$, we expand the left hand side of (6.6) according to the composition formula (see Theorem 3.14 and the notation thereof) from which we get

$$(P(h) - z)\left(\sum_{k=0}^{N-1} h^k Op_h\big(q_{z,-2-k}\big)\right) = \sum_{k=0}^{N-1} h^k Op_h(c_{z,k}) + h^N Op_h\big(r_{z,-N}(h)\big), \tag{6.7}$$

with

$$c_{z,0} = (p_2 - z)q_{z,-2} \tag{6.8}$$

$$c_{z,k} = (p_2 - z)q_{z,-2-k} + \sum_{\substack{j+l+m=k\\ l<k}} \big(p_{2-j}\# q_{z,-2-l}\big)_m, \tag{6.9}$$

for $1 \leq k \leq N-1$, and with

$$r_{z,-N}(h) = \sum_{j=0}^{2} \sum_{k=0}^{N-1} h^{o(k,j)} r^{\#}_{J(k,j)}\big(p_{2-j}, q_{z,-2-k}, h\big), \tag{6.10}$$

where

$$J(k,j) = \max\big(N-k-j, 0\big), \qquad o(k,j) = k+j-N+J(k,j). \tag{6.11}$$

Note that $o(k,j) \geq 0$ (for all terms but one we have $o(k,j) = k+j-N$; we have introduced this notation only to deal with the term corresponding to $j=2$ and $k=N-1$).

By comparing the right hand sides of (6.6) and (6.7), we see that we have to require first that $c_{z,0} = 1$, that is

$$q_{z,-2} = (p_2 - z)^{-1}, \tag{6.12}$$

and that $c_{z,k} = 0$ for $k \geq 1$, that is

$$q_{z,-2-k} = -\frac{1}{p_2 - z} \sum_{\substack{j+l+m=k \\ l<k}} \big(p_{2-j} \# q_{z,-2-l}\big)_m, \qquad k \geq 1. \tag{6.13}$$

This defines the functions $q_{z,-2-k}$ inductively since the right hand side of (6.13) depends only on $q_{z,-2}, \ldots, q_{z,-2-(k-1)}$. So defined, it is easy to check by induction that $q_{z,-2-k}$ belongs to S^{-2-k} (we shall review this fact more precisely below) and, using (6.11), this implies in turn that

$$r^{\#}_{J(k,j)}\big(p_{2-j}, q_{z,-2-k}, h\big) \in S^{-j-k-J(k,j)} \subset S^{-N}.$$

In particular, this implies that $r_{z,-N}(h) \in S^{-N}$.

We have proved a large part of the following theorem.

Theorem 6.3. *There exist symbols $d_{jk} \in S^{2j-k}$, which are polynomial in ξ and independent of z and h, such that the symbols*

$$\begin{aligned} q_{z,-2-k} &= \sum_{j=1}^{2k} \frac{d_{jk}}{(p_2 - z)^{1+j}}, \qquad k \geq 1, \\ q_{z,-2} &= \frac{1}{p_2 - z}, \end{aligned} \tag{6.14}$$

satisfy for all $z \in \mathbb{C} \setminus [0, +\infty)$, all $N \geq 1$ and all $h \in (0,1]$,

$$(P(h) - z)\left(\sum_{k=0}^{N-1} h^k Op_h(q_{z,-2-k})\right) = I + h^N Op_h(r_{z,-N}(h)),$$

with a remainder such that, for all α, β, there exist $C_{\alpha\beta}$ and $M_{\alpha\beta}$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta r_{z,-N}(x,\xi,h)\right| \leq C_{\alpha\beta}\langle\xi\rangle^{-N-|\beta|}\left(\frac{\langle z\rangle}{d(z,\mathbb{R}^+)}\right)^{M_{\alpha\beta}}, \tag{6.15}$$

for all $z \in \mathbb{C}\setminus[0,+\infty)$, all $h \in (0,1]$ and all $x,\xi \in \mathbb{R}^n$. Finally, the maps

$$z \mapsto q_{z,-2-k} \in S^{-2-k}, \qquad z \mapsto r_{z,-N}(h) \in S^{-N},$$

are continuous on $\mathbb{C}\setminus[0,+\infty)$.

Let us comment on this theorem. If one fixes $N \geq 0$ (as large as we wish) and defines

$$Q_z(h) = \sum_{k=0}^{N-1} h^k Op_h(q_{z,-2-k}), \qquad R_z(h) = Op_h(r_{z,-N}(h)),$$

we have

$$(P(h)-z)Q_z(h) = I + h^N R_z(h), \tag{6.16}$$

and it follows from Corollary 5.2 and (6.15) that there exist $C, M \geq 0$ such that

$$||R_z(h)||_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C\left(\frac{\langle z\rangle}{d(z,\mathbb{R}^+)}\right)^M, \qquad h \in (0,1],\ z \in \mathbb{C}\setminus[0,+\infty).$$

This means in particular that, for a given z, $h^N R_z(h)$ is small when h is small, hence that the right hand side of (6.16) is close to the identity in the operator norm on $L^2(\mathbb{R}^n)$. This is a first justification that $Q_z(h)$ is an approximate inverse of $P(h)-z$. Another justification is in term of regularity as follows. By Corollary 5.6, we know that for all $s \in \mathbb{R}$,

$$Q_z(h) : H^s \to H^{s+2}, \qquad R_z(h) : H^s \to H^{s+N}, \tag{6.17}$$

for the symbol of $Q_z(h)$ belongs to S^{-2} and the one of $R_z(h)$ to S^{-N}. Since N is as large as we wish, $R_z(h)$ can be considered as a smoothing operator and one can thus interpret (6.16) by saying that we have inverted $P(h)-z$ modulo a smoothing operator. It is customary in PDE to treat smoothing operators as residual and we will see an illustration of this fact in Corollary 6.4 below.

The dependence on z in Theorem 6.3 is a more technical aspect whose interest will become clearer in the next chapter. We only mention that the continuous dependence of the symbols on z and the bound (6.15) will be important when we use the Helffer-Sjöstrand formula (see in particular Proposition 4.17).

Proof. The proof of (6.14) follows by induction using the form of $(a\#b)_j$ (see Theorem 3.14) and Exercise 3.7. We omit the complete verification of this and only record that (6.13) implies

$$q_{z,-3} = -\frac{p_1}{(p_2-z)^2} + \frac{1}{i}\frac{\nabla_x p_2 \cdot \nabla_\xi p_2}{(p_2-z)^3},$$

which is indeed of the form (6.14) for $k = 1$. To prove (6.15) one observes that Proposition 6.2 and (6.14) imply that the seminorms of $q_{z,-2-k}$ in S^{-2-k} are bounded by (non negative) powers of $\langle z \rangle / d(z, \mathbb{R}^+)$ hence the same holds for the seminorms of $r^{\#}_{\nu(k,j)}\big(p_{2-j}, q_{z,-2-k}, h\big)$ in S^{-N} by symbolic calculus. The continuous dependence on z follows from Proposition 6.2 and (6.14) for $q_{z,-2-k}$. This implies in turn the continuous dependence of $r_{z,-N}(h)$ on z, using (6.10) and the continuous dependence of $r^{\#}_J(a, b, h)$ on a and b as stated in Theorem 3.14. $\square$

