Strichartz estimates and the Isozaki-Kitada parametrix on asymptotically hyperbolic manifolds

Jean-Marc Bouclet
Toulouse 3 University

Tokyo,
2010-04-13

Strichartz estimates

Let $(\mathcal{M}^n, G)$ be a riemannian manifold

- Laplace operator Δ_G,
- riemannian measure $dG = \det G(x)^{1/2} dx$.

We are interested in the unitary group

$$e^{it\Delta_G} : L^2(\mathcal{M}, dG) \to L^2(\mathcal{M}, dG),$$

which solves the Schrödinger equation

$$i\partial_t u + \Delta_G u = 0, \qquad u_{|t=0} = u_0,$$

ie

$$u(t) = e^{it\Delta_G} u_0.$$

Strichartz estimates

Let $(\mathcal{M}^n, G)$ be a riemannian manifold

- Laplace operator Δ_G,
- riemannian measure $dG = \det G(x)^{1/2} dx$.

We are interested in the unitary group

$$e^{it\Delta_G} : L^2(\mathcal{M}, dG) \to L^2(\mathcal{M}, dG),$$

which solves the Schrödinger equation

$$i\partial_t u + \Delta_G u = 0, \qquad u_{|t=0} = u_0,$$

ie

$$u(t) = e^{it\Delta_G} u_0.$$

Strichartz estimates

Let $(\mathcal{M}^n, G)$ be a riemannian manifold

- Laplace operator Δ_G,
- riemannian measure $dG = \det G(x)^{1/2} dx$.

We are interested in the unitary group

$$e^{it\Delta_G} : L^2(\mathcal{M}, dG) \to L^2(\mathcal{M}, dG),$$

which solves the Schrödinger equation

$$i\partial_t u + \Delta_G u = 0, \qquad u_{|t=0} = u_0,$$

ie

$$u(t) = e^{it\Delta_G} u_0.$$

Strichartz estimates

Let $(\mathcal{M}^n, G)$ be a riemannian manifold

- Laplace operator Δ_G,
- riemannian measure $dG = \det G(x)^{1/2} dx$.

We are interested in the unitary group

$$e^{it\Delta_G} : L^2(\mathcal{M}, dG) \to L^2(\mathcal{M}, dG),$$

which solves the Schrödinger equation

$$i\partial_t u + \Delta_G u = 0, \qquad u_{|t=0} = u_0,$$

ie

$$u(t) = e^{it\Delta_G} u_0.$$

Strichartz estimates

Let $(\mathcal{M}^n, G)$ be a riemannian manifold

- Laplace operator Δ_G,
- riemannian measure $dG = \det G(x)^{1/2} dx$.

We are interested in the unitary group

$$e^{it\Delta_G} : L^2(\mathcal{M}, dG) \to L^2(\mathcal{M}, dG),$$

which solves the Schrödinger equation

$$i\partial_t u + \Delta_G u = 0, \qquad u_{|t=0} = u_0,$$

ie

$$u(t) = e^{it\Delta_G} u_0.$$

Strichartz estimates

Question: How to guarantee that

$$u(t) \in L^q(\mathcal{M}, dG)$$

for (some) $q > 2$?

Motivation: non linear equations.

Strichartz estimates

Question: How to guarantee that

$$u(t) \in L^q(\mathcal{M}, dG)$$

for (some) $q > 2$?

Motivation: non linear equations.

Strichartz estimates

Question: How to guarantee that

$$u(t) \in L^q(\mathcal{M}, dG)$$

for (some) $q > 2$?

Motivation: non linear equations.

Strichartz estimates

1st possible answer: prove **Sobolev embeddings**

$$||v||_{L^q(\mathcal{M},dG)} \lesssim ||(1-\Delta_G)^{\sigma/2} v||_{L^2(\mathcal{M},dG)} =: ||v||_{H^\sigma},$$

with

$$\sigma > n\left(\frac{1}{2}-\frac{1}{q}\right).$$

Rem: we know they hold on many reasonable manifolds.

For the original problem:

$$||u(t)||_{L^q(\mathcal{M},dG)} \lesssim ||u(t)||_{H^\sigma} = ||u_0||_{H^\sigma}.$$

- Advantage: $[t \mapsto u(t)] \in C(\mathbb{R}, L^q(\mathcal{M}, dG))$,
- Drawback: requires u_0 to be (too much) smooth.

Strichartz estimates

1st possible answer: prove **Sobolev embeddings**

$$||v||_{L^q(\mathcal{M},dG)} \lesssim ||(1-\Delta_G)^{\sigma/2} v||_{L^2(\mathcal{M},dG)} =: ||v||_{H^\sigma},$$

with

$$\sigma > n\left(\frac{1}{2} - \frac{1}{q}\right).$$

Rem: we know they hold on many reasonable manifolds.

For the original problem:

$$||u(t)||_{L^q(\mathcal{M},dG)} \lesssim ||u(t)||_{H^\sigma} = ||u_0||_{H^\sigma}.$$

- Advantage: $[t \mapsto u(t)] \in C(\mathbb{R}, L^q(\mathcal{M}, dG))$,
- Drawback: requires u_0 to be (too much) smooth.

Strichartz estimates

1st possible answer: prove **Sobolev embeddings**

$$||v||_{L^q(\mathcal{M},dG)} \lesssim ||(1-\Delta_G)^{\sigma/2} v||_{L^2(\mathcal{M},dG)} =: ||v||_{H^\sigma},$$

with

$$\sigma > n\left(\frac{1}{2} - \frac{1}{q}\right).$$

Rem: we know they hold on many reasonable manifolds.

For the original problem:

$$||u(t)||_{L^q(\mathcal{M},dG)} \lesssim ||u(t)||_{H^\sigma} = ||u_0||_{H^\sigma}.$$

- Advantage: $[t \mapsto u(t)] \in C(\mathbb{R}, L^q(\mathcal{M}, dG))$,
- Drawback: requires u_0 to be (too much) smooth.

Strichartz estimates

1st possible answer: prove **Sobolev embeddings**

$$||v||_{L^q(\mathcal{M},dG)} \lesssim ||(1-\Delta_G)^{\sigma/2} v||_{L^2(\mathcal{M},dG)} =: ||v||_{H^\sigma},$$

with

$$\sigma > n\left(\frac{1}{2}-\frac{1}{q}\right).$$

Rem: we know they hold on many reasonable manifolds.

For the original problem:

$$||u(t)||_{L^q(\mathcal{M},dG)} \lesssim ||u(t)||_{H^\sigma} = ||u_0||_{H^\sigma}.$$

- Advantage: $[t \mapsto u(t)] \in C(\mathbb{R}, L^q(\mathcal{M}, dG))$,
- Drawback: requires u_0 to be (too much) smooth.

Strichartz estimates

1st possible answer: prove **Sobolev embeddings**

$$||v||_{L^q(\mathcal{M},dG)} \lesssim ||(1-\Delta_G)^{\sigma/2} v||_{L^2(\mathcal{M},dG)} =: ||v||_{H^\sigma},$$

with

$$\sigma > n\left(\frac{1}{2}-\frac{1}{q}\right).$$

Rem: we know they hold on many reasonable manifolds.

For the original problem:

$$||u(t)||_{L^q(\mathcal{M},dG)} \lesssim ||u(t)||_{H^\sigma} = ||u_0||_{H^\sigma}.$$

- Advantage: $[t \mapsto u(t)] \in C(\mathbb{R}, L^q(\mathcal{M}, dG))$,
- Drawback: requires u_0 to be (too much) smooth.

Strichartz estimates

1st possible answer: prove **Sobolev embeddings**

$$||v||_{L^q(\mathcal{M},dG)} \lesssim ||(1-\Delta_G)^{\sigma/2} v||_{L^2(\mathcal{M},dG)} =: ||v||_{H^\sigma},$$

with

$$\sigma > n\left(\frac{1}{2} - \frac{1}{q}\right).$$

Rem: we know they hold on many reasonable manifolds.

For the original problem:

$$||u(t)||_{L^q(\mathcal{M},dG)} \lesssim ||u(t)||_{H^\sigma} = ||u_0||_{H^\sigma}.$$

- Advantage: $[t \mapsto u(t)] \in C(\mathbb{R}, L^q(\mathcal{M}, dG))$,
- Drawback: requires u_0 to be (too much) smooth.

Strichartz estimates

1st possible answer: prove **Sobolev embeddings**

$$||v||_{L^q(\mathcal{M},dG)} \lesssim ||(1-\Delta_G)^{\sigma/2} v||_{L^2(\mathcal{M},dG)} =: ||v||_{H^\sigma},$$

with

$$\sigma > n\left(\frac{1}{2}-\frac{1}{q}\right).$$

Rem: we know they hold on many reasonable manifolds.

For the original problem:

$$||u(t)||_{L^q(\mathcal{M},dG)} \lesssim ||u(t)||_{H^\sigma} = ||u_0||_{H^\sigma}.$$

- Advantage: $[t \mapsto u(t)] \in C(\mathbb{R}, L^q(\mathcal{M}, dG))$,
- Drawback: requires u_0 to be (too much) smooth.

Strichartz estimates

1st possible answer: prove **Sobolev embeddings**

$$||v||_{L^q(\mathcal{M},dG)} \lesssim ||(1-\Delta_G)^{\sigma/2} v||_{L^2(\mathcal{M},dG)} =: ||v||_{H^\sigma},$$

with

$$\sigma > n\left(\frac{1}{2} - \frac{1}{q}\right).$$

Rem: we know they hold on many reasonable manifolds.

For the original problem:

$$||u(t)||_{L^q(\mathcal{M},dG)} \lesssim ||u(t)||_{H^\sigma} = ||u_0||_{H^\sigma}.$$

- Advantage: $[t \mapsto u(t)] \in C(\mathbb{R}, L^q(\mathcal{M}, dG))$,
- Drawback: requires u_0 to be (too much) smooth.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L_t^p L_x^q} := \left(\int_0^1 ||u(t)||^p_{L^q(\mathcal{M},dG)} dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2,\infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p,q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L^p_t L^q_x} := \left(\int_0^1 ||u(t)||^p_{L^q(\mathcal{M},dG)} dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2,\infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p,q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L^p_t L^q_x} := \left(\int_0^1 ||u(t)||^p_{L^q(\mathcal{M}, dG)} dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2, \infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p, q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L^p_t L^q_x} := \left(\int_0^1 ||u(t)||^p_{L^q(\mathcal{M},dG)} dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2,\infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p,q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L_t^p L_x^q} := \left(\int_0^1 ||u(t)||^p_{L^q(\mathcal{M},dG)} dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2,\infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p,q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L^p_t L^q_x} := \left(\int_0^1 ||u(t)||^p_{L^q(\mathcal{M},dG)} dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2,\infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p,q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L_t^p L_x^q} := \left(\int_0^1 ||u(t)||^p_{L^q(\mathcal{M}, dG)} dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2, \infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p, q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L_t^p L_x^q} := \left(\int_0^1 ||u(t)||_{L^q(\mathcal{M},dG)}^p dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2,\infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p,q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L_t^p L_x^q} := \left(\int_0^1 ||u(t)||^p_{L^q(\mathcal{M},dG)} dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2,\infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p,q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L^p_t L^q_x} := \left(\int_0^1 ||u(t)||^p_{L^q(\mathcal{M},dG)} dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2,\infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p,q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

2nd possible answer : prove **Strichartz estimates**

$$||u||_{L^p_t L^q_x} := \left(\int_0^1 ||u(t)||^p_{L^q(\mathcal{M},dG)} dt \right)^{1/p} \lesssim ||u_0||_{H^s},$$

with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q) \neq (2,\infty), \qquad p \geq 2,$$

and some $s \geq 0$ (**loss** of derivatives).

Pairs (p,q) as above are called admissible pairs.

Some cases where Strichartz estimates hold:

1. $\mathcal{M} = \mathbb{R}^n$ (flat): no loss $s = 0$,
   due to Strichartz, Ginibre-Velo, Keel-Tao.
2. "General" $\mathcal{M}$: loss $1/p$,
   due to Burq-Gérard-Tzvetkov.

Strichartz estimates

Important fact Strichartz estimates are twice better than Sobolev embeddings:

- Sobolev
$$\sigma_q = n\left(\frac{1}{2} - \frac{1}{q}\right),$$
- Strichartz
$$s_q \leq \frac{n}{2}\left(\frac{1}{2} - \frac{1}{q}\right).$$

Interest solve non linear Schrödinger equations at "low" regularity.

Strichartz estimates

Important fact Strichartz estimates are twice better than Sobolev embeddings:

- Sobolev

$$\sigma_q = n\left(\frac{1}{2} - \frac{1}{q}\right),$$

- Strichartz

$$s_q \le \frac{n}{2}\left(\frac{1}{2} - \frac{1}{q}\right).$$

Interest solve non linear Schrödinger equations at "low" regularity.

Strichartz estimates

Important fact Strichartz estimates are twice better than Sobolev embeddings:

- Sobolev
$$\sigma_q = n\left(\frac{1}{2} - \frac{1}{q}\right),$$
- Strichartz
$$s_q \leq \frac{n}{2}\left(\frac{1}{2} - \frac{1}{q}\right).$$

Interest solve non linear Schrödinger equations at "low" regularity.

