A first course in quantum chaos
(Preliminary version)

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Foreword

These notes are an expanded version of a course given in winter 2015 in Toulouse and which was meant as a continuation of a course of Jean-Pierre Otal on Riemannian geometry and ergodic geodesic flows. We thought together it would be nice to supplement such a course by presenting the subject of Quantum Chaos (or quantum ergodicity) and the famous Shnirelman Theorem.

The main goal of the course itself was both to introduce students to the beautiful subject of Quantum Chaos and to motivate some fundamental results of microlocal analysis, in particular the Egorov Theorem which describes the relationship between the geodesic flow and the semiclassical Schrödinger equation.

This expanded version can be considered at two levels. At the first level, we define the objects involved in the Shnirelman Theorem (Theorem 1.18), namely the Laplace Beltrami operator and its eigenfunctions on the quantum side, and the ergodicity of the geodesic flow on the classical side, and then prove the theorem by using the existence of a pseudo-differential quantization (Theorem 2.6) as a black box. This is the purpose of Chapters 1 and 2 which correspond essentially to what was thaught in class (on the board); we had some time left to present bits of proofs of Theorem 2.6 but definitely not all of them!

At the second level, the objective is to demystify the proof of the Shnirelman Theorem to beginners in the field by giving a complete and elementary[1] proof of Theorem 2.6. This rests on a fairly minimal and elementary approach of the pseudo-differential calculus on $\mathbb{R}^n$ (mostly with compactly supported symbols) and its use in the construction of a quantization on a manifold. With such tools, we can then prove the local Weyl law and the Egorov Theorem which are the main microlocal inputs in the proof of the Shnirelman Theorem.

In principle, this course is designed for students with a standard background on analysis (functional analysis, distributions theory) and differential geometry at the master level, although we realize that more familiarity with any of these subjects, e.g. on differential topology and Riemannian geometry, is helpful. In a few places in these notes, the reader will find informal references to Otal's course; this is mostly for the convenience of our own students to emphasize the relationships between our courses but, in principle, this document is mostly self contained.

[1] one can always do simpler but, for the sake of completeness, it is hard to avoid certain technicalities such as the invariance of pseudo-differential operators under coordinates change

Chapter 1

The Shnirelman Theorem

In this chapter, we introduce the objects involved in the Shnirelman Theorem namely, on the spectral side, the Laplace-Beltrami operator and its eigenfunctions and, on the dynamical side, the geodesic flow and the notion of ergodicity. The theorem itself is stated in Section 1.4 (see Theorem 1.18).

1.1 Memento on differential geometry

In this section, we record, in a rather compact form, some notation, conventions and usual formulas that will be used in these notes. We refer to the lectures by J.-P. Otal and standard textbooks on differential geometry for a more detailed presentation.

We consider a smooth manifold M of dimension n.

Charts on M. If $\kappa : U \subset M \to V \subset \mathbb{R}^n$ is a local diffeomorphism (or chart), the local coordinates $(x_1, \dots, x_n)$ are the components of κ, i.e. $\kappa(m) = \big(x_1(m), \dots, x_n(m)\big)$. If $\tilde{\kappa} : \tilde{U} \to \tilde{V}$ is another chart (with components $\big(\tilde{x}_1, \dots, \tilde{x}_n\big)$), the associated transition maps are

$$\tilde{\kappa} \circ \kappa^{-1} : \kappa(U \cap \tilde{U}) \to \tilde{\kappa}(U \cap \tilde{U}), \qquad \kappa \circ \tilde{\kappa}^{-1} : \tilde{\kappa}(U \cap \tilde{U}) \to \kappa(U \cap \tilde{U}).$$

Letting $\chi = \tilde{\kappa} \circ \kappa^{-1}$, one denotes

$$(\partial \tilde{x} / \partial x) := \left(\frac{\partial \tilde{x}_i}{\partial x_j} \right) = d\chi(x), \qquad (\partial x / \partial \tilde{x}) = \left(\frac{\partial x_i}{\partial \tilde{x}_j} \right) = d\chi(x)^{-1}, \tag{1.1}$$

that are respectively the Jacobian matrices of the transition maps $\tilde{\kappa} \circ \kappa^{-1}$ and $\kappa \circ \tilde{\kappa}^{-1}$. Our convention is that i labels the rows and j labels the columns.

Tangent bundle. Given local coordinates $\kappa = (x_1, \dots, x_n)$ on $U \subset M$ and φ smooth on U, one sets

$$\partial_{x_j} \varphi := \partial_j (\kappa_* \varphi), \qquad \kappa_* \varphi := \varphi \circ \kappa^{-1},$$

and, so defined, $\big(\partial_{x_1}, \ldots, \partial_{x_n}\big)$ is a local basis of the tangent space. More precisely, for any $m \in U$, the n maps $\varphi \mapsto (\partial_{x_j}\varphi)(\kappa(m))$ form a basis of T_mM. If one considers another system of coordinates $\big(\tilde{x}_1, \ldots, \tilde{x}_n\big)$, then two decompositions of the same vector

$$\mathrm{v} = \sum v_j \partial_{x_j} = \sum \tilde{v}_k \partial_{\tilde{x}_k}$$

are related by

$$\tilde{v}_k = \sum_j \frac{\partial \tilde{x}_k}{\partial x_j} v_j,$$

or, in matrix form (viewing $v = (v_1, \ldots, v_n)^T$ and $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_n)^T$ as rows),

$$\tilde{v} = (\partial \tilde{x}/\partial x) v.$$

The chart of TM associated to κ is the map $TU = \pi_{TM}^{-1}(U) \to V \times \mathbb{R}^n$ defined by

$$TU \ni \mathrm{v} \mapsto \big(x_1(m), \ldots, x_n(m), v_1, \ldots, v_n\big), \qquad m = \pi_{TM}(\mathrm{v}), \quad \mathrm{v} = \sum v_j \partial_{x_j},$$

where $\pi_{TM} : TM \to M$ is the projection. The transition map between two such systems is thus of the form

$$(x, v) \mapsto (\tilde{x}, \tilde{v}) = \big(\chi(x), d\chi(x)v\big).$$

Cotangent bundle. The cotangent bundle is the dual to TM in the sense that

$$T^*M = \sqcup_{m \in M} T_m^*M$$

with $T_m^*M := (T_mM)^*$ the dual to the tangent space at m. Given coordinates on M, one denotes by $(dx_1, \ldots, dx_n)$ the dual basis to $(\partial_{x_1}, \ldots, \partial_{x_n})$. Then two decompositions of the same covector

$$T_m^*M \ni X^* = \sum \xi_j dx_j = \sum \tilde{\xi}_k d\tilde{x}_k$$

are related by

$$\tilde{\xi}_k = \sum_j \frac{\partial x_j}{\partial \tilde{x}_k} \xi_j,$$

or in matrix form (with $\xi = (\xi_1, \ldots, \xi_n)^T$ and $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_n)^T$)

$$\tilde{\xi} = \big((\partial \tilde{x}/\partial x)^{-1}\big)^T \xi. \tag{1.2}$$

The local coordinates on T^*M associated to the chart $\kappa : U \to V$ on M are defined by

$$T^*U \ni X^* \mapsto (x_1(m), \ldots, x_n(m), \xi_1, \ldots, \xi_n), \qquad m = \pi_{T^*M}(X^*), \quad X^* = \sum \xi_j dx_j,$$

where $\pi_{T^*M} : T^*M \to M$ is the projection. The transition maps are then

$$(x, \xi) \mapsto (\tilde{x}, \tilde{\xi}) = \big(\chi(x), (d\chi(x)^{-1})^T \xi\big). \tag{1.3}$$

Also, if a is a smooth function on T^*M, we set

$$(\kappa_* a)(x,\xi) := a(X^*) \qquad \text{if } \ x = \kappa\big(\pi_{T^*M}(X^*)\big), \quad X^* = \sum \xi_j dx_j. \tag{1.4}$$

Notice that, strictly speaking, $\kappa_* a$ is meaningless since κ_* can be applied to functions on M rather than T^*M. Rigorously, this means we consider the pushforward by the coordinate diffeomorphism associated to κ on T^*U, but we use the above abuse of notion for simplicity.

The sections of T^*M are the 1-forms on M. In particular, the differential of a smooth function φ on M is intrinsically defined and reads in local coordinates $d\varphi = \sum_j (\partial_{x_j}\varphi) dx_j$.

Transition maps on $T(T^*M)$ To describe the symplectic structure on T^*M, it is useful to record the form of the transition maps between charts on $T(T^*M)$. Given a chart $\kappa : U \to V$ on M, one considers the associated chart on T^*M, i.e. $T^*U \to V \times \mathbb{R}^n$ as above, and then associates to it the corresponding chart on $T(T^*M)$, i.e. $T(T^*U) \to V \times \mathbb{R}^n \times \mathbb{R}^{2n}$. Then, the transition map between two such charts associated respectively to κ and $\tilde{\kappa}$ is of the form

$$\big(x, \xi, X, Y\big) \mapsto \big(\tilde{x}, \tilde{\xi}, \tilde{X}, \tilde{\Xi}\big)$$

where, viewing $X, \Xi, \tilde{X}, \tilde{\Xi}$ as rows,

$$\tilde{x} = \chi(x), \qquad \tilde{\xi} = (d\chi(x)^{-1})^T \xi, \qquad \begin{pmatrix} \tilde{X} \\ \tilde{\Xi} \end{pmatrix} = \begin{pmatrix} (\partial\tilde{x}/\partial x) & 0_n \\ \Sigma(x,\xi) & (\partial x/\partial\tilde{x})^T \end{pmatrix} \begin{pmatrix} X \\ \Xi \end{pmatrix} \tag{1.5}$$

where $\Sigma(x,\xi) = (\Sigma_{ij})$ with

$$\Sigma_{ij} = -\sum_{\lambda,\mu,\nu} \frac{\partial x_\lambda}{\partial \tilde{x}_i} \frac{\partial^2 \tilde{x}_\mu}{\partial x_j \partial x_\lambda} \frac{\partial x_\nu}{\partial \tilde{x}_\mu} \xi_\nu. \tag{1.6}$$

In other words, two decompositions[1] of a same vector of $T(T^*M)$ of the form

$$\sum_j X_j \partial_{x_j} + \Xi_j \partial_{\xi_j} = \sum_k \tilde{X}_k \partial_{\tilde{x}_k} + \tilde{\Xi}_k \partial_{\tilde{\xi}_k},$$

are related by

$$\tilde{X}_k = \sum_j \frac{\partial \tilde{x}_k}{\partial x_j} X_j, \qquad \tilde{\Xi}_k = -\sum_j \left(\sum_{\lambda,\mu,\nu} \frac{\partial x_\lambda}{\partial \tilde{x}_k} \frac{\partial^2 \tilde{x}_\mu}{\partial x_j \partial x_\lambda} \frac{\partial x_\nu}{\partial \tilde{x}_\mu} \xi_\nu \right) X_j + \sum_j \frac{\partial x_j}{\partial \tilde{x}_k} \Xi_j.$$

Symplectic structure on T^*M. The cotangent bundle T^*M is equipped with a natural 1-form (i.e. a section of $T^*(T^*M)$) called the **Liouville form**, which we denote by ξdx. It is defined in local coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ as

$$\xi dx = \sum_j \xi_j d(x_j \circ \pi_{T^*M}),$$

[1] here ∂_{x_j} is a short hand for $\partial_{x_j \circ \pi_{T^*M}}$ since, strictly speaking, the coordinates induced on T^*M by those on M are $\big(x_1 \circ \pi_{T^*M}, \ldots, x_n \circ \pi_{T^*M}, \xi_1, \ldots, \xi_n\big)$

which one also writes usually (and below) as $\sum_j \xi_j dx_j$, but which is better understood written as above to emphasize it is a section of $T^*(T^*M)$ and not an element of T^*M. The invariance of this definition follows from the fact that, given another coordinate system $(\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{\xi}_1, \ldots, \tilde{\xi}_n)$, one has $\sum \xi_j dx_j = \sum \tilde{\xi}_k d\tilde{x}_k$. Equivalently, by using the notation of the previous paragraph, this last equality can be written

$$\sum_j \xi_j X_j = \sum_k \tilde{\xi}_k \tilde{X}_k$$

and follows from the transformations rules (1.2) and (1.5). One can then define the **symplectic form** σ as the exterior derivative of the Liouville form

$$\sigma := d(\xi dx) = \sum_j d\xi_j \wedge dx_j.$$

It is a 2-form on T^*M (i.e. a section of $\Lambda^2 T^*(T^*M)$) which is closed. To see more explicitly the action of σ, we record that given $X^* \in T^*M$ and two vectors $W, W' \in T_{X^*}(T^*M)$, which we write

$$W = \sum X_j \partial x_j + \Xi_j \partial_{\xi_j}, \qquad W' = \sum X_j' \partial_{x_j} + \Xi_j' \partial_{\xi_j},$$

one has

$$\sigma_{X^*}(W, W') = \sum_j \Xi_j X_j' - X_j \Xi_j' = \begin{pmatrix} X \\ \Xi \end{pmatrix} \cdot \begin{pmatrix} 0_n & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} X' \\ \Xi' \end{pmatrix}. \tag{1.7}$$

One can also check directly that σ is invariantly defined by $\sum_j d\xi_j \wedge dx_j$, in the sense that $\sum_j d\xi_j \wedge dx_j = \sum_k d\tilde{\xi}_k \wedge d\tilde{x}_k$. To prove this equality it suffices to observe that, if we set

$$S = \begin{pmatrix} (\partial \tilde{x}/\partial x) & 0_n \\ \Sigma(x,\xi) & (\partial x/\partial \tilde{x})^T \end{pmatrix}, \qquad J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0 \end{pmatrix}, \tag{1.8}$$

the invariance is a consequence of the identity

$$S^T J S = J,$$

(one says S is a symplectic matrix) which can be checked by hand by using the form of S, in particular (1.6).

1.2 The Laplace-Beltrami operator

Everywhere in this course we consider a compact (connected) Riemannian manifold (M, g). We recall that this means that at any point $m \in M$ the tangent space $T_m M$ is equipped with a scalar product $g(m)$ (also denoted by $\langle\cdot,\cdot\rangle_{g(m)}$) smoothly varying with m. More explicitly, if $\kappa = (x_1,\dots,x_n) : U \subset M \to V \subset \mathbb{R}^n$ is a local chart then

$$g(m) = \sum_{j,k=1}^{n} g_{jk}(x) dx_j dx_k,$$

where $x = \kappa(m)$ and $(g_{jk}(\cdot))$ is a matrix of smooth real valued functions on V which is positive definite at every point. At each point, this matrix is the Gram matrix of the scalar product in the basis $\big(\partial_{x_1},\dots,\partial_{x_n}\big)$. In other words, if $\mathrm{v} = \sum v_j \partial_{x_j}$ and $\mathrm{w} = \sum w_j \partial_{x_j}$ are two vectors in $T_m M$, one has

$$\langle \mathrm{v}, \mathrm{w}\rangle_{g(m)} = \sum_{j,k=1}^{n} g_{jk}(x) v_j w_k.$$

Exercise 1.1. *If $\tilde\kappa = (\tilde x_1,\dots,\tilde x_n)$ is another coordinates system in which*

$$g(m) = \sum_{\tilde j,\tilde k=1}^{n} \tilde g_{\tilde j\tilde k}(\tilde x) d\tilde x_{\tilde j} d\tilde x_{\tilde k},$$

make sure you understand that, as a product of matrices,

$$\big(\partial x/\partial \tilde x\big)^T \big(g_{jk}(x)\big)\big(\partial x/\partial \tilde x\big) = \big(\tilde g_{\tilde j\tilde k}(\tilde x)\big). \tag{1.9}$$

Here we use the notation (1.1).

We also introduce the standard notation

$$(g^{jk}(x)) := (g_{jk}(x))^{-1} \tag{1.10}$$

which will be used further in the text. This inverse matrix is the matrix of the natural scalar product on $T^*_m M$ in the basis $(dx_1,\dots,dx_n)$.

There is a natural measure associated to g, called the **Riemannian measure**, which is locally of the form

$$d\mathrm{vol}_g = |g(x)| dx_1 \cdots dx_n, \qquad |g(x)| := \sqrt{\det\big(g_{jk}(x)\big)} \tag{1.11}$$

in the sense that, whenever φ is a continuous function compactly supported in U,

$$\int_M \varphi(m) d\mathrm{vol}_g = \int_V \big(\kappa_*\varphi)(x) |g(x)| dx, \tag{1.12}$$

where we recall that $\kappa_*\varphi = \varphi \circ \kappa^{-1}$.

Exercise 1.2. *Use the transformation rule (1.9) to check that (1.12) is independent of the choice of the chart.*

The **Laplace-Beltrami operator**, or more simply the **Laplacian**, is a differential operator of order 2 acting on functions on M and naturally associated to g. One can define it as follows. For any smooth function φ on M and any $m \in M$, the differential of φ at m, $d\varphi(m)$, belongs to T_m^*M, the dual to the tangent space T_mM. Thanks to the inner product $g(m)$, $d\varphi(m)$ can be represented by a vector, denoted by $\nabla_g\varphi(m)$, in the sense that

$$d\varphi(m)\cdot \mathrm{v} = \langle \nabla_g\varphi(m), \mathrm{v}\rangle_{g(m)} \qquad \text{for all } \ \mathrm{v} \in T_mM,$$

the left hand side standing for $d\varphi(m)(\mathrm{v})$. The vector $\nabla_g\varphi(m) \in T_mM$ is called the **gradient** of φ at m.

Exercise 1.3. *Check that, in local coordinates, if $d\varphi = \sum_j \partial_{x_j}\varphi dx_j$ then*

$$\nabla_g\varphi = \sum (\nabla_g\varphi)_k \partial_{x_k} \qquad \textit{with} \qquad (\nabla_g\varphi)_k = \sum_j g^{jk}(x)\partial_{x_j}\varphi,$$

where we recall that $\partial_{x_j}\varphi(x)$ is the usual short hand for $\partial(\kappa_*\varphi)/\partial x_j$.

Then, Δ_g can be defined as the (unique) differential operator on M which satisfies

$$-\int_M \varphi(m)(\Delta_g\psi)(m)d\mathrm{vol}_g = \int \langle \nabla_g\varphi(m), \nabla_g\psi(m)\rangle_{g(m)} d\mathrm{vol}_g, \tag{1.13}$$

for all $\varphi, \psi \in C^\infty(M)$. To justify this fact and see what this operator looks like, we work in local coordinates and assume that at least one of the two functions φ, ψ is compactly supported in U. Then, using Exercise 1.3,

$$\begin{aligned}
\int_M \langle \nabla_g\varphi(m), \nabla_g\psi(m)\rangle_{g(m)} d\mathrm{vol}_g &= \int_V \sum_{j,k=1}^n g_{jk}(x)(\nabla_g\varphi)_j(\nabla_g\psi)_k |g(x)|dx \\
&= \int_V \sum_{j,k=1}^n g^{jk}(x)(\partial_{x_j}\varphi)(\partial_{x_k}\psi)|g(x)|dx.
\end{aligned}$$

Integrating by part, we find

$$\begin{aligned}
\int_M \langle \nabla_g\varphi, \nabla_g\psi\rangle_g d\mathrm{vol}_g &= -\int_V \sum_{j,k} (\kappa_*\varphi)\partial_{x_j}\left(|g(x)|g^{jk}(x)(\partial_{x_k}\psi)\right)dx \\
&= -\int_V (\kappa_*\varphi)\sum_{j,k} |g(x)|^{-1}\partial_{x_j}\left(|g(x)|g^{jk}(x)(\partial_{x_k}\psi)\right)|g(x)|dx \\
&= -\int_M \varphi(m)(\Delta_g\psi)(m)d\mathrm{vol}_g
\end{aligned}$$

provided we define $\Delta_g\psi$ by

$$\kappa_*(\Delta_g\psi) = \sum_{j,k} |g(x)|^{-1}\frac{\partial}{\partial x_j}\left(|g(x)|g^{jk}(x)\frac{\partial}{\partial x_k}(\kappa_*\psi)\right). \tag{1.14}$$

Exercise 1.4. *Check directly that (1.14) is invariantly defined, namely if $\tilde{\kappa} : \tilde{U} \to \tilde{V}$ is another chart, with components $(\tilde{x}_1, \ldots, \tilde{x}_n)$, then*

$$\kappa^* \sum_{j,k} |g(x)|^{-1} \partial_{x_j} \Big(|g(x)| g^{jk}(x) \partial_{x_k} (\kappa_* \psi) \Big) = \tilde{\kappa}^* \sum_{j,k} |\tilde{g}(\tilde{x})|^{-1} \partial_{\tilde{x}_j} \Big(|\tilde{g}(\tilde{x})| \tilde{g}^{jk}(\tilde{x}) \partial_{\tilde{x}_k} (\tilde{\kappa}_* \psi) \Big)$$

for all ψ supported in $U \cap \tilde{U}$.

Exercise 1.5. *Check that in local coordinates, the Laplace-Beltrami operator also reads*

$$\sum_{j,k} g^{jk}(x) \partial_{x_j} \partial_{x_k} - \sum_{i,j,k} g^{jk}(x) \Gamma^i_{jk}(x) \partial_{x_i} \tag{1.15}$$

where $\Gamma^i_{jk}(x)$ are the Christoffel symbols of g given by

$$\Gamma^i_{jk}(x) = \frac{1}{2} \sum_l g^{il}(x) \big(\partial_{x_j} g_{kl}(x) + \partial_{x_k} g_{lj}(x) - \partial_{x_l} g_{jk}(x) \big). \tag{1.16}$$

We next introduce the inner product on $L^2(M) = L^2(M, d\mathrm{vol}_g)$,

$$\langle \varphi, \psi \rangle_M = \int_M \overline{\varphi(m)} \psi(m) d\mathrm{vol}_g.$$

The above integral has a clear sense say for $\varphi, \psi \in C(M)$; the space $L^2(M)$ itself can be defined as the closure of $C(M)$ (or even $C^\infty(M)$) for the L^2 norm. Then, it follows from (1.13) that

$$\langle \varphi, \Delta_g \psi \rangle_M = \langle \Delta_g \varphi, \psi \rangle_M, \qquad \text{for all } \varphi, \psi \in C^\infty(M). \tag{1.17}$$

One says that Δ_g is formally **self-adjoint** on $C^\infty(M)$. To justify the terminology, we note that such a relation is the analogue of the fact that $AX \cdot Y = X \cdot AY$ when A is a real symmetric matrix of size N and $X, Y \in \mathbb{R}^N$. This analogy goes further since, much as symmetric matrices can be diagonalized in an orthonormal basis, we have the following result.

Theorem 1.6. *There exists an orthonormal basis $(e_j)_{j \in \mathbb{N}}$ of $L^2(M)$ composed of smooth functions and a sequence $(\lambda_j)_{j \in \mathbb{N}}$ of real numbers such that*

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \qquad \lim_{j \to \infty} \lambda_j = +\infty, \tag{1.18}$$

and

$$-\Delta_g e_j = \lambda_j e_j.$$

The functions e_j are called the **eigenfunctions** and the numbers λ_j the **eigenvalues** of $-\Delta_g$. The sequence $(\lambda_j)_{j \in \mathbb{N}}$ is called the **spectrum**. Let us point out that the eigenvalues are counted according to their **multiplicities**; in other words,

$$\mathrm{mult}(\lambda_N) := \#\{ j \in \mathbb{N} \mid \lambda_j = \lambda_N \} = \dim \mathrm{Ker}(-\Delta_g - \lambda_N I) \geq 1.$$

The multiplicity $\mathrm{mult}(\lambda_N)$ is finite for each N (since $\lambda_j \to \infty$ as $j \to \infty$) but it can grow as $N \to \infty$. We also note in passing that $\lambda_1 = 0$ has multiplicity 1 (hence the first strict inequality in (1.18)) since

$$0 = -\int_M e_1 (\Delta_g e_1) d\mathrm{vol}_g = \int_M |\nabla_g e_1|^2_{g(m)} d\mathrm{vol}_g \qquad \Rightarrow \qquad \nabla_g e_1 = 0 \qquad \Rightarrow \qquad de_1 = 0,$$

which in turns implies that e_1 is constant since M is connected. However, we will not use this information since the Shnirelman Theorem deals with eigenfunctions in the limit $j \to \infty$.

The most simple illustration of Theorem 1.6 is given by the circle: identifying $L^2(\mathbb{T})$ with $L^2(0, 2\pi)$, the functions $e_j(\theta) := e^{\mathrm{i}j\theta}/\sqrt{2\pi}$ form an orthonormal basis of $L^2(\mathbb{T})$, by the L^2 theory of Fourier series, and are eigenfunctions of the natural Laplacian on $\mathbb{T}$, i.e.

$$-\frac{d^2}{d\theta^2} e_j = j^2 e_j.$$

The eigenvalues are j^2 and have multiplicity 2 if $j \neq 0$, illustrating that multiplicities can be larger than 1 (note however that the boundedness of the multiplicities is something quite specific to one dimensional domains).

We won't prove Theorem 1.6, mostly by lack of time but also since its proof is not essential to understand the Shnirelman Theorem (for a proof see e.g. [6, Ch. 5, Sec. 1] or [9, Th. 14.7]. See also [1]).

The spectrum and eigenfunctions of the Laplace-Beltrami operator are fundamental objects but they are in general very hard (not to say impossible) to compute, up to some very rare cases such as the torus $\mathbb{T}^n$ or the sphere $\mathbb{S}^n$. The Shnirelman Theorem will give an example of the kind of qualitative information on the eigenfunctions which can be recovered from geometric properties of the manifold.