We now give an application to the elliptic regularity. Theorem 6.3 means that one can construct a *right* parametrix for $P(h) - z$, that is an approximate inverse of $P(h) - z$ to the right. One can also obtain a *left* parametrix using the following argument. We fix $z = -1$ and $h = 1$ to simplify the notation and since this will be sufficient to prove Corollary 6.4. Using that P^* is of the same form as P, ie a second order differential operator with symbol in S^2 and the same principal symbol as P, one can find pseudo-differential operators $\overline{Q}$ and $\overline{R}$ with symbols in S^{-2} and S^{-N} respectively such that

$$\big(P^* + 1\big)\overline{Q} = I + \overline{R}.$$

By taking the adjoint (using Theorem 3.16) in this identity, we get

$$\widetilde{Q}\big(P + 1\big) = I + \widetilde{R}, \tag{6.18}$$

where, for all s,

$$\widetilde{Q} = \overline{Q}^* : H^s \to H^{s+2}, \qquad \widetilde{R} = \overline{R}^* : H^s \to H^{s+N},$$

since the symbols of $\widetilde{Q}$ and $\widetilde{R}$ are respectively in S^{-2} and S^{-N}. With this at hand, it is easy to prove the following elliptic regularity result.

Corollary 6.4 (Global elliptic regularity)**.** *Let B be a first order (pseudo-)differential operator on $\mathbb{R}^n$ with symbol in S^1. Assume that $u \in L^2(\mathbb{R}^n)$ and $f \in H^s(\mathbb{R}^n)$ satisfy*

$$Pu = f + Bu. \tag{6.19}$$

Then u belongs to H^{s+2}. In particular, if f belongs to $\cap_s H^s$, then u is smooth.

Proof. Let us set $B_1 = B + 1$. The equation (6.19) is then equivalent to

$$(P + 1)u = f + B_1 u. \tag{6.20}$$

We consider $\widetilde{Q}, \widetilde{R}$ defined as above with N such that $N \geq s + 2$. Applying $\widetilde{Q}$ to both sides of (6.20) and using (6.18), we get

$$u = \widetilde{Q} f + \widetilde{Q} B_1 u - \widetilde{R} u, \tag{6.21}$$

where $\widetilde{Q} f \in H^{s+2}$, $\widetilde{R} u \in H^N \subset H^{s+2}$ and $\widetilde{Q} B_1 u \in H^1$ since $B_1 u \in H^{-1}$. Therefore,

$$u \in H^{s+2} + H^1 \subset H^{\min(s+2,1)}.$$

If $s+2 \leq 1$ we are done. Otherwise, we have $u \in H^1$ which is an improvement with respect to the initial assumption that $u \in L^2$. It implies that $\widetilde{Q}B_1 u \in H^2$ and thus (6.21) yields

$$u \in H^{s+2} + H^2 \subset H^{\min(s+2,2)}.$$

Repeating this argument a finite number of steps, we see that u belongs to H^{s+2}. $\square$

6.2 Localization of the parametrix

In this short section, we explain how to localize the construction of Section 6.1. This will be useful to prove the local version of Corollary 6.4 (see Theorem 6.7 below) and to construct a parametrix for $-\Delta_g$ in coordinates patches which will be the central tool of the next chapter.

Let V be an open subset of $\mathbb{R}^n$. Assume that we are given a second order differential operator P_V on V with smooth coefficients of the form

$$P_V = -\sum_{j,k=1}^{n} g^{jk}(x)\partial_j\partial_k + \sum_{j=1}^{n} c_j(x)\partial_j + c_0(x)$$

such that, for all $x \in V$,

$$(g^{jk}(x)) \text{ is a positive definite matrix.} \tag{6.22}$$

Of course, the Laplace-Beltrami operator written in local coordinates is of this form (see (2.3) and (2.5)), which is the reason why we consider operators of this form. Notice however that no self-adjointness will be required in this part.

Proposition 6.5. *Let $V_0 \Subset V$. Then, one can find an elliptic operator P globally defined on $\mathbb{R}^n$, of the form (6.1) and such that (6.2) and (6.3) hold, with the property that*

$$P_V = P \qquad \text{on } V_0. \tag{6.23}$$

This proposition means that, locally (ie in V_0), one can assume that P_V is the restriction of an operator globally defined on $\mathbb{R}^n$.

Exercise 6.6. *Let $G(x)$ be a real symmetric matrix with coefficients depending continuously on $x \in V$ and such that $G(x)$ is positive definite for each x. Show that for all $V_0 \Subset V$ there exists $C, c > 0$ such that*

$$c \leq G(x) \leq C,$$

for all $x \in V_0$.

Proof of Proposition 6.5. Let $\chi \in C_0^\infty(V)$ with values in $[0,1]$ and with $\chi \equiv 1$ on V_0. Set

$$P = -(1-\chi)\Delta + \chi P_V.$$

Obviously, P is a second order operator with symbol in S^2 and which satisfy (6.23). Its principal symbol is $(1-\chi(x))|\xi|^2+\chi(x)\sum_{jk} g^{jk}(x)\xi_j\xi_k$. By Exercise 6.6, there exists $c>0$ such that

$$(1-\chi(x))|\xi|^2+\chi(x)\sum_{jk} g^{jk}(x)\xi_j\xi_k \geq (1-\chi(x))|\xi|^2+\chi(x)c|\xi|^2 \geq \min(c,1)|\xi|^2,$$

and this completes the proof. □

One obtains the following local elliptic regularity result.

Theorem 6.7. *Let B_V be a first order differential operator on V with smooth coefficients and $f_V \in C^\infty(V)$. Assume that $v \in L^2_{\mathrm{loc}}(V)$ satisfies*

$$P_V v = f_V + B_V v, \tag{6.24}$$

in the distributions sense on V. Then $v \in C^\infty(V)$.

Proof. It suffices to show that, for all $k \in \mathbb{N}$ and all $\chi \in C_0^\infty(V)$, $\chi v \in H^k(\mathbb{R}^n)$. This is true for $k=0$ by assumption. Assume that this is true for k and let us show that it is true for $k+1$. Fix $\chi \in C_0^\infty(V)$ and multiply (6.24) by χ. Then

$$P_V \chi v = \chi f_V + \chi B_V v - [\chi, P_V]v,$$

where

$$[\chi, P_V] := \chi P_V - P_V \chi,$$

By the Leibnitz rule, this is a first order differential operator with coefficients supported in $\mathrm{supp}(\nabla\chi) \subset \mathrm{supp}(\chi)$. Then, if $\widetilde{\chi} \in C_0^\infty(V)$ is equal to 1 near the support of χ, we have

$$\chi B_V v - [\chi, P_V]v = \chi B_V \widetilde{\chi} v - [\chi, P_V]\widetilde{\chi} v \in H^{k-1}(\mathbb{R}^n),$$

since $\widetilde{\chi} v \in H^k(\mathbb{R}^n)$ by the induction assumption and $\chi B_V - [\chi, P_V]$ is a first order differential operator. If we set

$$u = \chi v \in L^2(\mathbb{R}^n), \qquad f = \chi f_V + \chi B_V \widetilde{\chi} v - [\chi, P_V]\widetilde{\chi} v \in H^{k-1}(\mathbb{R}^n),$$

and use Proposition 6.5 with $V_0 = \mathrm{supp}(\chi)$, then

$$Pu = f,$$

which, by Corollary 6.4, implies that $u \in H^{k+1}(\mathbb{R}^n)$. This completes the proof of the induction, hence of the theorem. □

In the next theorem, we give a local version of Theorem 6.3.