Strichartz estimates

Important fact Strichartz estimates are twice better than Sobolev embeddings:

- Sobolev
$$\sigma_q = n\left(\frac{1}{2} - \frac{1}{q}\right),$$
- Strichartz
$$s_q \leq \frac{n}{2}\left(\frac{1}{2} - \frac{1}{q}\right).$$

Interest solve non linear Schrödinger equations at "low" regularity.

Strichartz estimates

Important fact Strichartz estimates are twice better than Sobolev embeddings:

- Sobolev
$$\sigma_q = n\left(\frac{1}{2} - \frac{1}{q}\right),$$
- Strichartz
$$s_q \leq \frac{n}{2}\left(\frac{1}{2} - \frac{1}{q}\right).$$

Interest solve non linear Schrödinger equations at "low" regularity.

Strichartz estimates

Further remarks:

- Strichartz estimates show only that $u(t) \in L^q$ for a.e. t...
- ... but this is sufficient for non linear applications.
- $s = 1/p$ is an *upper bound* on the possible losses.
- There are Strichartz estimates for other dispersive equations (*e.g.* wave equations).

Strichartz estimates

Further remarks:

- Strichartz estimates show only that $u(t) \in L^q$ for a.e. t...
- ... but this is sufficient for non linear applications.
- $s = 1/p$ is an *upper bound* on the possible losses.
- There are Strichartz estimates for other dispersive equations (*e.g.* wave equations).

Strichartz estimates

Further remarks:

- Strichartz estimates show only that $u(t) \in L^q$ for a.e. t...
- ... but this is sufficient for non linear applications.
- $s = 1/p$ is an *upper bound* on the possible losses.
- There are Strichartz estimates for other dispersive equations (*e.g.* wave equations).

Strichartz estimates

Further remarks:

- Strichartz estimates show only that $u(t) \in L^q$ for a.e. t...
- ... but this is sufficient for non linear applications.
- $s = 1/p$ is an *upper bound* on the possible losses.
- There are Strichartz estimates for other dispersive equations (*e.g.* wave equations).

Strichartz estimates

Further remarks:

- Strichartz estimates show only that $u(t) \in L^q$ for a.e. t...
- ... but this is sufficient for non linear applications.
- $s = 1/p$ is an *upper bound* on the possible losses.
- There are Strichartz estimates for other dispersive equations (*e.g.* wave equations).

Strichartz estimates

The general question we want to address is :

When can one prove Strichartz estimates without losses ?

Strichartz estimates are a high frequency problem

Observation: for $\psi_0 \in C_0^\infty(\mathbb{R})$, set

$$U_{\psi_0}(t) = e^{it\Delta_G}\psi_0(\Delta_G).$$

Then

$$||U_{\psi_0}(\cdot)u_0||_{L_t^p L_x^q} \leq \sup_t ||\psi(\Delta_G)e^{it\Delta_G}u_0||_{L^q} \lesssim ||u_0||_{L^2},$$

by Sobolev embeddings.

Interpretation: Losses in the Strichartz estimates may only come from high frequency effects.

Strichartz estimates are a high frequency problem

Observation: for $\psi_0 \in C_0^\infty(\mathbb{R})$, set

$$U_{\psi_0}(t) = e^{it\Delta_G}\psi_0(\Delta_G).$$

Then

$$||U_{\psi_0}(\cdot)u_0||_{L_t^p L_x^q} \leq \sup_t ||\psi(\Delta_G)e^{it\Delta_G}u_0||_{L^q} \lesssim ||u_0||_{L^2},$$

by Sobolev embeddings.

Interpretation: Losses in the Strichartz estimates may only come from high frequency effects.

Strichartz estimates are a high frequency problem

Observation: for $\psi_0 \in C_0^\infty(\mathbb{R})$, set

$$U_{\psi_0}(t) = e^{it\Delta_G}\psi_0(\Delta_G).$$

Then

$$||U_{\psi_0}(\cdot)u_0||_{L_t^p L_x^q} \leq \sup_t ||\psi(\Delta_G)e^{it\Delta_G}u_0||_{L^q} \lesssim ||u_0||_{L^2},$$

by Sobolev embeddings.

Interpretation: Losses in the Strichartz estimates may only come from high frequency effects.

Strichartz estimates are a high frequency problem

Observation: for $\psi_0 \in C_0^\infty(\mathbb{R})$, set

$$U_{\psi_0}(t) = e^{it\Delta_G}\psi_0(\Delta_G).$$

Then

$$||U_{\psi_0}(\cdot)u_0||_{L_t^p L_x^q} \leq \sup_t ||\psi(\Delta_G)e^{it\Delta_G}u_0||_{L^q} \lesssim ||u_0||_{L^2},$$

by Sobolev embeddings.

Interpretation: Losses in the Strichartz estimates may only come from high frequency effects.

Strichartz estimates are a high frequency problem

By semiclassical correspondance,

- "High frequency waves travel along the geodesic flow".

$\Rightarrow$ The losses should be related to the geodesic flow.

- More precisely: for the Schrödinger equation, solutions localized at frequency $1/h$ travel at speed $1/h$: for initial data spectrally localized at frequency $\sim 1/h \to \infty$, ie

$$u_0^h = \psi(h^2\Delta_G)u_0, \qquad \text{for some } \psi \in C_0^\infty(\mathbb{R}\setminus 0),$$

we have

$$\Phi^T_{\text{geodesic}}\left(WF_{\text{s-cl}}(u_0^h)\right) = WF_{\text{s-cl}}\left(e^{ihT\Delta_G}u_0^h\right)$$

$\Rightarrow$ Schrödinger group for $|t| \leq 1 \leftrightarrow$ geodesic flow for $|T| \leq 1/h$.

Strichartz estimates are a high frequency problem

By semiclassical correspondance,

- "High frequency waves travel along the geodesic flow".

$\Rightarrow$ The losses should be related to the geodesic flow.

- More precisely: for the Schrödinger equation, solutions localized at frequency $1/h$ travel at speed $1/h$: for initial data spectrally localized at frequency $\sim 1/h \to \infty$, ie

$$u_0^h = \psi(h^2 \Delta_G) u_0, \qquad \text{for some } \psi \in C_0^\infty(\mathbb{R} \setminus 0),$$

we have

$$\Phi^T_{\mathrm{geodesic}} \left(WF_{\mathrm{s-cl}}(u_0^h) \right) = WF_{\mathrm{s-cl}} \left(e^{ihT\Delta_G} u_0^h \right)$$

$\Rightarrow$ Schrödinger group for $|t| \leq 1 \leftrightarrow$ geodesic flow for $|T| \leq 1/h$.

Strichartz estimates are a high frequency problem

By semiclassical correspondance,

- "High frequency waves travel along the geodesic flow".

$\Rightarrow$ The losses should be related to the geodesic flow.

- More precisely: for the Schrödinger equation, solutions localized at frequency $1/h$ travel at speed $1/h$: for initial data spectrally localized at frequency $\sim 1/h \to \infty$, ie

$$u_0^h = \psi(h^2 \Delta_G) u_0, \qquad \text{for some } \psi \in C_0^\infty(\mathbb{R} \setminus 0),$$

we have

$$\Phi^T_{\mathrm{geodesic}}\left(WF_{\mathrm{s-cl}}(u_0^h)\right) = WF_{\mathrm{s-cl}}\left(e^{ihT\Delta_G} u_0^h\right)$$

$\Rightarrow$ Schrödinger group for $|t| \leq 1 \leftrightarrow$ geodesic flow for $|T| \leq 1/h$.

Strichartz estimates are a high frequency problem

By semiclassical correspondance,

- "High frequency waves travel along the geodesic flow".

$\Rightarrow$ The losses should be related to the geodesic flow.

- More precisely: for the Schrödinger equation, solutions localized at frequency $1/h$ travel at speed $1/h$: for initial data spectrally localized at frequency $\sim 1/h \to \infty$, ie

$$u_0^h = \psi(h^2\Delta_G)u_0, \qquad \text{for some } \psi \in C_0^\infty(\mathbb{R} \setminus 0),$$

we have

$$\Phi^T_{\mathrm{geodesic}}\left(WF_{\mathrm{s-cl}}(u_0^h)\right) = WF_{\mathrm{s-cl}}\left(e^{ihT\Delta_G}u_0^h\right)$$

$\Rightarrow$ Schrödinger group for $|t| \leq 1 \leftrightarrow$ geodesic flow for $|T| \leq 1/h$.

Strichartz estimates are a high frequency problem

By semiclassical correspondance,

- "High frequency waves travel along the geodesic flow".

$\Rightarrow$ The losses should be related to the geodesic flow.

- More precisely: for the Schrödinger equation, solutions localized at frequency $1/h$ travel at speed $1/h$: for initial data spectrally localized at frequency $\sim 1/h \to \infty$, ie

$$u_0^h = \psi(h^2 \Delta_G) u_0, \qquad \text{for some } \psi \in C_0^\infty(\mathbb{R} \setminus 0),$$

we have

$$\Phi^T_{\text{geodesic}} \left(WF_{\text{s-cl}}(u_0^h) \right) = WF_{\text{s-cl}} \left(e^{ihT\Delta_G} u_0^h \right)$$

$\Rightarrow$ Schrödinger group for $|t| \leq 1 \leftrightarrow$ geodesic flow for $|T| \leq 1/h$.

Strichartz estimates are a high frequency problem

By semiclassical correspondance,

- "High frequency waves travel along the geodesic flow".

$\Rightarrow$ The losses should be related to the geodesic flow.

- More precisely: for the Schrödinger equation, solutions localized at frequency $1/h$ travel at speed $1/h$: for initial data spectrally localized at frequency $\sim 1/h \to \infty$, ie

$$u_0^h = \psi(h^2 \Delta_G) u_0, \qquad \text{for some } \psi \in C_0^\infty(\mathbb{R} \setminus 0),$$

we have

$$\Phi^T_{\mathrm{geodesic}} \left(WF_{\mathrm{s-cl}}(u_0^h) \right) = WF_{\mathrm{s-cl}} \left(e^{ihT\Delta_G} u_0^h \right)$$

$\Rightarrow$ Schrödinger group for $|t| \leq 1 \leftrightarrow$ geodesic flow for $|T| \leq 1/h$.

Strichartz estimates are a high frequency problem

By semiclassical correspondance,

- "High frequency waves travel along the geodesic flow".

$\Rightarrow$ The losses should be related to the geodesic flow.

- More precisely: for the Schrödinger equation, solutions localized at frequency $1/h$ travel at speed $1/h$: for initial data spectrally localized at frequency $\sim 1/h \to \infty$, ie

$$u_0^h = \psi(h^2 \Delta_G) u_0, \qquad \text{for some } \psi \in C_0^\infty(\mathbb{R} \setminus 0),$$

we have

$$\Phi^T_{\text{geodesic}}\left(WF_{\text{s-cl}}(u_0^h)\right) = WF_{\text{s-cl}}\left(e^{ihT\Delta_G} u_0^h\right)$$

$\Rightarrow$ Schrödinger group for $|t| \leq 1 \leftrightarrow$ geodesic flow for $|T| \leq 1/h$.

Strichartz estimates are a high frequency problem

By semiclassical correspondance,

- "High frequency waves travel along the geodesic flow".

$\Rightarrow$ The losses should be related to the geodesic flow.

- More precisely: for the Schrödinger equation, solutions localized at frequency $1/h$ travel at speed $1/h$: for initial data spectrally localized at frequency $\sim 1/h \to \infty$, ie

$$u_0^h = \psi(h^2\Delta_G)u_0, \qquad \text{for some } \psi \in C_0^\infty(\mathbb{R}\setminus 0),$$

we have

$$\Phi^T_{\text{geodesic}}\left(WF_{\text{s-cl}}(u_0^h)\right) = WF_{\text{s-cl}}\left(e^{ihT\Delta_G}u_0^h\right)$$

$\Rightarrow$ Schrödinger group for $|t| \leq 1 \leftrightarrow$ geodesic flow for $|T| \leq 1/h$.

Strichartz estimates are a high frequency problem

By semiclassical correspondance,

- "High frequency waves travel along the geodesic flow".

$\Rightarrow$ The losses should be related to the geodesic flow.

- More precisely: for the Schrödinger equation, solutions localized at frequency $1/h$ travel at speed $1/h$: for initial data spectrally localized at frequency $\sim 1/h \to \infty$, ie

$$u_0^h = \psi(h^2 \Delta_G) u_0, \qquad \text{for some } \psi \in C_0^\infty(\mathbb{R} \setminus 0),$$

we have

$$\Phi^T_{\mathrm{geodesic}} \left(WF_{\mathrm{s-cl}}(u_0^h) \right) = WF_{\mathrm{s-cl}} \left(e^{ihT\Delta_G} u_0^h \right)$$

$\Rightarrow$ Schrödinger group for $|t| \leq 1 \leftrightarrow$ geodesic flow for $|T| \leq 1/h$.

Strichartz estimates are a high frequency problem

By semiclassical correspondance,

- "High frequency waves travel along the geodesic flow".

$\Rightarrow$ The losses should be related to the geodesic flow.

