Littlewood-Paley decompositions on manifolds with ends

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Abstract

For certain non compact Riemannian manifolds with ends which may or may not satisfy the doubling condition on the volume of geodesic balls, we obtain Littlewood-Paley type estimates on (weighted) L^p spaces, using the usual square function defined by a dyadic partition.

(French translation) Pour certaines variétés riemanniennes à bouts, satisfaisant ou non la condition de doublement de volume des boules géodésiques, nous obtenons des décompositions de Littlewood-Paley sur des espaces L^p (à poids), en utilisant la fonction carrée usuelle définie via une partition dyadique.

Keywords. Littlewood-Paley decomposition, square function, manifolds with ends, semiclassical analysis.

Mots-clefs. Décomposition de Littlewood-Paley, fonction carrée, variétés à bouts, analyse semi-classique.

Class. Math. : 42B20, 42B25, 58J40

1 Introduction

1.1 Motivation and description of the results

Let $(\mathcal{M}, g)$ be a Riemannian manifold, Δ_g be the Laplacian on functions and dg be the Riemannian measure. Consider a dyadic partition of unit, namely choose $\varphi_0 \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(0, +\infty)$ such that

$$1 = \varphi_0(\lambda) + \sum_{k \geq 0} \varphi(2^{-k}\lambda), \qquad \lambda \geq 0. \tag{1.1}$$

The existence of such a partition is standard. In this paper, we are basically interested in getting estimates of $||u||_{L^p(\mathcal{M},dg)}$ in terms of $\varphi(-2^{-k}\Delta_g)u$, either through the following square function

$$S_{-\Delta_g} u(\underline{x}) := \left(|\varphi_0(-\Delta_g)u(\underline{x})|^2 + \sum_{k \geq 0} |\varphi(-2^{-k}\Delta_g)u(\underline{x})|^2 \right)^{1/2}, \qquad \underline{x} \in \mathcal{M}, \tag{1.2}$$

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or, at least, through

$$\left(\sum_{k\geq 0}||\varphi(-2^k\Delta_g)u||^2_{L^p(\mathcal{M},dg)}\right)^{1/2},$$

and a certain remainder term. For the latter, we have typically in mind estimates of the form

$$||u||_{L^p(\mathcal{M},dg)} \lesssim \left(\sum_{k\geq 0}||\varphi(-2^k\Delta_g)u||^2_{L^p(\mathcal{M},dg)}\right)^{1/2} + ||u||_{L^2(\mathcal{M},dg)}, \tag{1.3}$$

for $p \geq 2$. In the best possible cases, we want to obtain the equivalence of norms

$$||S_{-\Delta_g}u||_{L^p(\mathcal{M},dg)} \approx ||u||_{L^p(\mathcal{M},dg)}, \tag{1.4}$$

which is well known, for $1 < p < \infty$, if $\mathcal{M} = \mathbb{R}^n$ and g is the Euclidean metric (see for instance [13, 12, 16]).

Such inequalities are typically of interest to localize at high frequencies the solutions (and the initial data) of partial differential equations involving the Laplacian such as the Schrödinger equation $i\partial_t u = \Delta_g u$ or the wave equation $\partial_t^2 u = \Delta_g u$, using that $\varphi(-h^2\Delta_g)$ commutes with Δ_g. For instance, estimates of the form (1.3) have been successfully used in [5] to prove Strichartz estimates for the Schrödinger equation on compact manifolds. The article [5] was the first source of inspiration of the present paper, a part which is to prove (1.3) for non compact manifolds. Another motivation came from the fact that, rather surprisingly, we were unable to find in the literature a reference for the equivalence (1.4) in reasonable cases such as asymptotically conical manifolds (the latter is certainly clear to specialists).

We point out that the equivalence (1.4) actually holds on compact manifolds, but (1.3) is sufficient to get Strichartz estimates. Moreover (1.3) is rather robust and still holds in many cases where (1.4) does not. For instance, on asymptotically hyperbolic manifolds where the volume of geodesic balls grows exponentially (with respect to their radii), (1.4) is not expected to hold, but, as a consequence of the results of the present paper, we have nevertheless (1.3). We will briefly recall the application of (1.3) to Strichartz estimates, and more precisely a spatially localized version thereof, after Theorem 1.7.

Littlewood-Paley inequalities on Riemannian manifolds are subjects of intensive studies. There is a vast literature in harmonic analysis studying continuous analogues of the square function (1.2), the so-called Littlewood-Paley-Stein functions defined via integrals involving the Poisson and heat semigroups [13]. An important point is to prove $L^p \to L^p$ bounds related to these square functions (see for instance [8] and [6]). However, as explained above, weaker estimates of the form (1.3) are often highly sufficient for applications to PDEs. Moreover, square functions of the form (1.2) are particularly convenient in microlocal analysis since they involve compactly supported functions of the Laplacian, rather than fast decaying ones. To illustrate heuristically this point, we consider the linear Schrödinger equation $i\partial_t u = \Delta_g u$: if the initial data is spectrally localized at frequency $2^{k/2}$, ie $\varphi(-2^{-k}\Delta_g)u(0,.) = u(0,.)$, there is microlocal finite propagation speed stating that the microlocal support (or wavefront set) of $u(t,.)$ is obtained by shifting the one of $u(0,.)$ along the geodesic flow at speed $\approx 2^{k/2}$. This property, which is very useful in the applications, fails if φ is not compactly supported (away from 0). Another similar interest of compactly supported spectral cutoffs for the Schrödinger equation is that high frequency asymptotics like the geometric optics approximation are much easier to obtain for spectrally localized data.

As far as dyadic decompositions associated to non constant coefficients operators are concerned, we have already mentioned [5]. We also have to quote the papers [7] and [10]. In [7], the authors develop a dyadic Littlewood-Paley theory for tensors on compact surfaces with limited regularity (but in low dimension) which is of great interest for nonlinear applications. In [10], the L^p equivalence of norms for dyadic square functions (including small frequencies) associated to Schrödinger operators are proved for a restricted range of p. See also the recent survey [9] for Schrödinger operators on $\mathbb{R}^n$.

In the present paper, we shall use the analysis of $\varphi(-h^2\Delta_g)$ for $h \in (0,1]$, obtained in [1], to derive Littlewood-Paley inequalities on manifolds with ends. We can summarize our results in a model case as follows (see Definition 1.1 for the general manifolds considered here). Assume for simplicity that a neighborhood of infinity of $(\mathcal{M}, g)$ is isometric to $\big((R,\infty)\times S, dr^2 + d\theta^2/w(r)^2\big)$, with $(S, d\theta^2)$ a compact manifold and $w(r) > 0$ a smooth bounded positive function. For instance $w(r) = r^{-1}$ corresponds to conical ends, and $w(r) = e^{-r}$ to hyperbolic ends. We first show that by considering the modified measure $\widetilde{dg} = w(r)^{1-n} dg \approx dr d\theta$ and the associated modified Laplacian $\widetilde{\Delta}_g = w(r)^{(1-n)/2}\Delta_g w(r)^{(n-1)/2}$, we always have the equivalence of norms

$$||S_{-\widetilde{\Delta}_g} u||_{L^p(\mathcal{M},\widetilde{dg})} \approx ||u||_{L^p(\mathcal{M},\widetilde{dg})},$$

for $1 < p < \infty$, the square function $S_{-\widetilde{\Delta}_g}$ being defined by changing Δ_g into $\widetilde{\Delta}_g$ in (1.2). By giving weighted version of this equivalence, we recover (1.4) when w^{-1} is of polynomial growth. Nevertheless, we emphasize that (1.4) can not hold in general for it implies that $\varphi(-\Delta_g)$ is bounded on $L^p(\mathcal{M}, dg)$ which may fail for instance in the hyperbolic case (see [15]). Secondly, we prove that more robust estimates of the form (1.3) always hold and can be spatially localized (see Theorem 1.7).

Here are the results.

Definition 1.1. *The manifold $(\mathcal{M}, g)$ is called almost asymptotic if there exist a compact set $\mathcal{K} \Subset \mathcal{M}$, a real number R, a compact manifold S, a function $r \in C^\infty(\mathcal{M},\mathbb{R})$ and a function $w \in C^\infty(\mathbb{R},(0,+\infty))$ with the following properties:*

1. *r is a coordinate near $\overline{\mathcal{M}\setminus\mathcal{K}}$ and*

$$r(\underline{x}) \to +\infty, \qquad \underline{x} \to \infty,$$

2. *for some $r_{\mathcal{K}} > 0$, there is a diffeomorphism*

$$\mathcal{M}\setminus\mathcal{K} \to (r_{\mathcal{K}}, +\infty)\times S, \tag{1.5}$$

through which the metric reads in local coordinates

$$g \quad = \quad G_{\rm unif}\left(r,\theta,dr,w(r)^{-1}d\theta\right) \tag{1.6}$$

with

$$G_{\rm unif}(r,\theta,V) := \sum_{1\le j,k\le n} G_{jk}(r,\theta)V_jV_k, \qquad V=(V_1,\dots,V_n)\in\mathbb{R}^n,$$

if $\theta = (\theta_1,\dots,\theta_{n-1})$ are local coordinates on S.

3. *The symmetric matrix $(G_{jk}(r,\theta))_{1\le j,k\le n}$ has smooth coefficients such that, locally uniformly with respect to θ,*

$$\left|\partial_r^j\partial_\theta^\alpha G_{jk}(r,\theta)\right| \quad \lesssim \quad 1, \qquad\qquad r > r_{\mathcal{K}}, \tag{1.7}$$

and is uniformly positive definite in the sense that, locally uniformly in θ,

$$G_{\rm unif}(r,\theta,V) \quad \approx \quad |V|^2, \qquad r > r_{\mathcal{K}},\ V\in\mathbb{R}^n. \tag{1.8}$$

4. The function w is smooth and satisfies, for all $k \in \mathbb{N}$,

$$\begin{aligned} w(r) &\lesssim 1, && (1.9) \\ w(r)/w(r') &\approx 1, \qquad \text{if } \ |r-r'| \leq 1 && (1.10) \\ \left| d^k w(r)/dr^k \right| &\lesssim w(r), && (1.11) \end{aligned}$$

for $r, r' \in \mathbb{R}$.

Typical examples are given by asymptotically conical manifolds for which $w(r) = r^{-1}$ (near infinity) or asymptotically hyperbolic ones for which $w(r) = e^{-r}$. We note that (1.10) is equivalent to the fact that, for some $C > 0$,

$$C^{-1}e^{-C|r-r'|} \leq \frac{w(r)}{w(r')} \leq Ce^{C|r-r'|}, \qquad r, r' \in \mathbb{R}. \tag{1.12}$$

In particular, this implies that $w(r) \gtrsim e^{-Cr}$.