Theorem 6.8. *Let $V_0 \Subset V$ and $\chi_0, \chi_1, \chi_2 \in C_0^\infty(V_0)$ such that*

$$\chi_1 \equiv 1 \quad \textit{near}\ \mathrm{supp}(\chi_0), \qquad \chi_2 \equiv 1 \quad \textit{near}\ \mathrm{supp}\big(\chi_1\big).$$

Let the symbols

$$q_{z,-2-k} \in S^{-2-k}, \qquad z \in \mathbb{C} \setminus [0, +\infty),$$

be defined as in Theorem 6.3 for the operator P of Proposition 6.5 and set, for fixed N,

$$Q_{N,z}^V(h) = \chi_1 \left(\sum_{k=0}^{N-1} h^k Op_h(q_{z,-2-k}) \right) \chi_0.$$

Then

$$\big(h^2 P_V - z\big)\, Q_{N,z}^V(h) = \chi_0 + h^N \chi_2 Op_h(r_{N,z,V}(h))\chi_0, \tag{6.25}$$

with $r_{N,z,V}(h) \in S^{-N}$ depends continuously on z and satisfies, for any seminorm $\mathcal{N}_M^{S^{-N}}$ of S^{-N},

$$\mathcal{N}_M^{S^{-N}}\big(r_{N,z,V}(h)\big) \le C_{M,N} \left(\frac{z}{d(z, \mathbb{R}^+)} \right)^{K(M,N)}, \tag{6.26}$$

for all

$$z \in \mathbb{C} \setminus [0, +\infty), \quad h \in (0, 1].$$

Apart from the technical estimates (6.26), the main result of this theorem is (6.25) whose right hand side is not of the form $I + \mathcal{O}(h^N)$ as in Theorem 6.3. In the applications, χ_0 will be one term of a partition of unity and we will get a parametrix in the same spirit as in Theorem 6.3 by summing the contributions of all terms of the partition.

Proof. The operator $\big(h^2 P_V - z\big)\, Q_{N,z}^V(h)$ reads

$$\chi_1 \big(h^2 P_V - z\big) \left(\sum_{k=0}^{N-1} h^k Op_h(q_{z,-2-k}) \right) \chi_0 + [h^2 P_V, \chi_1] \left(\sum_{k=0}^{N-1} h^k Op_h(q_{z,-2-k}) \right) \chi_0,$$

where, as in the proof of Theorem 6.7, $[h^2 P_V, \chi_1] = h^2(P_V \chi_1 - \chi_1 P_V)$ is a first order differential operator. Since $h^2 P_V = P(h)$ on the support of χ_1, Theorem 6.3 shows that

$$\begin{aligned} \chi_1 \big(h^2 P_V - z\big) \left(\sum_{k=0}^{N-1} h^k Op_h(q_{z,-2-k}) \right) \chi_0 \quad &= \quad \chi_1 \big(I + h^N Op_h(r_{z,-N}(h))\big) \chi_0, \\ &= \quad \chi_0 + h^N \chi_1 Op_h(r_{z,-N}(h))\chi_0. \end{aligned}$$

On the other hand, since the coefficients of $[h^2 P_V, \chi_1]$ vanish where $\chi_1 \equiv 1$ hence near $\mathrm{supp}(\chi_0)$, we see that all terms of the expansion of the symbol of $[h^2 P_V, \chi_1] Op_h(a)\chi_0$ vanish, whatever the symbol a is. By picking $\widetilde{\chi}_0$ which is equal to one near $\mathrm{supp}(\chi_0)$ and such that $\chi_1 \equiv 1$ near $\mathrm{supp}(\widetilde{\chi}_0)$ the symbolic calculus and the support properties allow to write, for all N,

$$\begin{aligned}
[h^2 P_V, \chi_1] Op_h(a) \chi_0 &= \chi_2 \left([h^2 P_V, \chi_1] Op_h(a) \widetilde{\chi}_0 \right) \chi_0, \\
&= h^N \chi_2 Op_h(r_N(h)) \chi_0
\end{aligned}$$

for some $r_N(h) \in S^{-N}$ depending continuously on a. Using this fact for

$$a = \sum_{k=0}^{N-1} h^k q_{z,-2-k},$$

and using the z dependence of such symbols implied by Proposition 6.2 and (6.14), we see that the corresponding $r_N(h)$ satisfies bounds of the form (6.26). The same holds for the remainder $r_{z,-N}(h)$ above by (6.15) so the result follows. $\square$

Chapter 7

Proof of the Weyl law

In this chapter, we use the construction of Chapter 6 (mainly Theorem 6.8) to prove Theorems 2.2 and 2.3 (or rather Theorem 2.5 which, as we have seen in Chapter 2, implies Theorem 2.3).

7.1 The resolvent of the Laplacian on a compact manifold

The purpose of this section is to prove Proposition 7.2 below, which is an analogue of Theorem 6.3 on a compact manifold.

By Proposition 1.5, we can choose a partition of unity on M associated to a finite atlas $(U_i, V_i, \kappa_i)_{i\in\mathcal{F}}$ (here $\mathcal{F}$ is a finite set), namely

$$1 = \sum_{i\in\mathcal{F}} \theta_i, \qquad \operatorname{supp}(\theta_i) \subset U_i. \tag{7.1}$$

For each $i \in \mathcal{F}$, define

$$\chi_{0,i} := \theta_i \circ \kappa_i^{-1} \in C_0^\infty(V_i) \subset C_0^\infty(\mathbb{R}^n), \tag{7.2}$$

and choose $\chi_{1,i}, \chi_{2,i} \in C_0^\infty(\mathbb{R}^n)$ such that

$$\chi_{1,i} \equiv 1 \ \text{ near } \ \operatorname{supp}(\chi_{0,i}), \qquad \operatorname{supp}(\chi_{1,i}) \subset V_i,$$

and

$$\chi_{2,i} \equiv 1 \ \text{ near } \ \operatorname{supp}(\chi_{1,i}), \qquad \operatorname{supp}(\chi_{2,i}) \subset V_i.$$

According to the notation (1.12), we also denote

$$-\kappa_{i*}\Delta_g\kappa_i^* =: P_i, \tag{7.3}$$

which is a differential operator on $V_i \subset \mathbb{R}^n$ and wet let $p_{2,i}$ be its principal symbol, namely

$$p_{2,i} = \sum_{j,k=1}^n g_i^{jk}(x)\xi_j\xi_k,$$

as follows from (2.5). Here and in the sequel, all the functions, operators, *etc...* are indexed by i which labels the charts.

By (2.5), each P_i satisfies the same assumptions as the operator P_V in Section 6.2. Therefore, if we fix an integer

$$N \geq 1, \tag{7.4}$$

Theorem 6.8 allows to can find pseudo-differential operators $Q_{i,N}(h,z)$ and $R_{i,N}(h,z)$ of the following form

$$Q_{i,N}(h,z) \quad = \quad \chi_{1,i} \left(\sum_{k=0}^{N-1} h^k Op_h(q_{z,-2-k,i}) \right) \chi_{0,i}, \tag{7.5}$$

where $q_{z,-2-k,i}$ is of the form (6.14) with p_2 replaced by $p_{2,i}$, and

$$R_{i,N}(h,z) \quad = \quad \chi_{2,i} Op_h(r_{N,z,V_i}(h)) \chi_{0,i}. \tag{7.6}$$

Their main property is that

$$\left(h^2 P_i - z \right) Q_{i,N}(h,z) = \chi_{0,i} + h^N R_{i,N}(h,z), \tag{7.7}$$

for all $z \in \mathbb{C} \setminus [0, +\infty)$ and all $h \in (0,1]$. The dependence on z and h of the symbols $q_{z,-2-k,i}$ and $r_{N,z,V_i}(h)$ is as in Theorem 6.8.