- More precisely: for the Schrödinger equation, solutions localized at frequency $1/h$ travel at speed $1/h$: for initial data spectrally localized at frequency $\sim 1/h \to \infty$, ie

$$u_0^h = \psi(h^2\Delta_G)u_0, \qquad \text{for some } \psi \in C_0^\infty(\mathbb{R} \setminus 0),$$

we have

$$\Phi^T_{\mathrm{geodesic}}\left(WF_{\mathrm{s-cl}}(u_0^h)\right) = WF_{\mathrm{s-cl}}\left(e^{ihT\Delta_G}u_0^h\right)$$

$\Rightarrow$ Schrödinger group for $|t| \leq 1 \leftrightarrow$ geodesic flow for $|T| \leq 1/h$.

Influence of the geometry

Compact manifolds (trapping case)

Theorem [Burq-Gérard-Tzvetkov] For a general $\mathcal{M}$,

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{H^{1/p}}.$$

Sharp for $p = 2$ and $\mathcal{M} = \mathbb{S}^3$.

Theorem [Bourgain] On $\mathcal{M} = \mathbb{T}^2$,

$$||u||_{L_t^4 L_x^4} \leq C_\epsilon ||u_0||_{H^\epsilon},$$

for all $\epsilon > 0$.

Rem: many closed geodesics on $\mathbb{S}^n$ / a few of them on $\mathbb{T}^n$.

Influence of the geometry

Compact manifolds (trapping case)

Theorem [Burq-Gérard-Tzvetkov] For a general $\mathcal{M}$,

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{H^{1/p}}.$$

Sharp for $p = 2$ and $\mathcal{M} = \mathbb{S}^3$.

Theorem [Bourgain] On $\mathcal{M} = \mathbb{T}^2$,

$$||u||_{L_t^4 L_x^4} \leq C_\epsilon ||u_0||_{H^\epsilon},$$

for all $\epsilon > 0$.

Rem: many closed geodesics on $\mathbb{S}^n$ / a few of them on $\mathbb{T}^n$.

Influence of the geometry

Compact manifolds (trapping case)

Theorem [Burq-Gérard-Tzvetkov] For a general $\mathcal{M}$,

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{H^{1/p}}.$$

Sharp for $p = 2$ and $\mathcal{M} = \mathbb{S}^3$.

Theorem [Bourgain] On $\mathcal{M} = \mathbb{T}^2$,

$$||u||_{L_t^4 L_x^4} \leq C_\epsilon ||u_0||_{H^\epsilon},$$

for all $\epsilon > 0$.

Rem: many closed geodesics on $\mathbb{S}^n$ / a few of them on $\mathbb{T}^n$.

Influence of the geometry

Compact manifolds (trapping case)

Theorem [Burq-Gérard-Tzvetkov] For a general $\mathcal{M}$,

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{H^{1/p}}.$$

Sharp for $p = 2$ and $\mathcal{M} = \mathbb{S}^3$.

Theorem [Bourgain] On $\mathcal{M} = \mathbb{T}^2$,

$$||u||_{L_t^4 L_x^4} \leq C_\epsilon ||u_0||_{H^\epsilon},$$

for all $\epsilon > 0$.

Rem: many closed geodesics on $\mathbb{S}^n$ / a few of them on $\mathbb{T}^n$.

Influence of the geometry

Non compact manifolds (1/2)

Theorem If $\mathcal{M}$ is asymptotically euclidean and non trapping

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}. \qquad \text{(no loss !)}$$

- flat case: Strichartz, Ginibre-Velo, Keel-Tao,
- flat outside a compact set: Staffilani-Tataru (+ Ivanovici for convex obstacles)
- Short range: Robbiano-Zuily
- Long range: Hassell-Tao-Wunsch, B-Tzvetkov

Influence of the geometry

Non compact manifolds (1/2)

Theorem If $\mathcal{M}$ is asymptotically euclidean and non trapping

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}. \qquad \text{(no loss !)}$$

- flat case: Strichartz, Ginibre-Velo, Keel-Tao,
- flat outside a compact set: Staffilani-Tataru (+ Ivanovici for convex obstacles)
- Short range: Robbiano-Zuily
- Long range: Hassell-Tao-Wunsch, B-Tzvetkov

Influence of the geometry

Non compact manifolds (1/2)

Theorem If $\mathcal{M}$ is asymptotically euclidean and non trapping

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}. \qquad \text{(no loss !)}$$

- flat case: Strichartz, Ginibre-Velo, Keel-Tao,
- flat outside a compact set: Staffilani-Tataru (+ Ivanovici for convex obstacles)
- Short range: Robbiano-Zuily
- Long range: Hassell-Tao-Wunsch, B-Tzvetkov

Influence of the geometry

Non compact manifolds (1/2)

Theorem If $\mathcal{M}$ is asymptotically euclidean and non trapping

$$||u||_{L^p_t L^q_x} \lesssim ||u_0||_{L^2}. \qquad \text{(no loss !)}$$

- flat case: Strichartz, Ginibre-Velo, Keel-Tao,
- flat outside a compact set: Staffilani-Tataru (+ Ivanovici for convex obstacles)
- Short range: Robbiano-Zuily
- Long range: Hassell-Tao-Wunsch, B-Tzvetkov

Influence of the geometry

Non compact manifolds (2/2)

Theorem For certain $\mathcal{M}$ with non positive constant Ricci curvature

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}.$$

- Hyperbolic space: Banica (+ Carles-Staffilani), Anker-Pierfelice,
- Damek-Ricci spaces: Pierfelice,

as well as for some spherically symmetric manifolds and radial data (Banica-Duyckaerts). Furthermore, these estimates hold in weighted L^q spaces.

Influence of the geometry

Non compact manifolds (2/2)

Theorem For certain $\mathcal{M}$ with non positive constant Ricci curvature

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}.$$

- Hyperbolic space: Banica (+ Carles-Staffilani), Anker-Pierfelice,
- Damek-Ricci spaces: Pierfelice,

as well as for some spherically symmetric manifolds and radial data (Banica-Duyckaerts). Furthermore, these estimates hold in weighted L^q spaces.

Influence of the geometry

Non compact manifolds (2/2)

Theorem For certain $\mathcal{M}$ with non positive constant Ricci curvature

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}.$$

- Hyperbolic space: Banica (+ Carles-Staffilani), Anker-Pierfelice,
- Damek-Ricci spaces: Pierfelice,

as well as for some spherically symmetric manifolds and radial data (Banica-Duyckaerts). Furthermore, these estimates hold in weighted L^q spaces.

Influence of the geometry

Non compact manifolds (2/2)

Theorem For certain $\mathcal{M}$ with non positive constant Ricci curvature

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}.$$

- Hyperbolic space: Banica (+ Carles-Staffilani), Anker-Pierfelice,
- Damek-Ricci spaces: Pierfelice,

as well as for some spherically symmetric manifolds and radial data (Banica-Duyckaerts). Furthermore, these estimates hold in weighted L^q spaces.

Influence of the geometry

Non compact manifolds (2/2)

Theorem For certain $\mathcal{M}$ with non positive constant Ricci curvature

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}.$$

- Hyperbolic space: Banica (+ Carles-Staffilani), Anker-Pierfelice,
- Damek-Ricci spaces: Pierfelice,

as well as for some spherically symmetric manifolds and radial data (Banica-Duyckaerts). Furthermore, these estimates hold in weighted L^q spaces.

Influence of the geometry

Non compact manifolds (2/2)

Theorem For certain $\mathcal{M}$ with non positive constant Ricci curvature

$$||u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}.$$

- Hyperbolic space: Banica (+ Carles-Staffilani), Anker-Pierfelice,
- Damek-Ricci spaces: Pierfelice,

as well as for some spherically symmetric manifolds and radial data (Banica-Duyckaerts). Furthermore, these estimates hold in weighted L^q spaces.

- The results for negatively curved manifolds hold under rigid conditions.

⇒ Can we obtain similar results in more general cases ? (*e.g.* avoid spherical symmetry, Lie group structure, constant curvature.)

- Understand which regions of the phase space may cause losses.
- In particular, how necessary is the non trapping condition (cf Tzvetkov-Takaoka, Burq-Guillarmou-Hassell) ?

- The results for negatively curved manifolds hold under rigid conditions.

$\Rightarrow$ Can we obtain similar results in more general cases ? (*e.g.* avoid spherical symmetry, Lie group structure, constant curvature.)

- Understand which regions of the phase space may cause losses.
- In particular, how necessary is the non trapping condition (cf Tzvetkov-Takaoka, Burq-Guillarmou-Hassell) ?

- The results for negatively curved manifolds hold under rigid conditions.

⇒ Can we obtain similar results in more general cases ? (*e.g.* avoid spherical symmetry, Lie group structure, constant curvature.)

- Understand which regions of the phase space may cause losses.
- In particular, how necessary is the non trapping condition (cf Tzvetkov-Takaoka, Burq-Guillarmou-Hassell) ?

- The results for negatively curved manifolds hold under rigid conditions.

$\Rightarrow$ Can we obtain similar results in more general cases ? (*e.g.* avoid spherical symmetry, Lie group structure, constant curvature.)

- Understand which regions of the phase space may cause losses.
- In particular, how necessary is the non trapping condition (cf Tzvetkov-Takaoka, Burq-Guillarmou-Hassell) ?

- The results for negatively curved manifolds hold under rigid conditions.

$\Rightarrow$ Can we obtain similar results in more general cases ? (*e.g.* avoid spherical symmetry, Lie group structure, constant curvature.)

- Understand which regions of the phase space may cause losses.
- In particular, how necessary is the non trapping condition (cf Tzvetkov-Takaoka, Burq-Guillarmou-Hassell) ?

Asymptotically hyperbolic manifolds

$(\mathcal{M}, G)$ is **asymptotically hyperbolic** if, outside some compact subset $\mathcal{K} \Subset \mathcal{M}$,

$$(\mathcal{M} \setminus \mathcal{K}, G) \simeq \Big((R_0, +\infty) \times S, dr^2 + e^{2r} g(r)\Big) \,,$$

where

- S is a compact manifold,
- for each r, $g(r)$ is a riemannian metric on S,
- for some fixed metric g on S and some $\nu > 0$,

$$\partial_r^k \left(g(r) - g\right) = \mathcal{O}(\langle r \rangle^{-\nu - k}).$$

Asymptotically hyperbolic manifolds

$(\mathcal{M}, G)$ is **asymptotically hyperbolic** if, outside some compact subset $\mathcal{K} \Subset \mathcal{M}$,

$$(\mathcal{M} \setminus \mathcal{K}, G) \simeq \left((R_0, +\infty) \times S, dr^2 + e^{2r} g(r)\right),$$

where

- S is a compact manifold,
- for each r, $g(r)$ is a riemannian metric on S,
- for some fixed metric g on S and some $\nu > 0$,

$$\partial_r^k (g(r) - g) = \mathcal{O}(\langle r \rangle^{-\nu - k}).$$

Asymptotically hyperbolic manifolds

$(\mathcal{M}, G)$ is **asymptotically hyperbolic** if, outside some compact subset $\mathcal{K} \Subset \mathcal{M}$,

$$(\mathcal{M} \setminus \mathcal{K}, G) \simeq \Big((R_0, +\infty) \times S, dr^2 + e^{2r} g(r) \Big) \,,$$

where

- S is a compact manifold,
- for each r, $g(r)$ is a riemannian metric on S,
- for some fixed metric g on S and some $\nu > 0$,

$$\partial_r^k \left(g(r) - g\right) = \mathcal{O}(\langle r \rangle^{-\nu - k}).$$

Asymptotically hyperbolic manifolds

$(\mathcal{M}, G)$ is **asymptotically hyperbolic** if, outside some compact subset $\mathcal{K} \Subset \mathcal{M}$,

$$(\mathcal{M} \setminus \mathcal{K}, G) \simeq \left((R_0, +\infty) \times S, dr^2 + e^{2r} g(r)\right),$$

where

- S is a compact manifold,
- for each r, $g(r)$ is a riemannian metric on S,
- for some fixed metric g on S and some $\nu > 0$,

$$\partial_r^k \left(g(r) - g\right) = \mathcal{O}(\langle r \rangle^{-\nu - k}).$$

Asymptotically hyperbolic manifolds

Some remarks:

- our definition is more general than conformally compact manifolds,
- it contains $\mathbb{H}^n$ and some of its infinite volume quotients,
- physically, asymptotically hyperbolic manifolds appear as spacelike hypersurfaces of black hole spacetimes (*e.g.* de Sitter-Reissner-Nordstöm black holes)
- long run motivation: (non linear) wave equations on spaces times with asymptotically hyperbolic ends.

Asymptotically hyperbolic manifolds

Some remarks:

- our definition is more general than conformally compact manifolds,
- it contains $\mathbb{H}^n$ and some of its infinite volume quotients,
- physically, asymptotically hyperbolic manifolds appear as spacelike hypersurfaces of black hole spacetimes (*e.g.* de Sitter-Reissner-Nordstöm black holes)
- long run motivation: (non linear) wave equations on spaces times with asymptotically hyperbolic ends.