We recall that, if $\theta = (\theta_1, \ldots, \theta_{n-1})$ are local coordinates on S and (r, θ) are the corresponding ones on $\mathcal{M} \setminus \mathcal{K}$, the Riemannian measure takes the following form near infinity

$$dg = w(r)^{1-n} b(r, \theta) dr d\theta_1 \ldots d\theta_{n-1} \tag{1.13}$$

with $b(r, \theta)$ bounded from above and from below for $r \gg 1$, locally uniformly with respect to θ. We also define the density

$$\widetilde{dg} = w(r)^{n-1} dg \tag{1.14}$$

and the operator

$$\widetilde{\Delta}_g = w(r)^{(1-n)/2} \Delta_g w(r)^{(n-1)/2}. \tag{1.15}$$

The multiplication by $w(r)^{(n-1)/2}$ is a unitary isomorphism between $L^2(\mathcal{M}, \widetilde{dg})$ and $L^2(\mathcal{M}, dg)$ so the operators Δ_g and $\widetilde{\Delta}_g$, which are respectively essentially self-adjoint on $L^2(\mathcal{M}, dg)$ and $L^2(\mathcal{M}, \widetilde{dg})$, are unitarily equivalent.

Let us denote by P either $-\Delta_g$ or $-\widetilde{\Delta}_g$. For $u \in C_0^\infty(\mathcal{M})$, we define the square function $S_P u$ related to the partition of unit (1.1) by

$$S_P u(\underline{x}) := \left(|\varphi_0(P) u(\underline{x})|^2 + \sum_{k \geq 0} |\varphi(2^{-k} P) u(\underline{x})|^2 \right)^{1/2}, \qquad \underline{x} \in \mathcal{M}. \tag{1.16}$$

Our first result is the following one.

Theorem 1.2. *For all $1 < p < \infty$, the following equivalence of norms holds*

$$||u||_{L^p(\mathcal{M}, \widetilde{dg})} \approx ||S_{-\widetilde{\Delta}_g} u||_{L^p(\mathcal{M}, \widetilde{dg})}.$$

This theorem implies in particular that $\varphi_0(-\widetilde{\Delta}_g)$ and $\varphi(-2^k \widetilde{\Delta}_g)$ are bounded on $L^p(\mathcal{M}, \widetilde{dg})$. For the Laplacian itself, it is known that compactly supported functions of Δ_g are in general not bounded on $L^p(\mathcal{M}, dg)$ (see [16]) so we can not hope to get the same property. We however have the following result.

Theorem 1.3. *For all $2 \leq p < \infty$ and all $M \geq 0$,*

$$||u||_{L^p(\mathcal{M},dg)} \lesssim ||S_{-\Delta_g}u||_{L^p(\mathcal{M},dg)} + ||(1-\Delta_g)^{-M}u||_{L^2(\mathcal{M},dg)}.$$

Using the fact that, for $p \geq 2$, $||(\sum_k |u_k|^2)^{1/2}||_{L^p} \leq (\sum_k ||u_k||_{L^p}^2)^{1/2}$, we obtain in particular:

Corollary 1.4. *For all $p \in [2,\infty)$,*

$$||u||_{L^p(\mathcal{M},\widetilde{dg})} \lesssim \left(\sum_{k\geq 0} ||\varphi(-2^k\widetilde{\Delta}_g)u||^2_{L^p(\mathcal{M},\widetilde{dg})} \right)^{1/2} + ||\varphi_0(-\widetilde{\Delta}_g)u||_{L^p(\mathcal{M},\widetilde{dg})}, \qquad (1.17)$$

$$||u||_{L^p(\mathcal{M},dg)} \lesssim \left(\sum_{k\geq 0} ||\varphi(-2^k\Delta_g)u||^2_{L^p(\mathcal{M},dg)} \right)^{1/2} + ||u||_{L^2(\mathcal{M},dg)}. \qquad (1.18)$$

Note the two different situations. In (1.18), we have an L^2 remainder which comes essentially from the Sobolev injection

$$(1-\Delta_g)^{-n/2-\epsilon} : L^2(\mathcal{M},dg) \to L^\infty(\mathcal{M}). \qquad (1.19)$$

The translation of (1.19) in terms of $\widetilde{\Delta}_g$ is that $w(r)^{(n-1)/2}(1-\widetilde{\Delta_g})^{-n/2-\epsilon}$ is bounded from $L^2(\mathcal{M},\widetilde{dg})$ to $L^\infty(\mathcal{M})$ which of course doesn't imply in general that $(1-\widetilde{\Delta_g})^{-n/2-\epsilon}$ is bounded from $L^2(\mathcal{M},\widetilde{dg})$ to $L^\infty(\mathcal{M})$. In particular, one can not clearly replace the last term of (1.17) by $||u||_{L^2(\mathcal{M},\widetilde{dg})}$. We may however notice that, by the results of [1], C_0^∞ functions of $\widetilde{\Delta}_g$ are bounded on $L^p(\mathcal{M},\widetilde{dg})$, for $1 < p < \infty$.

Actually, we have a result which is more general than Theorem 1.2. Consider a temperate weight $W : \mathbb{R} \to (0,+\infty)$, that is a positive function such that, for some $C, M > 0$,

$$W(r') \leq CW(r)(1+|r-r'|)^M, \qquad r,r' \in \mathbb{R}. \qquad (1.20)$$

Theorem 1.5. *For all $1 < p < \infty$, we have the equivalence of norms*

$$||W(r)u||_{L^p(\mathcal{M},\widetilde{dg})} \approx ||W(r)S_{-\widetilde{\Delta}_g}u||_{L^p(\mathcal{M},\widetilde{dg})}.$$

This is a weighted version of Theorem 1.2. Then, using that

$$L^p(\mathcal{M},dg) = w(r)^{\frac{n-1}{p}} L^p(\mathcal{M},\widetilde{dg}), \qquad p \in [1,\infty), \qquad (1.21)$$

and that products or (real) powers of weight functions are weight functions, we deduce the following result.

Corollary 1.6. *If w is a temperate weight, then for all $1 < p < \infty$, we have the equivalence of norms*

$$||W(r)u||_{L^p(\mathcal{M},dg)} \approx ||W(r)S_{-\Delta_g}u||_{L^p(\mathcal{M},dg)}.$$

Naturally, this result holds with $W = 1$ and we obtain (1.4) if w is a temperate weight. In particular, in the case of asymptotically euclidean manifolds, this provides a justification of Lemma 3.1 of [4].

As noted previously, Theorems 1.2 and 1.3 are interesting to localize some PDEs in frequency. In practice, it is often interesting to localize the datas both spatially and spectrally. For the latter, one requires additional knowledge on the spectral cutoffs, typically commutator estimates. Such estimates are rather straightforward consequences of the analysis of [1] and allow to prove the following localization property.

Theorem 1.7. *Let $\chi \in C^\infty(\mathcal{M})$ be constant outside a compact set (typically χ or $1-\chi$ compactly supported). Assume that $p \in [2,\infty)$ and that*

$$0 \le \frac{n}{2} - \frac{n}{p} \le 1. \tag{1.22}$$

Then

$$||\chi u||_{L^p(\mathcal{M},dg)} \lesssim \left(\sum_{k\ge 0} ||\chi\varphi(-2^k\Delta_g)u||^2_{L^p(\mathcal{M},dg)} \right)^{1/2} + ||u||_{L^2(\mathcal{M},dg)}. \tag{1.23}$$

This theorem could be generalized by considering for instance more general cutoffs, or even differential operators. We give only this simple version, which will be used in [2] to prove Strichartz estimates at infinity using semi-classical methods in the spirit of [3]. To make such applications of Theorem 1.7 clearer, we recall very briefly the interest of the estimate (1.23) for the proof of (spatially localized) Strichartz estimates. We follow [5]. If $u(t) = e^{it\Delta_g}u_0$ is the solution to the homogeneous linear Schrödinger equation with initial condition $u_0 \in L^2(\mathcal{M},dg)$, and if $(p_1,p_2) \ne (2,\infty)$ is a Schrödinger admissible pair of exponents, ie such that

$$\frac{2}{p_1} + \frac{n}{p_2} = \frac{n}{2}, \qquad p_1 \ge 2, \;\; p_2 \ge 2,$$

then p_2 satisfies (1.22) and (1.23) gives

$$||\chi e^{it\Delta_g}u_0||_{L^{p_2}(\mathcal{M},dg)} \lesssim \left(\sum_{k\ge 0} ||\chi e^{it\Delta_g}\varphi(-2^k\Delta_g)u_0||^2_{L^{p_2}(\mathcal{M},dg)} \right)^{1/2} + ||u_0||_{L^2(\mathcal{M},dg)}. \tag{1.24}$$

If we are able to prove Strichartz estimates for spectrally localized data, ie

$$\left(\int_0^1 ||\chi e^{it\Delta_g}\varphi(-2^k\Delta_g)u_0||^{p_1}_{L^{p_2}(\mathcal{M},dg)} dt \right)^{1/p_1} \le C||u_0||_{L^2(\mathcal{M},dg)}, \tag{1.25}$$

with C independent of k (and u_0 of course), then we can assume that the term u_0 is spectrally localized too, by applying a spectral cutoff $\tilde{\varphi}(-2^{-k}\Delta_g)$ with $\tilde{\varphi}\varphi = \varphi$. Proving (1.25) is a different topic, but we point out that the spectral localization simplifies significantly the analysis. We then automatically obtain the non spectrally localized version

$$\left(\int_0^1 ||\chi e^{it\Delta_g}u_0||^{p_1}_{L^{p_2}(\mathcal{M},dg)} dt \right)^{1/p_1} \le C||u_0||_{L^2(\mathcal{M},dg)},$$

by squaring (1.25) with a spectrally localized right hand side, and summing over k; the L^2 norms in the right hand side can be summed by almost orthogonality, and for the left hand side, one uses (1.24) to get

$$\left(\int_0^1 ||\chi e^{it\Delta_g}u_0||^{2\frac{p_1}{2}}_{L^{p_2}(\mathcal{M},dg)} dt \right)^{\frac{2}{p_1}} \lesssim \sum_k \left(\int_0^1 ||\chi e^{it\Delta_g}\varphi(-2^k\Delta_g)u_0||^{2\frac{p_1}{2}}_{L^{p_2}(\mathcal{M},dg)} dt \right)^{\frac{2}{p_1}} + ||u_0||^2_{L^2(\mathcal{M},dg)},$$

since $p_1 \ge 2$ and where the sum converges since each term is controlled by $||\widetilde{\varphi}(-2^k\Delta_g)u_0||^2_{L^2}$, using (1.25).

1.2 Outline of the proofs

In this subsection, we summarize the analysis of the next sections by giving the main tools leading to our Littlewood-Paley estimates and, as an illustration, by proving Theorem 1.3 and Corollary 1.6, in the slightly simpler situation where $W = 1$. The first tool relies on the results of [1] and the second one is the core of the present paper.

For simplicity, we denote by A_k the operators

$$A_0 = \varphi_0(P), \qquad A_k = \varphi(2^{1-k}P), \qquad k \geq 1, \tag{1.26}$$

and by

$$f_k : [0,1] \to \{-1, 1\}, \qquad k \geq 0,$$

the usual Rademacher sequence (see Section 5).