To built an operator on M, let us define

$$\mathcal{Q}_N(h,z) = \sum_i \kappa_i^* Q_{i,N}(h,z) \kappa_{i*}, \qquad \mathcal{R}_N(h,z) = \sum_i \kappa_i^* R_{i,N}(h,z) \kappa_{i*}, \tag{7.8}$$

seen as operators acting on $C^\infty(M)$. Here we slightly abuse the notation since κ_{i*} is only defined on $C^\infty(U_i)$ (or even $L^2(U_i)$) but not outside U_i. The precise meaning of these expressions is the following. Denote by e_{U_i} the extension map by 0 outside U_i, by r_{U_i} the restriction map to $U_i \subset M$ and similarly e_{V_i}, r_{V_i} for $V_i \subset \mathbb{R}^n$. Then, for a pseudo-differential operator A on $\mathbb{R}^n$, we set

$$\kappa_i^* A \kappa_{i*} := e_{U_i} \kappa_i^* \left(r_{V_i} A e_{V_i} \right) \kappa_{i*} r_{U_i}, \tag{7.9}$$

which is now perfectly defined. For a general operator A, there is no reason in general why $\kappa_i^* A \kappa_{i*} \varphi$ should belong to $C^\infty(M)$ for all $\varphi \in C^\infty(M)$. Indeed, if $\varphi \in C^\infty(M)$ then $\kappa_{i*} r_{U_i} \varphi$ is simply the map $\varphi \circ \kappa_i^{-1} : V_i \to \mathbb{R}$ which is smooth, but its extension by 0 outside V_i is in general not a smooth function on $\mathbb{R}^n$. Nevertheless this the case if $\varphi \circ \kappa_i^{-1}$ belongs to $C_0^\infty(V_i)$. Similarly, if u is a Schwartz function, then $r_{V_i} Au$ is smooth on V_i hence $\kappa_i^*(r_{V_i} Au)$ is smooth on U_i, but its extension by 0 will in general not be smooth on M. This will however be the case if we know that $\mathrm{supp}(Au) \Subset V_i$.

With this discussion in mind, it is not hard to prove the following property which we leave as an exercise.

Exercise 7.1. *Fix cutoffs $\chi_i, \widetilde{\chi}_i \in C_0^\infty(V_i)$ and a symbol $a \in S^{\mu,m}$ (for some $\mu, m \in \mathbb{R}$). Check that*

$$e_{U_i}\kappa_i^* r_{V_i}\left(\widetilde{\chi}_i Op_h(a)\chi_i\right) e_{V_i}\kappa_{i*} r_{U_i}$$

maps $C^\infty(M)$ in $C^\infty(M)$.

All this means that, in (7.8), one has to consider that the (natural) extension and restriction operators have been dropped from the notation for simplicity, but the exact definition is as in (7.9).

The operators defined in (7.8) have the following properties.

Proposition 7.2. *For all $z \in \mathbb{C} \setminus [0, +\infty)$ and all $h \in [0, 1)$,*

1. *$\mathcal{Q}_N(h, z)$ and $\mathcal{R}_N(h, z)$ map $C^\infty(M)$ into $C^\infty(M)$,*
2. *$\mathcal{Q}_N(h, z)$ and $\mathcal{R}_N(h, z)$ have bounded closures on $L^2(M)$ which are compact,*
3. *in the sense of operators on $C^\infty(M)$,*

$$(-h^2\Delta_g - z)\mathcal{Q}_N(h, z) = I + h^N \mathcal{R}_N(h, z). \tag{7.10}$$

Furthermore, there exist $C = C(N)$ and $M = M(N)$ such that

$$||\mathcal{R}_N(h, z)\varphi||_{L^2(M)} \leq C \left(\frac{\langle z \rangle}{d(z, \mathbb{R}^+)}\right)^M ||\varphi||_{L^2(M)}, \tag{7.11}$$

for all $z \in \mathbb{C} \setminus [0, +\infty)$, all $h \in [0, 1)$ and all $\varphi \in C^\infty(M)$.

Proof. Item 1 follows from Exercise 7.1 using the cutoffs $\chi_{0,i}, \chi_{1,i}$ and $\chi_{2,i}$ involved in (7.5) and (7.6). Item 2 is a consequence of Corollary 5.12, since the symbols involved in (7.5) and (7.6) are compactly supported in x and of negative order in ξ (using that $N \geq 1$ by (7.4) for $\mathcal{R}_N(h, z)$) and using (1.10) to translate estimates on $L^2(\mathbb{R}^n)$ to $L^2(M)$ (see the proof of Proposition 7.3 below for more details on this point). Item 3 is a consequence of (7.1), (7.2) and (7.7). The estimate on $\mathcal{R}_N(h, z)$ follows from Theorem 5.1, the last estimate of Theorem 6.8 with $N = 0$ and (1.10) as above. □

The following proposition will be useful to deal with trace class estimates.

Proposition 7.3 (Further properties of $\mathcal{R}_N(h, z)$)**.** *Assume that $N > n$. Then there exist Hilbert Schmidt operators*

$$A_{i,N}(h, z) : L^2(M) \to L^2(\mathbb{R}^n), \qquad B_{i,N}(h) : L^2(M) \to L^2(\mathbb{R}^n)$$

such that,

$$\mathcal{R}_N(h, z) = \sum_i A_{i,N}(h, z)^* B_{i,N}(h),$$

and, for some constant C, M,

$$||A_{i,N}(h,z)||_{\mathrm{HS}} \leq Ch^{-n/2}\left(\frac{\langle z\rangle}{d(z,\mathbb{R}^+)}\right)^M, \tag{7.12}$$

$$||B_{i,N}(h)||_{\mathrm{HS}} \leq Ch^{-n/2}, \tag{7.13}$$

for all $h \in [0,1)$ and all $z \in \mathbb{C}\setminus[0,+\infty)$. In addition, the map

$$z \mapsto A_{i,N}(h,z)$$

is continuous from $\mathbb{C}\setminus[0,+\infty)$ to the Hilbert-Schmidt class.