Asymptotically hyperbolic manifolds

Some remarks:

- our definition is more general than conformally compact manifolds,
- it contains $\mathbb{H}^n$ and some of its infinite volume quotients,
- physically, asymptotically hyperbolic manifolds appear as spacelike hypersurfaces of black hole spacetimes (*e.g.* de Sitter-Reissner-Nordstöm black holes)
- long run motivation: (non linear) wave equations on spaces times with asymptotically hyperbolic ends.

Asymptotically hyperbolic manifolds

Some remarks:

- our definition is more general than conformally compact manifolds,
- it contains $\mathbb{H}^n$ and some of its infinite volume quotients,
- physically, asymptotically hyperbolic manifolds appear as spacelike hypersurfaces of black hole spacetimes (*e.g.* de Sitter-Reissner-Nordstöm black holes)
- long run motivation: (non linear) wave equations on spaces times with asymptotically hyperbolic ends.

Asymptotically hyperbolic manifolds

Theorem [B.] *If* $(\mathcal{M}, G)$ *is asymptotically hyperbolic, there exists* $\chi \in C_0^\infty(\mathcal{M})$ *($\chi \equiv 1$ on a large enough compact subset), such that, for all admissible pair* (p, q)*,*

$$||(1-\chi)u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}.$$

Comments:

- The cutoff localizes the solution near spatial infinity,
- The Strichartz estimates hold without loss and no condition on the geodesic flow.
- If the geodesic flow is non trapping, one can show that one may take $\chi \equiv 0$.

Asymptotically hyperbolic manifolds

Theorem [B.] *If* $(\mathcal{M}, G)$ *is asymptotically hyperbolic, there exists* $\chi \in C_0^\infty(\mathcal{M})$ *(*$\chi \equiv 1$ *on a large enough compact subset), such that, for all admissible pair* (p, q)*,*

$$||(1-\chi)u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}.$$

Comments:

- The cutoff localizes the solution near spatial infinity,
- The Strichartz estimates hold without loss and no condition on the geodesic flow.
- If the geodesic flow is non trapping, one can show that one may take $\chi \equiv 0$.

Asymptotically hyperbolic manifolds

Theorem [B.] *If* $(\mathcal{M}, G)$ *is asymptotically hyperbolic, there exists* $\chi \in C_0^\infty(\mathcal{M})$ *(*$\chi \equiv 1$ *on a large enough compact subset), such that, for all admissible pair* (p, q)*,*

$$||(1-\chi)u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}.$$

Comments:

- The cutoff localizes the solution near spatial infinity,
- The Strichartz estimates hold without loss and no condition on the geodesic flow.
- If the geodesic flow is non trapping, one can show that one may take $\chi \equiv 0$.

Asymptotically hyperbolic manifolds

Theorem [B.] *If* $(\mathcal{M}, G)$ *is asymptotically hyperbolic, there exists* $\chi \in C_0^\infty(\mathcal{M})$ *(*$\chi \equiv 1$ *on a large enough compact subset), such that, for all admissible pair* (p, q)*,*

$$||(1-\chi)u||_{L_t^p L_x^q} \lesssim ||u_0||_{L^2}.$$

Comments:

- The cutoff localizes the solution near spatial infinity,
- The Strichartz estimates hold without loss and no condition on the geodesic flow.
- If the geodesic flow is non trapping, one can show that one may take $\chi \equiv 0$.

Asymptotically hyperbolic manifolds

Some formulas (near infinity):

- Laplacian

$$\Delta_G = \partial_r^2 + e^{-2r}\Delta_{g(r)} + c(r,s)\partial_r + (n-1)\partial_r,$$

where

$$c(r,s) = \frac{\partial_r \mathrm{det} g(r,s)}{2\mathrm{det} g(r,s)}.$$

using local coordinates $\theta_1, \ldots, \theta_{n-1}$, the principal symbol is

$$\begin{aligned} p(r,\theta,\rho,\eta) &= \rho^2 + e^{-2r} q(r,\theta,\eta) \\ &= \rho^2 + q\left(r,\theta, e^{-r}\eta\right), \end{aligned}$$

with $q(r,.,.)$ the principal symbol of $-\Delta_{g(r)}$.

- Measure

$$dG = e^{(n-1)r} dr dg(r),$$

with $dg(r)$ the riemannian measure on S relatively to $g(r)$.

Asymptotically hyperbolic manifolds

Some formulas (near infinity):

- Laplacian

$$\Delta_G = \partial_r^2 + e^{-2r}\Delta_{g(r)} + c(r,s)\partial_r + (n-1)\partial_r,$$

where

$$c(r,s) = \frac{\partial_r \mathrm{det} g(r,s)}{2\mathrm{det} g(r,s)}.$$

using local coordinates $\theta_1, \ldots, \theta_{n-1}$, the principal symbol is

$$\begin{aligned} p(r,\theta,\rho,\eta) &= \rho^2 + e^{-2r} q(r,\theta,\eta) \\ &= \rho^2 + q\left(r,\theta,e^{-r}\eta\right), \end{aligned}$$

with $q(r,.,.)$ the principal symbol of $-\Delta_{g(r)}$.

- Measure

$$dG = e^{(n-1)r} dr dg(r),$$

with $dg(r)$ the riemannian measure on S relatively to $g(r)$.

Asymptotically hyperbolic manifolds

Some formulas (near infinity):

- Laplacian

$$\Delta_G = \partial_r^2 + e^{-2r}\Delta_{g(r)} + c(r,s)\partial_r + (n-1)\partial_r,$$

where

$$c(r,s) = \frac{\partial_r \mathrm{det} g(r,s)}{2\mathrm{det} g(r,s)}.$$

using local coordinates $\theta_1, \ldots, \theta_{n-1}$, the principal symbol is

$$\begin{aligned} p(r,\theta,\rho,\eta) &= \rho^2 + e^{-2r} q(r,\theta,\eta) \\ &= \rho^2 + q\left(r,\theta,e^{-r}\eta\right), \end{aligned}$$

with $q(r,.,.)$ the principal symbol of $-\Delta_{g(r)}$.

- Measure

$$dG = e^{(n-1)r} dr dg(r),$$

with $dg(r)$ the riemannian measure on S relatively to $g(r)$.

Asymptotically hyperbolic manifolds

Some formulas (near infinity):

- Laplacian

$$\Delta_G = \partial_r^2 + e^{-2r}\Delta_{g(r)} + c(r,s)\partial_r + (n-1)\partial_r,$$

where

$$c(r,s) = \frac{\partial_r \mathrm{det} g(r,s)}{2\mathrm{det} g(r,s)}.$$

using local coordinates $\theta_1, \ldots, \theta_{n-1}$, the principal symbol is

$$\begin{aligned} p(r,\theta,\rho,\eta) &= \rho^2 + e^{-2r} q(r,\theta,\eta) \\ &= \rho^2 + q\left(r,\theta,e^{-r}\eta\right), \end{aligned}$$

with $q(r,.,.)$ the principal symbol of $-\Delta_{g(r)}$.

- Measure

$$dG = e^{(n-1)r} dr dg(r),$$

with $dg(r)$ the riemannian measure on S relatively to $g(r)$.

Asymptotically hyperbolic manifolds

Some formulas (near infinity):

- Laplacian

$$\Delta_G = \partial_r^2 + e^{-2r}\Delta_{g(r)} + c(r,s)\partial_r + (n-1)\partial_r,$$

where

$$c(r,s) = \frac{\partial_r \det g(r,s)}{2\det g(r,s)}.$$

using local coordinates $\theta_1, \ldots, \theta_{n-1}$, the principal symbol is

$$\begin{aligned} p(r,\theta,\rho,\eta) &= \rho^2 + e^{-2r}q(r,\theta,\eta) \\ &= \rho^2 + q\left(r,\theta,e^{-r}\eta\right), \end{aligned}$$

with $q(r,.,.)$ the principal symbol of $-\Delta_{g(r)}$.

- Measure

$$dG = e^{(n-1)r}drdg(r),$$

with $dg(r)$ the riemannian measure on S relatively to $g(r)$.

Asymptotically hyperbolic manifolds

Some formulas (near infinity):

- Laplacian

$$\Delta_G = \partial_r^2 + e^{-2r}\Delta_{g(r)} + c(r,s)\partial_r + (n-1)\partial_r,$$

where

$$c(r,s) = \frac{\partial_r \mathrm{det} g(r,s)}{2\mathrm{det} g(r,s)}.$$

using local coordinates $\theta_1, \ldots, \theta_{n-1}$, the principal symbol is

$$\begin{aligned} p(r,\theta,\rho,\eta) &= \rho^2 + e^{-2r} q(r,\theta,\eta) \\ &= \rho^2 + q\left(r,\theta,e^{-r}\eta\right), \end{aligned}$$

with $q(r,.,.)$ the principal symbol of $-\Delta_{g(r)}$.

- Measure

$$dG = e^{(n-1)r} dr dg(r),$$

with $dg(r)$ the riemannian measure on S relatively to $g(r)$.

Proof: spectral localization

Consider a dyadic partition of unity

$$I = \psi_0(\Delta_G) + \sum_{k=0}^{\infty} \psi(2^{-k}\Delta_G),$$

for some suitable

$$\psi_0 \in C_0^\infty(\mathbb{R}), \qquad \psi \in C_0^\infty(\mathbb{R} \setminus 0).$$

Lemma [B. 2010] For any fixed $\chi \in C_0^\infty(\mathcal{M})$ and all $q \in [2, \infty)$,

$$||(1-\chi)v||_{L^q} \lesssim \left(\sum_{h^2 = 2^{-k}} ||(1-\chi)\psi(h^2\Delta_G)v||_{L^q}^2 \right)^{1/2} + ||v||_{L^2}.$$

Rem These are Littlewood-Paley type estimates. The proof uses Calderon-Zygmund theory with some modification since the measure dG is non doubling.

Proof: spectral localization

Consider a dyadic partition of unity

$$I = \psi_0(\Delta_G) + \sum_{k=0}^{\infty} \psi(2^{-k}\Delta_G),$$

for some suitable

$$\psi_0 \in C_0^\infty(\mathbb{R}), \qquad \psi \in C_0^\infty(\mathbb{R} \setminus 0).$$

Lemma [B. 2010] For any fixed $\chi \in C_0^\infty(\mathcal{M})$ and all $q \in [2, \infty)$,

$$||(1-\chi)v||_{L^q} \lesssim \left(\sum_{h^2 = 2^{-k}} ||(1-\chi)\psi(h^2\Delta_G)v||_{L^q}^2 \right)^{1/2} + ||v||_{L^2}.$$

Rem These are Littlewood-Paley type estimates. The proof uses Calderon-Zygmund theory with some modification since the measure dG is non doubling.

Proof: spectral localization

Consider a dyadic partition of unity

$$I = \psi_0(\Delta_G) + \sum_{k=0}^{\infty} \psi(2^{-k}\Delta_G),$$

for some suitable

$$\psi_0 \in C_0^\infty(\mathbb{R}), \qquad \psi \in C_0^\infty(\mathbb{R} \setminus 0).$$

Lemma [B. 2010] For any fixed $\chi \in C_0^\infty(\mathcal{M})$ and all $q \in [2, \infty)$,

$$||(1-\chi)v||_{L^q} \lesssim \left(\sum_{h^2 = 2^{-k}} ||(1-\chi)\psi(h^2\Delta_G)v||_{L^q}^2 \right)^{1/2} + ||v||_{L^2}.$$

<u>Rem</u> These are Littlewood-Paley type estimates. The proof uses Calderon-Zygmund theory with some modification since the measure dG is non doubling.

Proof: spectral localization

Consider a dyadic partition of unity

$$I = \psi_0(\Delta_G) + \sum_{k=0}^{\infty} \psi(2^{-k}\Delta_G),$$

for some suitable

$$\psi_0 \in C_0^\infty(\mathbb{R}), \qquad \psi \in C_0^\infty(\mathbb{R} \setminus 0).$$

Lemma [B. 2010] For any fixed $\chi \in C_0^\infty(\mathcal{M})$ and all $q \in [2, \infty)$,

$$||(1-\chi)v||_{L^q} \lesssim \left(\sum_{h^2=2^{-k}} ||(1-\chi)\psi(h^2\Delta_G)v||_{L^q}^2 \right)^{1/2} + ||v||_{L^2}.$$

<u>Rem</u> These are Littlewood-Paley type estimates. The proof uses Calderon-Zygmund theory with some modification since the measure dG is non doubling.

Proof: spectral localization

Consequence It suffices to show that, for some χ,

$$||(1-\chi)\psi(h^2\Delta_G)e^{it\Delta_G}u_0||_{L_t^pL_x^q} \leq C||u_0||_{L^2},$$

for all u_0 and all $h \in (0,1]$.

$\Rightarrow$ **Strategy**: to find a suitable decomposition

$$(1-\chi)\psi(h^2\Delta_G) = \sum_{j=1}^{N} A_j(h),$$

such that, for each j,

$$||A_j(h)e^{it\Delta_G}u_0||_{L_t^pL_x^q} \leq C||u_0||_{L^2},$$

for all u_0 and all $h \in (0,1]$.

Proof: spectral localization

Consequence It suffices to show that, for some χ,

$$||(1-\chi)\psi(h^2\Delta_G)e^{it\Delta_G}u_0||_{L_t^pL_x^q} \leq C||u_0||_{L^2},$$

for all u_0 and all $h \in (0,1]$.