1.3 The geodesic flow

In this section, we recall the definition of the geodesic flow on (M, g), in particular its Hamiltonian formulation on the cotangent bundle. We first recall briefly the Riemannian formulation: for any $m \in M$ and $\mathrm{v} \in T_m M$, there exists a unique smooth curve $\gamma : \mathbb{R} \to M$ which solves the differential equation

$$\gamma(0) = m, \qquad \dot{\gamma}(0) = \mathrm{v}, \qquad D_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0, \tag{1.19}$$

where $D_{\dot{\gamma}(t)} \dot{\gamma}(t)$ is the covariant derivative of the vector field $\dot{\gamma}$ along the curve γ. That γ is defined on $\mathbb{R}$ is non trivial (it solves a nonlinear ODE) but it follows from the compactness of M (see your course or any textbook on Riemannian geometry). In local coordinates, this equation reads

$$\ddot{x}_i(t) + \sum_{j,k} \Gamma^i_{jk}(x(t)) \dot{x}_j(t) \dot{x}_k(t) = 0, \tag{1.20}$$

where $x(t) = \kappa(\gamma(t))$. The **geodesic flow** $(G^t)_{t\in\mathbb{R}}$ on the tangent bundle is then the family of maps $G^t : TM \to TM$ associating to any element $\mathrm{v} \in T_mM$ the value at time t of the solution to (1.20), that is

$$\mathrm{v} \in T_mM \mapsto \dot\gamma(t) \in T_{\gamma(t)}M,$$

i.e. $\pi_{TM}(G^t(\mathrm{v})) = \gamma(t)$ and $G^t(\mathrm{v}) = \dot\gamma(t) \in T_{\gamma(t)}M$.

A property of the equation (1.19) is that the Riemannian norm is preserved along the flow, i.e. $||\dot\gamma(t)||_{g(\gamma(t))} = ||\mathrm{v}||_{g(m)}$ for all t. This implies that one can restrict G^t to the **unit sphere bundle** SM (also denoted by UM) defined as the subset of vectors in TM of norm 1.

In this course, we will adopt the Hamiltonian description of the geodesic flow, i.e. on the cotangent bundle rather than on the tangent bundle. Both formulations are equivalent, but the Hamiltonian point of view is more natural in our context. Let us recall some definitions. The **Hamiltonian** associated to the metric g is the function

$$p : T^*M \to \mathbb{R}$$

which is invariantly defined by the following formula in local coordinates

$$(\kappa_* p)(x,\xi) = \sum_{j,k} g^{jk}(x)\xi_j\xi_k, \tag{1.21}$$

(we refer to (1.4) for the abuse of notation $\kappa_* p$). In other words, in each fiber of T^*M, p is the quadratic form arising from the natural scalar product induced by $g(m)$. By looking at the top order term in the expression of the Laplace-Beltrami operator in (1.15) (i.e. the one with derivatives of higher order) and by replacing ∂_{x_j} by $\mathrm{i}\xi_j$, we recover (1.21). For this reason, p is also called the **principal symbol** of (minus) the Laplacian[2].

The differential dp is a 1-form on T^*M and, using the symplectic form σ on T^*M, one can identify dp with a vector field on T^*M, namely the unique vector field W_p such that, for all vector field W on T^*M

$$\sigma(W, W_p) = dp \cdot W, \tag{1.22}$$

i.e. at each $X^* \in T^*M$, $\sigma_{X^*}(W(X^*), W_p(X^*)) = dp(X^*) \cdot W(X^*)$. In local coordinates on T^*M, W_p is of the form

$$W_p = \sum_j (\partial_{\xi_j} p)\partial_{x_j} - (\partial_{x_j} p)\partial_{\xi_j}. \tag{1.23}$$

One can then consider the differential equation associated to this vector field

$$\dot\gamma^* = W_p(\gamma^*), \tag{1.24}$$

[2]we will not do any further use of the notion of principal symbol in these notes; we thus record this terminology very briefly, mainly to help a reader who will find it in the literature

whose solutions are curves $t \mapsto \gamma^*(t)$ on T^*M. More explicitly in local coordinates, if we write $(x_1^t, \ldots, x_n^t, \xi_1^t, \ldots, \xi_n^t)$ the coordinates of $\gamma^*(t)$, the equation (1.24) reads

$$\dot{x}_i^t = 2\sum_{k=1}^n g^{ik}(x^t)\xi_k^t, \qquad \dot{\xi}^t = -\sum_{j,k=1}^n (\partial_i g^{jk})(x^t)\xi_j^t\xi_k^t. \tag{1.25}$$

Exercise 1.7. *The two questions are independent.*

1. *Check that*

$$\frac{d}{dt}\left(\sum_{j,k=1}^n g^{jk}(x^t)\xi_j^t\xi_k^t\right) = 0. \tag{1.26}$$

2. *Recover the Riemannian formulation (1.20) from the Hamiltonian one (1.25).*

   Hints: use (and check) that $\dot{x}_j^t = \frac{1}{2}\sum g_{jk}(x^t)\xi_k^t$ and that, if $G(x) = \big(g_{jk}(x)\big)$, then $\partial_{x_i}\big(G(x)^{-1}\big) = -G(x)^{-1}\big(\partial_{x_i}G(x)\big)G(x)^{-1}$. Recall also (1.10) and (1.16).

The **geodesic flow on the cotangent bundle** $(\Phi^t)_{t\in\mathbb{R}}$ is the flow of the vector field W_p, i.e. given $X_0^* \in T^*M$, $t \mapsto \Phi^t(X_0^*)$ is the solution to (1.24) such that $\gamma^*(0) = X_0^*$. That it is defined for all $t \in \mathbb{R}$ follows from (1.26), i.e. equivalently

$$p \circ \Phi^t = p, \tag{1.27}$$

and from the fact that for any compact set K of $\mathbb{R}$, $p^{-1}(K)$ is compact (the trajectory $t \mapsto \Phi^t(X_0^*)$ is confined in the compact set $p^{-1}\big(\{p(X_0^*)\}\big)$ hence cannot blow up in finite time). This last property follows from the compactness of M and the positive definiteness of the matrix $(g^{jk}(x))$ in (1.21).

Exercise 1.8. *Check the details.*

The property (1.27) is called the **conservation of the energy**. It is nothing but the Hamiltonian expression of the preservation of the Riemannian norm along the flow, as mentionned previously. Much as G^t acts on the unit sphere bundle SM, Φ^t acts on the unit **cosphere bundle** S^*M which is defined as

$$S^*M = p^{-1}(1).$$

That S^*M is preserved by the flow is a direct consequence of (1.27).

We next recall the notion of **ergodicity of the geodesic flow**. One first needs to recall that T^*M is equipped with a natural measure $|dxd\xi|$ defined locally as $dx_1\cdots dx_n d\xi_1\cdots d\xi_n$. The invariance of this definition follows from the fact that the transitions maps (1.3) on T^*M have their Jacobian equal to 1; indeed, if $\kappa : U \to V$ and $\tilde{\kappa} : \tilde{U} \to \tilde{V}$ are local charts on M and $a \in C_0^\infty(T^*M)$ such that a vanishes outside the intersection of coordinates patches $\pi_{T^*M}^{-1}(U \cap \tilde{U})$, one has

$$\int\cdots\int_{V\times\mathbb{R}^n} (\kappa_* a)(x,\xi)dx_1\cdots d\xi_n = \int\cdots\int_{\tilde{V}\times\mathbb{R}^n} (\tilde{\kappa}_* a)(\tilde{x},\tilde{\xi})d\tilde{x}_1\cdots d\tilde{\xi}_n,$$

where we use the (abuse of) notation (1.4) for κ_* and $\tilde{\kappa}_*$. One then defines the common value of these integrals as $\int a|dxd\xi|$ and extends this definition to all functions in $C_0^\infty(T^*M)$ by mean of a partition of unity on M.

There is then a natural measure on S^*M called the **Liouville measure**[3] which we denote by L_g or dL_g. It is defined by mean of "polar coordinates" on T^*M as follows. The Liouville measure can be defined as the unique measure on S^*M such that

$$\int_{T^*M} a|dxd\xi| = \int_0^\infty \left(\int_{S^*M} a(\rho\omega) dL_g(\omega) \right) \rho^{n-1} d\rho, \tag{1.28}$$

say for all $a \in C_0^\infty(T^*M)$. We point out that the precise knowledge of the construction of dL_g is useless for our purposes (basically, it consists in introducing polar coordinates on each cotangeant space as can be done on any Euclidean space). To see that it is unique, and also to compute certain integrals, we essentially only need to observe that, for $\rho > 0$ and $a \in C_0^\infty(T^*M)$,

$$\frac{d}{d\rho} \left(\int_{p \leq \rho^2} a|dxd\xi| \right) = \rho^{n-1} \int_{S^*M} a(\rho\omega) dL_g(\omega). \tag{1.29}$$

Exercise 1.9. *Prove formula (1.29) by cheking first that*

$$\int_{p \leq \rho^2} a|dxd\xi| = \int_0^\rho \mu^{n-1} \left(\int_{S^*M} a(\mu\omega) dL_g(\omega) \right) d\mu.$$

Hint: formally this is (1.28) with a replaced by $a\mathbb{1}_{[0,\rho^2]}(p)$, but the latter is not smooth. Justify the formula by approximating $\mathbb{1}_{[0,\rho^2]}(p)$ by a family of smooth functions $\psi((p-\rho^2)/\epsilon)$ with $\epsilon \to 0^+$ and $\psi \in C^\infty(\mathbb{R})$ equal to 1 on $(-\infty, 0]$ and vanishing on $[1, \infty)$.

We illustrate the interest of (1.29) with the following formula which will be useful in the proof of the Shnirelman theorem (see (2.17) and (2.23)).

Proposition 1.10. *Let $f \in C_0^\infty(\mathbb{R})$, $\psi \in C^\infty(M)$ and consider $a = \psi \times f \circ p$ (i.e. $a = (\psi \circ \pi_{T^*M})(f \circ p)$). Then*

$$\int_{S^*M} a(\rho\omega) dL_g = n|\mathbb{B}^n| f(\rho^2) \int_M \psi(m) d\mathrm{vol}_g,$$

where $|\mathbb{B}^n|$ is the Lebesgue measure of the unit (Euclidean) ball on $\mathbb{R}^n$.

Proof. By using a partition of unity, we may assume that ψ is supported in a coordinate patch. Then

$$\int_{p \leq \rho^2} a|dxd\xi| = \int \int_{\xi \cdot G(x)\xi \leq \rho^2} (\kappa_* \psi)(x) f(\xi \cdot G(x)\xi) dx d\xi$$

[3] this terminology is also used for the natural measure on SM but we will not use it in these notes

where $G(x) = (g^{jk}(x))$. One can rewrite $\xi \cdot G(x)\xi$ as $|G(x)^{1/2}\xi|^2$ with $G(x)^{1/2}$ the unique positive definite square root of $G(x)$. Then, using the change of variable $G(x)^{1/2}\xi = \zeta$ whose Jacobian is $\det\big(G(x)^{1/2}\big)^{-1}$, that is exactly $|g(x)|$ (see (1.11)), we find

$$\int_{p\leq\rho^2} a|dxd\xi| = \left(\int_{|\zeta|\leq\rho} f(|\zeta|^2)d\zeta\right)\left(\int(\kappa_*\psi)(x)|g(x)|dx\right)$$

where the second factor in the right hand side is precisely $\int_M \psi d\mathrm{vol}_g$. On the other hand, we can easily differentiate the first factor with respect ot ρ by using polar coordinates on $\mathbb{R}^n$ for it reads

$$\int_0^\rho \int_{\mathbb{S}^{n-1}} f(r^2)r^{n-1}drd\mathrm{vol}_{\mathbb{S}^{n-1}}$$

hence its derivative is $\rho^{n-1}f(\rho^2)$ times the the measure of $\mathbb{S}^{n-1}$ which is exactly $n|\mathbb{B}^n|$ since $|\mathbb{B}^n| = \int_0^1 r^{n-1}\int_{\mathbb{S}^{n-1}} d\mathrm{vol}_{\mathbb{S}^{n-1}}dr$. □

A straightforward consequence of Proposition 1.10 is that the Liouville measure is **finite** and more precisely that

$$L_g(S^*M) = n|\mathbb{B}_n|\mathrm{vol}_g(M).$$

It is also **invariant** by the geodesic flow Φ^t, i.e. $L_g(B) = L_g\big(\Phi^t(B)\big)$ for every Borel subset $B \subset S^*M$ and $t \in \mathbb{R}$ (for completeness this fact is proved in Proposition 1.15).

Definition 1.11. *The geodesic flow is* **ergodic** *if the Borel subsets of S^*M which are invariant by Φ^t have either zero or full Liouville measure.*

For a function a on T^*M (resp. S^*M), we denote by $[a]_T$ its time average along the geodesic flow,

$$[a]_T(\omega) := \frac{1}{T}\int_0^T a\circ\Phi^t(\omega)dt.$$

It is a function on T^*M (resp. S^*M ; note that the definition still makes sense since Φ^t preserves S^*M). For a function on S^*M, we denote by

$$⨍_{S^*M} a := \frac{1}{L_g(S^*M)}\int_{S^*M} adL_g$$

the (phase space) average of a over S^*M.

A consequence of the ergodicity is the following famous result.

Theorem 1.12 (Birkhoff's Ergodic Theorem)**.** *Let $a \in C^\infty(S^*M)$. For L_g almost every $\omega \in S^*M$, $[a]_T(\omega)$ has a limit $a^+(\omega)$ as $T \to +\infty$. If in addition the geodesic flow is ergodic, then for L_g almost every ω, one has*

$$a_+(\omega) = ⨍_{S^*M} a.$$

We leave as an exercise for the reader to check that the above theorem implies the following result.

Corollary 1.13. *If* $(\Phi^t)_{t\in\mathbb{R}}$ *is ergodic, then for every* $a \in C^\infty(S^*M, \mathbb{R})$ *one has*

$$\int_{S^*M} \left([a]_T - \fint_{S^*M} a \right)^2 dL_g \to 0, \qquad T \to +\infty. \tag{1.30}$$

Notice that a direct proof of Corollary 1.13 (i.e. without using Theorem 1.12) is given in Zworski's book [9].

Remark. As far as the Shnirelman Theorem is concerned, one could consider directly (1.30) as an assumption (instead of the ergodicity), since this is this property which will be used in the proof (see page 36). The course by J.-P. Otal and the Birkoff Ergodic Theorem show you that manifolds with an ergodic geodesic flow (e.g. compact quotients of the hyperbolic plane) satisfy this property.

Exercise 1.14 (The Poisson bracket and its invariance by the geodesic flow)**.** *Given two functions* $a, b \in C^\infty(T^*M)$ *one defines their* **Poisson bracket** *by*

$$\{a, b\} := \sigma(W_a, W_b) \tag{1.31}$$

(see (1.7) and (1.23)) that is, in local coordinates,

$$\{a, b\} = \sum_{j=1}^{n} \frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j}. \tag{1.32}$$

It is a function on T^*M*. The purpose of this exercise is to show that*

$$\{a \circ \Phi^t, b \circ \Phi^t\} = \{a, b\} \circ \Phi^t \tag{1.33}$$

for all $t \in \mathbb{R}$ *(recall that the flow is complete by compactness of* M*). In Questions 1 and 2 below, we choose local coordinates and use the same notation for the Hamiltonian vector field or the geodesic flow and their components in local coordinates, namely*

$$W_p = (\partial_{\xi_1} p, \dots, \partial_{\xi_n} p, -\partial_{x_1} p, \dots, -\partial_{x_n} p)^T$$

for the (components of the) Hamiltonian vector field and

$$\Phi^t = (x_1^t, \dots, x_n^t, \xi_1^t, \dots, \xi_n^t)^T$$

for the (components of the) geodesic flow.

1. *Check that* $JdW_p + dW_p^T J = 0$ *(see (1.8) for* J*).*
2. *Check that*
$$\frac{d}{dt}\left((d\Phi^t)^T J (d\Phi^t)\right) = 0.$$

3. *Check that for all $X^* \in T^*M$, $W, W' \in T_{X^*}(T^*M)$ and $t \in \mathbb{R}$,*

$$\sigma_{\Phi^t(X^*)}\big(d\Phi^t_{|X^*}W, d\Phi^t_{|X^*}W'\big) = \sigma_{X^*}(W, W').$$

Remark: *one says that the symplectic form is invariant by the flow, or that Φ^t is symplectic.*

4. *Prove (1.33) by using Question 3.*

Proposition 1.15 (Measure preserving properties of the geodesic flow)**.** *For all $t \in \mathbb{R}$ and all functions $a \in C_0^\infty(T^*M)$, $b \in C^\infty(S^*M)$,*

$$\int_{T^*M} a \circ \Phi^t |dxd\xi| = \int_{T^*M} a|dxd\xi| \tag{1.34}$$

and

$$\int_{S^*M} b \circ \Phi^t \; dL_g = \int_{S^*M} b \; dL_g. \tag{1.35}$$

Furthermore, for any Borel subset $B \subset S^*M$, $L_g(\Phi^t(B)) = L_g(B)$.

Proof. We prove first (1.34). Since the equality is true for $t = 0$ and $\mathbb{R}$ is connected, it suffices to check that the left hand side is locally constant in t. By the group property of the flow, it then suffices to show that for any $a \in C_0^\infty(T^*M)$ there exists $t_0 > 0$ such that (1.34) holds for $|t| < t_0$. By using a partition of unity on M, we may assume that a is supported in $\pi_{T^*M}^{-1}(K)$ for some compact subset K of an open coordinate patch U. By compactness of supp(a) and continuity of the flow, there exists $t_0 > 0$ such that $\Phi^t(\text{supp}(a))$ is contained in $\pi_{T^*M}^{-1}(U)$ for $|t| < t_0$. The interest of this reduction is that we can compute everything in single patch which we do now. As in Exercise 1.14, we keep the notation Φ^t for its expression in local coordinates. Then using that $\frac{d}{dt}\text{det}M_t = \text{det}M_t \; \text{tr}\big(M_t^{-1}\dot{M}_t\big)$ whenever M_t is an invertible matrix depending smoothly on t, we find

$$\frac{d}{dt}\det(d\Phi^t) = \det(d\Phi^t) \; \text{tr}\big((d\Phi^t)^{-1} dW_{p|\Phi^t} d\Phi^t\big).$$

On the other hand, using that[4]

$$dW_p = \begin{pmatrix} \frac{\partial^2 p}{\partial x_j \partial \xi_i} & \frac{\partial^2 p}{\partial \xi_j \partial \xi_i} \\ -\frac{\partial^2 p}{\partial x_j \partial x_i} & -\frac{\partial^2 p}{\partial \xi_j \partial x_i} \end{pmatrix}$$

we have $\text{tr}(dW_p) = 0$ hence $\frac{d}{dt}\det(d\Phi^t) = 0$. Since $d\Phi^0 = I$, the determinant is equal to 1 and we get (1.34) by the change of variable $(x', \xi') = \Phi^t(x, \xi)$.

We can now prove (1.35). We start by observing that (1.34) implies that for any ρ

$$\int_{p^{-1}([0,\rho^2])} a \circ \Phi^t |dxd\xi| = \int_{p^{-1}([0,\rho^2])} a|dxd\xi|.$$

[4] i labels the rows and j the columns

This follows as Exercise 1.9 together with the observation that $p^{-1}([0,\rho^2])$ is invariant by Φ^t, which follows from (1.27). Then, by differentiating the above identity at $\rho = 1$ and using (1.29), we get (1.35) when $b = a_{|S^*M}$. Since any smooth function on S^*M can be written as such a restriction, we have proved (1.35). Finally, the invariance $L_g(\Phi^t(B)) = L_g(B)$ is exactly (1.35) with, formally, $b = \mathbb{1}_B$. This can be justified using standard arguments, by approximating $\mathbb{1}_B$ in $L^1(S^*M, dL_g)$ by a sequence of smooth functions (in local coordinates dL_g is the Lebesgue measure multiplied by a positive smooth function). $\square$

Exercise 1.16. *For a function $a \in C^\infty(T^*M)$ and $\rho > 0$, we denote by a_ρ the function on S^*M defined by $a_\rho(\omega) = a(\rho\omega)$.*

1. *Check that for all $\rho \geq 0$, $\omega \in S^*M$ and $t \in \mathbb{R}$*

$$\Phi^t(\rho\omega) = \rho\Phi^{\rho t}(\omega).$$

2. *Show that for all $a \in C_0^\infty(T^*M)$, $\omega \in S^*M$ and $\rho > 0$,*

$$[a_\rho]_T(\omega) = [a]_{T/\rho}(\rho\omega),$$

and thus that, if the geodesic flow is ergodic, for all $\rho > 0$ and almost every $\omega \in S^*M$ one has

$$[a]_T(\rho\omega) \to \fint_{S^*M} a_\rho, \qquad T \to +\infty.$$

1.4 Statement of the theorem

Definition 1.17. *A subset S of $\mathbb{N}$ is of density 1 if*

$$D_N(S) := \frac{1}{N}\#S \cap [1,N] \to 1, \qquad N \to \infty,$$

or equivalently[5] *if*

$$\liminf_{N\to\infty} D_N(S) = 1.$$

In the sequel, we let

$$\fint_M \psi := \frac{1}{\mathrm{vol}_g(M)} \int_M \psi(m) d\mathrm{vol}_g,$$

be the (space) average of a (continuous) function ψ on M.

[5] the equivalence following from the obvious fact that $D_N(S) \leq 1$ hence so does its lim sup

Theorem 1.18 (Shnirelman's theorem)**.** *Assume that the geodesic flow on M is ergodic. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions of Δ_g. Then there exists $S\subset\mathbb{N}$ of density* 1 *such that, for all $\psi\in C(M)$,*

$$\langle e_j,\psi e_j\rangle_M \to \fint_M \psi \qquad \textit{as } j\to\infty \textit{ in } S. \tag{1.36}$$

This result can be rephrased as the following weak convergence of probability measures

$$|e_j(m)|^2 d\mathrm{vol}_g \rightharpoonup \frac{1}{\mathrm{vol}_g(M)} d\mathrm{vol}_g, \qquad \text{as } j\to\infty \text{ in } S.$$

One says loosely that "almost all eigenfunctions are equidistributed" over the manifold, "almost" refering to the fact that j runs over a set of density 1 and "equidistributed" refering to the fact that the limit is the uniform measure on M.

Some references: Theorem 1.18 first appeared in the paper [4] of Shnirelman. Complete proofs were given later by Zelditch [8] and Colin de Verdière [2] (we do not comment on other forms of this theorem for manifolds with boundary or toral automorphisms). Since then, it has attracted a lot of activity, particularly around the question of Unique Quantum Ergodicity, i.e. whether or not one can take $S=\mathbb{N}$. This was answered positively in the special case of quotients of the hyperbolic plane (and for a special type of eigenfunctions) by Lindenstrauss [5] but remains open for manifolds with variable (negative) curvature. The proof of Theorem 1.18 displayed in the next chapter follows closely the presentation of Zworski's book [9] (up to the fact that we try to use a fairly elementary pseudo-differential calculus rather than the complete theory developed in [9]).

In the rest of this section, we give a brief description of the quantum mechanical interpretation of Shnirelman's theorem, to justify in particular the name of quantum chaos.

We present first some elementary facts on *classical* vs *quantum* mechanics. To keep the discussion at a reasonable length and since we don't assume any prior knowledge of the reader on quantum mechanics, we will be fairly sketchy and only stress on the main elementary aspects illustrating our purpose.

Let us start with a well known example of classical mechanics, the harmonic oscillator, that is a particle of mass m connected to a spring moving (horizontally) along an axis. Its motion is governed by Newton's law (mass $\times$ acceleration = sum of forces), i.e. the ordinary differential equation

$$m\ddot{x} = -kx, \tag{1.37}$$

where $x=x(t)\in\mathbb{R}$ is the position on the axis of the particle at time t. This system can be described equivalently by its Hamiltonian, or total energy, namely the function

$$H_{\mathrm{o}}(x,\xi) = \frac{1}{2m}\xi^2 + \frac{k}{2}x^2$$

from which the following (system of) first order differential equation(s)

$$\dot{x} = \frac{\partial H_o}{\partial \xi}(x,\xi) = \frac{\xi}{m}, \qquad \dot{\xi} = -\frac{\partial H_o}{\partial x}(x,\xi) = -kx \tag{1.38}$$

allows clearly to recover (1.37). The solutions to this autonomous system are given by a one parameter group of diffeomorphisms $(\Phi_o^t)_{t\in\mathbb{R}}$, called the (classical) flow and defined by $\Phi_o^t(x_0,\xi_0) := (x(t),\xi(t))$ if $(x(t),\xi(t))$ is the solution to (1.38) at time t satisfying $(x(0),\xi(0)) = (x_0,\xi_0)$.

Exercise 1.19. *Compute explicitly* Φ_o^t.

Let us note that the first equation in (1.38) yields $\xi = m\dot{x}$ so that ξ can be interpreted as the momentum of the particle and the first term of the Hamiltonian as its kinetic energy. The second term is the potential energy and the associated force $-kx$ is minus its derivative with respect to x (one says the force derives from the potential $kx^2/2$). As is well known, the total energy of the particule is constant (in time): if $(x(t),\xi(t))$ is a solution to the above system, then

$$\begin{aligned} \frac{d}{dt}H_o(x(t),\xi(t)) &= \frac{\partial H_o}{\partial x}(x(t),\xi(t))\dot{x}(t) + \frac{\partial H_o}{\partial \xi}(x(t),\xi(t))\dot{\xi}(t) \\ &= 0. \end{aligned} \tag{1.39}$$

In other words, $H_o = H_o \circ \Phi_o^t$, i.e. the Hamiltonian is invariant by the flow.

The above properties of the harmonic oscillator remain valid for any smooth function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, or more generally $H : T^*M \to \mathbb{R}$. In this context, $\mathbb{R}^n \times \mathbb{R}^n$ or T^*M are called the classical **phase space**. The associated Hamilton motion equations are the system of $2n$ equations

$$\dot{x} = \frac{\partial H}{\partial \xi}(x,\xi), \qquad \dot{\xi} = -\frac{\partial H}{\partial x}(x,\xi), \tag{1.40}$$

generating a one parameter group of diffeomorphisms under which H is invariant. A special case of interest is when $H = p$ (see (1.21)), in which case the Hamiltonian trajectories are geodesics (compare (1.40) and the vector field (1.23)). For this reason, we will see *the geodesic flow as an object of classical mechanics* and sometimes rephrase some of its mathematical properties in physical terms; for instance, the preservation of the cosphere bundle by the geodesic flow is exactly the conservation of the energy.