1st tool. Parametrix of the A_k. It consists in getting a decomposition of the operators A_k of the form

$$A_k = \Psi_k + R_k, \qquad k \geq 0, \tag{1.27}$$

where Ψ_k is a properly supported pseudo-differential operator (with kernel supported close to the diagonal) and R_k is a remainder satisfying good properties. More explicitly, we will use the following properties of the sequence $(R_k)_{k\geq 0}$:

1. If w is a temperate weight, then for all $p_1 \in (1, 2]$

$$\sum_k ||R_k||_{L^{p_1}(\mathcal{M},dg)\to L^{p_1}(\mathcal{M},dg)} < \infty. \tag{1.28}$$

This actually holds for all $1 < p_1 < \infty$ but we shall not need this property for $p_1 > 2$.

2. In the general case, if w is not necessarily temperate, then for all $M \geq 0$,

$$\sum_k ||(1-\Delta_g)^M R_k (1-\Delta_g)^M||_{L^2(\mathcal{M},dg)\to L^2(\mathcal{M},dg)} < \infty. \tag{1.29}$$

This means more precisely that, for all $M \geq 0$, we can split A_k according to (1.27) with $\Psi_k = \Psi_k(M)$ and $R_k = R_k(M)$ both depending on M with $(R_k(M))_{k\geq 0}$ satisfying (1.29).

This description is sufficient here. In Section 2, we will recall more precisely the results of [1] that will be used in this paper.

2nd tool: Singular integral estimates on the diagonal term. Using the precise description of the operators Ψ_k, we will show that, for all $p_1 \in (1, 2]$,

$$||\sum_{k=0}^{\overline{k}} f_k(t)\Psi_k||_{L^{p_1}(\mathcal{M},dg)\to L^{p_1}(\mathcal{M},dg)} \lesssim_{p_1} 1, \qquad t \in [0,1], \ \overline{k} \in \mathbb{N}. \tag{1.30}$$

As usual, this will come from an interpolation between a trivial L^2 bound and a non trivial weak L^1 bound. The L^2 bound, ie (1.30) with $p_1 = 2$, is obtained by writing $\Psi_k = A_k - R_k$ and by using the almost orthogonality of the operators A_k and the L^2 summability of the R_k (ie (1.29) with $M = 0$). The weak L^1 estimate will use a suitable Calderón-Zygmund decomposition which we now describe.

We first explain how to transfer the analysis to an open subset of $\mathbb{R}^n$ of the form

$$\Omega = (r_{\mathcal{K}}, +\infty)_r \times \mathbb{R}_\theta^{n-1}, \tag{1.31}$$

equipped with the measure

$$\nu = w(r)^{1-n} dr d\theta,$$

where $d\theta$ is the Lebesgue measure on $\mathbb{R}^{n-1}$. We simply use that $\mathcal{M}$ can be covered by finitely many coordinate patches $\mathcal{U}_\iota$ such that

$$\Psi_k = \sum_\iota \Psi_{k,\iota},$$

where each $\Psi_{k,\iota}$ has a Schwartz kernel supported in $\mathcal{U}_\iota \times \mathcal{U}_\iota$. Thus, for each ι, the operators $\sum_{k \leq \bar{k}} f_k(t) \Psi_{k,\iota}$ are fully described in the single chart $\mathcal{U}_\iota$ and, using local coordinates, we are left with operators acting on Ω and with kernels supported on $\Omega \times \Omega$. The measure ν is the expression of dg in coordinates, up to factor bounded from above and below (see (1.13)). Note that this kind of local charts are typically relevant in the neighborhood of infinity of $\mathcal{M}$ but compactly supported charts, and the expression of dg therein, can be artificially put under this form too.

Our Calderón-Zygmund decomposition for a function $u \in L^1(\Omega, d\nu)$ will be of the form $u = \tilde{u} + \sum_j u_j$ where $\tilde{u} \in L^\infty(\Omega, d\nu) \cap L^1(\Omega, d\nu)$ is the good part, and the u_j form the bad parts which will be supported either in 'small balls' of the form

$$\mathcal{B}(r_0, \theta_0, t_0) = \left\{ (r, \theta) \mid |r - r_0| + \frac{|\theta - \theta_0|}{w(r_0)} \leq t_0 \right\}, \tag{1.32}$$

with $t_0 \leq 1$, or in 'large cylinders' of the form

$$\mathcal{C}(r_0, \theta_0, t_0) = \left\{ (r, \theta) \mid |r - r_0| \leq 1 \ \text{ and } \ \frac{|\theta - \theta_0|}{w(r_0)} \leq t_0 \right\}, \tag{1.33}$$

with $t_0 > 1$. The reason for considering those sets is that the measure $d\nu$ is non doubling in general, in the sense that $\nu(\mathcal{B}(r_0, \theta_0, Dt_0))$ can not be estimated by $D^m \nu(\mathcal{B}(r_0, \theta_0, t_0))$ (for some m) uniformly with respect to (r_0, θ_0, t_0) if we allow large t_0. This is easily seen when $w(r) = e^{-r}$ for instance. We shall however exploit that, if we set

$$\mathcal{B}_D^*(r_0, \theta_0, t_0) = \left\{ (r, \theta) \mid |r - r_0| + \frac{|\theta - \theta_0|}{w(r_0)} \leq Dt_0 \right\}, \tag{1.34}$$

$$\mathcal{C}_D^*(r_0, \theta_0, t_0) = \left\{ (r, \theta) \mid |r - r_0| \leq 2 \ \text{ and } \ \frac{|\theta - \theta_0|}{w(r_0)} \leq Dt_0 \right\}, \tag{1.35}$$

we have the doubling property on the sets of small balls and large cylinders ie

$$\mathcal{Q} := \{ \mathcal{B}(r, \theta, t) \mid (r, \theta) \in \Omega, \ 0 < t \leq 1 \} \cup \{ \mathcal{C}(r, \theta, t) \mid (r, \theta) \in \Omega, \ t > 1 \}. \tag{1.36}$$

This is the meaning of the following proposition.

Proposition 1.8. *For all 'doubling' parameter $D > 1$, there exists $C = C(n, w, D)$ such that*

$$\nu(Q_D^*) \leq C(n, w, D) \nu(Q), \qquad Q \in \mathcal{Q},$$

where $Q_D^* = \mathcal{B}_D^*(r_0, \theta_0, t_0)$ if $Q = \mathcal{B}(r_0, \theta_0, t_0)$ or $\mathcal{C}_D^*(r_0, \theta_0, t_0)$ if $Q = \mathcal{C}(r_0, \theta_0, t_0)$.

Proof. Consider a cylinder $Q = \mathcal{C}(r_0, \theta_0, t_0)$. If ω_{n-1} is the volume of the unit euclidean ball on $\mathbb{R}^{n-1}$, then

$$\nu(Q) = \int_{r_0-1}^{r_0+1} w(r)^{1-n} dr \int_{|\theta-\theta_0|\leq t_0 w(r_0)} d\theta = \omega_{n-1} t_0^{n-1} \int_{r_0-1}^{r_0+1} \left(\frac{w(r_0)}{w(r)}\right)^{n-1} dr \tag{1.37}$$

and, similarly,

$$\nu(Q_D^*) = \omega_{n-1} D^{n-1} t_0^{n-1} \int_{r_0-2}^{r_0+2} \left(\frac{w(r_0)}{w(r)}\right)^{n-1} dr.$$

The result in this case follows from the fact that, by (1.12), $w(r_0)/w(r)$ is bounded from above and below, uniformly with respect to r_0 and $r \in [r_0 - 2, r_0 + 2]$. The case of balls is similar : if $Q = \mathcal{B}(r_0, \theta_0, t_0)$ with $t_0 \leq 1$, then

$$\nu(Q) = \iint_{|r-r_0|+|\theta-\theta_0|/w(r_0)\leq t_0} w(r)^{1-n} dr d\theta = \iint_{|s|+|\alpha|\leq t_0} \left(\frac{w(r_0)}{w(r_0+s)}\right)^{n-1} dr d\alpha \tag{1.38}$$

and

$$\nu(Q_D^*) = D^n \iint_{|s|+|\alpha|\leq t_0} \left(\frac{w(r_0)}{w(r_0+Ds)}\right)^{n-1} dr d\alpha.$$

The result comes again from (1.12) which shows that $w(r_0)/w(r_0 + sD)$ and $w(r_0)/w(r_0 + s)$ are uniformly bounded from above and below, with respect to $|s| \leq t_0 \leq 1$ and $r_0 \in \mathbb{R}$. $\square$

That the elements of $\mathcal{Q}$ will be enough for our purpose relies strongly on the proper support property of the operators.

We shall then obtain the weak L^1 estimate by adapting suitably the usual proof whose crucial point is the following. Let us denote by $K = K_{\overline{k},t}$ the expression in local coordinates of the kernel of $\sum_{k\leq\overline{k}} f_k(t)\Psi_{k,\iota}$, with respect to $d\nu$. If, for each $Q \in \mathcal{Q}$ centered at (r_0, θ_0), we define by $\Psi_{\mathcal{Q}}$ the operator with kernel $K(r, \theta, r', \theta') - K(r, \theta, r_0, \theta_0)$, then we will see that, at least for some $D > 1$,

$$\sup_{Q\in\mathcal{Q}} || \left(1 - \chi_{Q_D^*}\right) \Psi_Q \chi_Q ||_{L^1(\Omega,d\nu)\to L^1(\Omega,d\nu)} < \infty, \tag{1.39}$$

where χ_E denotes the characteristic function of the set E. More precisely this supremum will also be uniformly bounded with respect to the parameters $\overline{k}$ and t of (1.30).

We now explain the simple derivation of Theorem 1.3 and Corollary 1.6 from (1.28)/(1.29) and (1.30) above. For a sequence of operators $B = (B_k)_{k\geq 0}$ (typically $(A_k), (\Psi_k)$ and (R_k)), it will be convenient to denote

$$S_B u := \big(\sum_{k\geq 0} |B_k u|^2\big)^{1/2},$$

say for $u \in C_0^\infty(\mathcal{M})$. For instance, S_A is exactly the square function $S_{-\Delta_g}$ in (1.2).

Proof of Corollary 1.6. Using (1.28) and (1.30), we get

$$|| \sum_{k=0}^{\overline{k}} f_k(t) A_k ||_{L^{p_1}(\mathcal{M},dg)\to L^{p_1}(\mathcal{M},dg)} \lesssim_{p_1} 1, \qquad t \in [0,1],\ \overline{k} \in \mathbb{N}, \tag{1.40}$$

first for all $p_1 \in (1,2]$ and then automatically for all $p_1 \in (1,\infty)$ by taking the adjoint, since $\sum_{k\leq \bar{k}} f_k(t)A_k$ is selfadjoint. Once we have (1.40), the proof goes exactly as the usual one on $\mathbb{R}^n$. We recall this proof to emphasize the difference with Theorem 1.3. By Khinchine's inequality, (1.40) implies that

$$||S_A u||_{L^p(\mathcal{M},dg)} \lesssim_p ||u||_{L^p(\mathcal{M},dg)}, \tag{1.41}$$

for all $p \in (1,\infty)$, which is the expected upper bound on $S_{-\Delta_g}$. To get the lower bound, one writes

$$(u_1,u_2) := \int_{\mathcal{M}} \overline{u_1} u_2 dg = \sum_{\substack{k_1,k_2\geq 0 \\ |k_1-k_2|\leq 1}} \int \overline{A_{k_1}u_1}\ A_{k_2}u_2\ dg, \tag{1.42}$$

using the partition of unity $\sum_k A_k = 1$ and its almost orthogonality, ie $A_{k_1}A_{k_2} = 0$ for $|k_1-k_2| \geq 2$. Then, if $p_1, p_2 \in (1,\infty)$ are conjugate exponents, Hölder's inequality in (1.42) and the upper bound (1.41) give

$$\begin{aligned} |(u_1,u_2)| &\leq 3||S_A u_1||_{L^{p_1}(\mathcal{M},dg)}||S_A u_2||_{L^{p_2}(\mathcal{M},dg)} \\ &\lesssim_{p_1} ||u_1||_{L^{p_1}(\mathcal{M},dg)}||S_A u_2||_{L^{p_2}(\mathcal{M},dg)}, \end{aligned}$$

from which one clearly deduces the lower bound $||u_2||_{L^{p_2}(\mathcal{M},dg)} \lesssim_{p_2} ||S_A u_2||_{L^{p_2}(\mathcal{M},dg)}$.