$\Rightarrow$ **Strategy**: to find a suitable decomposition

$$(1-\chi)\psi(h^2\Delta_G) = \sum_{j=1}^{N} A_j(h),$$

such that, for each j,

$$||A_j(h)e^{it\Delta_G}u_0||_{L_t^pL_x^q} \leq C||u_0||_{L^2},$$

for all u_0 and all $h \in (0,1]$.

Proof: spectral localization

Consequence It suffices to show that, for some χ,

$$||(1-\chi)\psi(h^2\Delta_G)e^{it\Delta_G}u_0||_{L_t^pL_x^q} \leq C||u_0||_{L^2},$$

for all u_0 and all $h \in (0,1]$.

$\Rightarrow$ **Strategy**: to find a suitable decomposition

$$(1-\chi)\psi(h^2\Delta_G) = \sum_{j=1}^{N} A_j(h),$$

such that, for each j,

$$||A_j(h)e^{it\Delta_G}u_0||_{L_t^pL_x^q} \leq C||u_0||_{L^2},$$

for all u_0 and all $h \in (0,1]$.

Proof: reduction to a semiclassical problem

The decomposition

$$(1-\chi)\psi(h^2\Delta_G) = \sum_{j=1}^{N} A_j(h),$$

will be obtained by *quantization* of a suitable partition of unity on the phase space $T^*\mathcal{M}$.

To deal with pseudo-differential operators, it is convenient to introduce

$$\widetilde{dG} = e^{-(n-1)r} dG,$$

since in local coordinates it is essentially

$$dr d\theta_1 \cdots d\theta_{n-1},$$

ie agrees with the Lebesgue measure (up to a harmless factor) and carries no exponential weight.

Proof: reduction to a semiclassical problem

The decomposition

$$(1-\chi)\psi(h^2\Delta_G) = \sum_{j=1}^{N} A_j(h),$$

will be obtained by *quantization* of a suitable partition of unity on the phase space $T^*\mathcal{M}$.
To deal with pseudo-differential operators, it is convenient to introduce

$$\widetilde{dG} = e^{-(n-1)r} dG,$$

since in local coordinates it is essentially

$$dr d\theta_1 \cdots d\theta_{n-1},$$

ie agrees with the Lebesgue measure (up to a harmless factor) and carries no exponential weight.

Proof: reduction to a semiclassical problem

The decomposition

$$(1-\chi)\psi(h^2\Delta_G) = \sum_{j=1}^{N} A_j(h),$$

will be obtained by *quantization* of a suitable partition of unity on the phase space $T^*\mathcal{M}$.
To deal with pseudo-differential operators, it is convenient to introduce

$$\widetilde{dG} = e^{-(n-1)r} dG,$$

since in local coordinates it is essentially

$$dr d\theta_1 \cdots d\theta_{n-1},$$

ie agrees with the Lebesgue measure (up to a harmless factor) and carries no exponential weight.

Proof: reduction to a semiclassical problem

We also set

$$P = -e^{(n-1)r/2} \Delta_G e^{-(n-1)r/2}$$

$\Rightarrow$ selfadjoint on $L^2(\mathcal{M}, \widetilde{dG})$ and unitarily equivalent to $-\Delta_G$.
The problem is then equivalent to find

$$(1-\chi)\psi(h^2 P) = \sum_{j=1}^{N} B_j(h),$$

such that

$$||e^{-(n-1)r/2} B_j(h) e^{-itP} \tilde{u}_0||_{L^p([0,1], L^q(\mathcal{M}, dG))} \leq C||\tilde{u}_0||_{L^2(\mathcal{M}, \widetilde{dG})}.$$

$\rightarrow$ if we can achieve this, then take

$$A_j(h) := e^{-(n-1)r/2} B_j(h) e^{(n-1)r/2}.$$

Proof: reduction to a semiclassical problem

We also set

$$P = -e^{(n-1)r/2}\Delta_G e^{-(n-1)r/2}$$

$\Rightarrow$ selfadjoint on $L^2(\mathcal{M}, \widetilde{dG})$ and unitarily equivalent to $-\Delta_G$.

The problem is then equivalent to find

$$(1-\chi)\psi(h^2P) = \sum_{j=1}^{N} B_j(h),$$

such that

$$||e^{-(n-1)r/2}B_j(h)e^{-itP}\tilde{u}_0||_{L^p([0,1],L^q(\mathcal{M},dG))} \leq C||\tilde{u}_0||_{L^2(\mathcal{M},\widetilde{dG})}.$$

$\rightarrow$ if we can achieve this, then take

$$A_j(h) := e^{-(n-1)r/2}B_j(h)e^{(n-1)r/2}.$$

Proof: reduction to a semiclassical problem

We also set

$$P = -e^{(n-1)r/2}\Delta_G e^{-(n-1)r/2}$$

$\Rightarrow$ selfadjoint on $L^2(\mathcal{M}, \widetilde{dG})$ and unitarily equivalent to $-\Delta_G$.
The problem is then equivalent to find

$$(1-\chi)\psi(h^2P) = \sum_{j=1}^{N} B_j(h),$$

such that

$$||e^{-(n-1)r/2}B_j(h)e^{-itP}\tilde{u}_0||_{L^p([0,1],L^q(\mathcal{M},dG))} \leq C||\tilde{u}_0||_{L^2(\mathcal{M},\widetilde{dG})}.$$

$\rightarrow$ if we can achieve this, then take

$$A_j(h) := e^{-(n-1)r/2}B_j(h)e^{(n-1)r/2}.$$

Proof: reduction to a semiclassical problem

We also set

$$P = -e^{(n-1)r/2}\Delta_G e^{-(n-1)r/2}$$

$\Rightarrow$ selfadjoint on $L^2(\mathcal{M}, \widetilde{dG})$ and unitarily equivalent to $-\Delta_G$.
The problem is then equivalent to find

$$(1-\chi)\psi(h^2P) = \sum_{j=1}^{N} B_j(h),$$

such that

$$||e^{-(n-1)r/2}B_j(h)e^{-itP}\tilde{u}_0||_{L^p([0,1],L^q(\mathcal{M},dG))} \leq C||\tilde{u}_0||_{L^2(\mathcal{M},\widetilde{dG})}.$$

$\rightarrow$ if we can achieve this, then take

$$A_j(h) := e^{-(n-1)r/2}B_j(h)e^{(n-1)r/2}.$$

Proof: reduction to a semiclassical problem

We also set

$$P = -e^{(n-1)r/2}\Delta_G e^{-(n-1)r/2}$$

$\Rightarrow$ selfadjoint on $L^2(\mathcal{M}, \widetilde{dG})$ and unitarily equivalent to $-\Delta_G$.
The problem is then equivalent to find

$$(1-\chi)\psi(h^2P) = \sum_{j=1}^{N} B_j(h),$$

such that

$$||e^{-(n-1)r/2}B_j(h)e^{-itP}\tilde{u}_0||_{L^p([0,1],L^q(\mathcal{M},dG))} \leq C||\tilde{u}_0||_{L^2(\mathcal{M},\widetilde{dG})}.$$

$\rightarrow$ if we can achieve this, then take

$$A_j(h) := e^{-(n-1)r/2}B_j(h)e^{(n-1)r/2}.$$

Proof: reduction to a semiclassical problem

By the **Keel-Tao** TT^* Theorem, the Strichartz estimates

$$||e^{-(n-1)r/2}B_j(h)e^{-itP}\tilde{u}_0||_{L^p([0,1],L^q(\mathcal{M},dG))} \leq C||\tilde{u}_0||_{L^2(\mathcal{M},\widetilde{dG})},$$

would follows from two types of estimates:

1. L^2 estimates,
2. $L^1 \to L^\infty$ estimates.

Proof: reduction to a semiclassical problem

By the **Keel-Tao** TT^* Theorem, the Strichartz estimates

$$||e^{-(n-1)r/2} B_j(h) e^{-itP} \tilde{u}_0||_{L^p([0,1],L^q(\mathcal{M},dG))} \leq C||\tilde{u}_0||_{L^2(\mathcal{M},\widetilde{dG})},$$

would follows from two types of estimates:

1. L^2 estimates,
2. $L^1 \to L^\infty$ estimates.

Proof: reduction to a semiclassical problem

1-<u>L^2 estimates</u>:

$$||B_j(h)||_{L^2(\mathcal{M},\widetilde{dG})\to L^2(\mathcal{M},\widetilde{dG})} \leq C,$$

for all $h \in (0,1]$.

2- <u>$L^1 \to L^\infty$ estimates</u> (dispersion estimates):

Proof: reduction to a semiclassical problem

1-L^2 estimates:

$$||B_j(h)||_{L^2(\mathcal{M},\widetilde{dG})\to L^2(\mathcal{M},\widetilde{dG})} \leq C,$$

for all $h \in (0,1]$.

2- $L^1 \to L^\infty$ estimates (dispersion estimates):

Proof: reduction to a semiclassical problem

1-$\underline{L^2 \text{ estimates}}$:

$$||B_j(h)||_{L^2(\mathcal{M},\widetilde{dG})\to L^2(\mathcal{M},\widetilde{dG})} \leq C,$$

for all $h \in (0,1]$.

2- $\underline{L^1 \to L^\infty \text{ estimates}}$ (dispersion estimates):

$$||e^{-(n-1)r/2}B_j(h)e^{itP}B_j(h)^*e^{-(n-1)r/2}||_{L^1(\mathcal{M},dG)\to L^\infty} \leq \frac{C}{|t|^{n/2}},$$

for $h \in (0,1]$ and $0 < |t| \leq 1$

Proof: reduction to a semiclassical problem

1-L^2 estimates:

$$||B_j(h)||_{L^2(\mathcal{M},\widetilde{dG})\to L^2(\mathcal{M},\widetilde{dG})} \leq C,$$

for all $h \in (0,1]$.

2- $L^1 \to L^\infty$ estimates (dispersion estimates):

$$||e^{-(n-1)r/2} B_j(h) e^{ihTP} B_j(h)^* e^{-(n-1)r/2}||_{L^1(\mathcal{M},dG)\to L^\infty} \leq \frac{C}{|hT|^{n/2}},$$

for all $h \in (0,1]$ and $0 < |T| \leq h^{-1}$.

Proof: a useful trick

Lemma Let $C(h) : L^2(\mathcal{M}, \widetilde{dG}) \to L^2(\mathcal{M}, dG)$ be a family of bounded operators. Then, the following properties are equivalent

$$||C(h)e^{iThP}C(h)^*||_{L^1\to L^\infty} \leq \frac{C}{|Th|^{n/2}}, \qquad h \in (0,1], \;\; 0 < |T| \leq h^{-1},$$

$$||C(h)e^{iThP}C(h)^*||_{L^1\to L^\infty} \leq \frac{C}{(Th)^{n/2}}, \qquad h \in (0,1], \;\; 0 < T \leq h^{-1},$$

$$||C(h)e^{iThP}C(h)^*||_{L^1\to L^\infty} \leq \frac{C}{|Th|^{n/2}}, \qquad h \in (0,1], \;\; -h^{-1} < T < 0.$$

$\Rightarrow$ It suffices to prove dispersion estimates in one sense of the time.

Proof: a useful trick

Lemma Let $C(h) : L^2(\mathcal{M}, \widetilde{dG}) \to L^2(\mathcal{M}, dG)$ be a family of bounded operators. Then, the following properties are equivalent

$$||C(h)e^{iThP}C(h)^*||_{L^1\to L^\infty} \leq \frac{C}{|Th|^{n/2}}, \qquad h \in (0,1], \;\; 0 < |T| \leq h^{-1},$$

$$||C(h)e^{iThP}C(h)^*||_{L^1\to L^\infty} \leq \frac{C}{(Th)^{n/2}}, \qquad h \in (0,1], \;\; 0 < T \leq h^{-1},$$

$$||C(h)e^{iThP}C(h)^*||_{L^1\to L^\infty} \leq \frac{C}{|Th|^{n/2}}, \qquad h \in (0,1], \;\; -h^{-1} < T < 0.$$

$\Rightarrow$ It suffices to prove dispersion estimates in <u>one sense</u> of the time.

Proof: a useful trick

Lemma Let $C(h) : L^2(\mathcal{M}, \widetilde{dG}) \to L^2(\mathcal{M}, dG)$ be a family of bounded operators. Then, the following properties are equivalent

$$||C(h)e^{iThP}C(h)^*||_{L^1\to L^\infty} \le \frac{C}{|Th|^{n/2}}, \qquad h \in (0,1], \;\; 0 < |T| \le h^{-1},$$

$$||C(h)e^{iThP}C(h)^*||_{L^1\to L^\infty} \le \frac{C}{(Th)^{n/2}}, \qquad h \in (0,1], \;\; 0 < T \le h^{-1},$$

$$||C(h)e^{iThP}C(h)^*||_{L^1\to L^\infty} \le \frac{C}{|Th|^{n/2}}, \qquad h \in (0,1], \;\; -h^{-1} < T < 0.$$

$\Rightarrow$ It suffices to prove dispersion estimates in one sense of the time.