Classical mechanics allows to describe the dynamics of systems at macroscopic scales. At microscopic (i.e. atomistic or molecular) scales, one has to use a quantum mechanical description by mean of wave functions. This means that a quantum particle moving in the Euclidean space[6] $\mathbb{R}^n$ is no longer described by $2n$ coordinates (position + speed (or momentum)) but by a function $\psi \in L^2(\mathbb{R}^n)$, so called state or wave function, which is normalized by $||\psi||_{L^2} = 1$. Information on the particle are recovered from ψ in a

[6]for simplicity, but one can also consider a n dimensional manifold

probabilistic fashion: for instance, the probability of the particle to be in a (Borel) set $B \subset \mathbb{R}^n$ is given by

$$\mathbb{P}(\text{particle} \in B) = \int_B |\psi(x)|^2 dx = \langle \psi, \mathbb{1}_B \psi \rangle_{L^2(\mathbb{R}^n)}$$

where $\mathbb{1}_B$ is the characteristic function of B and $\langle ., . \rangle_{L^2(\mathbb{R}^n)}$ the L^2 inner product. The quantum Hamiltonians are defined by self-adjoint operators on $L^2(\mathbb{R}^n)$. A typical example is the one of Schrödinger operators

$$\hat{H} := -\frac{\hbar^2}{2m}\Delta + V(x),$$

where m is the mass of the particle, $\hbar$ is the Planck constant, Δ is the usual Laplacian and V a real valued function acting on $L^2(\mathbb{R}^n)$ (or a subspace of $L^2(\mathbb{R}^n)$) as a multiplication operator. For instance, if one considers $V(x) = -K/|x|$ in $\mathbb{R}^3$ then, with appropriate physical constants, $\hat{H}$ gives a good description of the hydrogen atom (i.e. an electron moving around a proton) in the sense that the operator $\hat{H}$ has a sequence of negative eigenvalues[7] proportional to $-1/n^2$, $n \in \mathbb{N}$ (see [7]), which fit the energy levels that are observed experimentaly. We note that the corresponding hamiltonian in classical mechanics describing the motion of a light body around a heavy one (e.g. a satellite around the earth) would be the fun ction $\frac{|\xi|^2}{2m} - \frac{K}{|x|}$. Naively, this is very similar to an electron moving around a proton but, at microscopic scales, quantum Hamiltonians give a better description than classical ones.

The relation between classical and quantum Hamiltonians is given by a **quantization** procedure: in the present context, the quantum Hamiltonian $\hat{H}$ is constructed from the classical Hamiltonian $H(x, \xi) := \frac{|\xi|^2}{2m} + V(x)$ by replacing the momentum variables ξ_j by the differentiation operators $\hbar\partial_{x_j}/\mathrm{i}$, that is formally

$$\hat{H} = H(x, -\mathrm{i}\hbar\nabla).$$

The quantization is the mapping $H \mapsto \hat{H}$, which goes from the set (or a subset) of functions on the classical phase space $T^*\mathbb{R}^n$ to a set of operators on $L^2(\mathbb{R}^n)$.

From the dynamical (i.e. time dependent) point of view, the evolution of a quantum particle in the initial state ψ is described by the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\Psi = \hat{H}\Psi, \qquad \Psi(t = 0, x) = \psi(x),$$

which is linear PDE. At least formally, viewing $\hat{H}$ as continuous operator on L^2 (this is not the case; the Laplacian forces $\hat{H}$ to be defined on a subspace of L^2), the solution is given by

$$\Psi(t, x) = \left(e^{-\mathrm{i}t\hat{H}/\hbar}\psi\right)(x)$$

[7] there is also a (continuous) spectrum on $[0, +\infty)$ which corresponds to "scattering states" but we will not discuss it in this short description

that is by mean of the family of operators $\left(e^{-\mathrm{i}t\hat{H}/\hbar}\right)_{t\in\mathbb{R}}$ which forms a group. We do not enter here into the detailed definition of such a group (see Section 2.2 when $\hat{H} = \Delta_g$) but rather point out the analogy with classical mechanics, in particular with the geodesic flow, where the dynamics is also described by mean of a one parameter group.

We now discuss the role of the Planck constant which has so far only played the role of a physical backing. Physically, it is a very small constant which shows up only at the quantum level. From a theoretical point of view, we can extrapolate the smallness of $\hbar$ by letting it go to zero; it turns out that this allows to describe the transition between quantum and classical mechanics. This is called the **semiclassical limit**. In some sense, as $\hbar \to 0$, the equations of quantum mechanics converge to their classical counterparts. From a purely mathematical point of view, many equations involving a small parameter can be seen as quantum mechanical ones, the small parameter being interpreted as a Planck constant. For instance, in the study of large eigenvalues of the Laplace-Beltrami operator, we can rewrite

$$-\Delta_g e_N = \lambda_N e_N \qquad \Longleftrightarrow \qquad -h_N^2 \Delta_g e_N = e_N$$

and interpret $h_N := \lambda_N^{-1/2}$ as a Planck constant (it goes to 0 as $N \to \infty$). We will see in the next chapter to which extent the Schrödinger group associated to the Laplace-Beltrami operator $e^{\mathrm{i}th_N^2\Delta_g/h_N}$ converges to the geodesic flow (see the Egorov Theorem inside Theorem 2.6). This is a purely mathematical fact, but it is of course reminiscent of the above statement on the convergence of quantum to classical mechanics as $\hbar \to 0$.

Given this rough description of classical and quantum mechanics, one can summarize what quantum chaos is. It consists in the study of "quantum systems", e.g. the eigenfunctions of the Laplace Beltrami operator on a manifold, whose dynamics of the "classical background", e.g. the geodesic flow on the cotangent bundle, is chaotic, i.e. ergodic. Note that this is not limited to Riemannian manifolds; other models exhibit similar features (certain bounded domains of $\mathbb{R}^n$, toral automorphisms, quantized baker's map,...).

Chapter 2

Proof of the Shnirelman theorem

In this chapter, we show how to get the Shnirelman theorem from the existence of a *quantization* on M.

2.1 Sets of density one

Proposition 2.1. *Let $(u_j)_{j\geq 1}$ be a sequence of complex numbers such that*

$$\frac{1}{N}\sum_{j=1}^{N}|u_j| \to 0 \qquad \text{as } N \to \infty.$$

Then there exists a set S of density 1 such that

$$u_j \to 0 \qquad \text{as } j \to \infty,\ j \in S.$$

To prove this proposition, we will use the following lemma.

Lemma 2.2.

1. *A finite intersection of sets of density 1 has density 1.*
2. *If $(S_k)_{k\geq 1}$ is a nonincreasing[1] sequence of sets and $(N_k)_{k\geq 1}$ an increasing sequence of $\mathbb{N}$ such that*

$$D_N(S_k) \geq 1 - \frac{1}{k}, \qquad N \geq N_k, \tag{2.1}$$

then one can find a set S of density 1 such that

$$S \cap [N_k, +\infty) \subset S_k, \qquad \text{for all } k \geq 2. \tag{2.2}$$

3. *If $(S_k)_{k\geq 1}$ is a nonincreasing of sets of density 1, one can find an increasing sequence $(N_k)_{k\geq 1}$ such that (2.2) holds.*

[1] $S_{k+1} \subset S_k$

Proof of Lemma 2.2. 1. By induction, the proof is reduced to the case of two sets S_1, S_2 of density 1. Using that $S_1 \subset (S_1 \cap S_2) \cup S_2^c$, one has

$$S_1 \cap [1, N] \subset \big((S_1 \cap S_2) \cap [1, N]\big) \cup \big(S_2^c \cap [1, N]\big).$$

By taking the cardinal and dividing by N, we obtain

$$D_N(S_1) \leq D_N(S_1 \cap S_2) + 1 - D_N(S_2).$$

This shows that $D_N(S_1 \cap S_2) \geq D_N(S_1) + D_N(S_2) - 1$ which yields the result by taking the liminf.
2. Observe that the intervals $[N_k, N_{k+1}) \cap \mathbb{N}$ form a partition of $[N_1, \infty) \cap \mathbb{N}$. Define

$$S := \bigcup_{l=1}^{\infty} S_l \cap [N_l, N_{l+1}).$$

The fact that $S_l \subset S_k$ for $l \geq k$ implies that $S \cap [N_k, +\infty) \subset S_k$. To see that S has density 1, we note first that for any integer $N \geq N_1$ there is a unique $k = k(N)$ such that $N \in [N_k, N_{k+1})$. Since N_k is increasing, $k(N)$ is non decreasing. It is also clearly unbounded so $k(N) \to \infty$ as $N \to \infty$. Observing finally that

$$S_{k(N)} \cap [N_1, N] = S_{k(N)} \cap [N_1, N_2) \cup \cdots \cup S_{k(N)} \cap [N_{k(N)}, N]$$

is contained in $S \cap [N_1, N]$ (since S_k is nonincreasing), we see that

$$D_N(S) = \frac{\# S \cap [1, N]}{N} \geq \frac{\# S \cap [N_1, N]}{N} \geq \frac{\# S_{k(N)} \cap [N_1, N]}{N} \geq D_N(S_{k(N)}) - \frac{N_1}{N}$$

whose right hand side goes to 1 as $N \to \infty$ by (2.1).
3. It suffices to choose the sequence $(N_k)_{k \geq 1}$ as an increasing sequence such that (2.1) holds and then to use the item 2. $\square$

The last lemma and its proof are essentially taken from [3].

Proof of Proposition 2.1. For all $k \geq 1$, we can find $N_k \geq 1$ such that

$$\frac{1}{N} \sum_{j=1}^{N} |u_j| \leq \frac{1}{(2k)^2}, \qquad N \geq N_k.$$

By the Tchebychev inequality, we then have for all $N \geq N_k$,

$$\# \left\{ 1 \leq j \leq N \mid |u_j| \geq \frac{1}{2k} \right\} \leq \frac{N}{2k}$$

and thus by taking the complement, we have

$$\# \left\{ 1 \leq j \leq N \mid |u_j| < \frac{1}{2k} \right\} \geq N \left(1 - \frac{1}{2k} \right).$$

Setting

$$S_k := \left\{ j \geq 1 \mid |u_j| < \frac{1}{2k} \right\}$$

the above inequality says precisely that

$$D_N(S_k) \geq 1 - \frac{1}{k}, \qquad N \geq N_k,$$

and we can assume that the sequence (N_k) is increasing (otherwise, it suffices to replace N_k by $\max(N_1, \ldots, N_k) + 1$). We next choose S of density 1 according to the item 2 of Lemma 2.2 and it remains to see that $u_j \to 0$ as $j \to \infty$ in S. Indeed, since $S \cap [N_k, \infty)$ is contained in S_k we have

$$\limsup_{\substack{j \to \infty \\ j \in S}} |u_j| \leq \frac{1}{2k}.$$

This is true for all k so we get the result. $\square$

Another application of Lemma 2.2 is that Theorem 1.18 is a consequence of the following one.

Theorem 2.3. *For all $\psi \in C^\infty(M)$ there exists S_ψ of density 1 such that (1.36) holds.*

Lemma 2.4. *There is a countable subset of $C^\infty(M)$ which is dense in $C(M)$.*

Proof. Pick a partition of unity $\sum_{i=1}^N \varphi_i = 1$ on M with functions supported in coordinates patches. We let κ_i be the corresponding diffeomorphisms. Each space $C(\mathrm{supp}(\varphi_i))$ is separable since, by the Stone-Weierstrass Theorem, polynomials with rational coefficients are dense in $C\big(\kappa_i\big(\mathrm{supp}(\varphi_i)\big)\big)$, the space of continuous functions on the compact set $\kappa_i\big(\mathrm{supp}(\varphi_i)\big)$ (if you are not familiar with this result prove it as an exercise). The result follows. $\square$

Proof that Theorem 2.3 $\Rightarrow$ Theorem 1.18. By Lemma 2.4, we pick first a countable family $(\psi_l)_{l \geq 1}$ of $C^\infty(M)$ which is dense in $C(M)$. For each ψ_l, Theorem 2.3 allows to choose a subset S_{ψ_l} of density 1 such that (1.36) holds for ψ_l. We define

$$S_k = \bigcap_{l=1}^{k} S_{\psi_l}$$

which is a nonincreasing sequence of subsets of density 1 by the item 1 of Lemma 2.2. We next choose S as in the item 3 of the same lemma. Then, for all $l \geq 1$, we have

$$\langle e_j, \psi_l e_j \rangle_M \to \fint \psi_l \qquad \text{as } j \to \infty \text{ in } S,$$

since, for j large enough, j belongs to S_l (by (2.2)) hence to S_{ψ_l}. Then, for any $\psi \in C(M)$ and any $\epsilon > 0$ we can find l such that $||\psi - \psi_l||_\infty < \epsilon$. Thus, for all $j \geq 1$,

$$\begin{aligned} \left| \langle e_j, \psi e_j \rangle_M - \fint_M \chi \right| &\leq \left| \langle e_j, (\psi - \psi_l) e_j \rangle_M \right| + \left| \langle e_j, \psi_l e_j \rangle_M - \fint_M \psi_l \right| + \fint_M |\psi - \psi_l| \\ &\leq \epsilon + \left| \langle e_j, \psi_l e_j \rangle_M - \fint_M \psi_l \right| + \epsilon, \end{aligned}$$

where the estimate of the first term of the right hand side uses that $||e_j||_{L^2(M)} = 1$. Using next that, for all j large enough in S, the second term in the right hand side is smaller than ϵ, we get the result. □

The above proof reduces Theorem 1.18 to Theorem 2.3. The proof of Theorem 2.3 rests on the construction of a quantization as we will see below.

2.2 Functional calculus

In this section, we introduce the functional calculus associated to Δ_g. It will play an important role in the properties of the quantization displayed in Theorem 2.6 (Section 2.3).

Let us denote by $\mathcal{B}(\mathbb{R})$ the algebra of bounded Borel functions on $\mathbb{R}$ (actually, piecewise continuous will be sufficient here). For any $f \in \mathcal{B}(\mathbb{R})$ and any $\varphi \in L^2(M)$, which can be uniquely written as

$$\varphi = \sum_{j \in \mathbb{N}} c_j e_j, \qquad c_j = \langle e_j, \varphi \rangle_M,$$

with convergence in $L^2(M)$, we set

$$f(-\Delta_g)\varphi := \sum_{j \in \mathbb{N}} f(\lambda_j) c_j e_j. \tag{2.3}$$

This defines an element in $L^2(M)$ since it is a sum of orthogonal terms such that

$$\sum_j |f(\lambda_j) c_j|^2 \leq \sup_{\mathbb{R}} |f|^2 \sum_j |c_j|^2 = \big(\sup |f|\big)^2 ||\varphi||^2_{L^2(M)} < \infty. \tag{2.4}$$

It is clear that the map $\varphi \mapsto f(-\Delta_g)\varphi$ is linear on $L^2(M)$. It is also continuous since, by (2.4),

$$||f(-\Delta_g)||_{\mathcal{L}_c(L^2)} \leq \sup_{\mathbb{R}} |f|, \tag{2.5}$$

where $\mathcal{L}_c(L^2)$ denotes the algebra of bounded (i.e. continuous) linear operators on $L^2(M)$ and $|| \cdot ||_{\mathcal{L}_c(L^2)}$ is the usual operator norm.

Proposition 2.5. *The map*

$$\mathcal{B}(\mathbb{R}) \ni f \mapsto f(-\Delta_g) \in \mathcal{L}_c(L^2) \tag{2.6}$$

is a continuous morphism of algebras. In particular

$$f_1(-\Delta_g)f_2(-\Delta_g) = (f_1 f_2)(-\Delta_g),$$

for all $f_1, f_2 \in \mathcal{B}(\mathbb{R})$. *Furthermore,*

$$f(-\Delta_g)^* = \overline{f}(-\Delta_g),$$

for all $f \in \mathcal{B}(\mathbb{R})$. *In particular, if* f *is real valued then* $f(-\Delta_g)$ *is selfadjoint. Finally*

$$f \geq 0 \qquad \Longrightarrow \qquad f(-\Delta_g) \geq 0. \tag{2.7}$$

Proof. Left to the reader as an exercise. □

The morphism (2.6) is usually called the **functional calculus** of the Laplacian. The operators of the form $f(-\Delta_g)$ are called **functions of the Laplacian**.

Let us comment more specifically on the functions of the Laplacian we shall use to prove the Shnirelman Theorem. We will consider the **Schrödinger group** which is the family of operators defined as

$$e^{\mathrm{i}\tau\Delta_g} := f_\tau(-\Delta_g) \qquad \text{with} \quad f_\tau(\lambda) := e^{-\mathrm{i}\tau\lambda}.$$

It follows from Proposition 2.5 that the family $(e^{\mathrm{i}\tau\Delta_g})_{\tau\in\mathbb{R}}$ is a unitary group on $L^2(M)$ in the sense that

$$\left(e^{\mathrm{i}\tau\Delta_g}\right)^* = e^{-\mathrm{i}\tau\Delta_g}, \qquad e^{\mathrm{i}0\Delta_g} = I, \qquad e^{\mathrm{i}\tau_1\Delta_g}e^{\mathrm{i}\tau_2\Delta_g} = e^{\mathrm{i}(\tau_1+\tau_2)\Delta_g}.$$

The terminology comes from the fact that $e^{\mathrm{i}\tau\Delta_g}$ solves the Schrödinger equation. We record here an elementary aspect of this fact (which is the only one we will use in this course): given an eigenfunction e_j, the function $\varphi(\tau, m) = (e^{\mathrm{i}\tau\Delta_g}e_j)(m)$ solves

$$\mathrm{i}\partial_\tau\varphi + \Delta_g\varphi = 0, \qquad \varphi_{|\tau=0} = e_j, \tag{2.8}$$

which is a trivial consequence of the fact that $\varphi = e^{-\mathrm{i}\tau\lambda_j}e_j$ and $\Delta_g e_j = -\lambda_j e_j$.

We will see (and use) below that there is a deep connection between the geodesic flow and the Schrödinger group.

Other useful functions of the Laplacian are the **semiclassical spectral localizations** which are of the form $f(-h^2\Delta_g)$ with f compactly supported and h a (small) parameter in $(0,1]$. For instance, if $f = \mathbb{1}_{[0,1]}$, the operator $\mathbb{1}_{[0,1]}(-h^2\Delta_g)$ is an orthogonal projector, since

$$\mathbb{1}_{[0,1]}(-h^2\Delta_g) = \mathbb{1}_{[0,1]}(-h^2\Delta_g)^*, \qquad \mathbb{1}_{[0,1]}(-h^2\Delta_g)^2 = \mathbb{1}_{[0,1]}(-h^2\Delta_g)$$

by Proposition 2.5. An important observation is that this operator is a finite rank operator since it is given by

$$\mathbb{1}_{[0,1]}(-h^2\Delta_g)\varphi = \sum_{\lambda_j \in [0,h^{-2}]} \langle e_j, \varphi \rangle_M e_j,$$

where the sum contains finitely many terms since $\lambda_j \to +\infty$. As a finite rank operator, one can compute its **trace** and one finds

$$\begin{aligned} \mathrm{tr}\big(\mathbb{1}_{[0,1]}(-h^2\Delta_g)\big) &= \sum_{\lambda_j \in [0,h^{-2}]} 1 && (2.9) \\ &= \#\big\{j \in \mathbb{N} \mid \lambda_j \in [0, h^{-2}]\big\}. && (2.10) \end{aligned}$$

We will see in the next section the interest of this observation. We note here that the number (2.10) can be written as the trace of a function of $h^2\Delta_g$ which is very important: it turns out that one can give non trivial information on functions of $h^2\Delta$ but getting relevant properties on other projections, e.g. on individual eigenfunctions, is very hard (not to say impossible). Actually, analyzing the orthogonal projection $\mathbb{1}_{[0,1]}(-h^2\Delta_g)$ is not so easy but we will see that we can get more easily information from $f(-h^2\Delta_g)$ with $f \in C_0^\infty(\mathbb{R})$ and this will be sufficient.

2.3 Main proof

In this section, we prove Theorem 2.3 by using as a black box the following theorem on a the existence of *quantization* that will be investigated further on in text.

Theorem 2.6 (Existence of a quantization)**.** *One can find a quantization, namely a family of linear maps*

$$Op_h : C_0^\infty(T^*M) \to \mathcal{L}_c\big(L^2(M)\big)$$

indexed by $h \in (0,1]$ and satisfying the following properties

- **Uniform L^2 bounds***: for all $a \in C_0^\infty(T^*M)$, there exists $C > 0$ such that*

$$||Op_h(a)||_{\mathcal{L}_c(L^2)} \leq C \qquad (2.11)$$

  for all $h \in (0,1]$.

- **Symbolic calculus***: for all $a, b \in C_0^\infty(T^*M)$ there exists $C > 0$ such that*

$$\big|\big|Op_h(a)Op_h(b) - Op_h\big(ab\big)\big|\big|_{\mathcal{L}_c(L^2)} \leq Ch \qquad (2.12)$$

  and

$$\big|\big|Op_h(a)^* - Op_h\big(\bar{a}\big)\big|\big|_{\mathcal{L}_c(L^2)} \leq Ch \qquad (2.13)$$

  for all $h \in (0,1]$.

- **Egorov Theorem**: *for all $T > 0$ and all $a \in C_0^\infty(T^*M)$, there exists $C > 0$ such that*

$$\Big|\Big| e^{-ith\Delta_g} Op_h(a) e^{ith\Delta_g} - Op_h\big(a \circ \Phi^t\big) \Big|\Big|_{\mathcal{L}_c(L^2)} \leq Ch, \tag{2.14}$$

for all $h \in (0,1]$ and $t \in [-T, T]$. Recall that $(\Phi^t)_{t\in\mathbb{R}}$ is the geodesic flow on T^*M.

- **Approximate functional calculus**: *for all $f \in C_0^\infty(\mathbb{R})$ and all $\psi \in C^\infty(M)$ there exists $C > 0$ such that*

$$\big|\big| \psi f(-h^2\Delta_g) - Op_h\big(\psi f \circ p\big) \big|\big|_{\mathcal{L}_c(L^2)} \leq Ch \tag{2.15}$$

for all $h \in (0,1]$. Recall that p is the principal symbol of the Laplacian (see (1.21)).

- **Local Weyl's law**: *for all $a \in C_0^\infty(T^*M)$ and all $E_1 \leq E_2$,*

$$(2\pi h)^n \sum_{E_1 \leq h^2\lambda_j \leq E_2} \langle e_j, Op_h(a) e_j \rangle_M \to \int_{p^{-1}([E_1,E_2])} a |dx d\xi|, \qquad h \to 0. \tag{2.16}$$

The proof of this theorem will be splitted into separate propositions or theorems which will be proved in the next chapters. For the convenience of the reader, we record in the next table the localization of the proofs of the different items.

Parts of the proof of Theorem 2.6

Item	(2.11)	(2.12)	(2.13)	(2.14)	(2.15)	(2.16)
Proof	Prop 3.15	Prop 3.17	Prop 3.18	Sec. 3.3	Chap. 5	Thm 4.18

In the rest of the present section, we use Theorem 2.6 to prove the Shnirelman Theorem, more precisely Theorem 2.3.

Let us pick $\psi \in C^\infty(M)$. We wish to show that $\langle e_j, \psi e_j \rangle_M \to \fint_M \psi$ along a subset of density 1. Using that $||e_j||_{L^2(M)} = 1$, we have

$$\fint_M \psi = \left(\fint_M \psi \right) \langle e_j, e_j \rangle_M = \big\langle e_j, (\fint_M \psi) e_j \big\rangle_M$$

so up to the replacement of ψ by $\psi - \fint_M \psi$, we may assume that

$$\fint_M \psi = 0. \tag{2.17}$$

Under this assumption and according to Proposition 2.1, it then suffices to prove that

$$\frac{1}{N} \sum_{j=1}^N \big| \langle e_j, \psi e_j \rangle_M \big|^2 \to 0, \qquad N \to \infty. \tag{2.18}$$

Summing from $j = 1$ to N, namely over the eigenvalues $\lambda_1, \dots, \lambda_N$ is not easy; it is more convenient to sum over all eigenvalues λ_j such that $\lambda_1 \le \lambda_j \le \lambda_N$, that is to possibly consider additional terms $\lambda_{N+1}, \dots$ which are equal to λ_N. Therefore, we introduce

$$h_N^{-2} := \lambda_N$$

and define

$$N_{h_N} := \#\{j \mid \lambda_j \le \lambda_N\} = \#\{j \mid h_N^2 \lambda_j \in [0,1]\}.$$

We will see that we can replace the proof of (2.18) by the proof of

$$\frac{1}{N_{h_N}} \sum_{h_N^2 \lambda_j \le 1} \left| \langle e_j, \psi e_j \rangle_M \right|^2 \to 0, \qquad h_N \to 0. \tag{2.19}$$

The interest is that we will be able to study this sum by mean of the local Weyl law (with $E_1 = 0$ and $E_2 = 1$). This reduction follows from Theorem 2.7 and Proposition 2.8 below.