Proof of Theorem 1.3. Using (1.30) and Khinchine's inequality, we get for all $p_1 \in (1,2]$

$$||S_\Psi u||_{L^{p_1}(\mathcal{M},dg)} \lesssim_{p_1} ||u||_{L^{p_1}(\mathcal{M},dg)}. \tag{1.43}$$

Denote by $p_2 \in [2,\infty)$ the conjugate exponent of p_1. For $u_1, u_2 \in C_0^\infty(\mathcal{M})$, we expand $A_{k_j}u_j = (\Psi_{k_j} + R_{k_j})u_j$ into the right hand side of (1.42) and using again the Hölder inequality, we obtain

$$\begin{aligned} |(u_1,u_2)| \leq\ & 3||S_\Psi u_1||_{L^{p_1}(\mathcal{M},dg)} \left(||S_\Psi u_2||_{L^{p_2}(\mathcal{M},dg)} + ||S_R u_2||_{L^{p_2}(\mathcal{M},dg)} \right) \\ & + ||u_1||_{L^{p_1}(\mathcal{M},dg)} \left(\sum_{|k_1-k_2|\leq 1} ||R_{k_1}^* \Psi_{k_2} u_2||_{L^{p_2}(\mathcal{M},dg)} + ||R_{k_1}^* R_{k_2} u_2||_{L^{p_2}(\mathcal{M},dg)} \right). \end{aligned}$$

By writing $R_{k_1}^*\Psi_{k_2} = R_{k_1}^* A_{k_2} - R_{k_1}^* R_{k_2}$, where A_{k_2} commutes with Δ_g, and combining the Sobolev injection (1.19) with (1.29) (which of course holds also for R_k^*) with M large enough, we have

$$\sum_{|k_1-k_2|\leq 1} ||R_{k_1}^* \Psi_{k_2} u_2||_{L^{p_2}(\mathcal{M},dg)} + ||R_{k_1}^* R_{k_2} u_2||_{L^{p_2}(\mathcal{M},dg)} \lesssim_{p_2} ||(1-\Delta_g)^{-M} u_2||_{L^2(\mathcal{M},dg)}.$$

We also note that (1.19) and (1.29) give

$$||S_R u_2||_{L^{p_2}(\mathcal{M},dg)} \lesssim_{p_2} ||(1-\Delta_g)^{-M} u_2||_{L^2(\mathcal{M},dg)}.$$

Now, since

$$||S_\Psi u_2||_{L^{p_2}(\mathcal{M},dg)} \leq ||S_{-\Delta_g} u_2||_{L^{p_2}(\mathcal{M},dg)} + ||S_R u_2||_{L^{p_2}(\mathcal{M},dg)},$$

and using the upper bound (1.43), all this implies that

$$|(u_1,u_2)| \lesssim_{p_2} ||u_1||_{L^{p_1}(\mathcal{M},dg)} \left(||S_{-\Delta_g} u_2||_{L^{p_2}(\mathcal{M},dg)} + ||(1-\Delta_g)^{-M} u_2||_{L^2(\mathcal{M},dg)} \right),$$

which yields

$$||u_2||_{L^{p_2}(\mathcal{M},dg)} \lesssim_{p_2} ||S_{-\Delta_g} u_2||_{L^{p_2}(\mathcal{M},dg)} + ||(1-\Delta_g)^{-M} u_2||_{L^2(\mathcal{M},dg)}.$$

2 Functional calculus

In this short section, we recall some results from [1] that will be used later in the paper. It will also serve as a motivation for the introduction of the operators studied in Sections 3 and 4.

We fix $\phi \in C_0^\infty(\mathbb{R})$ and consider a semiclassical parameter $h \in (0,1]$. In the applications ϕ will be either φ_0 or φ and h^2 will be of the form 2^{-k} (if $\phi = \varphi_0$ we shall only consider $h = 1$).

In the following theorem, P will denote either $-\Delta_g$ or $-\widetilde{\Delta}_g$ when similar statements hold for both operators (this notation is already used in (1.16)). Otherwise we will give separate statements for each operator.

We recall that $S^m = S^m(\mathbb{R}^d \times \mathbb{R}^d)$ denotes the space of symbols of order m, ie functions a such that $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim \langle \xi \rangle^{m-|\beta|}$, and that $S^{-\infty} = \cap_m S^m$.

Theorem 2.1. *[1] For all $N \in \mathbb{N}$, $\phi(h^2 P)$ can be decomposed into*

$$\phi(h^2P) = \sum_{j<N} h^j \Phi_j(P,h) + h^N \mathcal{R}_N(P,h),$$

where

- *each $\Phi_j(P,h)$ is a finite sum*[1] *of operators whose kernels (with respect to $w(r)^{1-n} dr d\theta$) are, in local coordinates, of the form*

$$h^{-n} \widehat{a}\left(r, \theta, \frac{r-r'}{h}, \frac{\theta-\theta'}{hw(r)}\right) \zeta(r-r', \theta-\theta') \tag{2.1}$$

  where $\zeta \in C_0^\infty(\mathbb{R}^d)$ is supported in the unit ball and $\widehat{a}$ is the Fourier transform with respect to ξ of a symbol $a(r,\theta,\xi) \in S^{-\infty}(\mathbb{R}^n_{r,\theta} \times \mathbb{R}^n_\xi)$ which is compactly supported with respect to ξ and furthermore such that

$$0 \notin \operatorname{supp}(\phi) \qquad \Rightarrow \qquad \operatorname{supp}(a) \subset \mathbb{R}^d \times \{c \leq |\xi| \leq C\}, \tag{2.2}$$

  for some $C > c > 0$. Furthermore, for each $j < N$ and $p \in [2,\infty]$, we have

$$||\Phi_j(-\Delta_g, h)||_{L^p(\mathcal{M},dg) \to L^p(\mathcal{M},dg)} \lesssim 1, \tag{2.3}$$

$$||\Phi_j(-\Delta_g, h)||_{L^2(\mathcal{M},dg) \to L^p(\mathcal{M},dg)} \lesssim h^{\frac{n}{p}-\frac{n}{2}}, \tag{2.4}$$

  for $h \in (0,1]$.

- *The remainder $\mathcal{R}_N(P,h)$ satisfies*

  1. *if $P = -\Delta_g$: for all $p \in [2,\infty]$ and all $M \geq 0$,*

$$\left|\left|(1-\Delta_g)^M \mathcal{R}_N(-\Delta_g, h)(1-\Delta_g)^M\right|\right|_{L^2(\mathcal{M},dg) \to L^p(\mathcal{M},dg)} \lesssim h^{-n\left(\frac{1}{2}-\frac{1}{p}\right)-4M}, \tag{2.5}$$

  2. *if $P = -\widetilde{\Delta}_g$: for all temperate weight W (see (1.20)) and all $1 < p < \infty$,*

$$\left|\left|W(r)^{-1} \mathcal{R}_N(-\widetilde{\Delta}_g, h) W(r)\right|\right|_{L^p(\mathcal{M},\widetilde{dg}) \to L^p(\mathcal{M},\widetilde{dg})} \lesssim 1, \tag{2.6}$$

  for all $h \in (0,1]$.

Operators with kernels of the form (2.1), and sums of such kernels, will play a great role in the sequel. We shall study some of their elementary properties in Section 4.

[1] the number of terms is simply the one of a finite cover of the manifold by coordinate charts

3 A Calderón-Zygmund type theorem

An important consequence of the usual Calderón-Zygmund theorem is that pseudo-differential operators of order 0 are bounded on $L^p(\mathbb{R}^n)$, for all $1 < p < \infty$ (see for instance [13, 12, 16]). The purpose of this section is to show a similar result for pseudo-differential operators with symbols of the form $a_w(r, \theta, \rho, \eta) = a(r, \theta, \rho, w(r)\eta)$, with $a \in S^0$, and with kernel cutoff outside a neighborhood of the diagonal to be properly supported. Recall that w may not be bounded from below (see Definition 1.1) so a_w doesn't belong to S^0 in general.

We use the same notation as in Subsection 1.2, namely

$$\Omega = (r_{\mathcal{K}}, +\infty) \times \mathbb{R}^{n-1}, \qquad \nu = w(r)^{1-n} dr d\theta,$$

and recall that ν (or $d\nu$) is essentially the expression of dg in charts near infinity, ie they coincide up to a positive factor which is bounded from above and below and thus irrelevant for L^p estimates. For convenience and with no loss of generality, we assume that $r_{\mathcal{K}} \in \mathbb{N}$ (see Appendix A).

The following proposition is a version of the Calderón-Zygmund covering lemma adapted to the measure $d\nu$ (and to the underlying metric $dr^2 + w(r)^{-2} d\theta^2$). We will use the balls $\mathcal{B}(r, \theta, t)$ and the cylinders $\mathcal{C}(r, \theta, t)$ introduced in Subsection 1.2.