Proof: a useful trick

Lemma Let $C(h) : L^2(\mathcal{M}, \widetilde{dG}) \to L^2(\mathcal{M}, dG)$ be a family of bounded operators. Then, the following properties are equivalent

$$||C(h)e^{iThP}C(h)^*||_{L^1 \to L^\infty} \leq \frac{C}{|Th|^{n/2}}, \qquad h \in (0,1], \ \ 0 < |T| \leq h^{-1},$$

$$||C(h)e^{iThP}C(h)^*||_{L^1 \to L^\infty} \leq \frac{C}{(Th)^{n/2}}, \qquad h \in (0,1], \ \ 0 < T \leq h^{-1},$$

$$||C(h)e^{iThP}C(h)^*||_{L^1 \to L^\infty} \leq \frac{C}{|Th|^{n/2}}, \qquad h \in (0,1], \ \ -h^{-1} < T < 0.$$

$\Rightarrow$ It suffices to prove dispersion estimates in <u>one sense</u> of the time.

Proof: pseudo-differential calculus

Given a symbol $a \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$, we recall that

$$Op_h(a)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

In other words, the Schwartz kernel of $Op_h(a)$ is

$$(2\pi h)^{-n} \int e^{\frac{i}{h}(x-y)\cdot\xi} a(x,\xi)d\xi.$$

Recall also the Calderon-Vaillancourt Theorem:

$$||Op_h(a)||_{L^2 \to L^2} \leq C, \qquad h \in (0, 1].$$

Proof: pseudo-differential calculus

Given a symbol $a \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$, we recall that

$$Op_h(a)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

In other words, the Schwartz kernel of $Op_h(a)$ is

$$(2\pi h)^{-n} \int e^{\frac{i}{h}(x-y)\cdot\xi} a(x,\xi)d\xi.$$

Recall also the Calderon-Vaillancourt Theorem:

$$||Op_h(a)||_{L^2\to L^2} \le C, \qquad h \in (0,1].$$

Proof: pseudo-differential calculus

Given a symbol $a \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$, we recall that

$$Op_h(a)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

In other words, the Schwartz kernel of $Op_h(a)$ is

$$(2\pi h)^{-n} \int e^{\frac{i}{h}(x-y)\cdot\xi} a(x, \xi)d\xi.$$

Recall also the Calderon-Vaillancourt Theorem:

$$||Op_h(a)||_{L^2\to L^2} \leq C, \qquad h \in (0, 1].$$

Proof: pseudo-differential calculus

Given a symbol $a \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$, we recall that

$$Op_h(a)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

In other words, the Schwartz kernel of $Op_h(a)$ is

$$(2\pi h)^{-n} \int e^{\frac{i}{h}(x-y)\cdot\xi} a(x,\xi)d\xi.$$

Recall also the Calderon-Vaillancourt Theorem:

$$||Op_h(a)||_{L^2\to L^2} \leq C, \qquad h \in (0,1].$$

Proof: pseudo-differential calculus

Assume (without loss of generality) that $\chi = \chi(r)$ and satisfies

$$\chi(r) = 1 \;\text{ for }\; r \leq R.$$

Proposition Up to "nice" remainder terms, $(1-\chi)\psi(h^2P)$ takes the following form in charts

$$Op_h\left(a_0 + ha_1 + \cdots + h^M a_M\right),$$

with

$$a_k(r,\theta,\rho,\eta) = b_k(r,\theta,\rho,e^{-r}\eta),$$

for some $b_k(r,\theta,\xi)$ compactly supported in ξ. More precisely,

$$\mathrm{supp}(a_k) \subset \{r \geq R\} \cap \{p(r,\theta,\rho,\eta) \in \mathrm{supp}(\psi)\}.$$

Proof: pseudo-differential calculus

Assume (without loss of generality) that $\chi = \chi(r)$ and satisfies

$$\chi(r) = 1 \text{ for } r \leq R.$$

Proposition Up to "nice" remainder terms, $(1 - \chi)\psi(h^2 P)$ takes the following form in charts

$$Op_h\left(a_0 + ha_1 + \cdots + h^M a_M\right),$$

with

$$a_k(r, \theta, \rho, \eta) = b_k(r, \theta, \rho, e^{-r}\eta),$$

for some $b_k(r, \theta, \xi)$ compactly supported in ξ. More precisely,

$$\text{supp}(a_k) \subset \{r \geq R\} \cap \{p(r, \theta, \rho, \eta) \in \text{supp}(\psi)\}.$$

Proof: pseudo-differential calculus

Assume (without loss of generality) that $\chi = \chi(r)$ and satisfies

$$\chi(r) = 1 \ \text{ for } \ r \leq R.$$

Proposition Up to "nice" remainder terms, $(1-\chi)\psi(h^2 P)$ takes the following form in charts

$$Op_h\left(a_0 + ha_1 + \cdots + h^M a_M\right),$$

with

$$a_k(r,\theta,\rho,\eta) = b_k(r,\theta,\rho,e^{-r}\eta),$$

for some $b_k(r,\theta,\xi)$ compactly supported in ξ. More precisely,

$$\text{supp}(a_k) \subset \{r \geq R\} \cap \{p(r,\theta,\rho,\eta) \in \text{supp}(\psi)\}.$$

Proof: The partition of unity

For $J \Subset (0, +\infty)$, we decompose the region of the phase space

$$\{r > R\} \cap \{p = p(r, \theta, \rho, \eta) \in J\} = \Gamma^+(R, J) \cup \Gamma^-(R, J),$$

with

$$\Gamma^\pm(R, J) = \left\{ r > R,\ p \in J,\ \pm \frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}.$$

They are the **outgoing** and **incoming** areas.

Proof: The partition of unity

For $J \Subset (0, +\infty)$, we decompose the region of the phase space

$$\{r > R\} \cap \{p = p(r, \theta, \rho, \eta) \in J\} = \Gamma^+(R, J) \cup \Gamma^-(R, J),$$

with

$$\Gamma^{\pm}(R, J) = \left\{ r > R,\ p \in J,\ \pm \frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}.$$

They are the **outgoing** and **incoming** areas.

Proof: The partition of unity

For $J \Subset (0, +\infty)$, we decompose the region of the phase space

$$\{r > R\} \cap \{p = p(r, \theta, \rho, \eta) \in J\} = \Gamma^+(R, J) \cup \Gamma^-(R, J),$$

with

$$\Gamma^{\pm}(R, J) = \left\{ r > R,\ p \in J,\ \pm\frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}.$$

They are the **outgoing** and **incoming** areas.

Proof: The partition of unity

For $\epsilon > 0$, set

$$\Gamma^+_{\mathrm{st}}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} > (1-\epsilon) \right\},$$

which we call **strongly outgoing areas** and

$$\Gamma^+_{\mathrm{inter}}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ (1-\epsilon) \geq \frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}$$

which we call **intermediate outgoing area** (+ similar definitions in the incoming case). Given an additional $\delta > 0$, we can cover

$$\Gamma^+_{\mathrm{inter}}(R, J, \epsilon) = \cup_{j=1}^{N-1} \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j \right\}$$

with

$$\cup_{j=1}^{N-1} K_j = [-1/2, 1-\epsilon], \qquad K_j = \text{ interval of length} \leq \delta.$$

Proof: The partition of unity

For $\epsilon > 0$, set

$$\Gamma^+_{\rm st}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} > (1-\epsilon) \right\},$$

which we call **strongly outgoing areas** and

$$\Gamma^+_{\rm inter}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ (1-\epsilon) \geq \frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}$$

which we call **intermediate outgoing area** (+ similar definitions in the incoming case). Given an additional $\delta > 0$, we can cover

$$\Gamma^+_{\rm inter}(R, J, \epsilon) = \cup_{j=1}^{N-1} \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j \right\}$$

with

$$\cup_{j=1}^{N-1} K_j = [-1/2, 1-\epsilon], \qquad K_j = \text{ interval of length} \leq \delta.$$

Proof: The partition of unity

For $\epsilon > 0$, set

$$\Gamma^{+}_{\mathrm{st}}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} > (1-\epsilon) \right\},$$

which we call **strongly outgoing areas** and

$$\Gamma^{+}_{\mathrm{inter}}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ (1-\epsilon) \geq \frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}$$

which we call **intermediate outgoing area** (+ similar definitions in the incoming case). Given an additional $\delta > 0$, we can cover

$$\Gamma^{+}_{\mathrm{inter}}(R, J, \epsilon) = \cup_{j=1}^{N-1} \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j \right\}$$

with

$$\cup_{j=1}^{N-1} K_j = [-1/2, 1-\epsilon], \qquad K_j = \text{ interval of length } \leq \delta.$$

Proof: The partition of unity

For $\epsilon > 0$, set

$$\Gamma^+_{\rm st}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} > (1-\epsilon) \right\},$$

which we call **strongly outgoing areas** and

$$\Gamma^+_{\rm inter}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ (1-\epsilon) \geq \frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}$$

which we call **intermediate outgoing area** (+ similar definitions in the incoming case). Given an additional $\delta > 0$, we can cover

$$\Gamma^+_{\rm inter}(R, J, \epsilon) = \cup_{j=1}^{N-1} \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j \right\}$$

with

$$\cup_{j=1}^{N-1} K_j = [-1/2, 1-\epsilon], \qquad K_j = \text{ interval of length } \leq \delta.$$

Proof: The partition of unity

For $\epsilon > 0$, set

$$\Gamma^+_{\mathrm{st}}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} > (1-\epsilon) \right\},$$

which we call **strongly outgoing areas** and

$$\Gamma^+_{\mathrm{inter}}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ (1-\epsilon) \geq \frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}$$

which we call **intermediate outgoing area** (+ similar definitions in the incoming case). Given an additional $\delta > 0$, we can cover

$$\Gamma^+_{\mathrm{inter}}(R, J, \epsilon) = \cup_{j=1}^{N-1} \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j \right\}$$

with

$$\cup_{j=1}^{N-1} K_j = [-1/2, 1-\epsilon], \qquad K_j = \text{ interval of length } \leq \delta.$$

Proof: The partition of unity

For $\epsilon > 0$, set

$$\Gamma^+_{\rm st}(R,J,\epsilon) = \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} > (1-\epsilon) \right\},$$

which we call **strongly outgoing areas** and

$$\Gamma^+_{\rm inter}(R,J,\epsilon) = \left\{ r > R,\ p \in J,\ (1-\epsilon) \geq \frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}$$

which we call **intermediate outgoing area** (+ similar definitions in the incoming case). Given an additional $\delta > 0$, we can cover

$$\Gamma^+_{\rm inter}(R,J,\epsilon) = \cup_{j=1}^{N-1} \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j \right\}$$

with

$$\cup_{j=1}^{N-1} K_j = [-1/2, 1-\epsilon], \qquad K_j = \text{ interval of length} \leq \delta.$$

Proof: The partition of unity

For $\epsilon > 0$, set

$$\Gamma^+_{\mathrm{st}}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} > (1 - \epsilon) \right\},$$

which we call **strongly outgoing areas** and

$$\Gamma^+_{\mathrm{inter}}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ (1 - \epsilon) \geq \frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}$$

which we call **intermediate outgoing area** (+ similar definitions in the incoming case). Given an additional $\delta > 0$, we can cover

$$\Gamma^+_{\mathrm{inter}}(R, J, \epsilon) = \cup_{j=1}^{N-1} \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j \right\}$$

with

$$\cup_{j=1}^{N-1} K_j = [-1/2, 1 - \epsilon], \qquad K_j = \text{ interval of length} \leq \delta.$$

Proof: The partition of unity

For $\epsilon > 0$, set

$$\Gamma^{+}_{\text{st}}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} > (1 - \epsilon) \right\},$$

which we call **strongly outgoing areas** and

$$\Gamma^{+}_{\text{inter}}(R, J, \epsilon) = \left\{ r > R,\ p \in J,\ (1 - \epsilon) \geq \frac{\rho}{p^{1/2}} > -\frac{1}{2} \right\}$$

which we call **intermediate outgoing area** (+ similar definitions in the incoming case). Given an additional $\delta > 0$, we can cover

$$\Gamma^{+}_{\text{inter}}(R, J, \epsilon) = \cup_{j=1}^{N-1} \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j \right\}$$

with

$$\cup_{j=1}^{N-1} K_j = [-1/2, 1 - \epsilon], \qquad K_j = \text{ interval of length} \leq \delta.$$

Proof: The partition of unity

For any $R \gg 1$, $\epsilon > 0$ and $\delta > 0$ we choose a partition of unity

$$\sum_{j=1}^{N} \chi_j^+ + \chi_j^- \equiv 1 \qquad \text{near}\{r > R\} \cap \{p \in J = \text{supp}(\psi)\}$$

such that

1. $$\text{supp}\left(\chi_N^{\pm}\right) \subset \Gamma_{\text{st}}^{\pm}(R, J, \epsilon),$$

2. for $j = 1, \dots, N-1$,

$$\text{supp}\left(\chi_j^{\pm}\right) \subset \left\{ r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j \right\},$$

and such that each $\chi_j^{\pm}$ has the form

$$\chi_j^{\pm}(r, \theta, \rho, \eta) = c_j^{\pm}(r, \theta, \rho, e^{-r}\eta), \qquad 1 \leq j \leq N,$$

with $c_j^{\pm}(r, \theta, \xi)$ compactly supported with respect to ξ.