Theorem 2.7 (Weyl's law)**.** *Let $E_1 \le E_2$ be real numbers and for $h \in (0,1]$ set*

$$N_h(E_1, E_2) = \#\{j \mid h^2 \lambda_j \in [E_1, E_2]\}.$$

Then

$$(2\pi h)^n N_h(E_1, E_2) \to \int_{p^{-1}([E_1,E_2])} |dx d\xi|, \qquad h \to 0. \tag{2.20}$$

In particular, we have

$$N_{h_N} \sim (2\pi h_N)^{-n} \int_{p^{-1}([0,1])} |dx d\xi|, \tag{2.21}$$

and

$$\{j \mid \lambda_j = \lambda_N\} = o(h_N^{-n}). \tag{2.22}$$

Proof. This is essentially a consequence of the local Weyl law (2.16). Let $f \in C_0^\infty(\mathbb{R})$ be equal to one on $[E_1, E_2]$. Then

$$\begin{aligned} N_h(E_1, E_2) = \sum_{\{j \mid h^2\lambda_j \in [E_1,E_2]\}} 1 \quad &= \sum_{h^2\lambda_j \in [E_1,E_2]} \langle e_j, e_j \rangle_M \\ &= \sum_{h^2\lambda_j \in [E_1,E_2]} \langle e_j, f(-h^2\Delta_g) e_j \rangle_M \\ &= \sum_{h^2\lambda_j \in [E_1,E_2]} \big(\langle e_j, Op_h(f \circ p) e_j \rangle_M + O(h) \big) \end{aligned}$$

using (2.15). Therefore, using that the sum contains $N_h(E_1, E_2)$ terms, we get

$$\big(1 + O(h)\big) N_h(E_1, E_2) = \sum_{h^2\lambda_j \in [E_1,E_2]} \langle e_j, Op_h(f \circ p) e_j \rangle_M.$$

Multiplying this identity by $(2\pi h)^n$ and letting h go to zero, we get (2.20) from (2.16) with $a = f \circ p$ (note that $f \circ p = 1$ on $p^{-1}([E_1, E_2])$). Then (2.21) (resp. (2.22)) is a special case obtained by considering $h = h_N$ and $E_1 = 0, E_2 = 1$ (resp. $E_1 = E_2 = 1$). For (2.22), we use additionally that $p^{-1}(\{1\})$ has zero measure on T^*M. $\square$

Proposition 2.8. *Let $(b_j)_{j\geq 1}$ be a bounded sequence. Then*

$$\left(\frac{1}{N}\sum_{1\leq j\leq N} b_j\right) - \left(\frac{1}{N_{h_N}}\sum_{\lambda_1\leq\lambda_j\leq\lambda_N} b_j\right) \to 0, \qquad N\to\infty.$$

Proof. Let us observe first that $N \leq N_{h_N}$ and that, according to (2.22),

$$N_{h_N} = \#\{j \mid \lambda_j < \lambda_N\} + o(h_N^{-n}) \leq N + o(h_N^{-n})$$

so that $N_{h_N} - N = o(h_N^{-n})$. In particular

$$\left(\frac{1}{N}\sum_{1\leq j\leq N} b_j\right) - \left(\frac{1}{N_{h_N}}\sum_{1\leq j\leq N} b_j\right) = \frac{N_{h_N}-N}{N_{h_N}}\left(\frac{1}{N}\sum_{1\leq j\leq N} b_j\right) \to 0, \qquad N\to\infty$$

since the bracket in the right hand side is bounded while $\frac{N_{h_N}-N}{N_{h_N}}$ goes to zero by (2.21) and the fact that $N_{h_N} - N = o(h_N^{-n})$. On the other hand

$$\left|\left(\frac{1}{N_{h_N}}\sum_{1\leq j\leq N} b_j\right) - \left(\frac{1}{N_{h_N}}\sum_{\lambda_1\leq\lambda_j\leq\lambda_N} b_j\right)\right| \leq \frac{1}{N_{h_N}}\sum_{\lambda_j=\lambda_N}|b_j| \to 0, \qquad N\to\infty,$$

by (2.21) and (2.22). This completes the proof. $\square$

Proof of Theorem 2.3. Let $f \in C_0^\infty(\mathbb{R})$ be equal to 1 on $[0,1]$. Using that, if $\lambda_j \leq \lambda_N$, $f(-h_N^2\Delta_g)e_j = e_j$ and (2.15), we have

$$\frac{1}{N_{h_N}}\sum_{\lambda_j\leq\lambda_N}\left|\langle e_j, \psi e_j\rangle_M\right|^2 = \frac{1}{N_{h_N}}\sum_{\lambda_j\leq\lambda_N}\left|\langle e_j, Op_{h_N}(a)e_j\rangle_M\right|^2 + O(h_N),$$

with

$$a = \psi f \circ p.$$

Note that, by Proposition 1.10 and (2.17), we have for all $\rho > 0$,

$$\int_{S^*M} a(\rho\omega)dL_g(\omega) = 0. \tag{2.23}$$

Using next that $e^{\mathrm{i}th\Delta_g}e_j = e^{-\mathrm{i}th\lambda_j}e_j$, we have

$$\begin{aligned}\langle e_j, Op_{h_N}(a)e_j\rangle_M &= \langle e^{\mathrm{i}th_N\Delta_g}e_j, Op_{h_N}(a)e^{\mathrm{i}th_N\Delta_g}e_j\rangle_M \\ &= \langle e_j, e^{-\mathrm{i}th_N\Delta_g}Op_{h_N}(a)e^{\mathrm{i}th_N\Delta_g}e_j\rangle_M.\end{aligned}$$

By fixing $T > 0$ and taking the mean in time, (2.14) implies that

$$\langle e_j, Op_{h_N}(a)e_j\rangle_M = \langle e_j, Op_{h_N}([a]_T)e_j\rangle_M + O_T(h_N),$$

where $O_T(h_N)$ is a quantity bounded (in modulus) by Ch_N, for some constant independent of j (such that $\lambda_j \leq \lambda_N$) but possibly depending on T. Using that each e_j is normalized in $L^2(M)$, we get from (2.11), (2.12), (2.13) and the Cauchy Schwartz inequality

$$\begin{aligned} |\langle e_j, Op_{h_N}(a)e_j\rangle_M|^2 &\leq ||Op_{h_N}([a]_T)e_j||^2_{L^2(M)} + O_T(h_N) \\ &\leq \langle e_j, Op_{h_N}(|[a]_T|^2)e_j\rangle_M + O_T(h_N). \end{aligned}$$

Note that to get the second line, we have used

$$||Op_{h_N}([a]_T)e_j||^2_{L^2(M)} = \langle e_j, Op_{h_n}([a]_T)^* Op_{h_N}([a]_T)e_j\rangle_M.$$

Thus, using the local Weyl law,

$$\limsup_{N\to\infty}\left(\frac{1}{N_{h_N}}\sum_{h_N^2\lambda_j\leq 1}\left|\langle e_j, \psi e_j\rangle_M\right|^2\right) \leq C\int_{p^{-1}([0,1])} |[a]_T|^2 |dxd\xi|. \tag{2.24}$$

Since T is arbitrary, we can let it go to infinity so that the right hand side goes to zero by Corollary 1.13, Exercise 1.16 and (2.23). This proves (2.19) and thus completes the proof.
□

Chapter 3

Quantization

3.1 Elementary pseudo-differential calculus

The prototype of operators on $\mathbb{R}^n$ for the quantization we wish to construct is given by pseudo-differential operators.

Definition 3.1. *For $a \in C_0^\infty(\mathbb{R}^{2n})$ and $h \in (0,1]$, the* **pseudo-differential operator** *$op_h[a]$ is defined by*

$$op_h[a]u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\cdot\xi} a(x, h\xi) \hat{u}(\xi) d\xi, \tag{3.1}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Here $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}y\cdot\xi} u(y) dy$ is the Fourier transform of u.

One has to think of $\mathbb{R}^{2n}$ as $T^*\mathbb{R}^n$. In this definition, we consider only symbols a in $C_0^\infty(\mathbb{R}^{2n})$ to avoid technical discussions which can be (mostly) avoided in this course. However (3.1) makes sense on much larger classes of symbols (see Chapter 5 for an example). In particular, if we take $a = 1$, the Fourier inversion formula yields

$$op_h[1] = I.$$

This formula justifies the normalization factor $(2\pi)^{-n}$ in (3.1).

A simple application of the Fubini Theorem and the change of variable $\xi \mapsto \xi/h$ show that

$$\begin{aligned} op_h[a]u(x) &= (2\pi)^{-n} \int\int e^{\mathrm{i}(x-y)\cdot\xi} a(x, h\xi) u(y) dy d\xi \\ &= \int \left((2\pi h)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi/h} a(x,\xi) d\xi \right) u(y) dy. \end{aligned}$$

In other words, $op_h[a]$ is an operator with an integral kernel, i.e.

$$op_h[a]u(x) = \int K_h(x,y) u(y) dy$$

with

$$K_h(x,y) = (2\pi h)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi/h} a(x,\xi) d\xi. \tag{3.2}$$

This suggests to introduce the following more general definition.

Definition 3.2. *For $A \in C_b^\infty(\mathbb{R}^{3n})$ with compact support in ξ (i.e.* $\mathrm{supp}(A) \subset \mathbb{R}^n \times \mathbb{R}^n \times \bar{B}(0,R)$ *for some $R > 0$), we let $OP_h(A)$ be the operator with integral kernel*

$$K_{A,h}(x,y) := (2\pi h)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi/h} A(x,y,\xi) d\xi. \tag{3.3}$$

Recall that $C_b^\infty(\mathbb{R}^{3n})$ is the space of smooth functions on $\mathbb{R}^{3n}$ which are bounded together with all their derivatives.

Remark. It is trivial but important to observe that

$$OP_h(A) = op_h[a], \qquad \text{if} \quad A(x,y,\xi) = a(x,\xi) \quad \text{with} \;\; a \in C_0^\infty(\mathbb{R}^{2n}). \tag{3.4}$$

To see that such operators are well defined on L^2, we will use the following lemma.

Lemma 3.3 (Fast decay of the Fourier transform)**.** *Let $A \in C_b^\infty(\mathbb{R}^{3n})$ be compactly supported in ξ. Denote by $\hat{A}$ be the Fourier transform of A with respect to ξ, i.e.*

$$\hat{A}(x,y,z) = \int e^{-\mathrm{i}\xi\cdot z} A(x,y,\xi) d\xi.$$

Then, for all $N > 0$ there exists $C > 0$ such that

$$|\hat{A}(x,y,z)| \leq C(1+|z|)^{-N},$$

for all $x, y, z \in \mathbb{R}^n$. In particular

$$K_{A,h}(x,y) = (2\pi h)^{-n} \hat{A}\left(x, y, \frac{y-x}{h}\right),$$

decays fast with respect to $\frac{x-y}{h}$.

Proof. The main observation is that

$$|z|^2 e^{-\mathrm{i}\xi\cdot z} = -\Delta_\xi e^{-\mathrm{i}\xi\cdot z}$$

so that, by iteration, $(1+|z|^2)^N e^{-\mathrm{i}\xi\cdot z} = (1-\Delta_\xi)^N e^{-\mathrm{i}\xi\cdot z}$. Then, by integration by part

$$(1+|z|^2)^N \hat{A}(x,y,z) = \int e^{-\mathrm{i}\xi\cdot z} (1-\Delta_\xi)^N A(x,y,\xi) d\xi.$$

Therefore, assuming that A (hence $(1-\Delta_\xi)^N A$) is supported in $\mathbb{R}^{2n} \times \bar{B}(0,R)$ we obtain

$$(1+|z|^2)^N \big|\hat{A}(x,y,z)\big| \leq ||(1-\Delta_\xi)^N A||_{L^\infty(\mathbb{R}^{3n})} \big|\bar{B}(0,R)\big| \tag{3.5}$$

whose right hand side is a constant independent of x, y, z. This shows that $|\hat{A}(x,y,z)| \leq C(1+|z|^2)^{-N}$ which implies the result since $(1+|z|^2)^{-1} \leq 2(1+|z|)^{-1}$. $\square$

Remark. If A depends on a parameter but is uniformly compactly supported in ξ and has uniformly bounded derivatives, it follows from (3.5) that the constant C in Lemma 3.3 can be chosen independent of the parameter.

It follows from Lemma 3.3 that, for all $u \in L^2(\mathbb{R}^n)$,

$$OP_h(A)u(x) = \int K_{A,h}(x,y)u(y)dy,$$

is a well defined continuous function on $\mathbb{R}^n_x$ (here $K_{A,h}$ is defined by (3.3)).

Exercise 3.4. *Check this ! Check also that, for all $a \in C_0^\infty(\mathbb{R}^{2n})$, $op_h[a]$ maps $L^2(\mathbb{R}^n)$ into $C_0^\infty(\mathbb{R}^n)$.*

Not only $OP_h(A)u$ is continuous (hence measurable) whenever u belongs to L^2, but we will see that it also belongs to L^2, making $OP_h(A)$ a linear map on L^2. Furthermore this map is continuous. This is a consequence of the following proposition.

Proposition 3.5 (Uniform L^2 boundedness)**.** *For a given $A \in C_b^\infty(\mathbb{R}^{3n})$ with compact support in ξ, there exists $C > 0$ such that*

$$\big|\big|OP_h(A)u\big|\big|_{L^2} \leq C||u||_{L^2},$$

for all $u \in L^2$ and $h \in (0,1]$.

Note that this proposition provides the estimate

$$\big|\big|OP_h(A)\big|\big|_{L^2 \to L^2} \leq C,$$

which is uniform in $h \in (0,1]$. It rests on the following elementary result.

Lemma 3.6 (Schur test)**.** *Let K be a measurable function on $\mathbb{R}^{2n}$ such that*

$$C_1 := \sup_{x \in \mathbb{R}^n} \int |K(x,y)|dy < \infty, \qquad C_2 := \sup_{y \in \mathbb{R}^n} \int |K(x,y)|dx < \infty. \tag{3.6}$$

Then the operator $\mathcal{K}$ with the integral kernel K satisfies

$$||\mathcal{K}||_{L^2 \to L^2} \leq \sqrt{C_1 C_2}. \tag{3.7}$$

Proof. By the Cauchy-Schwarz inequality for $|K(x,y)u(y)| = |K(x,y)|^{1/2}\times|K(x,y)|^{1/2}|u(y)|$, we find first

$$\begin{aligned} \int |K(x,y)u(y)|dy &\leq \left(\int |K(x,y)|dy\right)^{1/2}\left(\int |K(x,y)||u(y)|^2dy\right)^{1/2} \\ &\leq \sqrt{C_1}\left(\int |K(x,y)||u(y)|^2dy\right)^{1/2}. \end{aligned}$$

Then, by the above inequality (used in the second line below), we get

$$\begin{aligned} \int \Big| \int K(x,y)u(y)dy\Big|^2 dx &\leq |\int \left(\int |K(x,y)||u(y)|dy\right)^2 dx \\ &\leq C_1 \int\int |K(x,y)||u(y)|^2 dydx \\ &\leq C_1C_2\int |u(y)|^2dy \end{aligned}$$

which yields the result. ☐

Proof of Proposition 3.5. It is a consequence of Lemma 3.6 together with the kernel estimate of Lemma 3.3. Indeed, choosing $N > n$, one has

$$\int |K_{A,h}(x,y)|dy \leq C(2\pi h)^{-n}\int \left(1+\left|\frac{x-y}{h}\right|\right)^{-N} dy = C(2\pi)^{-n}\int (1+|z|)^{-N}dz$$

whose right hand side is finite (since the integral is finite) and independent of h and x. The same estimate holds for $\int |K_{A,h}(x,y)|dx$ so the result follows. ☐

As a direct consequence of (3.4) and Proposition 3.5, we get

Corollary 3.7. *For all $a \in C_0^\infty(\mathbb{R}^{2n})$, there exists $C > 0$ such that*

$$\big|\big|op_h[a]\big|\big|_{L^2\to L^2} \leq C,$$

for all $h \in (0,1]$.

The next proposition will be useful to justify the results of symbolic calculus.

Proposition 3.8. *Let $A, B \in C_b^\infty(\mathbb{R}^{3n})$ be compactly supported in ξ. Assume that*

$$A(x,x,\xi) = B(x,x,\xi), \tag{3.8}$$

for all $x, \xi \in \mathbb{R}^n$. Then there exists $C > 0$ such that, for all $h \in (0,1]$,

$$\big|\big|OP_h(A) - OP_h(B)\big|\big|_{L^2\to L^2} \leq Ch.$$

Proof. The integral kernel of $OP_h(A) - OP_h(B)$ reads

$$(2\pi h)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi/h}\big(A(x,y,\xi) - B(x,y,\xi)\big)d\xi. \tag{3.9}$$

By the Taylor formula

$$A(x,y,\xi) - A(x,x,\xi) = \sum_{j=1}^{n}(y_j - x_j)\int_0^1 (\partial_{y_j}A)(x, x + s(y-x), \xi)ds,$$

where each integral belongs to $C_b^\infty(\mathbb{R}^{3n})$ and is compactly supported in ξ. Thus, using (3.8), one can write

$$A(x,y,\xi) - B(x,y,\xi) = \sum_{j=1}^{n}(y_j - x_j)C_j(x,y,\xi)$$

with $C_j \in C_b(\mathbb{R}^{3n})$ compactly supported in ξ. Using that

$$(y_j - x_j)e^{\mathrm{i}(x-y)\cdot\xi/h} = \mathrm{i}h\partial_{\xi_j}e^{\mathrm{i}(x-y)\cdot\xi/h}$$

and integrations by part in ξ, we see that (3.9) is the sum

$$-\mathrm{i}h\sum_j (2\pi h)^{-n}\int e^{\mathrm{i}(x-y)\cdot\xi/h}\partial_{\xi_j}C_j(x,y,\xi)d\xi.$$

Therefore

$$OP_h(A) - OP_h(B) = -\mathrm{i}h\sum_j OP_h(\partial_{\xi_j}C_j)$$

has norm bounded by Ch according to Proposition 3.5. The result follows. □

Proposition 3.9 (Elementary composition formula)**.** *For all $a, b \in C_0^\infty(\mathbb{R}^{2n})$, there exists $C > 0$ such that*

$$\big|\big|op_h[a]op_h[b] - op_h[ab]\big|\big|_{L^2\to L^2} \le Ch,$$

for all $h \in (0,1]$.

Lemma 3.10. *Let $\widetilde{op}_h[b]$ be the operator with kernel $(2\pi h)^{-n}\int e^{\mathrm{i}(x-y)\cdot\xi/h}b(y,\xi)d\xi$. Then*

$$\widehat{\widetilde{op}_h[b]u}(\xi) = \int e^{-\mathrm{i}y\cdot\xi}b(y,h\xi)u(y)dy,$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let $\varphi_h(\xi) = \int e^{-\mathrm{i}y\cdot\xi} b(y, h\xi) u(y) dy$. Then

$$\widetilde{op}_h[b]u(x) = (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\xi} \varphi_h(\xi) d\xi = (2\pi)^{-n} \hat{\varphi}_h(-x)$$

thus, by the Fourier inversion formula

$$\widehat{\widetilde{op}_h[b]u}(\xi) = (2\pi)^{-n} \int e^{-\mathrm{i}x\cdot\xi} \hat{\varphi}_h(-x) dx = \varphi_h(\xi)$$

which is exactly the result. □

Proof of Proposition 3.9. Observe first that

$$\begin{aligned} ||op_h[a]op_h[b] - op_h[a]\widetilde{op}_h[b]||_{L^2\to L^2} &\leq ||op_h[a]||_{L^2\to L^2} ||op_h[b] - \widetilde{op}_h[b]||_{L^2\to L^2} \\ &\leq Ch, \end{aligned} \tag{3.10}$$

using Proposition 3.7 and Proposition 3.8. On the other hand, using the definition (3.1) together with Lemma 3.10

$$\begin{aligned} op_h[a]\widetilde{op}_h[b]u(x) &= (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\xi} a(x, h\xi) \left(\int e^{-\mathrm{i}y\cdot\xi} b(y, h\xi) u(y) dy \right) d\xi \\ &= \int \left((2\pi)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi} a(x, h\xi) b(y, h\xi) d\xi \right) u(y) dy \end{aligned}$$

that is

$$op_h[a]\widetilde{op}_h[b] = OP_h(A), \qquad A(x, y, \xi) = a(x, \xi) b(y, \xi).$$

Since $A(x, x, \xi) = a(x, \xi) b(x, \xi)$, Proposition 3.8 implies that

$$\big|\big| OP_h(A) - op_h[ab] \big|\big|_{L^2\to L^2} \leq Ch.$$

Together with (3.10), this yields the result. □

In the following proposition, we provide an elementary justification of the fact that pseudodifferential operators are invariant by conjugation by diffeomorphisms.

Proposition 3.11 (Invariance mod h). *Let $\chi : V \to W$ be a diffeomorphism between open subsets of $\mathbb{R}^n$. Let $\psi \in C_0^\infty(V)$. For $a \in C_0^\infty(W \times \mathbb{R}^n)$ define*

$$a_\chi(x, \xi) = a\big(\chi(x), (d\chi(x)^T)^{-1}\xi\big).$$

Then a_χ belongs to $C_0^\infty(V \times \mathbb{R}^n)$ and

$$\big|\big| \big(\chi^* op_h[a] \chi_*\big)\psi - op_h[a_\chi]\psi \big|\big|_{L^2\to L^2} \leq Ch, \qquad h \in (0, 1].$$

The constant C remains bounded as long as a and ψ belong respectively to bounded subsets of $C_0^\infty(W \times \mathbb{R}^n)$ and $C_0^\infty(V)$.

Let us comment that the operator $\big(\chi^* op_h[a]\chi_*\big)\psi$, namely

$$u \mapsto \big(\chi^* op_h[a]\chi_*\big)(\psi u)$$

is defined globally, i.e. for all $u \in L^2(\mathbb{R}^n)$, since the cutoff ψ guarantees that $\chi_*(\psi u) = (\psi \circ \chi^{-1})(u \circ \chi^{-1})$ is well defined on $\mathbb{R}^n$ (it is implicitly continued by 0 outside the support of $\psi \circ \chi^{-1} \in C_0^\infty(W)$) even if u is not supported where χ^{-1} is defined. Also $\chi^*(op_h[a]v)$ is well defined for all $v \in L^2(\mathbb{R}^n)$ (in particular for $v = \chi_*(\psi u)$) since the compactness of $\mathrm{supp}(a)$ in $W \times \mathbb{R}^n$ ensures that $op_h[a]v$ is supported in a compact subset of W hence can be composed with χ.

Lemma 3.12. *The integral kernel of $\chi^* op_h[a]\chi_*\psi$ is*

$$K_{\chi,h}(x,y) := (2\pi h)^{-n} \int e^{\mathrm{i}(\chi(x)-\chi(y))\cdot\eta/h} a(\chi(x),\eta)\psi(y)|\det(d\chi(y))|d\eta. \tag{3.11}$$

Proof. A simple calculation using that

$$\chi^* op_h[a]\chi_*\psi u(x) = (2\pi h)^{-n} \int\int e^{\mathrm{i}(\chi(x)-t)\cdot\eta/h} a(\chi(x),\eta)(\psi u)(\chi^{-1}(t))dtd\eta$$

and using the change of variable $\chi^{-1}(t) = y$. $\square$

Lemma 3.13. *Let $\theta \in C_0^\infty(\mathbb{R}^n)$ be such that $\theta = 1$ near 0. Then, for all $N \geq 0$, $(1-\theta(x-y))K_{\chi,h}(x,y)$ reads*

$$(-1)^N h^{2N} \frac{1-\theta(x-y)}{|\chi(x)-\chi(y)|^{2N}} (2\pi h)^{-n} \int e^{\mathrm{i}(\chi(x)-\chi(y))\cdot\eta/h} (\Delta_\eta^N a)(\chi(x),\eta)\psi(y)|\det(d\chi(y))|dy.$$

In particular, if $\mathcal{K}_{\chi,\theta,h}$ is the operator with kernel $(1-\theta(x-y))K_{\chi,h}(x,y)$, there exists $C > 0$ such that

$$\big|\big|\mathcal{K}_{\chi,\theta,h}\big|\big|_{L^2 \to L^2} \leq Ch, \tag{3.12}$$

for all $h \in (0,1]$.

Proof. The first observation is that, due to the support of a and ψ, we may assume that both x and y belong to a compact subset of V (otherwise $a(\chi(x),\eta)\psi(y)$ vanishes). Then, by injectivity of χ and the fact that $1-\theta(x-y)$ vanishes near the diagonal $x = y$, we know that $\chi(x)-\chi(y)$ does not vanish on the support of $(1-\theta(x-y))a(\chi(x),\eta)\psi(y)$. In particular, on such a support, we can use the identity

$$\frac{-h^2}{|\chi(x)-\chi(y)|^2}\Delta_\eta e^{\mathrm{i}(\chi(x)-\chi(y))\cdot\eta/h} = e^{\mathrm{i}(\chi(x)-\chi(y))\cdot\eta/h}.$$

Using N times this identity and integrating by part in (3.11), we get the expected formula for $(1-\theta(x,y))K_{\chi,\theta,h}(x,y)$. Let us now prove (3.12). The support of $(\Delta_\eta^N a)(\chi(x),\eta)\psi(y)$ ensures that there exist $R>0$, $C>0$ and $V_1 \Subset V$ such that

$$\left|(\Delta_\eta^N a)(\chi(x),\eta)\psi(y)|\det(d\chi(y))|\right| \leq C\mathbb{1}_{V_1}(x)\mathbb{1}_{V_1}(y)\mathbb{1}_{B(0,R)}(\eta).$$

On the other hand, on the support of the above function, we have

$$\left|\frac{1-\theta(x-y)}{|\chi(x)-\chi(y)|^{2N}}\right| \leq C,$$

since the denominator does not vanish on the support of $1-\theta(x-y)$. Therefore, one can find a constant (depending on N) such that

$$\left|(1-\theta(x-y))K_{\chi,h}(x,y)\right| \leq Ch^{N-n}\mathbb{1}_{V_1}(x)\mathbb{1}_{V_1}(y),$$

for all $x,y\in\mathbb{R}^n$ and $h\in(0,1]$. Choosing $N\geq n+1$ and using Lemma 3.6, the result follows. □

Proof of Proposition 3.11. By (3.12), it suffices to consider the operator with kernel $\theta(x-y)K_{\chi,h}(x,y)$ for some suitable $\theta\in C_0^\infty(\mathbb{R}^n)$ equal to 1 near 0. Let us introduce

$$A(x,y) := \int_0^1 d\chi(x+s(y-x))^T ds.$$

If we let $W_0 \Subset W$ be a compact subset of W such that $\mathrm{supp}(a)\subset W_0\times\mathbb{R}^n$, $A(x,y)$ is well defined for all $x\in\chi^{-1}(W_0)$ and $y\in\mathrm{supp}(\psi)$ such that $x-y$ is small enough (to guarantee that the segment $[x,y]$ is contained in V where χ is defined). Thus, by choosing θ with a small enough support,

$$\eta\mapsto A(x,y)\eta \quad \text{is invertible for all } x\in\chi^{-1}(W_0),\ y\in\mathrm{supp}(\psi) \quad \text{such that } x-y\in\mathrm{supp}(\theta),$$

since $A(x,y)$ is close to $d\chi(x)^T$ if $x-y$ is small. Then, writing

$$\big(\chi(x)-\chi(y)\big)\cdot\eta = (x-y)\cdot A(x,y)\eta,$$

and using the change of variable $A(x,y)\eta=\xi$

$$\theta(x-y)K_{\chi,h}(x,y) = (2\pi h)^{-n}\int e^{\mathrm{i}(x-y)\cdot\xi/h}F(x,y,\xi)dy$$

with

$$F(x,y,\xi) = \theta(x-y)a\left(\chi(x),A(x,y)^{-1}\xi\right)|\det(d\chi(y))|\left|\det A(x,y)\right|^{-1}\psi(y).$$

Setting $G(x,y,\xi)=a_\chi(x,\xi)\psi(y)$, we find

$$F(x,x,\xi) = a_\chi(x,\xi)\psi(x) = G(x,x,\xi),$$

so using Proposition 3.8 and the fact that $op_h[a_\chi]\psi = OP_h(G)$, the result follows. □

3.2 Definition of the quantization on M

Let us pick a finite atlas on M, composed of charts $\kappa_l : M \supset U_l \to V_l \subset \mathbb{R}^n$, $l = 1, \dots, L$, and consider a partition of unity on M of the form

$$\sum_{l=1}^{L} \Psi_l^2 = 1 \qquad \text{with} \qquad \Psi_l \in C_0^\infty(U_l). \tag{3.13}$$

The existence of such a partition is classical (for completeness, we refer to Proposition A.1 for a construction). We recall that κ_l^* and κ_{l*} denote respectively the pullback and pushforward operators defined as

$$\kappa_l^* u = u \circ \kappa_l, \qquad \kappa_{l*}\varphi = \varphi \circ \kappa_l^{-1},$$

when u is a function on V_l and φ a function on U_l. One also defines

$$\psi_l = \kappa_{l*}\Psi_l \in C_0^\infty(V_l) \subset C_0^\infty(\mathbb{R}^n). \tag{3.14}$$

We also recall that we use the notation κ_{l*} for functions on T^*M (see (1.4)). In particular, we use this notation in the following definition.