Proposition 3.1. *There exists $C_0 = C_0(n, w)$ such that, for all $\lambda > 0$, any $u \in L^1(\Omega, d\nu)$ can be decomposed as*

$$u = \tilde{u} + \sum_{j \in \mathbb{N}} u_j, \tag{3.1}$$

for some $\tilde{u} \in L^1(\Omega, d\nu) \cap L^\infty(\Omega, d\nu)$ and some sequence $(u_j)_{j \in \mathbb{N}}$ of $L^1(\Omega, d\nu)$ such that

$$||\tilde{u}||_{L^1(\Omega, d\nu)} + \sum_j ||u_j||_{L^1(\Omega, d\nu)} \leq C_0 ||u||_{L^1(\Omega, d\nu)}, \tag{3.2}$$

$$||\tilde{u}||_{L^\infty(\Omega)} \leq C_0 \lambda, \tag{3.3}$$

and such that, for some sequence of subsets $(Q_j)_{j \in \mathbb{N}}$ of Ω of the form

$$Q_j = \begin{cases} \mathcal{B}(r_j, \theta_j, t_j) & \text{with } t_j < 1, \\ \text{or} & \\ \mathcal{C}(r_j, \theta_j, t_j) & \text{with } t_j \geq 1, \end{cases} \qquad \text{for some } \ (r_j, \theta_j) \in \Omega, \tag{3.4}$$

we also have

$$\int u_j d\nu = 0, \qquad \text{supp}(u_j) \ \subset \ Q_j, \tag{3.5}$$

$$\sum_j \nu(Q_j) \leq C_0 \lambda^{-1} ||u||_{L^1(\Omega, d\nu)}. \tag{3.6}$$

Proof. See Appendix A . □

Consider next a smooth function K of the form

$$K(r, \theta, r', \theta') = \kappa \left(r, \theta, r - r', \frac{\theta - \theta'}{w(r)} \right)$$

with κ smooth on $\mathbb{R}^{2d}$ and satisfying

$$\left|\partial_{\hat{\xi}}\kappa(r,\theta,\hat{\xi})\right| \leq |\hat{\xi}|^{-1-n}, \qquad (r,\theta)\in\Omega,\ \hat{\xi}\in\mathbb{R}^n\setminus\{0\}, \tag{3.7}$$

$$\operatorname{supp}(\kappa)\subset\Omega\times\{|\hat{\xi}|<1\}. \tag{3.8}$$

We then define the operator Ψ by

$$(\Psi u)(r,\theta) = \int_\Omega K(r,\theta,r',\theta')u(r',\theta')d\nu(r',\theta'). \tag{3.9}$$

The assumption (3.8) states that this operator is properly supported. Using the notation of Theorem 2.1 with $h=2^{-k/2}$, we shall see that kernels of the form

$$K_{\overline{k},t}(r,\theta,r',\theta') = \sum_{k=0}^{\overline{k}} f_k(t)2^{kn/2}\widehat{a}\left(r,\theta,2^{k/2}(r-r'),2^{k/2}\frac{\theta-\theta'}{w(r)}\right)\zeta\left(r-r',\frac{\theta-\theta'}{w(r)}\right), \tag{3.10}$$

satisfy the conditions (3.7) and (3.8) uniformly with respect to $\overline{k}$ and t. This will be checked in Section 4. We will also see that the kernel (3.10) is very close to the expression in local coordinates of the kernel of $\sum_{k\leq\overline{k}} f_k(t)\Psi_k$ (see (1.27) for Ψ_k).

In the rest of the present section, we consider the problem of the boundedness of Ψ on $L^p(\Omega,d\nu)$.

Theorem 3.2. *There exists C such that, for all Ψ as above satisfying the additional condition*

$$||\Psi||_{L^2(\Omega,d\nu)\to L^2(\Omega,d\nu)}\leq 1, \tag{3.11}$$

we have: for all $u\in L^1(\Omega,d\nu)$ and all $\lambda>0$

$$\nu\left(\{|\Psi u|>\lambda\}\right)\leq C\lambda^{-1}||u||_{L^1(\Omega,d\nu)}.$$

In other words, B is of weak type $(1,1)$ relatively to $d\nu$. The proof is very close to the usual one for singular integrals on $\mathbb{R}^n$ and rests on Proposition 3.4 below (which was already stated in (1.39)). The main difference with the case of $\mathbb{R}^n$ is that the sets Q do not need to describe the whole set of dyadic cubes of arbitrary sides, but only the set $\mathcal{Q}$ of small balls and large cylinders defined in (1.36).

We first need to recall the following well known lemma on singular integrals.

Lemma 3.3. *There exists a constant c_n such that, for all $t>0$, for all $\tilde{K}\in C^1(\mathbb{R}^{2n})$ satisfying*

$$|\partial_y\tilde{K}(x,y)|\leq|x-y|^{-n-1}, \qquad x\neq y,\quad x,y\in\mathbb{R}^n, \tag{3.12}$$

and for all continuous function

$$Y:\{|x|>2t\}\to\{|y|<t\},$$

we have

$$\int_{|x|>2t}|\tilde{K}(x,Y(x))-\tilde{K}(x,0)|dx\leq c_n. \tag{3.13}$$

Note that, in the usual form of this lemma, the function Y is simply given by $Y(x) = y$ with $|y| < t$ independent of x. Of course, if (3.12) is replaced by $|\partial_y \tilde{K}(x,y)| \leq C|x-y|^{-n-1}$ one has to replace c_n by $c_n C$ in the final estimate. For completeness, we recall the simple proof.

Proof. By the Taylor formula and (3.12), the left hand side of (3.13) is bounded by

$$\int_{|x|>2t} t||x|-t|^{-n-1}dx = \mathrm{vol}(\mathbb{S}^{n-1}) \int_{2t}^{\infty} tr^{n-1}(r-t)^{-n-1}dr$$

where the change of variable $\tilde{r} = r/t$ shows that the last integral is finite and independent of t. $\square$

If $Q = \mathcal{B}(r_0, \theta_0, t_0)$ or $\mathcal{C}(r_0, \theta_0, t_0)$, we define the operator Ψ_Q by

$$(\Psi_Q u)(r,\theta) = \int (K(r,\theta,r',\theta') - K(r,\theta,r_0,\theta_0))\, u(r',\theta')d\nu(r',\theta').$$

We have the following result.

Proposition 3.4. *There exists $D(n,w) > 1$ such that, for all $D \geq D(n,w)$ and for all operator Ψ of the form (3.9),*

$$\sup_{Q\in\mathcal{Q}} ||(1-\chi_{Q_D^*})\Psi_Q\chi_Q||_{L^1(\Omega,d\nu)\to L^1(\Omega,d\nu)} \leq c_n,$$

with the same c_n as in Lemma 3.3.

The following lemma states that large cylinders do not contribute to the supremum.

Lemma 3.5. *There exists $D'(n,w) > 1$ such that, for all $D \geq D'(n,w)$ and all Q of the form $\mathcal{C}(r_0,\theta_0,t_0)$ with $t_0 > 1$,*

$$(1-\chi_{Q_D^*})\Psi\chi_Q = (1-\chi_{Q_D^*})\Psi_Q\chi_Q = 0.$$

Proof. It is sufficient to show that, if D is large enough,

$$(1-\chi_{Q_D^*})(r,\theta)K(r,\theta,r',\theta')\chi_Q(r',\theta') = 0.$$

This will give the statement for Ψ and then automatically for Ψ_Q since $\chi_Q(r_0,\theta_0) = 1$. If $(r,\theta,r',\theta') \in \mathrm{supp}(K)$ with $(r',\theta') \in Q$, we have $|r-r_0| \leq |r-r'| + |r'-r_0| \leq 2$. Therefore, if we consider $(r,\theta) \notin Q_D^*$, we necessarily have

$$\frac{|\theta-\theta_0|}{w(r_0)} > Dt_0,$$

so that $|\theta-\theta'| > (D-1)t_0 w(r_0)$. Since $|r-r_0| \leq 2$, there exists C depending only on w and n such that $|\theta-\theta'| > C(D-1)t_0 w(r)$. Thus $\frac{|\theta-\theta'|}{w(r)} > C(D-1)t_0 > 1$ if D is large enough so that K must actually vanish and we get the result. $\square$

Proof of Proposition 3.4. By Lemma 3.5, we only have to consider $Q = \mathcal{B}(r_0,\theta_0,t_0)$ with $t_0 \leq 1$. For $D > 1$, let us set

$$I_{Q,D}(r',\theta') = \int_{\Omega\setminus Q_D^*} |K(r,\theta,r',\theta') - K(r,\theta,r_0,\theta_0)|\, d\nu(r,\theta),$$

so that the Schur Lemma yields the estimate

$$||(1-\chi_{Q_D^*})\Psi_Q\chi_Q||_{L^1(\Omega,d\nu)\to L^1(\Omega,d\nu)} \leq \sup_{(r',\theta')\in Q} I_{Q,D}(r',\theta').$$

Using the change of variables

$$(r,\theta)\mapsto(\tilde{r},\tilde{\theta}) = \left(r-r_0, \frac{\theta-\theta_0}{w(r)}\right),$$

we have

$$I_{Q,D}(r',\theta') = \int_{|\tilde{r}|+\frac{w(r_0+\tilde{r})}{w(r_0)}|\tilde{\theta}|>Dt_0} \left|K_{r_0,\theta_0}\left(\tilde{r},\tilde{\theta},r'-r_0,\frac{\theta'-\theta_0}{w(\tilde{r}+r_0)}\right) - K_{r_0,\theta_0}(\tilde{r},\tilde{\theta},0,0)\right| d\tilde{r}d\tilde{\theta},$$

with

$$K_{r_0,\theta_0}(\tilde{r},\tilde{\theta},r',\theta') = \kappa\left(\tilde{r}+r_0, w(\tilde{r}+r_0)\tilde{\theta}+\theta_0, \tilde{r}-r', \tilde{\theta}-\theta'\right).$$

Now observe that for any $(r',\theta')\in\mathcal{B}(r_0,\theta_0,t_0)$ and $(\tilde{r},\tilde{\theta})\in\mathbb{R}^d$, we have

$$K_{r_0,\theta_0}\left(\tilde{r},\tilde{\theta},r'-r_0,\frac{\theta'-\theta_0}{w(\tilde{r}+r_0)}\right)\neq 0 \qquad \Rightarrow \qquad |\tilde{r}|\leq|\tilde{r}-(r'-r_0)|+|r'-r_0|\leq 2.$$

In particular, we have $\frac{w(r_0+\tilde{r})}{w(r_0)}\approx 1$ so there exists $C_1\geq 1$, depending only on n and w, such that

$$I_{Q,D}(r',\theta') \leq \int_{|\tilde{r}|+|\tilde{\theta}|>\frac{D}{C_1}t_0} \left|K_{r_0,\theta_0}\left(\tilde{r},\tilde{\theta},r'-r_0,\frac{\theta'-\theta_0}{w(\tilde{r}+r_0)}\right) - K_{r_0,\theta_0}(\tilde{r},\tilde{\theta},0,0)\right| d\tilde{r}d\tilde{\theta}.$$

Similarly, there exists C_2 depending only on n and w such that

$$|r'-r_0| + \frac{|\theta'-\theta_0|}{w(\tilde{r}+r_0)} \leq |r'-r_0| + C_2\frac{|\theta'-\theta_0|}{w(r_0)} \leq C_2 t_0,$$

thus, if $D\geq 2C_1C_2$, (3.7) and Lemma 3.3 show that, for all $(r',\theta')\in\mathcal{B}(r_0,\theta_0,t_0)$,

$$I_{Q,D}(r',\theta')\leq c_n,$$

and the result follows. $\square$

Proof of Theorem 3.2. We use the decomposition (3.1) and set $v=\sum_j u_j$. We have

$$\nu\left(\{|\Psi u|>\lambda\}\right) \leq \nu\left(\{|\Psi\tilde{u}|>\lambda/2\}\right) + \nu\left(\{|\Psi v|>\lambda/2\}\right).$$

Since

$$||\Psi\tilde{u}||^2_{L^2(\Omega,d\nu)} \leq ||\tilde{u}||^2_{L^2(\Omega,d\nu)} \leq C_0^2\lambda||u||_{L^1(\Omega,d\nu)},$$

the second inequality being due to (3.3) and (3.2), the Tchebychev inequality yields

$$\nu\left(\{|\Psi\tilde{u}|>\lambda/2\}\right) \leq 4\lambda^{-2}||\Psi\tilde{u}||^2_{L^2(\Omega,d\nu)} \leq 4C_0^2\lambda^{-1}||u||_{L^1(\Omega,d\nu)}.$$