Proof: The partition of unity

For any $R \gg 1$, $\epsilon > 0$ and $\delta > 0$ we choose a partition of unity

$$\sum_{j=1}^{N} \chi_j^+ + \chi_j^- \equiv 1 \qquad \text{near}\{r > R\} \cap \{p \in J = \text{supp}(\psi)\}$$

such that

1. $$\text{supp}\left(\chi_N^{\pm}\right) \subset \Gamma_{\text{st}}^{\pm}(R, J, \epsilon),$$

2. for $j = 1, \dots, N-1$,
$$\text{supp}\left(\chi_j^{\pm}\right) \subset \left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\},$$

and such that each $\chi_j^{\pm}$ has the form

$$\chi_j^{\pm}(r, \theta, \rho, \eta) = c_j^{\pm}(r, \theta, \rho, e^{-r}\eta), \qquad 1 \le j \le N,$$

with $c_j^{\pm}(r, \theta, \xi)$ compactly supported with respect to ξ.

Proof: The partition of unity

For any $R \gg 1$, $\epsilon > 0$ and $\delta > 0$ we choose a partition of unity

$$\sum_{j=1}^{N} \chi_j^+ + \chi_j^- \equiv 1 \qquad \text{near}\{r > R\} \cap \{p \in J = \text{supp}(\psi)\}$$

such that

1. $$\text{supp}\left(\chi_N^{\pm}\right) \subset \Gamma_{\text{st}}^{\pm}(R, J, \epsilon),$$

2. for $j = 1, \ldots, N-1$,

$$\text{supp}\left(\chi_j^{\pm}\right) \subset \left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\},$$

and such that each $\chi_j^{\pm}$ has the form

$$\chi_j^{\pm}(r, \theta, \rho, \eta) = c_j^{\pm}(r, \theta, \rho, e^{-r}\eta), \qquad 1 \leq j \leq N,$$

with $c_j^{\pm}(r, \theta, \xi)$ compactly supported with respect to ξ.

Proof: The partition of unity

The operators $B_j(h)$ will be pseudo-differential operators obtained after decomposition of

$$(1-\chi)\psi(h^2 P) \approx Op_h(a(h)),$$

according to our partition of unity, ie

$$a(h) = \sum_j \chi_j^+ a(h) + \sum_j \chi_j^- a(h),$$

($\rightarrow$ there are actually $2N$ operators $B_j(h)$).
For any choice of ϵ and δ, the bound

$$||B_j(h)||_{L^2 \rightarrow L^2} \leq C, \qquad h \in (0,1],$$

follows easily from the Calderon-Vaillancourt Theorem.
The tricky part is the proof of dispersion estimates.

Proof: The partition of unity

The operators $B_j(h)$ will be pseudo-differential operators obtained after decomposition of

$$(1-\chi)\psi(h^2P) \approx Op_h(a(h)),$$

according to our partition of unity, ie

$$a(h) = \sum_j \chi_j^+ a(h) + \sum_j \chi_j^- a(h),$$

($\rightarrow$ there are actually $2N$ operators $B_j(h)$).
For any choice of ϵ and δ, the bound

$$||B_j(h)||_{L^2\rightarrow L^2} \leq C, \qquad h \in (0,1],$$

follows easily from the Calderon-Vaillancourt Theorem.
The tricky part is the proof of dispersion estimates.

Proof: The partition of unity

The operators $B_j(h)$ will be pseudo-differential operators obtained after decomposition of

$$(1-\chi)\psi(h^2 P) \approx Op_h(a(h)),$$

according to our partition of unity, ie

$$a(h) = \sum_j \chi_j^+ a(h) + \sum_j \chi_j^- a(h),$$

($\rightarrow$ there are actually $2N$ operators $B_j(h)$).
For any choice of ϵ and δ, the bound

$$||B_j(h)||_{L^2 \rightarrow L^2} \leq C, \qquad h \in (0,1],$$

follows easily from the Calderon-Vaillancourt Theorem.
The tricky part is the proof of dispersion estimates.

Proof: The partition of unity

The operators $B_j(h)$ will be pseudo-differential operators obtained after decomposition of

$$(1-\chi)\psi(h^2 P) \approx Op_h(a(h)),$$

according to our partition of unity, ie

$$a(h) = \sum_j \chi_j^+ a(h) + \sum_j \chi_j^- a(h),$$

($\rightarrow$ there are actually $2N$ operators $B_j(h)$).
For any choice of ϵ and δ, the bound

$$||B_j(h)||_{L^2 \rightarrow L^2} \leq C, \qquad h \in (0,1],$$

follows easily from the Calderon-Vaillancourt Theorem.
The tricky part is the proof of dispersion estimates.

Proof: An Isozaki-Kitada parametrix

Proposition For $R \gg 1$, $0 < \epsilon \ll 1$, and any χ_+ such that

$$\chi_+(r, \theta, \rho, \eta) = c(r, \theta, \rho, e^{-r}\eta),$$

with $c(r, \theta, \xi)$ compactly supported with respect to ξ, and

$$\text{supp}(\chi_+) \subset \Gamma^+_{\text{st}}(R, J, \epsilon)$$

we have a parametrix valid for times $0 \leq T \leq h^{-1}$ of the form

$$e^{-iThP} Op_h(\chi_+) \approx \mathcal{H}_+(a_+(h)) e^{-iThD_r^2} \mathcal{H}_+(b_+(h))^*$$

where $\mathcal{H}_+(a)$ denotes an FIO with kernel of the form

$$(2\pi h)^{-n} \int\int e^{\frac{i}{h}(S_+(r,\theta,\rho,\eta) - r'\rho - \theta'\cdot\eta)} a(r, \theta, \rho, e^{-r}\eta) d\rho d\eta$$

with phase

$$S_+(r, \theta, \rho, \eta) = r\rho + \theta \cdot \eta + \frac{e^{-2r} q(\theta, \eta)}{4\rho} + \mathcal{O}(\epsilon^{1/2} e^{-2r} |\eta|^2).$$

Proof: An Isozaki-Kitada parametrix

Proposition For $R \gg 1$, $0 < \epsilon \ll 1$, and any χ_+ such that

$$\chi_+(r,\theta,\rho,\eta) = c(r,\theta,\rho,e^{-r}\eta),$$

with $c(r,\theta,\xi)$ compactly supported with respect to ξ, and

$$\text{supp}(\chi_+) \subset \Gamma^+_{\text{st}}(R,J,\epsilon)$$

we have a parametrix valid for times $0 \leq T \leq h^{-1}$ of the form

$$e^{-iThP} Op_h(\chi_+) \approx \mathcal{H}_+(a_+(h)) e^{-iThD_r^2} \mathcal{H}_+(b_+(h))^*$$

where $\mathcal{H}_+(a)$ denotes an FIO with kernel of the form

$$(2\pi h)^{-n} \int\int e^{\frac{i}{h}(S_+(r,\theta,\rho,\eta) - r'\rho - \theta'\cdot\eta)} a(r,\theta,\rho,e^{-r}\eta) d\rho d\eta$$

with phase

$$S_+(r,\theta,\rho,\eta) = r\rho + \theta\cdot\eta + \frac{e^{-2r}q(\theta,\eta)}{4\rho} + \mathcal{O}(\epsilon^{1/2}e^{-2r}|\eta|^2).$$

Proof: An Isozaki-Kitada parametrix

Proposition For $R \gg 1$, $0 < \epsilon \ll 1$, and any χ_+ such that

$$\chi_+(r,\theta,\rho,\eta) = c(r,\theta,\rho,e^{-r}\eta),$$

with $c(r,\theta,\xi)$ compactly supported with respect to ξ, and

$$\mathrm{supp}(\chi_+) \subset \Gamma^+_{\mathrm{st}}(R,J,\epsilon)$$

we have a parametrix valid for times $0 \leq T \leq h^{-1}$ of the form

$$e^{-iThP}Op_h(\chi_+) \approx \mathcal{H}_+(a_+(h))e^{-iThD_r^2}\mathcal{H}_+(b_+(h))^*$$

where $\mathcal{H}_+(a)$ denotes an FIO with kernel of the form

$$(2\pi h)^{-n}\int\int e^{\frac{i}{h}(S_+(r,\theta,\rho,\eta)-r'\rho-\theta'\cdot\eta)}a(r,\theta,\rho,e^{-r}\eta)d\rho d\eta$$

with phase

$$S_+(r,\theta,\rho,\eta) = r\rho + \theta\cdot\eta + \frac{e^{-2r}q(\theta,\eta)}{4\rho} + \mathcal{O}(\epsilon^{1/2}e^{-2r}|\eta|^2).$$

Proof: An Isozaki-Kitada parametrix

Proposition For $R \gg 1$, $0 < \epsilon \ll 1$, and any χ_+ such that

$$\chi_+(r, \theta, \rho, \eta) = c(r, \theta, \rho, e^{-r}\eta),$$

with $c(r, \theta, \xi)$ compactly supported with respect to ξ, and

$$\mathrm{supp}(\chi_+) \subset \Gamma^+_{\mathrm{st}}(R, J, \epsilon)$$

we have a parametrix valid for times $0 \leq T \leq h^{-1}$ of the form

$$e^{-iThP} Op_h(\chi_+) \approx \mathcal{H}_+(a_+(h)) e^{-iThD_r^2} \mathcal{H}_+(b_+(h))^*$$

where $\mathcal{H}_+(a)$ denotes an FIO with kernel of the form

$$(2\pi h)^{-n} \int\int e^{\frac{i}{h}(S_+(r,\theta,\rho,\eta) - r'\rho - \theta'\cdot\eta)} a(r, \theta, \rho, e^{-r}\eta) d\rho d\eta$$

with phase

$$S_+(r, \theta, \rho, \eta) = r\rho + \theta \cdot \eta + \frac{e^{-2r} q(\theta, \eta)}{4\rho} + \mathcal{O}(\epsilon^{1/2} e^{-2r} |\eta|^2).$$

Proof: An Isozaki-Kitada parametrix

Proposition For $R \gg 1$, $0 < \epsilon \ll 1$, and any χ_+ such that

$$\chi_+(r, \theta, \rho, \eta) = c(r, \theta, \rho, e^{-r}\eta),$$

with $c(r, \theta, \xi)$ compactly supported with respect to ξ, and

$$\mathrm{supp}(\chi_+) \subset \Gamma^+_{\mathrm{st}}(R, J, \epsilon)$$

we have a parametrix valid for times $0 \leq T \leq h^{-1}$ of the form

$$e^{-iThP} Op_h(\chi_+) \approx \mathcal{H}_+(a_+(h)) e^{-iThD_r^2} \mathcal{H}_+(b_+(h))^*$$

where $\mathcal{H}_+(a)$ denotes an FIO with kernel of the form

$$(2\pi h)^{-n} \int\int e^{\frac{i}{h}(S_+(r,\theta,\rho,\eta) - r'\rho - \theta'\cdot\eta)} a(r, \theta, \rho, e^{-r}\eta) d\rho d\eta$$

with phase

$$S_+(r, \theta, \rho, \eta) = r\rho + \theta \cdot \eta + \frac{e^{-2r} q(\theta, \eta)}{4\rho} + \mathcal{O}(\epsilon^{1/2} e^{-2r} |\eta|^2).$$

Proof: An Isozaki-Kitada parametrix

Proposition For $R \gg 1$, $0 < \epsilon \ll 1$, and any χ_+ such that

$$\chi_+(r,\theta,\rho,\eta) = c(r,\theta,\rho,e^{-r}\eta),$$

with $c(r,\theta,\xi)$ compactly supported with respect to ξ, and

$$\mathrm{supp}(\chi_+) \subset \Gamma^+_{\mathrm{st}}(R,J,\epsilon)$$

we have a parametrix valid for times $0 \leq T \leq h^{-1}$ of the form

$$e^{-iThP}Op_h(\chi_+) \approx \mathcal{H}_+(a_+(h))e^{-iThD_r^2}\mathcal{H}_+(b_+(h))^*$$

where $\mathcal{H}_+(a)$ denotes an FIO with kernel of the form

$$(2\pi h)^{-n}\int\int e^{\frac{i}{h}(S_+(r,\theta,\rho,\eta)-r'\rho-\theta'\cdot\eta)}a(r,\theta,\rho,e^{-r}\eta)d\rho d\eta$$

with phase

$$S_+(r,\theta,\rho,\eta) = r\rho + \theta\cdot\eta + \frac{e^{-2r}q(\theta,\eta)}{4\rho} + \mathcal{O}(\epsilon^{1/2}e^{-2r}|\eta|^2).$$

Proof: An Isozaki-Kitada parametrix

The dispersion estimates for the operators $B_j(h)$ localized in strongly outgoing (or incoming areas) reduces to estimate the L^∞ norm of the kernel. Up to remainders, it reduces to oscillatory integrals of the form

$$e^{-(n-1)\frac{r+r'}{2}} \int\int e^{\frac{i}{h}\Phi_t} \overline{a}(r,\theta,\rho,e^{-r}\eta) b(r',\theta',\rho,e^{-r'}\eta) \frac{d\rho d\eta}{(2\pi h)^n},$$

with

$$\begin{aligned} \Phi_t &= S_+(r,\theta,\rho,\eta) - t\rho^2 - S_+(r',\theta',\rho,\eta) \\ &\approx (r-r')\rho + (\theta-\theta')\cdot\eta - T\left(\rho^2 + \frac{e^{-2r'}q(\theta',\eta) - e^{-2r}q(\theta,\eta)}{4\rho T}\right) \end{aligned}$$

$\Rightarrow$ dispersion estimates via **stationary phase** (using that $r - r' \approx 2T\rho$ and $|e^{r'}(\theta-\theta')| \lesssim T$ after **non stationary phase**).