Definition 3.14. *For $h \in (0,1]$ and $a \in C_0^\infty(T^*M)$, we set*

$$Op_h(a) = \sum_{l=1}^{L} \big(\kappa_l^* op_h[\kappa_{l*}(\Psi_l a)]\kappa_{l*}\big)\Psi_l. \tag{3.15}$$

We repeat the same kind of comments as those after Proposition 3.11 to justify that this definition is meaningful. On one hand, the cutoff Ψ_l on the right hand side allows to localize functions inside U_l hence to apply the operator κ_{l*}. On the other hand the cutoff Ψ_l on a ensures that $\kappa_{l*}(\Psi_l a)$ is globally defined (it belongs to $C_0^\infty(\mathbb{R}^{2n})$) and that the range of $op_h[\kappa_{l*}(\Psi_l a)]$ is contained in $C_0^\infty(V_l)$ so that one can apply the operator κ_l^*.

The above formula defines $Op_h(a)$ on $C^\infty(M)$. However, according to the next proposition, we can extend it to $L^2(M)$.

Proposition 3.15. *For all $a \in C_0^\infty(T^*M)$ there exists $C > 0$ such that*

$$||Op_h(a)\varphi||_{L^2(M)} \leq C||\varphi||_{L^2(M)}$$

for all $h \in (0,1]$ and $\varphi \in C^\infty(M)$. In particular $Op_h(a)$ has a unique linear continuous extension to $L^2(M)$ and (2.11) holds true.

Proof. This is mostly a consequence of Corollary 3.7. We provide some detail to explain ho to pass from $L^2(\mathbb{R}^n)$ to $L^2(M)$. We consider a single term in the sum (3.15). Then

$$\big|\big|\big(\kappa_l^* op_h[\kappa_{l*}(\Psi_l a)]\kappa_{l*}\big)\Psi_l\varphi\big|\big|_{L^2(M)}^2 = \int_{\mathbb{R}^n} \big|op_h[\kappa_{l*}(\Psi_l a)](\kappa_{l*}\Psi_l\varphi)(x)\big|^2 |g_l(x)| dx$$

where $|g_l(x)|$ is the Riemannian density. Since $\kappa_{l*}(\Psi_l a)$ is supported in a compact subset of $V_l \times \mathbb{R}^n$, we integrate only over a compact subset of V_l. In particular, $|g_l(x)|$ is bounded there so

$$\begin{aligned} \left|\left|\left(\kappa_l^* op_h[\kappa_{l*}(\Psi_l a)]\kappa_{l*}\right)\Psi_l\varphi\right|\right|_{L^2(M)} &\leq C\,||op_h[\kappa_{l*}(\Psi_l a)]\kappa_{l*}(\Psi_l\varphi)||_{L^2(\mathbb{R}^n)} \\ &\leq C'\,||\kappa_{l*}(\Psi_l\varphi)||_{L^2(\mathbb{R}^n)} \end{aligned} \tag{3.16}$$

the second line following from Corollary 3.7. Finally, writing $1 = |g_l(x)|^{-1}|g_l(x)|$ and using that $|g_l(x)|^{-1}$ is bounded on the support of $\kappa_{l*}\Psi_l$, we have

$$||\kappa_{l*}(\Psi_l\varphi)||^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left|(\kappa_{l*}\Psi_l\varphi)(x)\right|^2 dx \leq C\int_{\mathbb{R}^n} \left|(\kappa_{l*}\Psi_l\varphi)(x)\right|^2 |g_l(x)|dx = C||\Psi_l\varphi||^2_{L^2(M)}.$$

We can thus replace $||\kappa_{l*}(\Psi_l\varphi)||_{L^2(\mathbb{R}^n)}$ in (3.16) by (a constant times) $||\Psi_l\varphi||_{L^2(M)}$, which is itself bounded by $||\varphi||_{L^2(M)}$, so the result follows. □

Proposition 3.16. *Let $\kappa : U \subset M \to V \subset \mathbb{R}^n$ be a local chart. If K is a compact subset of U and $\tilde{\Psi} \in C_0^\infty(U)$ is equal to 1 near K, then for all $a \in C_0^\infty(T^*M)$ such that*

$$\mathrm{supp}(a) \subset \pi^{-1}_{T^*M}(K), \tag{3.17}$$

we have

$$\left|\left|Op_h(a) - \left(\kappa^*\left(op_h[\kappa_* a]\right)\kappa_*\right)\tilde{\Psi}\right|\right|_{\mathcal{L}_c(L^2)} \leq Ch, \qquad h \in (0,1].$$

The constant C remains bounded as long as a belongs to a bounded subset of $C_0^\infty(T^*M)$ such that (3.17) holds.

Proof. Write first

$$\begin{aligned} \left(\kappa^*\left(op_h[\kappa_* a]\right)\kappa_*\right)\tilde{\Psi} &= \sum_l \Psi_l^2 \kappa^*\left(op_h[\kappa_* a](\kappa_*\tilde{\Psi})\right)\kappa_* \\ &= \sum_l \kappa^*\left((\kappa_*\Psi_l)^2 op_h[\kappa_* a](\kappa_*\tilde{\Psi})\right)\kappa_* \end{aligned} \tag{3.18}$$

where $\kappa_*\Psi_l$ makes only sense on $V_l \cap V$ for those l such that $U_l \cap U$ is non empty, but the assumption on the support of a allows to set to zero the terms such that $U_l \cap U$ is empty. Furthermore, for the other terms, $(\kappa_*\Psi_l)^2(\kappa_* a)$ is smooth on $\mathbb{R}^{2n}$. We next wish to write for each l

$$(\kappa_*\Psi_l)^2 op_h[\kappa_* a](\kappa_*\tilde{\Psi}) = (\kappa_*\Psi_l) op_h[\kappa_* a](\kappa_*\Psi_l)(\kappa_*\tilde{\Psi}) + O_{\mathcal{L}_c(L^2(\mathbb{R}^n))}(h), \tag{3.19}$$

namely to move one of the cutoffs $\kappa_*\Psi_l$ from the left to the right, up to a remainder. One has to be a little careful there, because $\kappa_*\Psi_l$ alone is not globally smooth on $\mathbb{R}^n$ (it is not well defined due to the part of $\mathrm{supp}(\Psi_l)$ which may not meet U). One can justify (3.19) as follows. Write the integral kernel of $(\kappa_*\Psi_l)^2 op_h[\kappa_* a]\kappa_*\tilde{\Psi}$ as

$$(\kappa_*\Psi_l)(x)^2 \left((2\pi h)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi/h} (\kappa_* a)(x,\xi)(\kappa_*\tilde{\Psi})(y) d\xi \right) \tag{3.20}$$

where the parenthese is smooth on $\mathbb{R}^{2n}$ and supported in $K_x \times K_y$ while $(\kappa_*\Psi_l)$ is only smooth (and defined) on V. Pick $\theta \in C_0^\infty(\mathbb{R}^n)$ which is equal to 1 near 0. Then, the operator with kernel $1-\theta(x-y)$ times (3.20) has $O(h^\infty)$ operator norm. On the other hand, on the support of the bracket of (3.20) i.e. on $K \times K$, one can write

$$((\kappa_*\Psi_l)(x) - (\kappa_*\Psi)_l(y))\, \theta(x-y) = \theta(x-y) \int_0^1 d(\kappa_*\Psi_l)(y+s(x-y))ds \cdot (x-y)$$

where the right hand side is smooth on $K \times K$ since the segment $[x,y]$ is contained in V whenever x,y belong to K and $x-y$ is small enough (which is guaranteed by taking the support of θ close enough to 0). This allows to replace $(\kappa_*\Psi_l)(x)^2$ by $(\kappa_*\Psi_l)(x)(\kappa_*\Psi_l)(y)$ in (3.20) up to an error of size $O(h)$ in operator norm, hence to justify (3.19). We are then left with the study of

$$\kappa^* \left(op_h[\kappa_*(\Psi_l a)](\kappa_*(\Psi_l\tilde{\Psi})) \right) \kappa_* = \kappa_l^* (\kappa\circ\kappa_l^{-1})^* \left(op_h[\kappa_*(\Psi_l a)](\kappa_*(\Psi_l\tilde{\Psi})) \right) (\kappa\circ\kappa_l^{-1})_* \kappa_{l_*}$$

where we can use both charts since $\Psi_l a$ and $\Psi_l\tilde{\Psi}$ are supported in the intersection of their domains. Using Proposition 3.11 with $\chi = \kappa\circ\kappa_l^{-1}$, the right hand side of the last displayed formula reads

$$\kappa_l^* \left(op_h[\kappa_{l*}(\Psi_l a)](\kappa_{l*}(\Psi_l\tilde{\Psi})) \right) \kappa_{l_*} + O_{\mathcal{L}_c(L^2)}(h) = \kappa_l^* \left(\psi_l op_h[\kappa_{l*}a]\psi_l(\kappa_{l*}\tilde{\Psi}) \right) \kappa_{l_*} + O_{\mathcal{L}_c(L^2)}(h).$$

It remains to see that, since $\widetilde{\Psi} = 1$ near the support of a, one has

$$\kappa_l^* \left(\psi_l op_h[\kappa_{l*}a]\psi_l(\kappa_{l*}\tilde{\Psi}) \right) \kappa_{l_*} = \kappa_l^* \left(\psi_l op_h[\kappa_{l*}a]\psi_l \right) \kappa_{l_*} + O_{\mathcal{L}_c(L^2)}(h)$$

which is proved similarly as (3.19). Thus (3.18), reads

$$\sum_l \kappa_l^* \big(\psi_l op_h[\kappa_{l*}a]\psi_l \big) \kappa_{l*} + O_{\mathcal{L}_c(L^2)}(h) = Op_h(a) + O_{\mathcal{L}_c(L^2)}(h),$$

which completes the proof. $\square$

One first application of Proposition 3.16 is to provide a simple proof of the approximate composition formula (2.12).

Proposition 3.17. *For all $a,b \in C_0^\infty(T^*M)$, there exists $C>0$ such that, for all $h \in (0,1]$,*

$$\big|\big| Op_h(a)Op_h(b) - Op_h(ab) \big|\big|_{\mathcal{L}_c(L^2)} \leq Ch.$$

Proof. By linearity of $Op_h(\cdot)$ and by using a partition of unity on M, we may assume that a is as in Proposition 3.16. Then, using that $||Op_h(b)||_{\mathcal{L}_c(L^2)}$ is bounded uniformly in h, we find

$$Op_h(a)Op_h(b) = \kappa^*\big(op_h[\kappa_* a]\big)\kappa_* Op_h(\tilde{\Psi} b) + O_{\mathcal{L}_c(L^2)}(h).$$

Now observe that $\tilde{\Psi} b$ is supported in the same coordinate patch as a. Then by picking $\tilde{\tilde{\Psi}} \in C_0^\infty(U)$ which is equal to 1 near the support of $\tilde{\Psi}$, Proposition 3.16 gives

$$Op_h(\tilde{\Psi} b) = \Big(\kappa^*\big(op_h[\kappa_*(\tilde{\Psi} b)]\kappa_*\Big)\,\tilde{\tilde{\Psi}} + O_{\mathcal{L}_c(L^2)}(h)$$

Using Proposition 3.9, we have

$$\big(\kappa^*\big(op_h[\kappa_* a]\big)\kappa_*\big)\,\Big(\kappa^*\big(op_h[\kappa_*(\tilde{\Psi} b)]\kappa_*\Big)\,\tilde{\tilde{\Psi}} = \Big(\kappa^*\big(op_h[\kappa_*(a\tilde{\Psi} b)]\big)\kappa_*\Big)\,\tilde{\tilde{\Psi}} + O_{\mathcal{L}_c(L^2)}(h).$$

Using the support property of a, we have $a\tilde{\Psi} b = ab$. Using again Proposition 3.16, we then find

$$\big(\kappa^*\big(op_h[\kappa_*(ab)]\kappa_*\big)\,\tilde{\tilde{\Psi}} = Op_h(ab) + O_{\mathcal{L}_c(L^2)}(h)$$

and the result follows. □

Proposition 3.18. *The property (2.13) is true.*

Proof. Let $T = \big(\kappa_l^* op_h[\kappa_{l*}(\Psi_l a)]\kappa_{l*}\big)\Psi_l$. For notational simplicity in the computation below, we drop the index l and, for $\phi, \varphi \in C^\infty(M)$, we set

$$v = \kappa_*\phi, \qquad u = \kappa_*\varphi, \qquad \psi = \kappa_*\Psi (= \kappa_{l*}\Psi_{l*}).$$

Then

$$\begin{aligned}
\langle \phi, T\varphi\rangle_M &= \int \overline{(\kappa_*\phi)(x)}\,\big(op_h[\kappa_*(\Psi a)]\kappa_*(\Psi\varphi)\big)(x)|g(x)|dx \\
&= \int \overline{v(x)}\left((2\pi)^{-n}\int e^{\mathrm{i}(x-y)\cdot\xi}\psi(x)(\kappa_* a)(x,h\xi)\psi(y)u(y)dy\right)|g(x)|dx \\
&= (2\pi)^{-n}\int\left(\overline{\int e^{\mathrm{i}(y-x)\cdot\xi}|g(x)|(\kappa_*\bar{a})(x,h\xi)\psi(x)v(x)dx}\right)\psi(y)u(y)dy.
\end{aligned}$$

Introducing the Riemannian measure by writing $dy = |g(y)|^{-1}|g(y)|dy$, and then swapping the names of the integration variables, we see that $\langle \phi, T\varphi\rangle_M$ reads

$$(2\pi)^{-n}\int\left(\overline{\int e^{\mathrm{i}(x-y)\cdot\xi}|g(y)|(\kappa_*\bar{a})(y,h\xi)\psi(y)v(y)dy}\right)|g(x)|^{-1}\psi(x)u(x)|g(x)|dx,$$

or equivalently

$$\langle \phi, T\varphi\rangle_M = \big\langle\big(\kappa^*(OP_h(A)\kappa_*\phi), \varphi\big\rangle_M$$

with

$$A(x,y,\xi) = |g(x)|^{-1}\psi(x)(\kappa_*\bar{a})(y,h\xi)|g(y)|\psi(y).$$

Using proposition (3.8) together with the fact that A and $\psi(x)(\kappa_*\bar{a})(x,h\xi)\psi(y)$ coincide when $x = y$, we find that

$$T^* = \kappa^* OP_h(A)\kappa_* = \big(\kappa^* op_h[\kappa_*(\Psi\bar{a})]\kappa_*\big)\Psi + O_{\mathcal{L}_c(L^2)}(h),$$

and the result follows. □

3.3 The Egorov Theorem

In this section, we prove a weak version of the Egorov Theorem, namely the property (2.14). The proof itself is given at the end of the section. We need first to establish some preparatory results. In passing, we collect some computations which will be useful in other sections.

We start with calcultation in local coordinates; we use a chart $\kappa : U \subset M \to V \subset \mathbb{R}^n$ and consider the local expression of Δ_g

$$-P := \sum_{j,k} g^{jk}(x)\partial_j\partial_k + \sum_i \Gamma_i(x)\partial_i,$$

where, according to Exercise 1.5, one has $\Gamma_i(x) = -\sum_{j,k} g^{jk}(x)\Gamma^i_{jk}(x)$. We won't however use the explicit form of $\Gamma_i(x)$. We also define

$$\begin{aligned} p_0(x,\xi) &:= \sum_{j,k} g^{jk}(x)\xi_j\xi_k, \\ p_1(x,\xi) &:= \sum_i \Gamma_i(x)\xi_i \end{aligned}$$

that is $p_0 = \kappa_* p$ according to (1.21).

We start with a local result.

Proposition 3.19. *Let $\psi \in C_0^\infty(V)$ and $b \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$*

$$h^2 P\left(\psi op_h[b]\right) = op_h\left[b_0 + hb_1 + h^2 b_2\right]$$

where

$$\begin{aligned} b_0 &= p_0\psi b \\ b_1 &= -\mathrm{i}\partial_\xi p_0 \cdot \partial_x(\psi b) - \mathrm{i}p_1\psi b \\ b_2 &= P(\psi b). \end{aligned}$$

Proof. Given any arbitrary $u \in C_0^\infty(\mathbb{R}^n)$, we apply $h^2 P$ to

$$op_h[\psi b]u(x) = (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\xi} \psi(x) b(x, h\xi)\hat{u}(\xi) d\xi$$

and use the Leibniz rule to distribute the x derivatives on $e^{ix\cdot\xi}$ and $\psi(x)b(x, h\xi)$. One then obtain a sum of the form $\sum h^k op_h[b_k]u$ with

$$\begin{aligned} b_0(x,\xi) &= \sum_{j,k} g^{jk}(x)\xi_j\xi_k\psi(x)b(x,\xi) \\ b_1(x,\xi) &= -2\mathrm{i}\sum_{j,k} g^{jk}(x)\xi_j\partial_{x_k}\big(\psi(x)b(x,\xi)\big) - \mathrm{i}\sum_k \Gamma_k(x)\xi_k\psi(x)b(x,\xi), \end{aligned}$$

and $b_2 = P(\psi b)$. This yields the result. □

We record in passing the following global consequence of this calculation.

Proposition 3.20. *For all $a \in C_0^\infty(T^*M)$, one has*

$$-h^2\Delta_g Op_h(a) = Op_h(pa) + O_{\mathcal{L}_c(L^2)}(h).$$

More generally, for all $k \in \mathbb{N}$,

$$(-h^2\Delta_g)^k Op_h(a) = Op_h(p^k a) + O_{\mathcal{L}_c(L^2)}(h).$$

Proof. We give the proof in the case $k = 1$ and leave its adaptation to the case of larger k as an exercise.

$$\begin{aligned} -h^2\Delta_g Op_h(a) &= \sum_{l=1}^{L} -h^2\Delta_g\big(\kappa_l^* op_h[\kappa_{l*}(\Psi_l a)]\kappa_{l*}\big)\Psi_l \\ &= \sum_{l=1}^{L} \big(\kappa_l^* h^2 P_l op_h[\kappa_{l*}(\Psi_l a)]\kappa_{l*}\big)\Psi_l \end{aligned}$$

where P_l is the expression of $-\Delta_g$ in the l-th chart, namely $-\Delta_g\kappa_l^* = \kappa_l^* P_l$. By Proposition 3.19 with $\psi b := \kappa_{l*}(\Psi_l a)$, one has

$$h^2 P_l op_h[\kappa_{l*}(\Psi_l a)] = op_h[\kappa_{l*}(\Psi_l pa)] + hop_h[b_{1,l} + hb_{2,l}]$$

for some smooth and compactly supported symbols $b_{1,l}, b_{2,l}$. One can then check that

$$\big|\big|\big(\kappa_l^* op_h[b_{1,l} + hb_{2,l}]\kappa_{l*}\big)\Psi_l\big|\big|_{\mathcal{L}_c(L^2(M))} \le C, \qquad h \in (0,1],$$

exactly as in the proof of Proposition 3.15. The result follows. □

In Proposition 3.19, we have composed a pseudo-differential operator with $h^2 P$ on the left. We now compute the composition with $h^2 P$ on the right.

Proposition 3.21. *Let $b \in C_0^\infty(\mathbb{R}^{2n})$ and $\psi \in C_0^\infty(V)$. Then*

$$op_h[b]\psi h^2 P = OP_h(A_0 + hA_1 + h^2 A_2)$$

where, for each j,

$$A_j(x,y,\xi) = b(x,\xi)c_j(y,\xi)$$

with

$$\begin{aligned} c_0 &= \psi p_0 \\ c_1 &= \mathrm{i}\left(\mathrm{div}_y\left(\psi\partial_\xi p_0\right) - \psi p_1\right) \\ c_2 &= -\sum_{j,k}\partial_j\partial_k\left(\psi g^{jk}\right) + \sum_i \partial_i\left(\psi\Gamma_i\right) \end{aligned}$$

Note that, using Propositions 3.5 and 3.8, we find in particular that

$$op_h[b]\psi h^2 P = op_h[bp_0]\psi + O_{\mathcal{L}_c(L^2(\mathbb{R}^n))}(h). \tag{3.21}$$

Proof of Proposition 3.21. It is stlightly less direct than the one Proposition 3.19. The first step is to observe that for any $u \in C_0^\infty(V)$

$$(\psi P u)(y) = -\sum_{j,k}\partial_j\partial_k\left(\psi(y)g^{jk}(y)u\right) - \sum_i \partial_i\left(\tilde{\Gamma}_i(y)u\right) + W(y)u \tag{3.22}$$

with

$$\begin{aligned} \tilde{\Gamma}_i(y) &= \psi(y)\Gamma_i(y) - 2\sum_j \partial_j\left(\psi(y)g^{ji}(y)\right) \\ W(y) &= -\sum_{j,k}\partial_j\partial_k\left(\psi(y)g^{jk}(y)\right) + \sum_i \partial_i\left(\psi(y)\Gamma_i(y)\right). \end{aligned}$$

The interest is to write ψP with derivatives to the left rather than to the right. Then, using the expression (3.2) for the kernel of $op_h[b]$, we can write

$$\left(op_h[b]\psi h^2 P u\right)(x) = (2\pi h)^{-n}\int\int e^{\mathrm{i}(x-y)\cdot\xi/h} b(x,\xi)(h^2 P u)(y)d\xi dy$$

and then integrate by part in y the derivatives coming from (3.22). This shows that the above integral reads

$$(2\pi h)^{-n}\int\int e^{\mathrm{i}(x-y)\cdot\xi/h} b(x,\xi)\left(\sum_{j=0}^{2} h^j c_j(y,\xi)\right)u(y)d\xi dy$$

which is exactly the result. $\square$

We then obtain a result similar to Proposition 3.20.

Proposition 3.22. *For all $a \in C_0^\infty(T^*M)$, one has*

$$Op_h(a)(-h^2\Delta_g) = Op_h(pa) + O_{\mathcal{L}_c(L^2)}(h).$$

More generally, for all $k \in \mathbb{N}$,

$$Op_h(a)(-h^2\Delta_g)^k = Op_h(p^k a) + O_{\mathcal{L}_c(L^2)}(h).$$

Proof. It is similar to the one of Proposition 3.20 using additionally (3.21). □

Definition 3.23. *The* **commutator** *of two linear operators A, B acting on the same vector space is the operator*

$$[A, B] = AB - BA.$$

This definition is relatively formal but is sufficient for our purpose below. Typically, we will consider the case when A, B are differential (or pseudo-differential) operators, for which the compositions AB and BA make clearly sense on the space of smooth and compactly supported functions. We suggest readers non familiar with commutators to work out the following exercise.

Exercise 3.24.

1. *If A, B, C are linear operators (such that all compositions make sense) check that*

$$[A, BC] = [A, B]C + B[A, C].$$

2. *Let ϕ be a smooth function, seen as a differential operator of order 0. Let α be a non zero multi-index. Show that*

$$\big[(h\partial_x)^\alpha, \phi(x)\big] = h \sum_{0 \neq \gamma \leq \alpha} h^{|\gamma|-1} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} (\partial^\gamma \phi(x))(h\partial_x)^{\alpha-\gamma}.$$

It follows in particular from Propositions 3.20 and 3.22 that

$$\begin{aligned} \big[-h^2\Delta_g, Op_h(a)\big] &= -h^2\Delta_g Op_h(a) + Op_h(a)h^2\Delta_g \\ &= Op_h(pa) - Op_h(pa) + O_{\mathcal{L}_c(L^2)}(h) \\ &= O_{\mathcal{L}_c(L^2)}(h). \end{aligned}$$

The next proposition gives a sharper description of this commutator.

Proposition 3.25. *For all $a \in C_0^\infty(T^*M)$,*

$$\big[-h^2\Delta_g, Op_h(a)\big] = \mathrm{i}h Op_h(\{a, p\}) + O_{\mathcal{L}_c(L^2)}(h^2), \qquad h \in (0, 1], \tag{3.23}$$

where $\{.,.\}$ is the Poisson bracket which has been introduced in Exercise 1.14.