We now consider v by studying the contribution of each function u_j. Using the notation of Propositions 3.4, we fix $D>D(n,w)$. Since u_j has zero mean and is supported in $Q_j=\mathcal{B}(r_j,\theta_j,t_j)$ or $\mathcal{C}(r_j,\theta_j,t_j)$, we have

$$\Psi u_j = \Psi_{Q_j}u_j,$$

which implies, by Proposition 3.4, that

$$\big|\big|\Psi u_j\big|\big|_{L^1(\Omega\backslash (Q_j)^*_D, d\nu)} = \big|\big|(1-\chi_{(Q_j)^*_D})\Psi_{Q_j} u_j\big|\big|_{L^1(\Omega, d\nu)} \leq c_n ||u_j||_{L^1(\Omega,d\nu)}. \tag{3.14}$$

Now, if we set

$$\mathcal{O} = \cup_j (Q_j)^*_D,$$

then Proposition 1.8 and (3.6) show that

$$\nu(\mathcal{O}) \leq C(n,w,D) C_0 \lambda^{-1} ||u||_{L^1(\Omega),d\nu}. \tag{3.15}$$

On the other hand, (3.14) imply that $||\Psi u_j||_{L^1(\Omega\backslash\mathcal{O},d\nu)} \leq c_n ||u_j||_{L^1(\Omega,d\nu)}$ so, using (3.2), we get

$$||\Psi v||_{L^1(\Omega\backslash\mathcal{O},d\nu)} \leq c_n C_0 ||u||_{L^1(\Omega,d\nu)}, \tag{3.16}$$

and then

$$\begin{aligned} \nu\left(\{|\Psi v| > \lambda/2\}\right) &\leq \nu(\mathcal{O}) + \nu\left(\{|\chi_{\Omega\backslash\mathcal{O}}\Psi v| > \lambda/2\}\right) \\ &\leq C\lambda^{-1} ||u||_{L^1(\Omega,d\nu)}, \end{aligned}$$

using (3.15) and (3.16). This completes the proof. □

The boundedness on L^p is then a classical consequence of the Marcinkiewicz interpolation theorem (see for instance [13, 16]) and we obtain the following corollary.

Corollary 3.6. *For all $p \in (1,2]$, there exists C_p such that, for all Ψ of the form (3.9), with κ satisfying (3.7) and (3.8), such that (3.11) holds, we have*

$$||\Psi||_{L^p(\Omega,d\nu)\to L^p(\Omega,d\nu)} \leq C_p.$$

4 Pseudo-differential operators

In this part, we study elementary properties of certain properly supported pseudo-differential operators. The main goal is to prove that kernels of the form (3.10) satisfy the assumptions (3.7) and (3.8). We will also see that the associated operator is bounded on $L^2(\Omega, d\nu)$ and hence on $L^p(\Omega, d\nu)$ for $1 < p \leq 2$ by Corollary 3.6. We shall even see that this remains true on weighted spaces.

To put it in a slightly more general framework, we consider a bounded sequence $(a_k)_{k\in\mathbb{N}}$ in $S^{-\infty}$, namely such that for all $j \in \mathbb{N}, \alpha \in \mathbb{N}^{n-1}, \beta \in \mathbb{N}^n$ and all $m > 0$,

$$\left|\partial_r^j \partial_\theta^\alpha \partial_\xi^\beta a_k(r,\theta,\xi)\right| \leq C_{j\alpha\beta m}(1+|\xi|)^{-m}, \tag{4.17}$$

with a constant independent of k. Assume that these symbols are supported in $\Omega \times \mathbb{R}^n$ where Ω is given by (1.31). We also use a cutoff $\zeta \in C_0^\infty(\mathbb{R}^n)$, supported in the unit ball, and such that $\zeta \equiv 1$ near 0.

For all $\overline{k} \geq 0$, consider the kernel

$$K_{(\overline{k})}(r,\theta,r',\theta') = \sum_{k=0}^{\overline{k}} 2^{kn/2} \hat{a}_k \left(r,\theta, 2^{k/2}(r-r'), 2^{k/2}\frac{\theta-\theta'}{w(r)}\right) \zeta(r-r',\theta-\theta'), \tag{4.18}$$

where $\hat{a}_k$ is the partial Fourier transform of a with respect to ξ. Notice that (3.10) is not exactly of this form, due to the form of the cutoff near the diagonal. Lemma 4.2 below will prove that this makes essentially no difference, as far as the L^p boundedness is concerned.

Example. The typical example of operator with kernel of the form (4.18) is given by $\sum_{k\leq\overline{k}} f_k(t)\Psi_k$, using (1.27). This follows clearly from Theorem 2.1 since we have to consider a_k of the form $f_k(t)a$ with a as in Theorem 2.1 (with $h = 2^{-k/2}$).

Consider next the associated operator

$$(\Psi_{(\overline{k})}u)(r,\theta) = \int_\Omega K_{(\overline{k})}(r,\theta,r',\theta')u(r',\theta')d\nu(r',\theta'). \tag{4.19}$$

Throughout this section, we fix a positive function W defined on $\mathbb{R}$ such that, for some $C > 0$,

$$W(r) \leq CW(r'), \qquad \text{for all } r, r' \in \mathbb{R} \text{ such that } |r - r'| \leq 1. \tag{4.20}$$

Temperate weights satisfy clearly this condition but also powers of w, although w may not be a temperate weight.

The section is devoted to the proof of the following result.

Proposition 4.1. *Let $(a_k)_{k\in\mathbb{N}}$ be a family of symbols supported in $\Omega \times \mathbb{R}^n$ satisfying (4.17). If moreover there exists $C > 0$ such that, for all $k \geq C$,*

$$(r,\theta,\rho,\eta) = (r,\theta,\xi) \in \operatorname{supp}(a_k) \Rightarrow C^{-1} \leq |\rho| + |\eta| \leq C, \tag{4.21}$$

then, for all positive function W satisfying (4.20) and all $p \in (1,2]$,

$$||W(r)\Psi_{(\overline{k})}W(r)^{-1}||_{L^p(\Omega,d\nu)\to L^p(\Omega,d\nu)} \lesssim 1, \qquad \overline{k} \in \mathbb{N}.$$

The proof is divided into the next four lemmas.

Lemma 4.2. *Denote by ϱ the function*

$$\varrho(r,\theta,r',\theta') = \zeta(r-r',\theta-\theta') - \zeta\left(r-r', \frac{\theta-\theta'}{w(r)}\right)$$

and by J_k the function

$$J_k(r,\theta,r',\theta') = 2^{kn/2}\hat{a}_k\left(r,\theta,2^{k/2}(r-r'),2^{k/2}\frac{\theta-\theta'}{w(r)}\right)\varrho(r,\theta,r',\theta').$$

Define the operator Γ_k by

$$\Gamma_k u(r,\theta) = \int_\Omega J_k(r,\theta,r',\theta')u(r',\theta')d\nu(r',\theta').$$

Then, for all $p \in [1,\infty]$,

$$\sum_{k\geq 0}\big|\big|W(r)\Gamma_k W(r)^{-1}\big|\big|_{L^p(\Omega,d\nu)\to L^p(\Omega,d\nu)} < \infty. \tag{4.22}$$

Note that (4.22) can be written equivalently as

$$\sum_{k\geq 0}\left|\left|W(r)w(r)^{\frac{1-n}{p}}\Gamma_k W(r)^{-1}w(r)^{\frac{n-1}{p}}\right|\right|_{L^p(\Omega,drd\theta)\to L^p(\Omega,drd\theta)}<\infty. \tag{4.23}$$

using the Lebesgue measure $drd\theta$ (with the convention that $(n-1)/p=0$ if $p=\infty$).

Proof. Let us prove (4.23). For all $\gamma\in\mathbb{R}$, (1.10) implies that Ww^{γ} also satisfies an estimate of the form (4.20). We may therefore replace $Ww^{(1-n)/p}$ by W with no loss of generality. Then

$$(W(r)\Gamma_k W(r)^{-1}u)(r,\theta)=\int\!\!\int \widetilde{J}_k(r,\theta,r',\theta')u(r',\theta')dr'd\theta'$$

with

$$\widetilde{J}_k(r,\theta,r',\theta')=w(r)^{1-n}J_k(r,\theta,r',\theta')\times\frac{W(r)}{W(r')}.$$

Since $\zeta\equiv 1$ near 0 and w is bounded, there exists $c>0$ such that,

$$|r-r'|+\frac{|\theta-\theta'|}{w(r)}\geq c,\qquad \text{on the support of } \varrho. \tag{4.24}$$

Integrating by part in the integral defining $\hat{a}_k$, one sees that, for all $N\geq 0$, J_k takes the following form

$$(-1)^N 2^{-(2N-n)k/2}\widehat{\Delta^N_{\rho,\eta}a}_k\left(r,\theta,2^{k/2}(r-r'),2^{k/2}\frac{\theta-\theta'}{w(r)}\right)\left(|r-r'|^2+\frac{|\theta-\theta'|^2}{w(r)^2}\right)^{-N}\varrho(r,\theta,r',\theta').$$

By the uniform estimates in k (4.17), (4.20) and (4.24), this implies that, for all N, there exists C_N such that

$$|\widetilde{J}_k(r,\theta,r',\theta')|\leq C_N 2^{-Nk}w(r)^{1-n}\left(1+|r-r'|+\frac{|\theta-\theta'|}{w(r)}\right)^{-N}$$

for all $(r,\theta),(r',\theta')\in\Omega$ and all $k\in\mathbb{N}$. The result follows then from the usual Schur Lemma. $\square$

By Lemma 4.2, the L^p boundedness of $W(r)\Psi_{(\overline{k})}W(r)^{-1}$ is thus equivalent to the one of $W(r)\widetilde{\Psi}_{(\overline{k})}W(r)^{-1}$ with $\widetilde{\Psi}_{(\overline{k})}$ defined similarly to (4.19) by the kernel

$$\widetilde{K}_{(\overline{k})}(r,\theta,r',\theta')=\sum_{k=0}^{\overline{k}}2^{kn/2}\hat{a}_k\left(r,\theta,2^{k/2}(r-r'),2^{k/2}\frac{\theta-\theta'}{w(r)}\right)\zeta\left(r-r',\frac{\theta-\theta'}{w(r)}\right).$$

We can then write

$$\left(W(r)\widetilde{\Psi}_{(\overline{k})}W(r)^{-1}u\right)(r,\theta)=\int_{\Omega}\widetilde{\kappa}_{(\overline{k})}\left(r,\theta,r-r',\frac{\theta-\theta'}{w(r)}\right)u(r',\theta')d\nu(r',\theta')$$

where $\widetilde{\kappa}_{(\overline{k})}$ is defined by

$$\widetilde{\kappa}_{(\overline{k})}(r,\theta,\hat{\rho},\hat{\eta})=\sum_{k\leq M}2^{kn/2}\hat{a}_k(r,\theta,2^{k/2}\hat{\rho},2^{k/2}\hat{\eta})\zeta(\hat{\rho},\hat{\eta})\times\frac{W(r)}{W(r-\hat{\rho})}.$$

To interpret this operator as an operator of the form (3.9), with a symbol satisfying (3.7), we need W to be smooth. We thus assume for a while that, for all $j \geq 0$,

$$|d^j W(r)/dr^j| \lesssim W(r). \tag{4.25}$$

We shall see further on that this smoothness condition can be removed.