Proof: An Isozaki-Kitada parametrix

The dispersion estimates for the operators $B_j(h)$ localized in strongly outgoing (or incoming areas) reduces to estimate the L^∞ norm of the kernel. Up to remainders, it reduces to oscillatory integrals of the form

$$e^{-(n-1)\frac{r+r'}{2}} \int \int e^{\frac{i}{h}\Phi_t} \overline{a}(r,\theta,\rho,e^{-r}\eta) b(r',\theta',\rho,e^{-r'}\eta) \frac{d\rho d\eta}{(2\pi h)^n},$$

with

$$\begin{aligned} \Phi_t &= S_+(r,\theta,\rho,\eta) - t\rho^2 - S_+(r',\theta',\rho,\eta) \\ &\approx (r-r')\rho + (\theta-\theta')\cdot\eta - T\left(\rho^2 + \frac{e^{-2r'}q(\theta',\eta) - e^{-2r}q(\theta,\eta)}{4\rho T}\right) \end{aligned}$$

$\Rightarrow$ dispersion estimates via **stationary phase** (using that $r - r' \approx 2T\rho$ and $|e^{r'}(\theta-\theta')| \lesssim T$ after **non stationary phase**).

Proof: An Isozaki-Kitada parametrix

The dispersion estimates for the operators $B_j(h)$ localized in strongly outgoing (or incoming areas) reduces to estimate the L^∞ norm of the kernel. Up to remainders, it reduces to oscillatory integrals of the form

$$e^{-(n-1)\frac{r+r'}{2}} \int\int e^{\frac{i}{h}\Phi_t}\overline{a}(r,\theta,\rho,e^{-r}\eta)b(r',\theta',\rho,e^{-r'}\eta)\frac{d\rho d\eta}{(2\pi h)^n},$$

with

$$\begin{aligned}\Phi_t &= S_+(r,\theta,\rho,\eta) - t\rho^2 - S_+(r',\theta',\rho,\eta)\\ &\approx (r-r')\rho + (\theta-\theta')\cdot\eta - T\left(\rho^2 + \frac{e^{-2r'}q(\theta',\eta) - e^{-2r}q(\theta,\eta)}{4\rho T}\right)\end{aligned}$$

$\Rightarrow$ dispersion estimates via **stationary phase** (using that $r - r' \approx 2T\rho$ and $|e^{r'}(\theta-\theta')| \lesssim T$ after **non stationary phase**).

Proof: An Isozaki-Kitada parametrix

The dispersion estimates for the operators $B_j(h)$ localized in strongly outgoing (or incoming areas) reduces to estimate the L^∞ norm of the kernel. Up to remainders, it reduces to oscillatory integrals of the form

$$e^{-(n-1)\frac{r+r'}{2}} \int\int e^{\frac{i}{h}\Phi_t} \overline{a}(r,\theta,\rho,e^{-r}\eta) b(r',\theta',\rho,e^{-r'}\eta) \frac{d\rho d\eta}{(2\pi h)^n},$$

with

$$\begin{aligned}
\Phi_t &= S_+(r,\theta,\rho,\eta) - t\rho^2 - S_+(r',\theta',\rho,\eta) \\
&\approx (r-r')\rho + (\theta-\theta')\cdot\eta - T\left(\rho^2 + \frac{e^{-2r'}q(\theta',\eta) - e^{-2r}q(\theta,\eta)}{4\rho T}\right)
\end{aligned}$$

$\Rightarrow$ dispersion estimates via **stationary phase** (using that $r - r' \approx 2T\rho$ and $|e^{r'}(\theta-\theta')| \lesssim T$ after **non stationary phase**).

Proof: An Isozaki-Kitada parametrix

The dispersion estimates for the operators $B_j(h)$ localized in strongly outgoing (or incoming areas) reduces to estimate the L^∞ norm of the kernel. Up to remainders, it reduces to oscillatory integrals of the form

$$e^{-(n-1)\frac{r+r'}{2}} \int\int e^{\frac{i}{h}\Phi_t} \overline{a}(r,\theta,\rho,e^{-r}\eta) b(r',\theta',\rho,e^{-r'}\eta) \frac{d\rho d\eta}{(2\pi h)^n},$$

with

$$\begin{aligned} \Phi_t &= S_+(r,\theta,\rho,\eta) - t\rho^2 - S_+(r',\theta',\rho,\eta) \\ &\approx (r-r')\rho + (\theta-\theta')\cdot\eta - T\left(\rho^2 + \frac{e^{-2r'}q(\theta',\eta) - e^{-2r}q(\theta,\eta)}{4\rho T}\right) \end{aligned}$$

$\Rightarrow$ dispersion estimates via **stationary phase** (using that $r - r' \approx 2T\rho$ and $|e^{r'}(\theta-\theta')| \lesssim T$ after **non stationary phase**).

Proof: An Isozaki-Kitada parametrix

The dispersion estimates for the operators $B_j(h)$ localized in strongly outgoing (or incoming areas) reduces to estimate the L^∞ norm of the kernel. Up to remainders, it reduces to oscillatory integrals of the form

$$e^{-(n-1)\frac{r+r'}{2}} \int\int e^{\frac{i}{h}\Phi_t} \bar{a}(r,\theta,\rho,e^{-r}\eta) b(r',\theta',\rho,e^{-r'}\eta) \frac{d\rho d\eta}{(2\pi h)^n},$$

with

$$\begin{aligned} \Phi_t &= S_+(r,\theta,\rho,\eta) - t\rho^2 - S_+(r',\theta',\rho,\eta) \\ &\approx (r-r')\rho + (\theta-\theta')\cdot\eta - T\left(\rho^2 + \frac{e^{-2r'}q(\theta',\eta) - e^{-2r}q(\theta,\eta)}{4\rho T}\right) \end{aligned}$$

$\Rightarrow$ dispersion estimates via **stationary phase** (using that $r - r' \approx 2T\rho$ and $|e^{r'}(\theta-\theta')| \lesssim T$ after **non stationary phase**).

Proof: Dispersion estimates in intermediate areas

Proposition For fixed $\epsilon > 0$ and $T_0 > 0$ ($\sim$ injectivity radius), we can choose $\delta > 0$ small enough such that

$$\Phi^T_{\text{geodesic}}\left(\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}\right) \cap \left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}$$

is empty for times $T \geq T_0$ (similar incoming case for $T \leq -T_0$).

Corollary If $B_j(h)$ is localized in

$$\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\},$$

then

$$||B_j(h)e^{-iThP}B_j(h)^*||_{L^2(\mathcal{M},\widetilde{dG}) \to L^2(\mathcal{M},\widetilde{dG})} = \mathcal{O}(h^\infty),\ \ T_0 \leq T \leq h^{-1}.$$

Proof: Dispersion estimates in intermediate areas

Proposition For fixed $\epsilon > 0$ and $T_0 > 0$ ($\sim$ injectivity radius), we can choose $\delta > 0$ small enough such that

$$\Phi^T_{\text{geodesic}}\left(\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}\right) \cap \left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}$$

is empty for times $T \geq T_0$ (similar incoming case for $T \leq -T_0$).

Corollary If $B_j(h)$ is localized in

$$\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\},$$

then

$$||B_j(h)e^{-iThP}B_j(h)^*||_{L^2(\mathcal{M},\widetilde{dG}) \to L^2(\mathcal{M},\widetilde{dG})} = \mathcal{O}(h^\infty),\quad T_0 \leq T \leq h^{-1}.$$

Proof: Dispersion estimates in intermediate areas

Proposition For fixed $\epsilon > 0$ and $T_0 > 0$ ($\sim$ injectivity radius), we can choose $\delta > 0$ small enough such that

$$\Phi^{T}_{\text{geodesic}}\left(\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}\right) \cap \left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}$$

is empty for times $T \geq T_0$ (similar incoming case for $T \leq -T_0$).

Corollary If $B_j(h)$ is localized in

$$\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\},$$

then

$$||B_j(h)e^{-iThP}B_j(h)^*||_{L^2(\mathcal{M},\widetilde{dG}) \to L^2(\mathcal{M},\widetilde{dG})} = \mathcal{O}(h^\infty),\ \ T_0 \leq T \leq h^{-1}.$$

Proof: Dispersion estimates in intermediate areas

Proposition For fixed $\epsilon > 0$ and $T_0 > 0$ ($\sim$ injectivity radius), we can choose $\delta > 0$ small enough such that

$$\Phi^T_{\text{geodesic}}\left(\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}\right) \cap \left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}$$

is empty for times $T \geq T_0$ (similar incoming case for $T \leq -T_0$).

Corollary If $B_j(h)$ is localized in

$$\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\},$$

then

$$||B_j(h)e^{-iThP}B_j(h)^*||_{L^2(\mathcal{M},\widetilde{dG}) \to L^2(\mathcal{M},\widetilde{dG})} = \mathcal{O}(h^\infty),\ \ T_0 \leq T \leq h^{-1}.$$

Proof: Dispersion estimates in intermediate areas

Proposition For fixed $\epsilon > 0$ and $T_0 > 0$ ($\sim$ injectivity radius), we can choose $\delta > 0$ small enough such that

$$\Phi^T_{\mathrm{geodesic}}\left(\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}\right) \cap \left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}$$

is empty for times $T \geq T_0$ (similar incoming case for $T \leq -T_0$).

Corollary If $B_j(h)$ is localized in

$$\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\},$$

then

$$||B_j(h)e^{-iThP}B_j(h)^*||_{L^2(\mathcal{M},\widetilde{dG}) \to L^2(\mathcal{M},\widetilde{dG})} = \mathcal{O}(h^\infty), \quad T_0 \leq T \leq h^{-1}.$$

Proof: Dispersion estimates in intermediate areas

Proposition For fixed $\epsilon > 0$ and $T_0 > 0$ ($\sim$ injectivity radius), we can choose $\delta > 0$ small enough such that

$$\Phi^T_{\text{geodesic}}\left(\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}\right) \cap \left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}$$

is empty for times $T \geq T_0$ (similar incoming case for $T \leq -T_0$).

Corollary If $B_j(h)$ is localized in

$$\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\},$$

then

$$||B_j(h) e^{-iThP} B_j(h)^*||_{L^2(\mathcal{M},\widetilde{dG}) \to L^2(\mathcal{M},\widetilde{dG})} = \mathcal{O}(h^\infty),\ \ T_0 \leq T \leq h^{-1}.$$

Proof: Dispersion estimates in intermediate areas

Proposition For fixed $\epsilon > 0$ and $T_0 > 0$ ($\sim$ injectivity radius), we can choose $\delta > 0$ small enough such that

$$\Phi^T_{\mathrm{geodesic}}\left(\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}\right) \cap \left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\}$$

is empty for times $T \geq T_0$ (similar incoming case for $T \leq -T_0$).

Corollary If $B_j(h)$ is localized in

$$\left\{r > R,\ p \in J,\ \frac{\rho}{p^{1/2}} \in K_j\right\},$$

then

$$||B_j(h) e^{-iThP} B_j(h)^*||_{L^2(\mathcal{M},\widetilde{dG}) \to L^2(\mathcal{M},\widetilde{dG})} = \mathcal{O}(h^\infty),\ \ T_0 \leq T \leq h^{-1}.$$

Proof: Dispersion estimates in intermediate areas

By Sobolev embeddings, we obtain that

$$||e^{-(n-1)r/2}B_j(h)e^{-iThP}B_j(h)^*e^{-(n-1)r/2}||_{L^1\to L^\infty} = \mathcal{O}\left(|Th|^{-n/2}\right)$$

for times $T_0 \leq T \leq h^{-1}$.

The dispersion estimate for times $|T| \leq T_0$ follows again from the non **Stationary Phase Theorem** in a FIO approximation of

$$e^{-(n-1)r/2}B_j(h)e^{-iThP}B_j(h)^*e^{-(n-1)r/2}.$$

Proof: Dispersion estimates in intermediate areas

By Sobolev embeddings, we obtain that

$$||e^{-(n-1)r/2}B_j(h)e^{-iThP}B_j(h)^*e^{-(n-1)r/2}||_{L^1\to L^\infty} = \mathcal{O}\left(|Th|^{-n/2}\right)$$

for times $T_0 \leq T \leq h^{-1}$.

The dispersion estimate for times $|T| \leq T_0$ follows again from the non **Stationary Phase Theorem** in a FIO approximation of

$$e^{-(n-1)r/2}B_j(h)e^{-iThP}B_j(h)^*e^{-(n-1)r/2}.$$