Remark 1. To be completely rigorous, the meaning of (3.23) is that the left hand side, seen as an operator acting on $C^\infty(M)$, has a continuous extension to $L^2(M)$ which coincides with the right hand side.

Remark 2. In the proof of (2.14) below, we will apply Proposition 3.25 with a replaced by $a \circ \Phi^t$ and t in a compact set. In this case, the remainder term $O_{\mathcal{L}_c(L^2)}(h^2)$ depends on t. However, the bound $O(h^2)$ is locally uniform in t, i.e. for all T there is C_T such that the norm of the remainder is bounded by $C_T h^2$ for $|t| \leq T$. This follows from the remark after Lemma 3.3 and the Schur test. We won't insist on this technical point below to focus on the main ideas.

Before proving in detail Proposition 3.25, we isolate first a calculation which allows to understand the main point on $\mathbb{R}^n$, without worrying about extra technical details due to the partition of unity on the manifold.

If $b \in C_0^\infty(\mathbb{R}^{2n})$, Propositions 3.19 and 3.21 allow to compute explicitly

$$\left[h^2 P, \psi op_h[a]\psi\right] = OP_h(C_0 + hC_1 + h^2 C_2) \tag{3.24}$$

with

$$\begin{aligned}
C_0 &= \psi(x)b(x,\xi)\big(p_0(x,\xi) - p_0(y,\xi)\big)\psi(y) \\
C_1 &= -\mathrm{i}\psi(x)\big(\partial_\xi p_0 \cdot \partial_x b(x,\xi) + (p_1(x,\xi) - p_1(y,\xi))b(x,\xi) + \mathrm{div}_y(\partial_\xi p_0(y,\xi))b(x,\xi)\big)\psi(y) \\
&\quad -\mathrm{i}\psi(x)b(x,\xi)\partial_\xi p_0(y,\xi) \cdot \partial_y \psi(y) - \mathrm{i}\partial_x \psi(x) \cdot \partial_\xi p_0(x,\xi)b(x,\xi)\psi(y)
\end{aligned}$$

and some computable $C_2 \in C_0^\infty(\mathbb{R}^{3n})$ which does not need to be explicited since, by Proposition 3.5,

$$||h^2 OP_h(C_2)||_{\mathcal{L}_c(L^2(\mathbb{R}^n))} \leq Ch^2.$$

One can also rewrite the operator $OP_h(C_0)$ by using the Taylor formula

$$p_0(x,\xi) - p_0(y,\xi) = \partial q_0(x,y,\xi) \cdot (x-y) \tag{3.25}$$

where

$$\partial q_0(x,y,\xi) := \int_0^1 (\partial_x p_0)(y + s(x-y),\xi) ds.$$

Indeed, by integration by part in ξ (e.g. as in the proof of Proposition 3.8), the kernel of $OP_h(C_0)$

$$\psi(x)\left((2\pi h)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi/h} b(x,\xi)\partial q_0(x,y,\xi) \cdot (x-y) d\xi\right)\psi(y)$$

can be written

$$\mathrm{i}h\psi(x)\left((2\pi h)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi/h} \mathrm{div}_\xi\left(b(x,\xi)\partial q_0(x,y,\xi)\right) d\xi\right)\psi(y)$$

which provides a factor ih. Using additionally that

$$\mathrm{div}_\xi \left(b(x,\xi)\partial q_0(x,y,\xi)\right)_{|x=y} = \partial_\xi b \cdot \partial_x p_0 + b\mathrm{div}_\xi(\partial_x p_0)$$

we conclude by using Proposition 3.8 that

$$OP_h(C_0) = \mathrm{i}h\psi op_h\left[\partial_\xi b \cdot \partial_x p_0 + b\mathrm{div}_\xi(\partial_x p_0)\right]\psi + O_{\mathcal{L}_c(L^2(\mathbb{R}^n))}(h^2). \tag{3.26}$$

Remark. Strictly speaking, the Taylor formula for p_0 (which is defined on $V \times \mathbb{R}^n$) only makes sense if the segment $[x,y]$ is contained in V. Since x,y belong to the support of ψ hence to V, this condition is satisfied if V is convex, but not in general. One can overcome this difficulty by considering $\tilde{\psi} \in C_0^\infty(V)$ which is equal to 1 near the support of ψ which allows to replace everywhere $p_0(x,\xi)$ by $\tilde{p}_0(x,\xi) := \tilde{\psi}(x)p_0(x,\xi)$ which is globally defined. This allows to justify completely the above argument and get in the end (3.26).

Using again Proposition 3.8, we can change y into x wherever we want in the expression of C_1 up to an error of size h at the level of the operator; in other words

$$OP_h(C_1) = -\mathrm{i}\ \psi op_h[\partial_\xi p_0 \cdot \partial_x b + b\mathrm{div}_x(\partial_\xi p_0)]\psi - \mathrm{i}op_h[b(\partial_x\psi^2)\cdot\partial_\xi p_0] + O_{\mathcal{L}_c(L^2(\mathbb{R}^n))}(h).$$

Observing that $\mathrm{div}_x(\partial_\xi p_0) = \mathrm{div}_\xi(\partial_x p_0)$ and taking into account the factor h in front of C_1 in (3.24), we conclude that

$$\begin{aligned}\left[h^2 P, \psi op_h[b]\psi\right] \quad &= \quad \mathrm{i}h\psi op_h[\partial_\xi b\cdot\partial_x p_0 - \partial_\xi p_0\cdot\partial_x b]\psi - \mathrm{i}h op_h[b(\partial_x\psi^2)\cdot\partial_\xi p_0] \\ &\qquad + O_{\mathcal{L}_c(L^2(\mathbb{R}^n))}(h^2).\end{aligned} \tag{3.27}$$

If the operator P was defined on $\mathbb{R}^n$, we could take $\psi \equiv 1$ so that $\partial_x(\psi^2) \equiv 0$ and (3.27) would read exactly

$$\left[h^2 P, op_h[b]\right] = \mathrm{i}h op_h[\{b, p_0\}] + \ O_{\mathcal{L}_c(L^2(\mathbb{R}^n))}(h^2)$$

that is the analogue of Proposition 3.25 on $\mathbb{R}^n$ rather than M. Intuitively, the reason why the above computations also cover the case of a manifold is that we will replace ψ by the functions (3.14) of the partition of unity (3.13) so that the sum of derivatives $\partial_x(\psi_l^2)$ will be zero thus discarding again the contribution of the second term of (3.27) after summation. In the proof below, we mostly consider this question.

Proof of Proposition 3.25. In the expression (3.15) of $Op_h(a)$, we let for simplicity

$$\psi_l = \kappa_{l*}\Psi_l, \qquad b_l = \kappa_{l*}a.$$

We also let $P_l = \kappa_{l*}(-\Delta_g)\kappa_l^*$ as in the proof of Proposition 3.20. Then

$$\left[-h^2\Delta_g, \kappa_l^*\psi_l op_h[b_l]\psi_l\kappa_{l*}\right] = \kappa_l^*\left[h^2 P_l, \psi_l op_h[b_l]\psi_l\right]\kappa_{l*}.$$

Also, letting $\tilde{\Psi}_l$ be smooth and compactly supported in the same patch as Ψ_l and equal to 1 near the support of Ψ_l, by setting $\psi_l = \kappa_{l*}\Psi_l$, one has

$$\left[h^2 P_l, \psi_l op_h[b_l]\psi_l\right] = \left[h^2 P_l, \psi_l op_h[b_l]\psi_l\right]\tilde{\psi}_l.$$

This follows from Exercise 3.24 using that $\psi_l \tilde{\psi}_l = \psi_l$ and $\psi_l [P, \tilde{\psi}_l] = 0$ Hence, using (3.27) and setting $p_{0,l} = \kappa_{l*} p$, we find

$$\big[h^2 P_l, \psi_l op_h[b_l]\psi_l\big] = \mathrm{i}h\psi op_h[\{b_l, p_{0,l}\}]\psi_l + \mathrm{i}h op_h[b_l\{\psi_l^2, p_{0,l}\}]\tilde{\psi}_l + \ O_{\mathcal{L}_c(L^2(\mathbb{R}^n))}(h^2)$$

where, by the invariance of the definition of the Poisson bracket under coordinates change,

$$\{b_l, p_{0,l}\} = \kappa_{l*}\{a, p\}, \qquad b_l\{\psi_l^2, p_{0,l}\} = \kappa_{l*}\big(a\{\Psi_l^2, p\}\big).$$

Furthermore, using Proposition 3.16, we have

$$\kappa_l^* \left(op_h[b_l\{\psi_l^2, p_{0,l}\}]\tilde{\psi}_l \right) \kappa_{l*} = Op_h\big(a\{\Psi_l^2, p\}\big) + O_{\mathcal{L}_c(L^2)}(h).$$

Therefore, after summation in l, we find

$$\big[-h^2\Delta_g, Op_h(a)\big] = \mathrm{i}hOp_h(\{a, p\}) + \mathrm{i}hOp_h\left(a\left\{\sum_l \Psi_l^2, p\right\}\right) + O_{\mathcal{L}_c(L^2)}(h^2)$$

where the second term in the right hand side vanishes since $\{1, p\} = 0$. This completes the proof. $\square$

The last step to prove (2.14) is the following one.

Proposition 3.26. *Let $a \in C_0^\infty(T^*M)$. Define $a_t = a \circ \Phi^t$ for all $t \in \mathbb{R}$. Then*

$$\partial_t a_t = \{p, a_t\}.$$

Proof. Using that Φ^t is the flow of the vector field W_p defined in (1.22) one has

$$\partial_t(a \circ \Phi^t) = \big(da \cdot W_p\big)_{|\Phi^t} = \sigma(W_p, W_a)_{|\Phi^t}.$$

Using the definition of the Poisson bracket (1.31), the above function equals precisely $\{p, a\} \circ \Phi^t$ and thus, using (1.33),

$$\partial_t a_t = \{p \circ \Phi^t, a \circ \Phi^t\}.$$

By conservation of energy, i.e. (1.27), one has $p \circ \Phi^t = p$ and the result follows. $\square$

Proof of (2.14). We start by giving the main idea and then comment on some minor technical details. One computes

$$\begin{aligned} \frac{d}{dt}\left(e^{\mathrm{i}th\Delta_g} Op_h(a_t) e^{-\mathrm{i}th\Delta_g}\right) &= e^{\mathrm{i}th\Delta_g}\left(\mathrm{i}h\Delta_h Op_h(a_t) + Op_h(\partial_t a_t) - Op_h(a_t)\mathrm{i}h\Delta_g\right) e^{-\mathrm{i}th\Delta_g} \\ &= e^{\mathrm{i}th\Delta_g}\left(Op_h(\partial_t a_t) - \frac{\mathrm{i}}{h}\big[-h^2\Delta_g, Op_h(a_t)\big]\right) e^{-\mathrm{i}th\Delta_g}. \quad (3.28) \end{aligned}$$

By integration of this identity, we find

$$e^{\mathrm{i}th\Delta_g} Op_h(a_t) e^{-\mathrm{i}th\Delta_g} - Op_h(a) = \int_0^t e^{\mathrm{i}sh\Delta_g} \left(Op_h(\partial_s a_s) - \frac{\mathrm{i}}{h} \big[- h^2 \Delta_g, Op_h(a_s) \big] \right) e^{-\mathrm{i}sh\Delta_g} ds.$$

By Propositions 3.25 and 3.26, one has

$$Op_h(\partial_s a_s) - \frac{\mathrm{i}}{h} \big[- h^2 \Delta_g, Op_h(a_s) \big] = Op_h(\{p, a_s\}) - Op_h(\{p, a_s\}) + O_{\mathcal{L}_c(L^2)}(h) \quad (3.29)$$

where the last term depends on s but, on every compact time interval J, its operator norm can be bounded by Ch, with C independent of s (but depending on J). By integration, also using that $e^{\pm \mathrm{i}sh\Delta_g}$ have norm 1, we find that for every T,

$$\left| \left| e^{\mathrm{i}th\Delta_g} Op_h(a_t) e^{-\mathrm{i}th\Delta_g} - Op_h(a) \right| \right|_{\mathcal{L}_c(L^2)} \leq C_T h, \qquad h \in (0,1], \ |t| \leq T. \quad (3.30)$$

By unitarity of $e^{\pm \mathrm{i}th\Delta_g}$, this estimate is equivalent to (2.14). This essentially completes the proof, up to the justification of some algebraic manipulations which are related to the remark after Proposition 3.25. To get (3.28), we have used that

$$\frac{d}{dt} e^{\mathrm{i}th\Delta_g} = e^{\mathrm{i}th\Delta_g} \mathrm{i}h\Delta_g, \qquad \frac{d}{dt} e^{-\mathrm{i}th\Delta_g} = -e^{\mathrm{i}th\Delta_g} \mathrm{i}h\Delta_g = -\mathrm{i}h\Delta_g e^{\mathrm{i}th\Delta_g}$$

which is formally obvious but has to be handled with care since Δ_g is not a bounded operator on $L^2(M)$. One way to proceed is to prove (3.28) in the weak sense, by testing it against finite linear combinations of eigenfunctions of Δ_g. Indeed for any two eigenfunctions e_j, e_k of Δ_g, one has

$$\begin{aligned} \left\langle e^{\mathrm{i}th\Delta_g} Op_h(a_t) e^{-\mathrm{i}th\Delta_g} e_j, e_k \right\rangle &= \left\langle Op_h(a_t) e^{-\mathrm{i}th\Delta_g} e_j, e^{-\mathrm{i}th\Delta_g} e_k \right\rangle \\ &= \left\langle Op_h(a_t) e^{\mathrm{i}th\lambda_j} e_j, e^{\mathrm{i}th\lambda_k} e_k \right\rangle \end{aligned}$$

and using (2.8) the derivative can be rigorously computed and is the sum of

$$\left\langle e^{\mathrm{i}th\Delta_g} Op_h(\partial_t a_t) e^{-\mathrm{i}th\Delta_g} e_j, e_k \right\rangle \quad (3.31)$$

and

$$\left\langle e^{\mathrm{i}th\Delta_g} Op_h(a_t)(-\mathrm{i}h\Delta_g) e^{-\mathrm{i}th\Delta_g} e_j, e_k \right\rangle + \left\langle Op_h(a_t) e^{-\mathrm{i}th\Delta_g} e_j, (-\mathrm{i}h\Delta_g) e^{\mathrm{i}th\Delta_g} e_k \right\rangle.$$

In these two terms, we can integrate by part the second Laplacian using (1.17) (recall also that we consider a complex Hilbert space so that moving i from one factor to the other turns it into $-$i) and we get

$$\left\langle e^{\mathrm{i}th\Delta_g} \big[\mathrm{i}h\Delta_g, Op_h(a_t) \big] e^{-\mathrm{i}th\Delta_g} e_j, e_k \right\rangle \quad (3.32)$$

which is meaningful since this equality is tested against smooth functions. Applying Propositions 3.25 and 3.26 and then integrating in time, we find

$$\left\langle \left(e^{\mathrm{i}th\Delta_g} Op_h(a_t) e^{-\mathrm{i}th\Delta_g} - Op_h(a) \right) e_j, e_k \right\rangle = \left\langle \left(\int_0^t hB_h(s)ds \right) e_j, e_k \right\rangle$$

for some $B_h(s) \in \mathcal{L}_c(L^2)$, locally uniformly bounded in s. By linearity, this identity remains true if one replaces the eigenfunctions by finite linear combinations of eigenfunctions and then by any L^2 functions, since one can use a density argument for the operators involved on both sides are continuous on $L^2(M)$. This then fully justifies (3.30). $\square$

Chapter 4

Proof of the local Weyl law

4.1 Hilbert-Schmidt operators

Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces over $\mathbb{C}$.

Lemma 4.1. *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. If $(e_j)_{j\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ are orthonormal bases of $\mathcal{H}$ and $\mathcal{K}$ respectively, then*

$$\sum_j ||Ae_j||_{\mathcal{K}}^2 = \sum_k ||A^* f_k||_{\mathcal{H}}^2.$$

Proof. For each j, we have

$$||Ae_j||_{\mathcal{K}}^2 = \sum_k \left|\left(f_k, Ae_j\right)_{\mathcal{K}}\right|^2 = \sum_k \left|\left(A^* f_k, e_j\right)_{\mathcal{K}}\right|^2.$$

Summing over j and swapping the summations with respect to j and k, we get

$$\sum_j ||Ae_j||_{\mathcal{K}}^2 = \sum_k \sum_j \left|\left(A^* f_k, e_j\right)_{\mathcal{K}}\right|^2 = \sum_k ||A^* f_k||_{\mathcal{H}}^2$$

which is precisely the result. $\square$

Definition 4.2. *An operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is* **Hilbert-Schmidt** *if*

$$||A||_{\mathrm{HS}} := \left(\sum_j ||Ae_j||_{\mathcal{K}}^2\right)^{1/2} < \infty,$$

for some orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $\mathcal{H}$. We denote by $S_2(\mathcal{H}, \mathcal{K})$ the set of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, we denote it by $S_2(\mathcal{H})$.

Note that $||A||_{\mathrm{HS}}$ is independent of the choice of the orthonormal basis by Lemma 4.1.

Proposition 4.3. *1. If $A \in S_2\big(\mathcal{H},\mathcal{K}\big)$, then $A^* \in S_2\big(\mathcal{K},\mathcal{H}\big)$ and*

$$||A||_{\mathrm{HS}} = ||A^*||_{\mathrm{HS}}.$$

2. For all $A \in S_2\big(\mathcal{H},\mathcal{K}\big)$, we have

$$||A||_{\mathcal{H}\to\mathcal{K}} \leq ||A||_{\mathrm{HS}}.$$

3. $S_2\big(\mathcal{H},\mathcal{K}\big)$ is a vector space and $||\cdot||_{\mathrm{HS}}$ is a norm thereon.

4. $S_2\big(\mathcal{H},\mathcal{K}\big)$ is complete.

5. If $\mathcal{H}_1$ and $\mathcal{K}_1$ are separable Hilbert spaces and

$$A \in S_2\big(\mathcal{H},\mathcal{K}\big), \qquad B \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}), \qquad C \in \mathcal{L}(\mathcal{K},\mathcal{K}_1)$$

then $CAB \in S_2\big(\mathcal{H}_1,\mathcal{K}_1\big)$ and

$$||CAB||_{\mathrm{HS}} \leq ||C||_{\mathcal{K}\to\mathcal{K}_1}||A||_{\mathrm{HS}}||B||_{\mathcal{H}_1\to\mathcal{H}}.$$

Proof. Item 1 follows directly from Lemma 4.1. To prove item 2, we fix $u \in \mathcal{H}$ and write

$$u = \lim_{N\to\infty} u_N, \qquad u_N = \sum_{j\leq N} \big(e_j, u\big)_{\mathcal{H}} e_j.$$

Then, by the triangle inequality and the Cauchy-Schwartz inequality,

$$\begin{aligned} ||Au_N||_{\mathcal{K}} &\leq \sum_{j\leq N} |\big(e_j,u\big)_{\mathcal{H}}|\big|\big|Ae_j\big|\big|_{\mathcal{K}} \\ &\leq \left(\sum_{j\leq N} |\big(e_j,u\big)_{\mathcal{H}}|^2\right)^{1/2} \left(\sum_{j\leq N} \big|\big|Ae_j\big|\big|_{\mathcal{K}}^2\right)^{1/2} \\ &\leq ||A||_{\mathrm{HS}}||u||_{\mathcal{H}}. \end{aligned}$$

Letting N go to infinity and using the continuity of A, we obtain $||Au||_{\mathcal{K}} \leq ||A||_{\mathrm{HS}}||u||_{\mathcal{H}}$ which yields the result. The proof of item 3 is a routine which we omit; we only point out that $||A||_{\mathrm{HS}} = 0$ only if $A = 0$ by item 2. Let us now prove item 4. Let (A_j) be a Cauchy sequence in $S_2\big(\mathcal{H},\mathcal{K}\big)$. By item 2, it is a Cauchy sequence in $\mathcal{L}(\mathcal{H},\mathcal{K})$ hence converges in operator norm to a bounded operator A. It remains to show that A is Hilbert-Schmidt and that $||A - A_j||_{\mathrm{HS}} \to 0$. Fix $\epsilon > 0$. Then for $J > 0$ large enough

$$||A_k - A_j||_{\mathrm{HS}} \leq \epsilon, \qquad j,k \geq J.$$

This implies in particular that,

$$\sum_{m\leq N} ||(A_k - A_j)e_m||_{\mathcal{K}}^2 \leq \epsilon^2, \qquad N \geq 0,\ j,k \geq J.$$

Thus, by letting k go to infinity for fixed N and j and then N to infinity, we see that A is Hilbert-Schmidt and that $||A - A_j||_{\mathrm{HS}} \leq \epsilon$ for $j \geq J$. Therefore $||A - A_j||_{\mathrm{HS}} \to 0$. We finally prove item 5. We first observe that

$$||CAe_j||^2_{\mathcal{K}_1} \leq ||C||^2_{\mathcal{K}\to\mathcal{K}_1}||Ae_j||^2_{\mathcal{K}}$$

hence by summing over j, we see that $CA \in S_2(\mathcal{H}, \mathcal{K}_1)$ and that $||CA||_{\mathrm{HS}} \leq ||C||_{\mathcal{K}\to\mathcal{K}_1}||A||_{\mathrm{HS}}$. To handle the case when $B \neq I_{\mathcal{H}}$, we observe that

$$CAB = \big(B^*(CA)^*\big)^*$$

which shows that CAB is Hilbert-Schmidt and that

$$||CAB||_{\mathrm{HS}} \leq ||B^*||_{\mathcal{H}\to\mathcal{H}_1}||CA||_{\mathrm{HS}} \leq ||C||_{\mathcal{K}\to\mathcal{K}_1}||A||_{\mathrm{HS}}||B||_{\mathcal{H}_1\to\mathcal{H}},$$

using item 1 and the fact that the operator norms of an operator and its adjoint coincide. □

Proposition 4.4. *Let $A \in S_2(\mathcal{H}, \mathcal{K})$ and $(f_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{K}$. Define*

$$\Pi_N = \textit{ orthogonal projection on } \operatorname{span}\{f_k \mid k \leq N\}.$$

Then

$$||\Pi_N A - A||_{\mathrm{HS}} \to 0, \qquad N \to \infty.$$

Proof. By item 1 of Proposition 4.3, it is equivalent to show that $||A^*\Pi_N - A^*||_{\mathrm{HS}} \to 0$. Writing this Hilbert-Schmidt norm in term of the orthonormal basis (f_k), we obtain

$$||A^*(\Pi_N - 1)||^2_{\mathrm{HS}} = \sum_{k>N} ||A^* f_k||^2_{\mathcal{H}}$$

which clearly goes to zero as $N \to \infty$. □

Remark. Of course if we consider a projection P_N onto the N first vectors of an orthonormal basis of $\mathcal{H}$, we also have $AP_N \to A$ in the Hilbert-Schmidt class. This follows directly from Proposition 4.4 by taking the adjoint.

Using that Π_N is a finite rank operator and the property 2 in Proposition 4.3, we derive automatically the following corollary.

Corollary 4.5. *Hilbert-Schmidt operators are compact.*

We conclude this section with an important example of Hilbert-Schmidt operator.
To any $K \in L^2(\mathbb{R}^{2n})$, we can associate the sesquilinear form

$$Q(v, u) = \int\int \overline{v(x)}K(x, y)u(y)dydx, \qquad u, v \in L^2(\mathbb{R}^n).$$

By the Cauchy-Schwarz inequality, Q is obviously continuous on $L^2(\mathbb{R}^n)^2$ hence there exists a unique bounded operator $A_K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that

$$Q(v, u) = \big(v, A_K u\big)_{L^2(\mathbb{R}^n)}.$$

Definition 4.6. *A_K is the operator with L^2* **kernel** *K.*

Proposition 4.7. *The operator A_K belongs to $S_2\big(L^2(\mathbb{R}^n)\big)$ and*

$$||A_K||_{\mathrm{HS}} = ||K||_{L^2(\mathbb{R}^{2n})}.$$

Proof. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$. Then the countable family

$$\big(e_k \otimes \overline{e_j}\big)_{(j,k)\in\mathbb{N}^2}, \qquad e_k \otimes \overline{e_j}(x,y) := e_k(x)\overline{e_j(y)}$$

is an orthonormal basis of $L^2(\mathbb{R}^{2n})$: that this is an orthonormal system is a simple calculation which we omit and proving that finite linear combinations are dense follows from the density of $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^{2n})$ and the density of finite linear combinations of (e_j) in $L^2(\mathbb{R}^n)$. Then

$$\begin{aligned} ||A_K||_{\mathrm{HS}}^2 = \sum_j ||A_K e_j||_{L^2(\mathbb{R}^n)}^2 &= \sum_j \sum_k \left|\big(e_k, A_K e_j\big)_{L^2(\mathbb{R}^n)}\right|^2 \\ &= \sum_j \sum_k \left|Q(e_k, e_j)\right|^2 \\ &= \sum_{(j,k)} \left|\big(e_k \otimes \overline{e_j}, K\big)_{L^2(\mathbb{R}^{2n})}\right|^2 = ||K||_{L^2(\mathbb{R}^{2n})}^2, \end{aligned}$$

completes the proof. □

4.2 Trace class operators

Definition 4.8 (Trace class operators)**.** *An operator $T : \mathcal{H} \to \mathcal{H}$ is* **trace class** *if it can be written*

$$T = \sum_{k=1}^{N} A_k^* B_k \tag{4.1}$$

for some N and some Hilbert-Schmidt operators

$$A_k, B_k : \mathcal{H} \to \mathcal{K}_k, \qquad k = 1, \ldots, N,$$

between a Hilbert space $\mathcal{K}_k$ and $\mathcal{H}$.

The interest of trace class operators is that one can define their trace.

Definition 4.9 (Trace of an operator)**.** *If $T : \mathcal{H} \to \mathcal{H}$ is trace class and $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, the* **trace** *of T is*

$$\mathrm{tr}(T) = \sum_{j\in\mathbb{N}} (e_j, Te_j). \tag{4.2}$$

Sometimes, if we need to specify the Hilbert space, we will denote $\mathrm{tr}_{\mathcal{H}}(T)$ for $\mathrm{tr}(T)$.