Proof. For the sake of clarity, we use the precise definition (7.9) (ie the right hand side of (7.9) rather than the left hand side) to distinguish properly what is defined on M and what is defined on $\mathbb{R}^n$. We can then write each term of $\mathcal{R}_N(h,z)$ as

$$e_{U_i}\kappa_i^*\left(r_{V_i}\chi_{2,i}Op_h(r_{N,z,V_i})\chi_{i,0}e_{V_i}\right)\kappa_{i*}r_{U_i} = A_{i,N}(h,z)B_{i,N}(h)$$

with

$$A_{i,N}(h,z) = e_{U_i}\kappa_i^* r_{V_i}\chi_{2,i}Op_h(r_{N,z,V_i})\langle x\rangle^{N/2}\langle hD\rangle^{N/2}$$

and

$$B_{i,N}(h) = \langle hD\rangle^{-N/2}\langle x\rangle^{-N/2}\chi_{i,0}e_{V_i}\kappa_{i*}r_{U_i}.$$

Using that $e_{U_i}\kappa_i^* r_{V_i}\chi_{2,i}$ maps $L^2(\mathbb{R}^n)$ to $L^2(M)$ and more precisely that

$$||e_{U_i}\kappa_i^* r_{V_i}\chi_{2,i}u||_{L^2(M)} \lesssim ||u||_{L^2(\mathbb{R}^n)}, \qquad u \in \mathcal{S}(\mathbb{R}^n)$$

by (1.9), and similarly that $\chi_{i,0}e_{V_i}\kappa_{i*}r_{U_i}$ maps $L^2(M)$ into $L^2(\mathbb{R}^n)$, the result follows from the fact that $\langle hD\rangle^{-N/2}\langle x\rangle^{-N/2}$ is Hilbert-Schmidt on $L^2(\mathbb{R}^n)$ by Theorem 5.10, and from the fact that

$$\chi_{2,i}Op_h(r_{N,z,V_i})\langle x\rangle^{N/2}\langle hD\rangle^{N/2} = Op_h(a_{N,z}(h)), \qquad a_{N,z}(h) \in S^{-N/2,-N/2}$$

(hence is Hilbert-Schmidt too), by symbolic calculus and the fact that $\chi_{2,i}r_{N,z,V_i}$ belongs to $S^{-N,-N}$ (the decay in x is due to $\chi_{2,i}$). In both cases, we also use item 5 of Proposition 4.3. The dependence on z of $A_{i,N}(h,z)$ follows from Theorem 3.14 and Theorem 6.8. $\square$

7.2 Diagonalization of Δ_g

In this section, we fix (for instance) $N = 1$ and $z = -1$. By Proposition 7.2, one can find $h > 0$ small enough and two operators $Q, R : C^\infty(M) \to C^\infty(M)$, with bounded closures $\overline{Q}$, $\overline{R}$ on $L^2(M)$, such that

$$\big(-h^2\Delta_g + 1\big)Q = I + R, \tag{7.14}$$

as well as

$$||\overline{R}||_{L^2(M)\to L^2(M)} \leq 1/2. \tag{7.15}$$

Moreover $\overline{Q}$ is compact on $L^2(M)$. All this follows by taking $Q := \mathcal{Q}(h,-1)$ and $R = h\mathcal{R}(h,-1)$ with h small enough.

Proposition 7.4. *1. The operator $I + \overline{R}$ is invertible on $L^2(M)$.*

2. The operator

$$K := \overline{Q}(I + \overline{R})^{-1}$$

maps $C^\infty(M)$ in $C^\infty(M)$.

3. As operators on $C^\infty(M)$,

$$\big(-h^2\Delta_g + 1\big)K = I. \tag{7.16}$$

Notice that item 1 is a direct consequence of (7.15). The main difficulty is to prove item 2. We will use the following lemma.

Lemma 7.5 (Elliptic regularity on M)**.** *Let $u \in L^2(M)$, $f \in C^\infty(M)$ and $\lambda \in \mathbb{C}$. Assume that, for all $\phi \in C^\infty(M)$,*

$$\big((-\Delta_g + \lambda)\phi, u\big)_{L^2(M)} = (\phi, f)_{L^2(M)}. \tag{7.17}$$

Then $u \in C^\infty(M)$.

Of course, this lemma means that if $u \in L^2(M)$ is such that $(-\Delta_g + \bar{\lambda})u$, taken in the distributions sense, is smooth on M then u is smooth on M. However since we have not discussed distributions on manifolds (and do not need to), we prefer to state the lemma in the form above.

Proof. Let (U, V, κ) be a coordinate chart and denote by P_V the expression of the Laplacian in these coordinates, ie $P_V = -\kappa_*\Delta_g\kappa^*$. According to Exercise 1.11, we can consider the expression of u in local coordinates which we denote $u_\kappa \in L^2_{\mathrm{loc}}(V)$. By specializing (7.17) to $\phi \in C_0^\infty(U)$ and using (7.17), we have

$$\int \overline{(P_V + \lambda)\kappa_*\phi(x)}u_\kappa(x)|g(x)|dx = \int \overline{\kappa_*\phi(x)}\kappa_* f(x)|g(x)|dx,$$

where we recall that $|g(x)|dx = \kappa^* d\mathrm{vol}_g$ (see 2.4). This implies that, in the distribution sense on $V \subset \mathbb{R}^n$,

$$(P_V + \lambda)^*\big(|g|u_\kappa\big) = |g|\kappa_* f,$$

where the formal adjoint is taken with respect to the Lebesgue measure dx, ie using (2.5),

$$P_V^* w(x) = \sum_{j,k=1}^n \frac{\partial^2}{\partial x_j \partial x_k}\big(g^{jk}(x)w(x)\big) - \frac{\partial}{\partial x_k}\left(w(x)|g(x)|^{-1}\frac{\partial}{\partial x_j}\Big(|g(x)|g^{jk}(x)\Big)\right).$$

A simple calculation using the Leibniz rule shows that

$$(P_V + \lambda)^* = P_V + \bar{\lambda} + B_V,$$

for some first order differential operator B_V (with smooth coefficients). Therefore, by the smoothness of $\kappa_* f$ and $|g|$, Theorem 6.7 imply that $|g|u_\kappa$ is smooth. Since $|g|$ does not vanish and U is arbitrary, we obtain that u is smooth on M. □

Proof of Proposition 7.4. Let us prove item 2. Fix $f \in C^\infty(M)$ and define

$$u = Kf.$$

We wish to show that $u \in C^\infty(M)$. By (7.14) and the formal selfadjointness of Δ_g with respect to $d\text{vol}_g$ (see (2.1)), we have

$$\big((-h^2\Delta_g + 1)\phi, Q\psi\big)_{L^2(M)} = (\phi, (I+R)\psi)_{L^2(M)},$$

for all $\phi, \psi \in C^\infty(M)$. If we replace ψ by a sequence $\psi_j \in C^\infty(M)$ which converges to $(I+\overline{R})^{-1} f$ in $L^2(M)$, we obtain in the limit

$$\big((-h^2\Delta_g + 1)\phi, u\big)_{L^2(M)} = (\phi, f)_{L^2(M)}. \tag{7.18}$$

Since this is true for all $\phi \in C^\infty(M)$, Lemma 7.5 implies that u is smooth. Item 3 is then a straightforward consequence of (7.18) since, as we know that $u = Kf$ is smooth, the formal selfadjointness of Δ_g yields

$$\big(\phi, (-h^2\Delta_g + 1)Kf\big)_{L^2(M)} = (\phi, f)_{L^2(M)}, \tag{7.19}$$

which implies the result since this holds for all $\phi, f \in C^\infty(M)$ which is dense in $L^2(M)$. □

Proposition 7.6. *1. K is a compact selfadjoint operator on $L^2(M)$.*

2. $\text{Ker}(K) = \{0\}$.

3. All eigenfunctions of K are smooth and are eigenfunctions of Δ_g.

4. $\sigma(K) \subset (0, +\infty)$.