Lemma 4.3. *Assume (4.20) and (4.25). There exists $C > 0$ such that, for all $\overline{k} \geq 0$,*

$$|\partial_{\hat{\rho},\hat{\eta}} \widetilde{\kappa}_{(\overline{k})}(r,\theta,\hat{\rho},\hat{\eta})| \leq C(|\hat{\rho}| + |\hat{\eta}|)^{-n-1}. \tag{4.26}$$

Proof. It is standard. We recall it for completeness. Thanks to the cutoff ζ, it is sufficient to consider the region where $|\hat{\rho}| + |\hat{\eta}| < 1$. By (4.17), $\hat{a}_k(r,\theta,.,.)$ is bounded in the Schwartz space as (r,θ) and k vary and, by (1.10), (1.11) and (4.25), $W(r)/W(r-\hat{\rho})$ is bounded on the support of ζ together with its derivatives. Thus, for all $N > 0$,

$$\begin{aligned}
|\partial_{\hat{\rho},\hat{\eta}} \kappa_{M,W}(r,\theta,\hat{\rho},\hat{\eta})| &\leq C_N \sum_{k\geq 0} 2^{\frac{k}{2}(n+1)}(1 + 2^{k/2}|\hat{\rho}| + 2^{k/2}|\hat{\eta}|)^{-N} \\
&\leq C_N \sum_{k \leq k_0} 2^{\frac{k}{2}(n+1)} + C_N \sum_{k > k_0} 2^{\frac{k}{2}(n+1)} 2^{\frac{(k_0-k)}{2}N} \approx C_N 2^{\frac{k_0}{2}(n+1)}
\end{aligned}$$

with $k_0 = k_0(\hat{\rho},\hat{\eta})$ such that $2^{-\frac{k_0+1}{2}} \leq |\hat{\rho}| + |\hat{\eta}| < 2^{-\frac{k_0}{2}}$. The result follows. □

We next consider the L^2 boundedness.

Lemma 4.4. *Assume (4.20), (4.25) and the existence of $C > 1$ such that, for all $k \geq C$, we have and (4.21). Then there exists $C' > 0$ such that, for all $\overline{k} \geq 0$,*

$$||W(r)\Psi_{(\overline{k})} W(r)^{-1}||_{L^2(\Omega,d\nu)\to L^2(\Omega,d\nu)} \quad \leq \quad C'. \tag{4.27}$$

Proof. The uniform boundedness of the family $(W(r)\Psi_{(\overline{k})}W(r)^{-1})_{\overline{k}\geq 0}$ on $L^2(\Omega, d\nu)$ is equivalent to uniform boundedness, on $\mathcal{L}(L^2(\mathbb{R}^n, drd\theta))$, of the family of pseudo-differential operators with kernels

$$\iint e^{-i(r-r')\rho - i(\theta-\theta')\cdot\eta} \widetilde{a}_{(\overline{k})}(r,r',\theta,\theta',\rho,\eta) d\rho d\eta$$

where

$$\widetilde{a}_{(\overline{k})}(r,r',\theta,\theta',\rho,\eta) = \frac{W(r)w(r)^{\frac{n-1}{2}}}{W(r')w(r')^{\frac{n-1}{2}}} \zeta(r-r',\theta-\theta') \sum_{k\leq \overline{k}} a_k(r,\theta,2^{-k/2}\rho, 2^{-k/2}w(r)\eta).$$

The function in front of the sum is smooth and bounded as well as its derivatives, by (1.10), (1.11), (4.20), (4.25) and the compact support of ζ. The result is then a consequence of the Calderón-Vaillancourt Theorem since all derivatives of $\tilde{a}_{(\overline{k})}$ are bounded, uniformly with respect to $\overline{k}$, which is a consequence of the estimate

$$|\partial_x^\alpha \partial_\xi^\beta \sum_{C<k\leq \overline{k}} a_k(x, 2^{-k/2}\xi)| \leq C_{\alpha\beta} \qquad \overline{k} > 0,\ (x,\xi) = (r,\theta,\rho,\eta) \in \mathbb{R}^{2n}.$$

This follows from the uniform estimates (4.17) and the fact that the above sum contains a finite number of terms, independent of x, ξ and $\overline{k}$ since, by (4.21), $2^{-k/2}|\xi|$ belongs to $[C^{-1}, C]$ (in particular $|\xi| \gtrsim 1$) and

$$2^{-k/2}|\xi| \in [C^{-1}, C] \qquad \Rightarrow \qquad k/2 \in \left[\ln_2|\xi| - \ln_2 C, \ln_2|\xi| - \ln_2 C^{-1}\right]$$

where the number of half integer points in the last interval is bounded. The proof is complete. $\square$

The following lemma shows that we can assume that W also satisfies (4.25).

Lemma 4.5. *We can find $\widetilde{W}$ satisfying (4.20), (4.25) and such that, for some $C > 1$,*

$$W(r)/C \leq \widetilde{W}(r) \leq CW(r). \tag{4.28}$$

Proof. Choose a non zero, non negative $\omega \in C_0^\infty(-1,1)$ and set $\widetilde{W}(r) = \int W(r-s)\omega(s)ds$. Since

$$C^{-1} \leq W(r-s)/W(r) \leq C, \qquad s \in (-1,1),$$

we obtain (4.28), which implies in turn that (4.20) holds for $\widetilde{W}$ since

$$\frac{\widetilde{W}(r)}{\widetilde{W}(r')} = \frac{\widetilde{W}(r)}{W(r)} \frac{W(r)}{W(r')} \frac{W(r')}{\widetilde{W}(r')}$$

is bounded if $|r - r'| \leq 1$. This implies that

$$|\widetilde{W}^{(j)}(r)| = |\int W(r-s)\omega^{(j)}(s)ds| \lesssim W(r) \lesssim \widetilde{W}(r),$$

which shows that (4.25) holds for $\widetilde{W}$. $\square$

Proof of Proposition 4.1. By (4.28), the result holds if and only if it holds with $\widetilde{W}$ instead of W. We may therefore assume that W satisfies (4.25). By Lemma 4.4, the estimate is true with $p = 2$. Then, by Lemma 4.2, it is also true for $\widetilde{\Psi}_{(\overline{k})}$ with $p = 2$. By Lemma 4.3, we can apply Corollary 3.6 to obtain the estimate for all $1 < p \leq 2$ with $\widetilde{\Psi}_{(\overline{k})}$ instead of $\Psi_{(\overline{k})}$ and we conclude using again Lemma 4.2. $\square$

5 Proofs of the main results

In this section, P and $d\mu$ denote either $-\Delta_g$ and dg or $-\widetilde{\Delta}_g$ and $\widetilde{dg}$. Using the partition of unit (1.1), we denote as in Subsection 1.2,

$$A_0 = \varphi_0(P), \qquad A_k = \varphi(2^{-(k-1)}P), \qquad k \geq 1,$$

so that, in the strong sense on $L^2(\mathcal{M}, d\mu)$, we have

$$\sum_{k \geq 0} A_k = 1, \tag{5.1}$$

and the square function (1.16) reads

$$S_P u(\underline{x}) = (\sum_k |A_k u(\underline{x})|^2)^{1/2}, \qquad \underline{x} \in \mathcal{M}.$$

In the next subsections, we will use the following classical result of harmonic analysis. Recall first the definition of the usual Rademacher sequence $(f_k)_{k\geq 0}$. For $k = 0$, f_0 is the function given on $[0,1)$ by

$$f_0(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1/2 \\ -1 & \text{if } 1/2 < t < 1 \end{cases},$$

and then extended on $\mathbb{R}$ as a 1 periodic function. If $k \geq 1$, $f_k(t) = f(2^k t)$, for all $t \in \mathbb{R}$. These functions are orthonormal in $L^2([0,1])$. Given a sequence of complex numbers $(a_k)_{k\geq 0}$, if we set

$$F(t) = \sum_{k\geq 0} a_k f_k(t),$$

then, for all $1 < p < \infty$, the key estimate related to the Rademacher functions is the following well known Khinchine inequality (see for instance [11, p. 54] or [13, p. 276]),

$$||F||_{L^2([0,1])} = \big(\sum_{k\geq 0} |a_k|^2\big)^{1/2} \leq C_p ||F||_{L^p([0,1])}. \tag{5.2}$$

As an immediate consequence of (5.2), we have the following result.

Proposition 5.1. *Let $(B_k)_{k\geq 0}$ be a family of operators from $C_0^\infty(\mathcal{M})$ to $L^p(\mathcal{M}, d\mu)$, for some $1 < p < \infty$. Define the associated square function $S_B u$ by*

$$S_B u(\underline{x}) = \big(\sum_{k\geq 0} \big|(B_k u)(\underline{x})\big|^2\big)^{1/2}, \qquad \underline{x} \in \mathcal{M}.$$

Then we have

$$||S_B u||_{L^p(\mathcal{M},d\mu)} \leq C_p \sup_{\overline{k}\in\mathbb{N}} \sup_{t\in[0,1]} \big|\big| \sum_{k\leq \overline{k}} f_k(t) B_k u \big|\big|_{L^p(\mathcal{M},d\mu)}. \tag{5.3}$$

In particular, if

$$\big|\big| \sum_{k\leq\overline{k}} f_k(t) B_k u \big|\big|_{L^p(\mathcal{M},d\mu)} \lesssim ||u||_{L^p(\mathcal{M},d\mu)}, \qquad t \in [0,1],\ u \in C_0^\infty(\mathcal{M}),\ \overline{k} \geq 0,$$

then

$$||S_B u||_{L^p(\mathcal{M},d\mu)} \lesssim ||u||_{L^p(\mathcal{M},d\mu)}, \qquad u \in C_0^\infty(\mathcal{M}).$$

5.1 Proof of Theorems 1.2 and 1.5

In this part $P = -\widetilde{\Delta}_g$, $d\mu = \widetilde{dg}$ and W is a temperate weight. Using Theorem 2.1, in particular (2.6) for the remainder, and Proposition 4.1 (see also the Example between (4.18) and (4.19)), we obtain the following proposition.