The following proposition says that this definition is meaninful.

Proposition 4.10. *Let $T : \mathcal{H} \to \mathcal{H}$ be trace class and $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then*

1. $\sum_{n\in\mathbb{N}} \left|(e_j, Te_j)\right| < \infty$. *In particular, the sum in (4.2) is convergent.*
2. *If $(f_k)_{k\in\mathbb{N}}$ is another orthonormal basis, then $\sum_{j\in\mathbb{N}}(e_j, Te_j) = \sum_{k\in\mathbb{N}}(f_k, Tf_k)$.*

The second item says that the trace of T is intrinsinc, i.e. does not depend on the orthonormal basis.

Proof of Proposition 4.10. It suffices to prove the result when $N = 1$ in (4.1) i.e. when $T = A^*B$, with $A, B : \mathcal{H} \to \mathcal{K}$ Hilbert-Schmidt. The first item follows from

$$\sum \left|(e_j, Te_j)_{\mathcal{H}}\right| = \sum \left|(Ae_j, Be_j)_{\mathcal{K}}\right| \leq \left(\sum ||Ae_j||_{\mathcal{K}}^2\right)^{1/2} \left(\sum ||Be_j||_{\mathcal{K}}^2\right)^{1/2}$$

the right hand side being finite since A and B are Hilbert-Schmidt. To prove the second item, we pick an orthonormal basis $(\varepsilon_m)_{l\in\mathbb{N}}$ of $\mathcal{K}$ and use that for any j,

$$\begin{aligned}
(e_j, Te_j)_{\mathcal{H}} &= (Ae_j, Be_j)_{\mathcal{K}} \\
&= \sum_m \overline{(\varepsilon_m, Ae_j)_{\mathcal{K}}}(\varepsilon_m, Be_j)_{\mathcal{K}} \\
&= \sum_m \overline{(A^*\varepsilon_m, e_j)_{\mathcal{H}}}(B^*\varepsilon_m, e_j)_{\mathcal{H}}
\end{aligned}$$

and thus, summing over j, we get

$$\begin{aligned}
\sum_j (e_j, Te_j)_{\mathcal{H}} &= \sum_j \sum_m \overline{(A^*\varepsilon_m, e_j)_{\mathcal{H}}}(B^*\varepsilon_m, e_j)_{\mathcal{H}} \\
&= \sum_m \sum_j \overline{(A^*\varepsilon_m, e_j)_{\mathcal{H}}}(B^*\varepsilon_m, e_j)_{\mathcal{H}} \\
&= \sum_m \sum_j (e_j, A^*\varepsilon_m)_{\mathcal{H}} \overline{(e_j, B^*\varepsilon_m)_{\mathcal{H}}} \\
&= \sum_m (B^*\varepsilon_m, A^*\varepsilon_m)_{\mathcal{H}}
\end{aligned}$$

which does not depend on $(e_j)_{j\in\mathbb{N}}$. Note that in the second line we have swapped the summations over m and j which is possible since $(A^*\varepsilon_m, e_j)_{\mathcal{H}}$ and $(B^*\varepsilon_m, e_j)_{\mathcal{H}}$ belong to $l^2(\mathbb{N}\times\mathbb{N})$ since

$$\sum_{m,j} \left|(A^*\varepsilon_m, e_j)_{\mathcal{H}}\right|^2 = \sum_m ||A^*\varepsilon_m||_{\mathcal{H}}^2 = ||A^*||_{\mathrm{HS}}^2 < \infty.$$

The proof is complete. □

In the next proposition, we give an example of calculation of a trace. We consider Hilbert-Schmidt operators with L^2 kernels, as in Definition 4.6.

Proposition 4.11. *Let $A = A_{K_1}$ and $B = A_{K_2}$ be operators on $L^2(\mathbb{R}^n)$ with L^2 kernels $K_1, K_2 \in L^2(\mathbb{R}^{2n})$. Then*

$$\mathrm{tr}_{L^2(\mathbb{R}^n)}\big(AB\big) = \iint K_1(x,y)K_2(y,x)dxdy.$$

Proof. Let us observe first that A^* is the operator with L^2 kernel $K_1^*(x,y) := \overline{K_1(y,x)}$. Observe also that

$$\big(K_1^*, K_2\big)_{L^2(\mathbb{R}^{2n})} = \iint K_1(x,y)K_2(y,x)dxdy.$$

The conclusion follows then from the following calculation. Given an orthonormal basis (e_j) of $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}
\mathrm{tr}(AB) \;=\; \sum_j \big(e_j, ABe_j\big)_{L^2(\mathbb{R}^n)} &= \sum_j \big(A^*e_j, Be_j\big)_{L^2(\mathbb{R}^n)} \\
&= \sum_j \sum_k \overline{\big(e_k, A^*e_j\big)}_{L^2(\mathbb{R}^n)} \big(e_k, Be_j\big)_{L^2(\mathbb{R}^n)} \\
&= \sum_j \sum_k \overline{\big(e_k \otimes \overline{e_j}, K_1^*\big)}_{L^2(\mathbb{R}^{2n})} \big(e_k \otimes \overline{e_j}, K_2\big)_{L^2(\mathbb{R}^{2n})} \\
&= \big(K_1^*, K_2\big)_{L^2(\mathbb{R}^{2n})},
\end{aligned}$$

using in the fourth line that $(e_k \otimes \overline{e_j})_{(j,k)}$ is an orthonormal basis of $L^2(\mathbb{R}^{2n})$ (see Proposition 4.7). ☐

A useful consequence of this proposition is the following one.

Proposition 4.12. *Let $K \in \mathcal{S}(\mathbb{R}^{2n})$ be a Schwartz function and A be the operator with kernel K,*

$$Au(x) = \int K(x,y)u(y)dy.$$

Then A is trace class on $L^2(\mathbb{R}^{2n})$ and

$$\mathrm{tr}(A) = \int K(x,x)dx.$$

Proof. Denote $\langle x \rangle = (1 + |x|^2)^{1/2}$. Let $A_2 = (1-\Delta)^n \langle x \rangle^{2n} A$, that is the operator with integral kernel

$$K_2(x,y) = (1-\Delta_x)^n \big(\langle x \rangle^{2n} K(x,y)\big),$$

which belongs to $\mathcal{S}(\mathbb{R}^{2n})$. Then $A = A_1 A_2$ with

$$A_1 = \langle x \rangle^{-2n}(1-\Delta)^{-n},$$

that is the operator with integral kernel

$$K_1(x,y) = \langle x \rangle^{-2n} (2\pi)^{-n} \int e^{\mathrm{i}(x-y)\cdot\xi} (1+|\xi|^2)^{-n} d\xi.$$

(If this is not clear to you check that $(1-\Delta)^n \langle x \rangle^{2n} \int K_1(x,y)u(y)dy = u(x)$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.) The function K_1 belongs to $L^2(\mathbb{R}^{2n})$ since the integral is an L^2 function of $x-y$ and $\langle x \rangle^{-2n}$ belongs to L^2. Obviously K_2 is L^2 since it is Schwartz so A_1 and A_2 are Hilbert-Schmidt. On the other hand, using that $A = A_1 A_2$, we have

$$K(x,z) = \int K_1(x,y) K_2(y,z) dy.$$

Using Proposition 4.11, we get the result. □

We conclude this section with applications to pseudo-differential operators.

Proposition 4.13. *Let $a \in C_0^\infty(\mathbb{R}^{2n})$ and $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^n)$. Then $\psi_1 op_h[a] \psi_2$ is both Hilbert-Schmidt and trace class on $L^2(\mathbb{R}^n)$. Furthermore*

$$||\psi_1 op_h[a] \psi_2||_{\mathrm{HS}} \leq C h^{-n/2}, \qquad h \in (0,1] \tag{4.3}$$

and

$$\mathrm{tr}_{L^2(\mathbb{R}^n)} \big(\psi_1 op_h[a] \psi_2\big) = (2\pi h)^{-n} \iint \psi_1(x) \psi_2(x) a(x,\xi) dx d\xi. \tag{4.4}$$

Proof. By (3.2), the integral kernel of $\psi_1 op_h[a] \psi_2$ is

$$K_h(x,y) \quad = \quad (2\pi)^{-n} \psi_1(x) \psi_2(y) h^{-n} \hat{a}\left(x, \frac{y-x}{h}\right)$$

where $\hat{a}$ is the Fourier transform of a with respect to ξ. The function K_h is smooth and compactly supported. It is in particular L^2 and its L^2 squared norm is bounded by

$$(2\pi h)^{-2n} \int \int \left| \hat{a}^2 \left(x, \frac{y-x}{h}\right) \right| dx dy ||\psi_1||_{L^\infty}^2 ||\psi_2||_{L^\infty}^2 = C h^{-n}$$

with

$$C = (2\pi)^{-2n} ||\psi_1||_{L^\infty}^2 ||\psi_2||_{L^\infty}^2 \int \int |\hat{a}(x,z)|^2 dx dz,$$

which yields (4.3) by application of Proposition 4.7. That the operator is trace class and the formula (4.4) follow from Proposition 4.12. □

We state a similar result for operators on M rather than $\mathbb{R}^n$.

Proposition 4.14. *For all $a \in C_0^\infty(T^*M)$, $Op_h(a)$ is Hilbert-Schmidt and trace class. Furthermore*

$$||Op_h(a)||_{\mathrm{HS}} \leq Ch^{-n/2}, \qquad h \in (0,1] \tag{4.5}$$

and

$$\mathrm{tr}_{L^2(M)}\big(Op_h(a)\big) = (2\pi h)^{-n} \int_{T^*M} a|dxd\xi|. \tag{4.6}$$

Proof. This result is of course a consequence of Proposition 4.13. Our purpose is to explain how to pass from $\mathbb{R}^n$ to M. Let us observe first that if $\kappa_l : U_l \subset M \to V_l \subset \mathbb{R}^n$ is a chart, the mapping

$$\mathcal{U}_l : L^2(U_l) \ni \varphi \mapsto |g_l|^{1/2}(\kappa_{l*}\varphi) \in L^2(V_l)$$

is unitary (here $|g_j|^{1/2}$ is a shorthand for for $|g(x)|^{1/2}$ - see (1.11) - in the l-th chart) since, by (1.12),

$$||\mathcal{U}_l\varphi||_{L^2(V_l,dx)} = ||\varphi||_{L^2(U_l,\mathrm{dvol}_g)}.$$

Furthermore, if B is an operator on $\mathbb{R}^n$ with kernel $K \in C_0^\infty(V_l \times V_l)$, then we can see B as an operator on $L^2(\mathbb{R}^n)$ or $L^2(V_l)$ (by restriction) and it is not hard to check that

$$||B||_{\mathrm{HS}(L^2(\mathbb{R}^n))} = ||B||_{\mathrm{HS}(L^2(V_l))}$$

by considering an othornormal basis of $L^2(\mathbb{R}^n)$ made of the union of an othonormal basis of $L^2(V_l)$ (made of functions supported in V) and of an orthonormal basis of the orthogonal complement $L^2(\mathbb{R}^n \setminus V_l)$, the contribution of which is zero in the Hilbert-Schmidt norm of B (on $L^2(\mathbb{R}^n)$) for all its elements belong to the null space of B. The same property holds for an operator A on $L^2(M)$ with a kernel supported on $U_l \times U_l$. As a first application, we consider the Hilbert-Schmidt norm of each operator $A_l := \big(\kappa_l^* op_h[\kappa_{l*}(\Psi_l a)]\kappa_{l*}\big)\Psi_l$ in the sum (3.15). One can see it as an operator on $L^2(U_l)$ and $L^2(M)$, and then one has

$$||A_l||_{\mathrm{HS}(L^2(M,\mathrm{dvol}_g))} = ||A_l||_{\mathrm{HS}(L^2(U_l,\mathrm{dvol}_g))} = ||\mathcal{U}_l A_l \mathcal{U}_l^*||_{L^2(V_l,dx)}.$$

Using that

$$\mathcal{U}_l A_l \mathcal{U}_l^* = |g_l|^{1/2}\psi_l op_h[(\kappa_{l*}a)]\psi_l|g_l|^{-1/2}$$

we see that (4.5) follows from (4.3). The same argument allows to show that A_l is of trace class since $\mathcal{U}_l A_l \mathcal{U}_l^*$ is trace class by Proposition 4.13 . Furthermore, using that $\mathcal{U}_l^*$ is unitary hence maps an orthonormal basis into an orthonormal basis, we also have

$$\begin{aligned}
\mathrm{tr}_{L^2(M,\mathrm{dvol}_g)}\big(A_l\big) = \mathrm{tr}_{L^2(U_l,\mathrm{dvol}_g)}(A_l) &= \mathrm{tr}_{L^2(V_l,dx)}\big(\mathcal{U}_l A_l \mathcal{U}_l^*\big) \\
&= (2\pi h)^{-n} \int\int \psi_l(x)^2(\kappa_{l*}a)(x,\xi)dxd\xi \\
&= (2\pi h)^{-n} \int_{T^*M} (\Psi_l \circ \pi_{T^*M})^2 a|dxd\xi|
\end{aligned}$$

the last two identities following from (4.4) and the definition of $|dxd\xi|$ on page 17. After summation over l, using (3.13), we get (4.6).

4.3 The local Weyl law

Definition 4.15. *For a given compact interval $J \subset \mathbb{R}$ we define*

$$\mathcal{N}_J(h) \quad = \quad \left\{ j \in \mathbb{N} \mid h^2 \lambda_j \in J \right\}$$

and

$$N_J(h) = \# \mathcal{N}_J(h).$$

One of the main results of this section is an asymptotic formula for $N_J(h)$. Before, proving it, we give first a rough estimate.

Proposition 4.16 (Rough bound on the eigenvalues distribution)**.** *For all compact interval $J \Subset \mathbb{R}$, there exists $C = C(J)$ such that*

$$N_J(h) \leq C h^{-n}, \tag{4.7}$$

for all $h \in (0, 1]$.

Proof. Let $f \in C_0^\infty(\mathbb{R})$ be equal to 1 on J. Then

$$N_J(h) = \sum_{j \in \mathcal{N}_J(h)} ||e_j||_{L^2}^2 = \sum_{j \in \mathcal{N}_J(h)} ||f(h^2 P) e_j||_{L^2}^2.$$

By (2.15) we have, for some C independent of h,

$$N_J(h) \quad \leq \quad \sum_{j \in \mathcal{N}_J(h)} \left|\left| Op_h(f \circ p) e_j \right|\right|_{L^2}^2 + C h N_J(h). \tag{4.8}$$

On the other hand, by the non-negativity of $Op_h(f \circ p)^* Op_h(f \circ p)$, we have

$$\begin{aligned} \sum_{j \in \mathcal{N}_J(h)} \left|\left| Op_h(f \circ p) e_j \right|\right|_{L^2}^2 \quad &= \quad \sum_{j \in \mathcal{N}_J(h)} \left\langle e_j, Op_h(f \circ p)^* Op_h(f \circ p) e_j \right\rangle_M \\ &\leq \quad \operatorname{tr}\big(Op_h(f \circ p)^* Op_h(f \circ p)\big) \;=\; \left|\left| Op_h(f \circ p) \right|\right|_{\mathrm{HS}}^2 \;=\; O(h^{-n}), \end{aligned}$$

the last estimate following from Proposition 4.14. Using (4.8), we obtain

$$(1 - Ch) N_J(h) \leq C h^{-n}.$$

This implies the result for $0 < h \leq h_0$ small enough. It also trivially true for $h \in [h_0, 1]$. This completes the proof. $\square$

A consequence of Proposition 4.16 is the following useful result.

Proposition 4.17. *For all $s > n/4$, $(1 - h^2 \Delta_g)^{-s}$ is Hilbert-Schmidt on $L^2(M)$ and one has the bound*

$$\left|\left| (1 - h^2 \Delta_g)^{-s} \right|\right|_{\mathrm{HS}} \leq C_s h^{-n/2},$$

for all $h \in (0, 1]$.

Proof. Write $\sum_j ||(1-h^2\Delta_g)^{-s}e_j||^2_{L^2(M)} = \sum (h^2\lambda_j+1)^{-2s}$ and decompose it as

$$\sum_{h^2\lambda_j<1}(h^2\lambda_j+1)^{-2s} + \sum_{k\geq 0}\left(\sum_{2^k\leq h^2\lambda_j<2^{k+1}}(h^2\lambda_j+1)^{-2s}\right).$$

By (4.7), this expression is of the form

$$O(h^{-n}) + \sum_{k\geq 0} O\big((2^{k/2}h^{-1})^n\big)O(2^{-2ks})$$

where, in the sum over k, the first O is for the number of terms in the sum over j (by (4.7) applied with $2^{-k/2}h$ instead of h) and the second O is for the terms themselves. Therefore

$$||(1-h^2\Delta_g)^{-s}||^2_{\mathrm{HS}} = O(h^{-n})\left(1+\sum_k 2^{(\frac{n}{2}-2s)k}\right)$$

the sum in the right hand side being convergent by the assumption on s. The result follows. □

Theorem 4.18 (Local Weyl's law)**.** *For all compact interval J and all $a\in C_0^\infty(T^*M)$,*

$$h^n\sum_{j\in\mathcal{N}_J(h)}\big\langle e_j, Op_h(a)e_j\big\rangle_M \longrightarrow (2\pi)^{-n}\int_{p^{-1}(J)} a\ |dxd\xi|,$$

as $h\to 0$.

Lemma 4.19. *Let U be a coordinate patch on M and $a\in C_0^\infty(T^*U)$ be real valued. Then there exist two real valued functions $b_1,b_2\in C_0^\infty(T^*U)$ such that*

$$a = b_2^2 - b_1^2.$$

Proof. Let $C>0$ such that $\sup|a|\leq C$. We can then choose $\chi\in C_0^\infty(T^*U)$ such that

$$\chi\equiv 1 \ \text{ in a neighborhood of } \operatorname{supp}(a).$$

If $\kappa: U\to V$ is a chart, this follows from the standard existence of cutoff functions on $V\times\mathbb{R}^n$. Write then

$$a = 2C\chi^2 + a - 2C\chi^2.$$

The function $2C\chi^2+a$ belongs to $C_0^\infty(T^*U)$. Our choice of C implies that it is non-negative. Therefore, its square root $b_2 := \sqrt{2C\chi^2+a}$ clearly belongs to $C_0^0(T^*U)$. Let us show it is smooth. On the open set $\operatorname{supp}(a)^c$, $b_2=\sqrt{2C}\chi$ which is smooth. On the the other hand, in a neighborhood of $\operatorname{supp}(a)$, we have

$$2C\chi^2 + a = 2C - a \geq C > 0$$

hence b_2 is smooth on this neighborhood. Therefore, b_2 is smooth on T^*M. Finally, we obviously have $-2C\chi^2 = -b_1^2$ with $b_1 := \sqrt{2C}\chi\in C_0^\infty(T^*U)$ so the result follows. □

Lemma 4.20. *Let $b \in C_0^\infty(T^*M)$ be real valued and set $B_h = Op_h(b)$. Let $f \in C_0^\infty(\mathbb{R})$. Then*

$$\left|\mathrm{tr}\big(B_h^* B_h f(-h^2\Delta_g)\big) - \mathrm{tr}\big(Op_h(b^2(f\circ p))\big)\right| \leq Ch^{1-n}$$

Proof. Let us write $B_h^* B_h f(-h^2\Delta_g) - Op_h(b^2(f\circ p))$ as

$$\big(B_h^* B_h f(-h^2\Delta_g) - Op_h(b^2(f\circ p))\big)\,(1-h^2\Delta_g)^n(1-h^2\Delta_g)^{-n}.$$

By Proposition 4.17, $(1-h^2\Delta_g)^{-n} = (1-h^2\Delta_g)^{-n/2}(1-h^2\Delta_g)^{-n/2}$ is a composition of two Hilbert-Schmidt operators whose product of Hilbert-Schmidt norms is $O(h^{-n})$. Therefore, it suffices to show that

$$\big|\big|\big(B_h^* B_h f(-h^2\Delta_g) - Op_h(b^2(f\circ p))\big)(1-h^2\Delta_g)^n\big|\big|_{\mathcal{L}_c(L^2)} \leq Ch.$$

On one hand, we have

$$B_h^* B_h f(-h^2\Delta_g)(1-h^2\Delta_g)^n = B_h^* B_h \tilde{f}(-h^2\Delta_g) \qquad \text{with} \;\; \tilde{f}(\lambda) = (\lambda^2+1)^n f(\lambda).$$

Moreover, using (2.11), (2.13), (2.12) and (2.15),

$$B_h^* B_h \tilde{f}(-h^2\Delta_g) = Op_h(b^2\tilde{f}\circ p) + O_{\mathcal{L}_c(L^2)}(h).$$

On the other hand, using Proposition 3.22, we have

$$\begin{aligned} Op_h(b^2(f\circ p))(1-h^2\Delta_g)^n &= Op_h(b^2(f\circ p)(p+1)^n) + O_{\mathcal{L}_c(L^2)}(h) \\ &= Op_h(b^2(\tilde{f}\circ p)) + O_{\mathcal{L}_c(L^2)}(h). \end{aligned}$$

The result follows. $\square$

Proof of Theorem 4.18. By linearity and a partition of unity argument, we may assume that a is supported in T^*U for some coordinate patch U. By linearity again, we may also assume that a is real valued and then, by Lemma 4.19, that $a = b^2$ for some $b \in C_0^\infty(T^*M)$. For simplicity, we write

$$A_h := Op_h(a), \qquad B_h := Op_h(b).$$

Then

$$\begin{aligned} \sum_{j\in\mathcal{N}_J(h)} \langle e_j, A_h e_j\rangle &= \mathrm{tr}\big(Op_h(b^2)\mathbb{1}_J(-h^2\Delta_g)\big) \\ &= \mathrm{tr}\big(B_h^* B_h \mathbb{1}_J(-h^2\Delta_g)\big) + O(h^{1-n}) \end{aligned}$$

using that

$$\begin{aligned} \left|\mathrm{tr}\left(\big(Op_h(b^2) - B_h^* B_h\big)\mathbb{1}_J(-h^2\Delta_g)\big)\right)\right| &\leq ||B_h^* B_h - Op_h(b^2)||_{\mathcal{L}_c(L^2)} ||\mathbb{1}_J(-h^2\Delta_g))||_{\mathrm{HS}}^2 \\ &\leq Ch^{1-n}. \end{aligned}$$

Fix next $f_1, f_2 \in C_0^\infty(\mathbb{R})$ such that

$$f_1 \leq \mathbb{1}_J \leq f_2.$$

Note that if J has empty interior (i.e. is reduced to a point) then $f_1 \equiv 0$. Then, by non-negativity of

$$\langle B_h^* B_h e_j, e_j \rangle = ||B_h e_j||_{L^2}^2$$

for all j, we have

$$\mathrm{tr}\big(B_h^* B_h f_1(-h^2\Delta_g)\big) \leq \mathrm{tr}\big(B_h^* B_h \mathbb{1}_J(-h^2\Delta_g)\big) \leq \mathrm{tr}\big(B_h^* B_h f_2(-h^2\Delta_g)\big).$$

By Lemma 4.20, we have

$$\mathrm{tr}\big(B_h^* B_h f_k(-h^2\Delta_g)\big) = \mathrm{tr}\big(Op_h(b^2 f_k \circ p)\big) + O(h^{1-n}), \qquad k = 1, 2,$$

hence, using (4.6), we have

$$h^n \mathrm{tr}\big(B_h^* B_h f_k(-h^2\Delta_g)\big) \longrightarrow (2\pi)^{-n} \int_{T^*M} b^2 f_k \circ p \, |dx d\xi|, \qquad h \to 0.$$

Therefore

$$(2\pi)^{-n} \int_{T^*M} a f_1 \circ p \, |dx d\xi| \leq \liminf_{h\to 0} \left(h^n \mathrm{tr}\big(A_h \mathbb{1}_J(-h^2\Delta_g)\big)\right)$$

and

$$\liminf_{h\to 0} \left(h^n \mathrm{tr}\big(A_h \mathbb{1}_J(-h^2\Delta_g)\big)\right) \leq (2\pi)^{-n} \int_{T^*M} a f_2 \circ p \, |dx d\xi|.$$

By approximating $\mathbb{1}_J$ by smooth functions (see Exercise 4.21 below), one can choose,for any $\epsilon > 0$, the functions f_1 and f_2 such that

$$\left| \int_{T^*M} a f_k \circ p \, |dx d\xi| - \int_{p^{-1}(J)} a \, |dx d\xi| \right| \leq \epsilon, \qquad k = 1, 2.$$

For all $\epsilon > 0$, we thus have the inequalities

$$(2\pi)^{-n} \int_{p^{-1}(J)} a \, |dx d\xi| - \epsilon \leq \liminf_{h\to 0} h^n \mathrm{tr}_{L^2(M)}\big(A_h \mathbb{1}_J(h^2 P)\big)$$

and

$$\limsup_{h\to 0} h^n \mathrm{tr}_{L^2(M)}\big(A_h \mathbb{1}_J(-h^2\Delta_g)\big) \leq (2\pi)^{-n} \int_{p-1(J)} a \, |dx d\xi| + \epsilon.$$

Since ϵ is arbitrary, we can let it go to zero and see that the liminf and the limsup are equal. This yields the result. $\square$

Exercise 4.21. *Let $J = [E_1, E_2]$ be a compact interval.*

1. *Let* $(f_{2,k})_{k\geq 1}$ *be a sequence of functions in* C_0^∞ *such that*

$$0 \leq f_{2,k} \leq 1, \qquad supp(f_{2,k}) \subset [E_1 - 1/k, E_2 + 1/k], \qquad f_{2,k} \equiv 1 \quad on \quad [E_1, E_2].$$

   Check that, as $k \to \infty$,

$$f_{2,k} \to \mathbb{1}_J \quad in\ L^1(\mathbb{R}), \qquad f_{2,k} \circ p \to \mathbb{1}_{p^{-1}(J)} \quad in\ L^1(T^*M, |dxd\xi|).$$

2. *Construct similarly a sequence* $f_{1,k}$ *in* $C_0^\infty(\mathbb{R})$ *approaching* $\mathbb{1}_J$ *from below and such that* $f_{1,k} \circ p \to \mathbb{1}_{p^{-1}(J)}$ *in* $L^1(T^*M, |dxd\xi|)$.