Proof. In item 1, we know that K is compact, so it remains to prove the selfadjointness. By density of $C^\infty(M)$ in $L^2(M)$, it suffices to show that

$$(f_1, Kf_2)_{L^2(M)} = (Kf_1, f_2)_{L^2(M)},$$

for all $f_1, f_2 \in C^\infty(M)$. This is a straightforward consequence of the formal selfadjointness of Δ_g on $C^\infty(M)$ and by taking $f = f_2$ and $\phi = Kf_1$ in (7.19). Let us now prove item 2. By (7.19), we have

$$\big((-h^2\Delta_g + 1)\phi, Kf\big)_{L^2(M)} = (\phi, f)_{L^2(M)}, \tag{7.20}$$

for all $\phi, f \in C^\infty(M)$. By density of $C^\infty(M)$ in $L^2(M)$ and continuity of K on $L^2(M)$, this is still true if $f \in L^2(M)$. In particular, if $f \in \mathrm{Ker}(K)$, one has $(\phi, f)_{L^2(M)} = 0$ for all $\phi \in C^\infty(M)$ hence $f = 0$ that is precisely the result. Let us now prove item 3. Assume that $u \in L^2(M)$ satisfies

$$Ku = \mu u,$$

for some $\mu \in \mathbb{R}$. By item 2, we can assume that $\mu \neq 0$. Using (7.20), it is easy to check that

$$\big((-h^2\Delta_g + 1 - \mu^{-1})\phi, u\big)_{L^2(M)} = 0,$$

for all $\phi \in C^\infty(M)$. Using Lemma 7.5, we see that u is smooth. Furthermore, this identity then shows that

$$(-h^2\Delta_g + 1 - \mu^{-1})u = 0,$$

ie that u is an eigenfunction of Δ_g. We finally prove item 4. By taking $\phi = Kf$ in (7.20) with $f \in C^\infty(M)$ and by the non positivity of Δ_g (see (2.2)), we obtain

$$\big((-h^2\Delta_g + 1)Kf, Kf\big)_{L^2(M)} = (Kf, f)_{L^2(M)} \geq 0,$$

from which the result follows by density of $C^\infty(M)$ in $L^2(M)$. □

Proof of Theorem 2.2. By item 1 and 4 of Proposition 7.6, one can consider the sequence

$$\mu_0 \geq \mu_1 \geq \mu_2 \geq \cdots > 0, \qquad \lim_{j\to\infty} \mu_j = 0$$

of eigenvalues of K. We let $(e_j)_{j\geq 0}$ be an associated orthonormal basis of eigenfunctions. By item 3 of Proposition 7.6, they are smooth and eigenfunctions of Δ_g. Furthermore, using (7.16), we have

$$-\Delta_j e_j = \lambda_j e_j$$

with $\lambda_j = h^{-2}(\mu_j^{-1} - 1)$. The sequence $(\lambda_j)_{j\geq 0}$ is non decreasing since $(\mu_j)_{j\geq 0}$ is non increasing and positive. Furthermore, $\lambda_j \to +\infty$. Since λ_j is non negative, as explained after the statement of Theorem 2.2, the proof is complete. □

7.3 Proof of the Weyl law

The purpose of this section is to prove Theorem 2.5 which, as shown in Chapter 2, implies the Weyl law stated in Theorem 2.3.

Let us fix $f \in C_0^\infty(\mathbb{R})$ and define $h = \lambda^{-1/2}$ so that

$$\lambda^{-n/2} f(-\Delta_g/\lambda) = h^n f(-h^2\Delta_g).$$

The first tool to analyze this operator is Proposition 4.22 which allows to write

$$f(-h^2\Delta_g) = \frac{1}{2\pi}\int \overline{\partial}\widetilde{f}(z)\big(-h^2\Delta_g - z\big)^{-1} L(dz), \tag{7.21}$$

for (any hence) some almost analytic extension $\widetilde{f} \in C_0^\infty(\mathbb{C})$ of f. The second tool is the following approximation of the resolvent, where $\mathcal{Q}_N(h,z)$ and $\mathcal{R}_N(h,z)$ are defined in Proposition 7.2.

Lemma 7.7. *For all $h \in (0,1]$ and $z \in \mathbb{C} \setminus [0,+\infty)$,*

$$\big(- h^2\Delta_g - z\big)^{-1} = \mathcal{Q}_N(h,z) - h^N\big(- h^2\Delta_g - z\big)^{-1}\mathcal{R}_N(h,z). \tag{7.22}$$

We recall that the resolvent is well defined by Definition 4.20 which uses Theorem 2.2 proved in the previous section.

Proof. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions of Δ_g and let $\mathcal{E}$ be the space of finite linear combinations of such eigenfunctions. Fix $\phi \in \mathcal{E}$ and define

$$\Phi = (-h^2\Delta_g - \bar{z})^{-1}\phi,$$

which is still an element of $\mathcal{E}$. We then fix an arbitrary $\psi \in C^\infty(M)$ and test the identity (7.10) against Φ and ψ. Using the easily verified fact that, since $\phi \in \mathcal{E}$,

$$(-h^2\Delta_g - \bar{z})(-h^2\Delta_g - \bar{z})^{-1}\phi = \phi,$$

and the fact that the adjoint of $(-h^2\Delta_g - \bar{z})^{-1}$ is $(-h^2\Delta_g - z)^{-1}$ by Proposition 2.4, we get

$$\big(\phi, \mathcal{Q}_N(h,z)\psi\big)_{L^2(M)} = \big(\phi, (-h^2\Delta_g - z)^{-1}\big(I + \mathcal{R}_N(h,z)\big)\psi\big)_{L^2(M)}.$$

By density of $\mathcal{E}$ and $C^\infty(M)$ in $L^2(M)$, the result follows. ☐

By combining the formula (7.21) and the decomposition (7.22), we see that, for any $N \geq 0$, one obtains that

$$f(-h^2\Delta_g) = F_N(f,h) + h^N T_N(f,h)$$

with

$$F_N(f,h) = \frac{1}{2\pi}\int \overline{\partial}\widetilde{f}(z)\mathcal{Q}_N(h,z)L(dz)$$

and

$$T_N(f,h) = -\frac{1}{2\pi}\int \overline{\partial}\widetilde{f}(z)\big(- h^2\Delta_g - z\big)^{-1}\mathcal{R}_N(h,z)L(dz).$$

More precisely, the integrals converge in operator norm on $L^2(M)$ by Proposition 4.17 and the a priori bounds (4.20) and (7.11).

Proposition 7.8. *If $N > n$, $T_N(f,h)$ is trace class and*

$$||T_N(f,h)||_{\mathrm{tr}} \lesssim h^{-n}, \qquad h \in (0,1].$$

Proof. By choosing $N > n$ and using Proposition 7.3, the second term in the right hand side of (7.22) is a sum of

$$h^N\big(-h^2\Delta_g - z\big)^{-1}A_{i,N}(h,z)^*B_{i,N}(h)$$

where $A_{i,N}(h,z)^*$ is Hilbert-Schmidt hence so is $\big(-h^2\Delta_g - z\big)^{-1}A_{i,N}(h,z)^*$ by item 5 of Proposition 4.3. Actually, by (4.20) and (7.12), one has

$$\big|\big|h^N\big(-h^2\Delta_g - z\big)^{-1}A_{i,N}(h,z)^*\big|\big|_{\mathrm{HS}} \lesssim h^{N-\frac{n}{2}}\frac{\langle z\rangle^M}{|\mathrm{Im}(z)|^{M+1}}.$$

We can then integrate in z, using that the space of Hilbert-Schmidt operators is complete (see item 4 of Proposition 4.3) and the fact that $\overline{\partial}\widetilde{f}(z)$ has compact support and satisifes $|\overline{\partial}\widetilde{f}(z)| \lesssim |\mathrm{Im}(z)|^{M+1}$. We obtain that

$$\left|\left|\int \overline{\partial}\widetilde{f}(z)h^N\big(-h^2\Delta_g - z\big)^{-1}A_{i,N}(h,z)^*L(dz)\right|\right|_{\mathrm{HS}} \lesssim h^{-n/2}. \tag{7.23}$$

Using next that $||B_{i,N}(h)||_{\mathrm{HS}} \lesssim h^{-n/2}$ (see Proposition 7.3), we get the result. $\square$

The following final proposition will complete the proof of Theorem 2.5.

Proposition 7.9. *For all N, $F_N(f,h)$ is trace class and*

$$\mathrm{tr}\big(F_N(f,h)\big) = (2\pi h)^{-n}\mathrm{vol}_g(M)\int f(|\eta|^2)d\eta + \mathcal{O}(h^{1-n}).$$

We isolate first the following formula.

Exercise 7.10. *We consider the operator (7.9) and assume additionally that*

$$A = \chi_1 B\chi_0,$$

with $\chi_0, \chi_1 \in C_0^\infty(V_i)$ and $\chi_0\chi_1 = \chi_0$, and also that B is trace class. Prove that

$$\mathrm{tr}_{L^2(M)}\left(e_{U_i}\kappa_i^*\left(r_{V_i}Ae_{V_i}\right)\kappa_{i*}r_{U_i}\right) = \mathrm{tr}_{L^2(\mathbb{R}^n)}(\chi_0 B).$$

Hint: use (and prove) that

$$\big(e_{V_i}\kappa_{i*}r_{U_i}\big)\big(e_{U_i}\kappa_i^*r_{V_i}\big) = \mathbb{1}_{V_i},$$

as a multiplication operator on $L^2(\mathbb{R}^n)$.

Proof of Proposition 7.9. Recall the structure of $\mathcal{Q}_N(h,z)$ which is given by (7.8) where each $Q_{i,N}(h,z)$ is of the form

$$\chi_{1,i}\sum_{k=0}^{N-1}h^k Op_h(q_{-2-k,z,i})\chi_{0,i}$$

where, on the support of $\chi_{1,i}$, one has

$$q_{-2,z,i} = \frac{1}{p_{2,i} - z}$$

with $p_{2,i}$ the principal symbol of $-\Delta_g$ in the i-th chart, and

$$q_{-2-k,z,i} = \sum_{j=1}^{2k} \frac{d_{jk,i}}{(p_{2,i} - z)^{1+j}},$$

with $d_{jk,i}$ polynomial in ξ and independent of z (see Theorem 6.3). Using Proposition 4.18 (and an elementary Fubini argument which we omit) we obtain that

$$\frac{1}{2\pi} \int \overline{\partial} \widetilde{f}(z) \chi_{1,i} Op_h(q_{-2,z,i}) L(dz) = \chi_{1,i} Op_h(f \circ p_{2,i}), \tag{7.24}$$

and similarly, for $k \geq 1$, that

$$\frac{1}{2\pi} \int \overline{\partial} \widetilde{f}(z) \chi_{1,i} Op_h(q_{-2-k,z,i}) L(dz) = \sum_{j=1}^{2k} \frac{(-1)^j}{j!} \chi_{1,i} Op_h(d_{jk,i} f^{(j)} \circ p_{2,i}). \tag{7.25}$$

Since $p_{2,i}$ is elliptic, $f^{(j)} \circ p_{2,i}$ has compact support in ξ. Thanks to the compact support of $\chi_{1,i}$, we see that the pseudo-differential operators in the left hand sides of (7.24) and (7.25) have compactly supported symbols on $\mathbb{R}^{2n}$. Therefore they are trace class by Theorem 5.13. Using (5.16) and Exercise 7.10, one can compute the traces and we see that the trace of all terms corresponding to $k \geq 1$ have a trace of order $h^{k-n} = \mathcal{O}(h^{1-n})$. This combined with Proposition 7.8 shows that

$$\operatorname{tr}\big(F_N(f,h)\big) = \sum_{i \in \mathcal{F}} (2\pi h)^{-n} \int \int \chi_{0,i}(x) f(p_{2,i}(x,\xi)) dx d\xi + \mathcal{O}(h^{1-n}).$$

It remains to remark that

$$p_{2,i}(x,\xi) = \big|H_i(x)\xi\big|^2 = \xi \cdot H_i(x)^2 \xi,$$

where $H_i(x)$ is a square matrix which is the positive definite square root of $(g_i^{jk}(x))$ (see (2.3)) in the i-th chart. Notice that

$$\det(H_i(x)) = |g_i(x)| = \det(g_i^{jk}(x))^{-1/2}.$$

Thus, using the change of variable $H_i(x)\xi = \eta$, we obtain

$$\begin{aligned} (2\pi h)^{-n} \int \int \chi_{0,i}(x) f(p_{2,i}(x,\xi)) dx d\xi &= (2\pi h)^{-n} \int \int \chi_{0,i}(x) f(|\eta|^2) |g_i(x)| dx d\eta \\ &= (2\pi h)^{-n} \left(\int f(|\eta|^2) d\eta \right) \int \theta_i d\mathrm{vol}_g \end{aligned}$$

using (1.5), (2.4) and (7.2) in the last line. Summing over i and using (7.1), the result follows. $\square$

Appendix A

Proof of the Peetre Theorem

In this section, we give a proof of Proposition 1.13. It is based on the following technical lemma.

Lemma A.1. *Let V be an open subset of $\mathbb{R}^n$ and $Q : C_0^\infty(V) \to C_0^\infty(V)$ a linear map such that*

$$\operatorname{supp}(Q\varphi) \subset \operatorname{supp}(\varphi), \qquad \varphi \in C_0^\infty(V). \tag{A.1}$$

Then, for all $x \in V$ and all $C > 0$, there exist a neighborhood W of x and an integer $k \geq 0$ such that, for all $\varphi \in C_0^\infty(V)$ and all $y \in W$,

$$\partial^\alpha \varphi(y) = 0 \ \text{ for all } |\alpha| \leq k \qquad \Longrightarrow \qquad |(Q\varphi)(y)| \leq C.$$

Proof. We argue by contradiction. There are then $x \in V$ and $C > 0$ such that for all $k \in \mathbb{N}$ and all neighborhood W of x we can find $y \in W$ and $\varphi_k \in C_0^\infty(V)$ such that $\partial^\alpha \varphi_k(y) = 0$ for all $|\alpha| \leq k$ and $|(P\varphi_k)(y)| > C$. In particular, we can choose a sequence of disjoint balls $B(y_k, r_k)$ and a sequence $\varphi_k \in C_0^\infty(V)$ such that

$$|y_k - x| \to 0, \qquad \partial^\alpha \varphi_k(y_k) = 0 \ \text{ for } |\alpha| \leq k, \qquad |(P\varphi_k)(y_k)| > C. \tag{A.2}$$