Hint: You may wish to use that $p^{-1}(\{E_1\})$ *and* $p^{-1}(\{E_2\})$ *have zero* $|dxd\xi|$ *measure.*

Chapter 5

Approximate functional calculus

In this part, we give a proof of the property (2.15). We give it in a separate chapter for it requires more than the pseudo-differential calculus with C_0^∞ symbols. However, it remains elementary enough to avoid using a general pseudo-differential calculus.

Let us introduce the following definition.

Definition 5.1 (Semiclassical resolvent of the Laplacian). *For $z \in \mathbb{C} \setminus \mathbb{R}$ and $h \in (0,1]$ we define*

$$(-h^2\Delta_g - z)^{-1} := f_z(-h^2\Delta_g),$$

with $f_z(\lambda) = (\lambda - z)^{-1}$, using (2.3).

Notice that, by (2.5), one has in particular

$$\begin{aligned} \left|\left|(-h^2\Delta_g - z)^{-1}\right|\right|_{\mathcal{L}_c(L^2)} &\leq ||f_z||_{L^\infty(\mathbb{R}_\lambda)} \\ &\leq \frac{1}{|\mathrm{Im}(z)|}. \end{aligned} \tag{5.1}$$

Exercise 5.2. *Show that the map $z \mapsto (-h^2\Delta_g - z)^{-1}$ is continuous from $\mathbb{C} \setminus \mathbb{R}$ to $\mathcal{L}_c(L^2)$. Hint. Use (and prove) that*

$$(-h^2\Delta_g - z)^{-1} - (-h^2\Delta_g - \zeta)^{-1} = -(z - \zeta)(-h^2\Delta_g - z)^{-1}(-h^2\Delta_g - \zeta)^{-1}.$$

The relationship between the semiclassical resolvent and C_0^∞ functions of $-h^2\Delta_g$ is given by the so called Helffer-Sjöstrand formula. To state it, we need another definition.

Definition 5.3. *Let $f \in C_0^\infty(\mathbb{R})$. A function $\widetilde{f} \in C_0^\infty(\mathbb{R}^2)$ is an* **almost analytic extension** *of f if*

1. *for all $N \geq 0$, $\overline{\partial}\widetilde{f}(x,y) = O(|y|^N)$, where $\overline{\partial} = \partial_x + \mathrm{i}\partial_y$,*
2. *$\widetilde{f}(x,0) = f(x)$.*

Note that the first condition is a condition at $y = 0$, that is on the real axis, saying that $\overline{\partial} \tilde{f}$ vanishes to infinite order at $y = 0$.

The next exercise gives a simple explicit way to construct almost analytic extensions.

Exercise 5.4. *Let $f \in C_0^\infty(\mathbb{R})$. Let $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R})$ such that*

$$\chi_1 \equiv 1 \quad \textit{near the support of } f, \qquad \chi_2 \equiv 1 \quad \textit{near } 0.$$

Check that

$$\widetilde{f}(x, y) := \chi_1(x)\chi_2(y)\frac{1}{2\pi}\int e^{i(x+iy)\xi}\chi_2(y\xi)\widehat{f}(\xi)d\xi$$

is an almost analytic extension of f.

In the sequel, for a continuous function $B(x, y)$ defined on $\mathbb{R}^2 \setminus \{y = 0\}$, or equivalently on $\mathbb{C} \setminus \mathbb{R}$, with values in a Banach space, we shall denote

$$\int_{|\mathrm{Im} z| \geq \epsilon} \overline{\partial}\widetilde{f}(z)B(z)L(dz) := \int_{|y|\geq\epsilon}\left(\int_{\mathbb{R}} \overline{\partial}\widetilde{f}(x, y)B(x, y)dx\right)dy, \qquad \epsilon > 0,$$

and

$$\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)B(z)L(dz) := \lim_{\epsilon\to 0}\int_{|\mathrm{Im} z|\geq\epsilon} \overline{\partial}\widetilde{f}(z)B(z)L(dz), \tag{5.2}$$

when the limit exists.

The main interest of almost analytic extensions is the following Cauchy type formula.

Proposition 5.5. *Let $f \in C_0^\infty(\mathbb{R})$ and $\widetilde{f} \in C_0^\infty(\mathbb{R}^2)$ be an almost analytic extension of f. Then, for all $\lambda \in \mathbb{R}$,*

$$f(\lambda) = \frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(\lambda - z)^{-1}L(dz). \tag{5.3}$$

Corollary 5.6 (Helffer-Sjöstrand formula)**.** *Let $f \in C_0^\infty(\mathbb{R})$ and $\widetilde{f}$ be an almost analytic extension of f. Then*

$$f(-h^2\Delta_g) = \frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(-h^2\Delta_g - z)^{-1}L(dz). \tag{5.4}$$

This corollary follows directly from Proposition 5.5 and the spectral theorem or, more precisely, the definition (2.3). We only note that the convergence of the integral follows from (5.1), which allows to bound the resolvent norm by $|\mathrm{Im}(z)|^{-1}$, and from the fact that $\bar{\partial}\tilde{f}(z) = O(|\mathrm{Im}(z)|)$. We let the interested reader work out the details in exercise (we also refer to [1]).

Proof of Proposition 5.5. For fixed ϵ, we integrate by part with $\overline{\partial}$ and use that $\overline{\partial}(\lambda - z)^{-1} = 0$ to get

$$\int_{|\mathrm{Im} z| \geq \epsilon} \overline{\partial} \widetilde{f}(z)(\lambda - z)^{-1} L(dz) = i \int_{\mathbb{R}} \widetilde{f}(x, -\epsilon) \frac{1}{\lambda - x + i\epsilon} - \widetilde{f}(x, \epsilon) \frac{1}{\lambda - x - i\epsilon} dx. \tag{5.5}$$

Then, using

$$\widetilde{f}(x, \pm\epsilon) = f(x) \pm \epsilon \partial_y \widetilde{f}(x, 0) + O(\epsilon^2 \langle x \rangle^{-2})$$

and the fact that

$$\frac{1}{|\lambda - x \pm i\epsilon|} \leq \frac{1}{\epsilon},$$

the right hand side of (5.5) can be written

$$2\epsilon \int_{\mathbb{R}} f(x) \frac{dx}{(x - \lambda)^2 + \epsilon^2} + i\epsilon \int_{\mathbb{R}} \partial_y \widetilde{f}(x, 0) \frac{2(x - \lambda)}{(x - \lambda)^2 + \epsilon^2} dx + O(\epsilon)$$

that is,

$$2 \int_{\mathbb{R}} f(\lambda + \epsilon t) \frac{dt}{1 + t^2} - i\epsilon \int_{\mathbb{R}} \partial_x \partial_y \widetilde{f}(\lambda + s, 0) \ln(s^2 + \epsilon^2) ds + O(\epsilon). \tag{5.6}$$

By dominated convergence, using that $\big| \ln(s^2 + \epsilon^2) \big| \lesssim \max \big(|\ln \epsilon|, 1 + |s| \big)$, (5.6) converges to $2\pi f(\lambda)$ as ϵ goes to zero. $\square$

The Helffer-Sjöstrand formula suggests that to get (2.15), which is a pseudo-differential approximation of $f(-h^2 \Delta_g)$, it suffices to find a pseudo-differential approximation of the semiclassical resolvent. This is what we check in the rest of this appendix. The proof of (2.15) itself is given at the end.

In the following definition, we use the notation $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Note also that below x belongs to $\mathbb{R}^n$ and has nothing to do with the notation x used above for $\mathrm{Re}(z)$.

Definition 5.7. *For $\mu \in \mathbb{R}$, $S^\mu = S^\mu(\mathbb{R}^{2n})$ is the space of smooth functions $a : \mathbb{R}^{2n} \to \mathbb{C}$ such that, for all multi-indices α, β*

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{a\alpha\beta} \langle \xi \rangle^{\mu - |\beta|},$$

i.e. equivalently such that $||\langle \xi \rangle^{-\mu + |\beta|} \partial_x^\alpha \partial_\xi^\beta a||_{L^\infty} < \infty$. Here and below $L^\infty = L^\infty(\mathbb{R}^{2n})$.

For such symbols, one can define $op_h[a]u$ for all u in $C_0^\infty(\mathbb{R}^n)$ (or in the Schwartz space) exactly as in (3.1), since $\hat{u}$ decays fast in ξ while $a(x, h\xi)$ grows at most polynomially. Note that constant functions belong to S^0 and, as already observed in Section 3.1, that the Fourier inversion formula gives directly

$$op_h[1] = I. \tag{5.7}$$

Apart from the Helffer-Sjöstrand formula, the main property we will use in this appendix is the following proposition, or rather Corollary 5.9 below.

Proposition 5.8. *Let $a \in S^{-m}(\mathbb{R}^n \times \mathbb{R}^n)$ with $0 < m < n$. Then $op_h[a]$ has a locally integrable integral kernel $K_{a,h}(x,y)$ such that,*

$$|K_{a,h}(x,y)| \leq C_a h^{-n} \left| \frac{x-y}{h} \right|^{m-n} \left(1 + \left| \frac{x-y}{h} \right| \right)^{-2n}, \tag{5.8}$$

with a constant C_a of the form

$$C_{n,m} \max_{|\beta| \leq N_{n,m}} \left| \left| \langle \xi \rangle^{m+|\beta|} \partial_\xi^\beta a \right| \right|_{L^\infty}. \tag{5.9}$$

Proposition 5.8 together with the Schur test (Lemma 3.6) imply directly

Corollary 5.9. *If $a \in S^{-m}$ with $0 < m < n$, then $op_h[a]$ is bounded on $L^2(\mathbb{R}^n)$ with an operator norm that can be estimated by (5.9) .*

Note that we consider only the case when $-n < -m < 0$ for this is the only one we are going to use, but the L^2 boundedness holds for other classes of symbols. In particular, any symbol of order $\mu \leq -n$ belongs to all spaces S^{-m} with $m \geq n$ hence the corresponding operator is also bounded on $L^2(\mathbb{R}^n)$.

To prove Proposition 5.8, we need the following lemma.

Lemma 5.10. *Let $\nu, N > 0$ be positive real numbers such that $N > \nu$. Then, there exists $C = C_{\nu N}$ such that, for all $\lambda > 0$,*

$$\sum_{k \geq 0} 2^{k\nu} \left(1 + \frac{\lambda}{2^{-k}} \right)^{-N} \leq C \frac{\lambda^{-\nu}}{(1+\lambda)^{N-\nu}}.$$

Proof. The estimate is equivalent to the fact that the sum is $O(\lambda^{-\nu})$ if $0 < \lambda \leq 1$ and $O(\lambda^{-N})$ if $\lambda > 1$. We thus prove these two estimates. Using that

$$\left(1 + \frac{\lambda}{2^{-k}} \right)^{-N} \leq \frac{2^{-kN}}{\lambda^N}$$

the sum is bounded by $\lambda^{-N} \sum_k 2^{-k(N-\nu)} = O(\lambda^{-N})$, which yields the result if $\lambda > 1$. If $0 < \lambda \leq 1$, we let k_0 be the integer part of $-\log \lambda / \log 2$ so that

$$2^{-k_0 - 1} < \lambda \leq 2^{-k_0}.$$

Then

$$(3/2)^{-N} \leq \left(1 + \frac{\lambda}{2^{-k}} \right)^{-N} \leq 1 \quad \text{if } k \leq k_0, \qquad \left(1 + \frac{\lambda}{2^{-k}} \right)^{-N} \leq 2^{-N(k-k_0-1)} \text{ if } k > k_0$$

so that

$$\sum_{k=0}^{k_0} 2^{k\nu} \left(1 + \frac{\lambda}{2^{-k}} \right)^{-N} \leq \sum_{k \leq k_0} 2^{k\nu} \lesssim 2^{k_0 \nu} \lesssim \lambda^{-\nu}$$

and

$$\sum_{k>k_0} 2^{k\nu}\left(1+\frac{\lambda}{2^{-k}}\right)^{-N} \leq \sum_{k>k_0} 2^{k\nu}2^{-N(k-k_0-1)} = 2^{\nu(k_0+1)}\sum_{k>k_0} 2^{-(N-\nu)(k-k_0-1)} \lesssim \lambda^{-\nu}.$$

This yields the result when $0<\lambda\leq 1$. $\square$

Exercise 5.11 (Dyadic partition of unity)**.** *Let $\chi_0 \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi_0(\eta)\equiv 1$ near $\eta=0$. Let $\chi(\eta):=\chi_0(\eta/2)-\chi_0(\eta)$. Check that $\chi\in C_0^\infty(\mathbb{R}^n\setminus 0)$ and that*

$$\chi_0(\xi)+\sum_{k=0}^{N-1}\chi(2^{-k}\xi)=\chi_0(2^{-N}\xi)\to 1. \tag{5.10}$$

as $N\to\infty$

Proof of Proposition 5.8. Formally, the Schwartz kernel of $op_h[a]$ reads

$$(2\pi h)^{-n}\int e^{\mathrm{i}\frac{x-y}{h}\cdot\xi}a(x,\xi)d\xi.$$

See for instance (3.2). The problem here is that a is not integrable in ξ. Using Exercise 5.11, this kernel can be rigorously written as the limit as $N\to\infty$, in the distributions sense, of

$$(2\pi h)^{-n}\int e^{\mathrm{i}\frac{x-y}{h}\cdot\xi}a(x,\xi)\chi_0(2^{-N}\xi)d\xi = h^{-n}\left(K_h(x,y)+\sum_{k=0}^{N-1}K_h^{(k)}(x,y)\right) \tag{5.11}$$

where, replacing $\frac{x-y}{h}$ by X, we have set

$$\begin{aligned}
K_h^{(k)}(x,y) &= (2\pi)^{-n}\int e^{\mathrm{i}X\cdot\xi}a(x,\xi)\chi(2^{-k}\xi)d\xi\\
&= 2^{nk}(2\pi)^{-n}\int e^{\mathrm{i}2^kX\cdot\eta}a(x,2^k\eta)\chi(\eta)d\eta\\
K_h(x,y) &= (2\pi)^{-n}\int e^{\mathrm{i}X\cdot\xi}a(x,\xi)\chi_0(\xi)d\xi.
\end{aligned}$$

We show that the right hand side of (5.11) converges in $L^1_{\mathrm{loc}}(\mathbb{R}^{2n})$. Proceeding exactly as for (3.5), we find that for any N,

$$\left|K_h^{(k)}(x,y)\right| \lesssim_N 2^{nk}\left(1+\frac{|X|}{2^{-k}}\right)^{-2N}\sup_\eta\left|(1-\Delta)^N\big(a(x,2^k\eta)\chi(\eta)\big)\right|$$

where the sup is bounded by a constant times $2^{-mk}\max_{|\beta|\leq 2N}||\langle\xi\rangle^{m+|\beta|}\partial_\xi^\beta a||_{L^\infty}$ since

$$
\begin{aligned}
\left|(1-\Delta_\eta)^N\big(a(x,2^k\eta)\chi(\eta)\big)\right| &\lesssim \sum_{|\beta|+|\alpha|=2N} \left|2^{k|\beta|}(\partial_\xi^\beta a)(x,2^k\eta)\right|\left|(\partial^\alpha\chi)(\eta)\right| \\
&\lesssim 2^{-mk}\sum_{|\beta|+|\alpha|=2N} 2^{k(|\beta|+m)}|(\partial_\xi^\beta a)(x,2^k\eta)|\left|(\partial^\alpha\chi)(\eta)\right| \\
&\lesssim 2^{-mk}\sum_{|\beta|+|\alpha|=2N} \langle 2^k\eta\rangle^{|\beta|+m}|(\partial_\xi^\beta a)(x,2^k\eta)|\left|(\partial^\alpha\chi)(\eta)\right|
\end{aligned}
$$

using that $|\eta|$ is bounded from below on the support of χ. Similarly,

$$
\begin{aligned}
\left|\int e^{iX\cdot\xi}a(x,\xi)\chi_0(\xi)d\xi\right| &\lesssim_N \left(1+\left|\frac{x-y}{h}\right|\right)^{-N} \\
&\lesssim \left|\frac{x-y}{h}\right|^{m-n}\left(1+\left|\frac{x-y}{h}\right|\right)^{-2n}
\end{aligned}
$$

by choosing $N\geq 3n-m$ and with an implicit multiplicative constant of the form (5.9). By Lemma 5.10, we thus see that, if $x\neq y$

$$
h^{-n}\left(|K_h(x,y)|+\sum_{k\geq 0}\left|K_h^{(k)}(x,y)\right|\right)\leq C_a h^{-n}\left|\frac{x-y}{h}\right|^{m-n}\left(1+\left|\frac{x-y}{h}\right|\right)^{-2n}.
$$

By dominated convergence, this implies the convergence of (5.11) in $L^1_{\mathrm{loc}}(\mathbb{R}^{2n})$ and then the estimate (5.8). $\square$

Our approximation of the semiclassical resolvent is the following one.

Proposition 5.12. *One can write*

$$
(-h^2\Delta_g - z)^{-1} = Op_h\big((p-z)^{-1}\big) + hR(h,z)
$$

where, for some $C>0$ and $M>0$,

$$
||R(h,z)||_{\mathcal{L}_c(L^2)}\leq C\left(\frac{1+|z|}{|\mathrm{Im}(z)|}\right)^M \tag{5.12}
$$

for all $h\in(0,1]$ and $z\in\mathbb{C}\setminus\mathbb{R}$.

We point out that in this statement we take Op_h of $(p-z)^{-1}$ which is not a C_0^∞ function on T^*M (it is smooth but not compactly supported) that is not as in Definition 3.14. We shall see in the proof below that this does not cause any problem and only uses that op_h can be defined on symbols in S^μ.

Exercise 5.13. *We let $p_0(x,\xi)=\sum g^{jk}(x)\xi_j\xi_k$ as in Section 3.3.*

1. *Show that, for any multi-index $\gamma \neq 0$ in $\mathbb{N}^{2n}$,*

$$\partial^\gamma \left(\frac{1}{p_0 - z} \right) = \text{ linear combination of } \frac{\partial^{\gamma_1} p_0 \cdots \partial^{\gamma_j} p_0}{(p_0 - z)^{1+j}},$$

with $1 \leq j \leq |\gamma|$, $\gamma_1 + \cdots + \gamma_j = \gamma$ and $\gamma_1, \ldots, \gamma_j \neq 0$.

2. *Show that for any compact subset $K \Subset V$, there exists C_K such that*

$$\left| \frac{\langle \xi \rangle^2}{p_0(x,\xi) - z} \right| \leq C_K \frac{1 + |z|}{|\mathrm{Im}(z)|}$$

for all $x \in K$, $\xi \in \mathbb{R}^n$ and $z \in \mathbb{C} \setminus \mathbb{R}$.
Hint: write $\frac{\langle \xi \rangle^2}{p_0 - z} = \frac{\langle \xi \rangle^2}{p_0 + 1} \frac{p_0 + 1}{p_0 - z}$ and use that $p_0(x,\xi) \geq c_K |\xi|^2$ for some $c_K > 0$ since the matrix $(g^{jk}(x))$ is positive definite for each x.

Proof of Proposition 5.12. We use the same notation as in the beginning of Section 3.3, for P, p_0, p_1, ψ, etc... We start by observing that the computation in Proposition 3.19 is still valid if the symbol is not compactly supported in ξ. In particular, we find that

$$\begin{aligned} (h^2 P - z) op_h \left[\frac{\psi}{p_0 - z} \right] \psi &= \big(op_h[\psi] + h op_h[q_{1,z}] + h^2 op_h[q_{2,z}] \big) \psi \\ &= \psi^2 + \big(h op_h[q_{1,z}] + h^2 op_h[q_{2,z}] \big) \psi \end{aligned} \tag{5.13}$$

where the function ψ to the right of the operators has to be understood as a multiplication operator and where

$$q_{1,z} = -\mathrm{i} \frac{p_1 \psi}{p_0 - z} - \mathrm{i} \partial_\xi p_0 \cdot \partial_x \left(\frac{\psi}{p_0 - z} \right), \qquad q_{2,z} = P \left(\psi (p_0 - z)^{-1} \right).$$

Note that to get the first term in (5.13), we have used that $op_h[\psi] = \psi op_h[1] = \psi$ by (5.7). Now, using Exercise 5.13, we observe that $q_{1,z} \in S^{-1}$ and $q_{2,z} \in S^{-2}$ (hence also belongs to S^{-1}) and that, for any N, there exist M and C such that, for $j = 1, 2$,

$$\max_{|\beta| \leq N} || \langle \xi \rangle^{1 + |\beta|} \partial_\xi^\beta q_{j,z} ||_{L^\infty} \leq C \left(\frac{1 + |z|}{|\mathrm{Im}(z)|} \right)^M .$$

Therefore, using Proposition 5.8 and (5.13) , we find that for some M,

$$(h^2 P - z) op_h \left[\frac{\psi}{p_0 - z} \right] \psi = \psi^2 + O_{\mathcal{L}_c(L^2(\mathbb{R}^n))} \left(h \left(\frac{1 + |z|}{|\mathrm{Im}(z)|} \right)^M \right).$$

Using this result with $\psi = \psi_l$ defined in (3.14) and pulling back the above identity on M, we find after summation over l,

$$(-h^2 \Delta_g - z) Op_h \big((p - z)^{-1} \big) = I + O_{\mathcal{L}_c(L^2(M))} \left(h \left(\frac{1 + |z|}{|\mathrm{Im}(z)|} \right)^M \right).$$

Applying $(-h^2\Delta_g - z)^{-1}$ to the left of the operators in both sides of this equality and using (5.1), we get the result. $\square$

Proof of the Property (2.15). We let $\tilde{f}$ be an almost analytic extension of f. We observe first that by linearity of Op_h and Proposition 5.5, we have

$$\begin{aligned} \frac{1}{2\pi}\int \bar{\partial}\tilde{f}(z) Op_h\big((p-z)^{-1}\big)L(dz) &= Op_h\left(\frac{1}{2\pi}\int \bar{\partial}\tilde{f}(z)(p-z)^{-1}L(dz)\right) \\ &= Op_h(f\circ p). \end{aligned}$$

Thus, using the Helffer-Sjöstrand formula and Proposition 5.12, we have

$$\begin{aligned} \big|\big|f(-h^2\Delta_g) - Op_h(f\circ p)\big|\big|_{\mathcal{L}_c(L^2)} &\leq h\int |\bar{\partial}\tilde{f}(z)|\big|\big|R(h,z)\big|\big|_{\mathcal{L}_c(L^2)} L(dz) \\ &\leq Ch \end{aligned}$$

since the integral can be bounded by

$$C\int_{\mathrm{supp}(\tilde{f})} |\mathrm{Im}(z)|^M \frac{1}{|\mathrm{Im}(z)|^M} L(dz)$$

using (5.12) and the fact that $\bar{\partial}\tilde{f}$ vanishes to infinite order on $\mathrm{Im}(z) = 0$. This completes the proof of (2.15) when $\psi = 1$. The general case follows from the easily verified fact that $Op_h(\psi f\circ p) = \psi Op_h(f\circ p)$. $\square$

Comment. In the proof of (2.15), we have deliberately omitted to specify in which sense the integrals are taken, in order stress on the simplicity of the argument. The scrupulous reader whishing to rewrite the proof in full details can take the integrals in (5.14) and in the Helffer-Sjöstrand formula in the weak sense (i.e. by testing the integrands against elements of $L^2(M)$).

Appendix A

Partition of unity

Proposition A.1 (Partition of unity)**.** *Assume that M is a smooth compact manifold and that we are given a finite open cover of M,*

$$M = \bigcup_{i=1}^{N} W_i, \qquad W_i \text{ open subset of } M.$$

Then there exist $\theta_i \in C_0^\infty(W_i)$, $i = 1, \dots, N$, such that

$$1 = \sum_{i=1}^{N} \theta_i \qquad \text{on } M.$$

Furthermore, each θ_i can be taken of the form $\theta_i = \varphi_i^2$ for some $\varphi_i \in C_0^\infty(W_i)$.

A useful application of this result is that, when $W_1, \dots, W_N$ are coordinate patches, each smooth function f on M can be written $\sum_i \theta_i f$, ie as a (finite) sum of functions supported in coordinate patches.

Proof. Since each W_i is open, for any $m \in W_i$ we can choose open subsets U_m^i and $\widetilde{U}_m^i$ contained in a coordinate patch at m such that

$$m \in U_m^i \Subset \widetilde{U}_m^i \subset W_i. \tag{A.1}$$

Then, by compactness, we obtain a finite open cover of M

$$M = \bigcup_{i=1}^{N} \left(U_{m_1^i}^i \cup \dots \cup U_{m_{n_i}^i}^i \right). \tag{A.2}$$

By pulling back cutoffs on $\mathbb{R}^n$, we can select

$$\phi_{m_k^i}^i \in C_0^\infty\big(\widetilde{U}_{m_k^i}^i\big) \qquad \text{such that} \qquad \phi_{m_k^i}^i = 1 \ \text{ on } U_{m_k^i}^i \qquad \text{and} \qquad \phi_{m_k^i}^i \geq 0 \ \text{ on } M. \tag{A.3}$$

We then introduce

$$\Phi_i = \left(\sum_{k=1}^{n_i} \phi^i_{m^i_k}\right)^2, \qquad \Phi = \sum_{i=1}^{N} \Phi_i.$$

Clearly Φ_i belongs to $C_0^\infty(W_i)$ by (A.1) and (A.3). By (A.3), we also have $\Phi_i \geq 1$ on each $U^i_{m^i_k}$ hence on their union (over k). Therefore (A.2) implies that $\Phi \geq 1$ on M so that we can define

$$\varphi_i = \frac{1}{\sqrt{\Phi}} \sum_{k=1}^{n_i} \phi^i_{m^i_k} \in C_0^\infty(W_i),$$

which obviously satisfies $\sum_{i=1}^{N} \varphi_i^2 = 1$. $\square$

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