Fix now $\chi \in C_0^\infty(B(0,1))$ such that $\chi \equiv 1$ on $B(0,1/2)$ and set

$$M_j := \#\{(\beta,\gamma) \mid |\beta + \gamma| \leq j\} \times \max_{|\beta+\gamma| \leq j} \frac{(\beta+\gamma)!}{\beta!\gamma!} \times \max_{|\beta| \leq j} ||\partial^\beta \chi||_{L^\infty}. \tag{A.3}$$

We consider even indices $k = 2j$. Using that $\partial^\alpha \varphi_{2j}(y_k) = 0$ for $|\alpha| \leq 2j$, the Taylor formula allows to find $0 < \delta_{2j} < \min(r_{2j}, 1)$ such that

$$\max_{|\gamma| \leq j} \sup_{y \in B(y_{2j}, \delta_{2j})} |\partial^\gamma \varphi_{2j}(y)| \leq \frac{1}{M_j} (\delta_{2j}/2)^j. \tag{A.4}$$

Then, if we define $\psi_j \in C_0^\infty(B(y_{2j}, \delta_{2j}))$ by

$$\psi_j(y) = \varphi_{2j}(y) \chi\left(\frac{y - y_{2j}}{\delta_{2j}}\right)$$

it is not hard to check, using the Leibnitz rule, (A.3) and (A.4), that

$$||\partial^\alpha \psi_j||_{L^\infty} \le 2^{-j}, \qquad |\alpha| \le j.$$

Therefore, the following function is smooth

$$\Psi := \sum_{j\ge 0} \psi_j$$

and belongs to $C_0^\infty(V)$. We then observe that, for each j, $y_{2j+1} \notin \mathrm{supp}(\Psi)$. Since the support is preseved by Q, we see that

$$Q\Psi(y_{2j+1}) = 0$$

hence that $Q\Psi(x) = 0$, using the first property in (A.2). On the other hand, since Ψ and φ_{2j} coincide near y_{2j}, we have $Q(\Psi - \varphi_{2j})(y_{2j}) = 0$ and therefore

$$\big|Q\Psi(y_{2j})\big| \ge C,$$

by the last property of (A.2). This implies that $|Q\Psi(x)| \ge C$ which yields a contradiction. □

Lemma A.2. *The operator $Q : C_0^\infty(V) \to C_0^\infty(V)$ is continuous.*

Proof. We prove first that Q is continuous from $C_0^\infty(V)$ to $C_0^0(V)$, by contradiction. Then, there exists $K \Subset V$ such that, for all $j \in \mathbb{N}$ we can find ψ_j such that

$$\psi_j \in C_0^\infty(K), \qquad ||Q\psi_j||_{L^\infty} > j \max_{|\alpha|\le j} \sup_K |\partial^\alpha \psi_j|.$$

Up to the replacement of ψ_j by $\psi_j/||Q\psi_j||_{L^\infty}$ we can assume that

$$||Q\psi_j||_{L^\infty} = 1 \qquad \text{and} \qquad \psi_j \to 0 \ \text{ in } C_0^\infty(K).$$

For each j, there exists $x_j \in K$ such that $|Q\psi_j(x_j)| = 1$ and, by possibly taking subsequences, we may assume that x_j converges to some $x \in K$. By Lemma A.1, we can find a neighborhood x, containing all x_j for j large enough, and an integer k such that, for all $\varphi \in C_0^\infty(V)$,

$$\partial^\alpha \varphi(x_j) = 0 \ \text{ for } |\alpha| \le k \qquad \Longrightarrow \qquad |Q\varphi(x_j)| \le 1/2.$$

(Choose $C = 1/2$.) Let $\chi \in C_0^\infty(V)$ be equal to 1 near x (hence near all x_j for j large) and set

$$\varphi_j(\cdot) = \psi_j(\cdot) - \sum_{|\alpha|\le k} \frac{1}{\alpha!} \partial^\alpha \psi_j(x_j)(\cdot - x_j)^\alpha \chi(\cdot).$$

Then $|Q\varphi_j(x_j)| \le 1/2$ for all j large enough. On the other hand, if we set $\chi_{j,\alpha}(\cdot) = (\cdot - x_j)^\alpha \chi(\cdot)$, we have

$$Q\varphi_j(x_j) = Q\psi_j(x_j) - \sum_{|\alpha|\le k} \frac{1}{\alpha!} \partial^\alpha \psi_j(x_j)(Q\chi_{j,\alpha})(x_j)$$

where the sequence $(Q\chi_{j,\alpha})(x_j)$ is bounded, by boundedness of the sequence x_j (expand $(\cdot - x_j)^\alpha$ according to the binomial formula), and where the term $\partial^\alpha \psi_j(x_j)$ go to zero. This implies that $\lim_{j\to\infty} |Q\psi_j(x_j)| \leq 1/2$ which yields a contradiction.

To complete the proof, it suffices to observe that, for any multiindex α, $\partial^\alpha Q$ preserves the supports hence is continuous from $C_0^\infty(V)$ to $C_0^0(V)$, ie Q is continuous from $C_0^\infty(V)$ to $C_0^{|\alpha|}(V)$ for all α. $\square$

Proof of Proposition 1.13. We consider $Q := \kappa_* P \kappa^*$ which is support preserving. Fix an open subset $K \Subset V$. By continuity of Q, there exists $C > 0$ and $m \in \mathbb{N}$ such that

$$||Q\varphi||_{L^\infty} \leq C \max_{|\alpha|\leq m} \sup_K |\partial^\alpha \varphi|,$$

for all $\varphi \in C_0^\infty(K)$. This estimate and the fact that Q is support preserving (ie non increasing) implies that, for any $x \in K$, the map

$$\varphi \mapsto (Q\varphi)(x),$$

is a distribution of order at most m which is supported at x. A classical result of Distributions Theory shows that it is a linear combination of the Dirac measure at x and its derivatives up to order m. In other words, there are (uniquely defined) complex numbers $a_\alpha(x)$ such that

$$(Q\varphi)(x) = \sum_{|\alpha|\leq m} a_\alpha(x) \partial^\alpha \varphi(x), \tag{A.5}$$

for all $\varphi \in C_0^\infty(K)$ and actually, by the support property, for all $\varphi \in C_0^\infty(V)$. To complete the proof, it suffices to show that the maps $x \mapsto a_\alpha(x)$ are smooth. This is a local statement so it suffices to show that these maps are smooth near any point of K. Pick $\chi_0 \in C_0^\infty(K)$ which is equal to 1 near such a point x. Then, for $\beta = 0$,

$$a_0 = \big(Q\chi_0\big)(x), \qquad \text{near } x,$$

hence a_0 is smooth on a neighborhood W of x. Consider next the operator

$$Q_0\varphi = Q\varphi - a_0\varphi,$$

which is continuous on $C_0^\infty(W)$ to itself. For $|\beta| = 1$, we set $\chi_\beta(y) = y^\beta \chi_1(y)$, with $\chi_1 \in C_0^\infty(W)$ which is equal to 1 near x. By (A.5),

$$a_\beta = Q_0\chi_\beta, \qquad \text{near } x,$$

which, as above, implies that a_β is smooth near x. Analogously, by considering successively the operators $Q_k = Q - \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$, $k = 1, \ldots, m-1$, one proves that all coefficients of Q are smooth near any $x \in K$. This completes the proof. $\square$

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