Proposition 5.2. *For all $N \geq 0$, we can write*

$$A_k = \Psi_k + R_k,$$

with Ψ_k such that, for all $1 < p \leq 2$,

$$\big|\big| \sum_{k\leq\overline{k}} f_k(t) W(r) \Psi_k u \big|\big|_{L^p(\mathcal{M},d\mu)} \lesssim ||W(r)u||_{L^p(\mathcal{M},d\mu)}, \qquad t \in [0,1],\ u \in C_0^\infty(\mathcal{M}),\ \overline{k} \geq 0,$$

and R_k such that, for all $1 < p < \infty$,

$$\big|\big|W(r) R_k W(r)^{-1}\big|\big|_{L^p(\mathcal{M},d\mu)\to L^p(\mathcal{M},d\mu)} \lesssim 2^{-Nk}, \qquad k \geq 0.$$

We can now prove Theorem 1.5 (Theorem 1.2 corresponds to the special case $W \equiv 1$). This proof is the standard one to establish the equivalence of norms of u and $S_P u$ for the usual Littlewood-Paley decomposition on $\mathbb{R}^n$ (see for instance [12, 13, 16]). This is a weighted version of the proof of Corollary 1.6 displayed in Subsection 1.2 for $W = 1$.

Proof of Theorem 1.5. Define $A_k^W = W(r) A_k W(r)^{-1}$. By Proposition 5.2, we have

$$\Big|\Big| \sum_{k \leq \overline{k}} f_k(t) A_k^W u \Big|\Big|_{L^p(\mathcal{M},d\mu)} \lesssim ||u||_{L^p(\mathcal{M},d\mu)}, \qquad \overline{k} \geq 0, \ t \in [0,1], \ u \in C_0^\infty(\mathcal{M}), \tag{5.4}$$

first for $1 < p \leq 2$, and then for all $1 < p < \infty$ by taking the adjoint in the above estimate and replacing W by W^{-1}. By Proposition 5.1, this implies that

$$||W(r) S_P u||_{L^p(\mathcal{M},d\mu)} \lesssim ||W(r) u||_{L^p(\mathcal{M},d\mu)}, \qquad u \in C_0^\infty(\mathcal{M}), \tag{5.5}$$

for $1 < p < \infty$. By the Cauchy-Schwarz inequality in the sum (1.42), Hölder's inequality and (5.5) with W^{-1} instead of W, we obtain

$$\begin{aligned} \left| \int_{\mathcal{M}} \overline{u_1} u_2 d\mu \right| &\leq 3 ||W(r) S_P u_1||_{L^p(\mathcal{M},d\mu)} ||W(r)^{-1} S_P u_2||_{L^{p'}(\mathcal{M},d\mu)} \\ &\lesssim ||W(r) S_P u_1||_{L^p(\mathcal{M},d\mu)} ||W(r)^{-1} u_2||_{L^{p'}(\mathcal{M},d\mu)} \end{aligned}$$

for $1 < p < \infty$, p' being its conjugate exponent. This then yields the lower bound

$$||W(r) u_1||_{L^p(\mathcal{M},d\mu)} \lesssim ||W(r) S_P u_1||_{L^p(\mathcal{M},d\mu)}, \qquad u_1 \in C_0^\infty(\mathcal{M}),$$

which completes the proof. □

5.2 Proof of Theorem 1.3

In this part $P = -\Delta_g$ and $d\mu = dg$. We refer to Subsection 1.2 for the proof of Theorem 1.3 itself and only record here the following proposition which is a direct consequence of Theorem 2.1 and Proposition 4.1 (see also the Example between (4.18) and (4.19)) and which completely justifies the tools used in Subsection 1.2.

Proposition 5.3. *For all $N, M \geq 0$, we can write*

$$A_k = \Psi_k + R_k,$$

with Ψ_k satisfying, for all $1 < p \leq 2$,

$$\Big|\Big| \sum_{k \leq \overline{k}} f_k(t) \Psi_k u \Big|\Big|_{L^p(\mathcal{M},d\mu)} \lesssim ||u||_{L^p(\mathcal{M},d\mu)}, \qquad t \in [0,1], \ u \in C_0^\infty(\mathcal{M}), \ \overline{k} \geq 0,$$

and R_k satisfying, for all $2 \leq p \leq \infty$,

$$\big|\big| (1 - \Delta_g)^M R_k (1 - \Delta_g)^M \big|\big|_{L^2(\mathcal{M},d\mu) \to L^2(\mathcal{M},d\mu)} \lesssim 2^{-Nk}, \qquad k \geq 0.$$

5.3 Proof of Theorem 1.7

The proof is formally the same as the one of [3, Prop. 4.5] excepted that it uses the following commutator estimates for properly supported operators. Fix $\widetilde{\varphi} \in C_0^\infty(0,+\infty)$ such that

$$\widetilde{\varphi}\varphi = \varphi. \tag{5.6}$$

By Theorem 2.1, we can write

$$\varphi(h^2P) = \Phi(h) + h^2\mathcal{R}(h), \qquad \widetilde{\varphi}(h^2P) = \widetilde{\Phi}(h) + h^2\widetilde{\mathcal{R}}(h),$$

where $\Phi(h)$ and $\widetilde{\Phi}(h)$ are finite sums of properly supported pseudo-differential operators of the form $h^j\Phi_j(-\Delta_g, h)$. By (1.22) and (2.5), we may assume that

$$||\mathcal{R}(h)||_{L^2(\mathcal{M},dg)\to L^2(\mathcal{M},dg)} \lesssim 1, \qquad ||\mathcal{R}(h)||_{L^2(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \lesssim h^{-1},$$

and, by (1.22), (2.3) and (2.4),

$$||\Phi(h)||_{L^q(\mathcal{M},dg)\to L^q(\mathcal{M},dg)} \lesssim 1, \qquad ||\Phi(h)||_{L^2(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \lesssim h^{-1},$$

for each $q \in [2,\infty]$. Of course, the same estimates holds for $\widetilde{\mathcal{R}}(h)$ and $\widetilde{\Phi}(h)$ respectively. These estimates imply easily that

$$||[h^2\mathcal{R}(h),\chi]||_{L^2(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \lesssim h, \tag{5.7}$$

and that

$$\left|\left|\left[\widetilde{\varphi}(-h^2\Delta_g), \left[\varphi(-h^2\Delta_g),\chi\right]\right] - \left[\widetilde{\Phi}(h), [\Phi(h),\chi]\right]\right|\right|_{L^2(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \lesssim h. \tag{5.8}$$

We have also the commutator estimates

$$\begin{aligned} ||[\Phi(h),\chi]||_{L^2(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} &\lesssim 1, &(5.9)\\ \left|\left|\left[\widetilde{\Phi}(h), [\Phi(h),\chi]\right]\right|\right|_{L^2(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} &\lesssim h, &(5.10)\end{aligned}$$

although they don't obviously follow from Theorem 2.1. We shall prove them below but show first how they lead to Theorem 1.7.

Proof of Theorem 1.7. By (1.18) in Corollary 1.4, we have

$$||\chi u||_{L^p(\mathcal{M},dg)} \lesssim \left(\sum_{k\geq 0} ||\varphi(-2^k\Delta_g)\chi u||^2_{L^p(\mathcal{M},dg)}\right)^{1/2} + ||u||_{L^2(\mathcal{M},dg)}. \tag{5.11}$$

Using (5.6), we can write $\varphi(-h^2\Delta_g)\chi$ as the sum of the following three terms

$$\begin{aligned} Q_1(h) &= \widetilde{\varphi}(-h^2\Delta_g)\chi\varphi(-h^2\Delta_g),\\ Q_2(h) &= \left[\varphi(-h^2\Delta_g),\chi\right]\widetilde{\varphi}(-h^2\Delta_g),\\ Q_3(h) &= \left[\widetilde{\varphi}(-h^2\Delta_g), \left[\varphi(-h^2\Delta_g),\chi\right]\right].\end{aligned}$$

Since $\widetilde{\varphi}(-h^2\Delta_g)\chi\varphi(-h^2\Delta_g) = \widetilde{\Phi}(h)\chi\varphi(-h^2\Delta_g) + h^2\widetilde{\mathcal{R}}(h)$ where $\widetilde{\Phi}(h)$ is uniformly bounded on $L^p(\mathcal{M},dg)$ and $||h^2\widetilde{\mathcal{R}}(h)||_{L^2(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \lesssim h$, we have

$$||Q_1(h)u||_{L^p(\mathcal{M},dg)} \lesssim ||\chi\varphi(-h^2\Delta_g)||_{L^p(\mathcal{M},dg)} + h||u||_{L^2(\mathcal{M},dg)}. \tag{5.12}$$

Using (5.7) and (5.9), we also have

$$||Q_2(h)u||_{L^p(\mathcal{M},dg)} \lesssim ||\widetilde{\varphi}(-h^2\Delta_g)u||_{L^2(\mathcal{M},dg)}, \tag{5.13}$$

and, by (5.8) and (5.10),

$$||Q_3(h)u||_{L^p(\mathcal{M},dg)} \lesssim h||u||_{L^2(\mathcal{M},dg)}. \tag{5.14}$$

The result then follows from (5.11), (5.12), (5.13), (5.14) and

$$\sum_{h^2=2^{-k},k\in\mathbb{N}} ||\widetilde{\varphi}(-h^2\Delta_g)u||^2_{L^2(\mathcal{M},dg)} \lesssim ||u||^2_{L^2(\mathcal{M},dg)}$$

by almost orthogonality, since $\widetilde{\varphi}$ is supported away from 0, and the Spectral Theorem. $\square$

It remains to prove (5.9) and (5.10).

Proof of (5.9). By working in local coordinates, the study of $\Phi(h)$ is reduced to operators with kernels of the form (2.1). If $\chi_1(r,\theta)$ is the expression in local coordinates of χ, the kernel of $[\Phi(h),\chi]$ is of the form

$$(2\pi h)^{-d}\left(\chi_1(r,\theta)-\chi_1(r',\theta')\right)\iint e^{\frac{i}{h}(r-r')\rho+\frac{i}{h}(\theta-\theta')\cdot\eta}a(r,\theta,\rho,w(r)\eta)d\rho d\eta\zeta(r-r',\theta-\theta'),$$

which, by expanding $\chi_1(r,\theta)-\chi_1(r',\theta')$ according to Taylor's formula and by integrating by parts in the integral above, is the sum of

$$h\times(2\pi h)^{-d}\iint e^{\frac{i}{h}(r-r')\rho+\frac{i}{h}(\theta-\theta')\cdot\eta}a_1(r,\theta,\rho,w(r)\eta)d\rho d\eta, \tag{5.15}$$

and of a remainder of the form

$$h^2\times(2\pi h)^{-d}\iint e^{\frac{i}{h}(r-r')\rho+\frac{i}{h}(\theta-\theta')\cdot\eta}A_2(r,\theta,r',\theta',\rho,w(r)\eta)d\rho d\eta, \tag{5.16}$$

with symbols $a_1\in S^{-\infty}(\mathbb{R}^{2n})$ and $A_2\in S^{-\infty}(\mathbb{R}^{3n})$, ie such that, for all $m>0$, $\gamma\in\mathbb{N}^{2n}$ and $\Gamma\in\mathbb{N}^{3n}$,

$$\begin{aligned}|\partial^\gamma a_1(r,\theta,\rho,\eta)| &\leq C_{m,\gamma}(1+|\rho|+|\eta|)^{-m},\\ |\partial^\Gamma A_2(r,\theta,r',\theta',\rho,\eta)| &\leq C_{m,\Gamma}(1+|\rho|+|\eta|)^{-m}.\end{aligned}$$

These estimates use that all the derivatives of χ_1 are bounded, which is clear if χ is constant outside a compact set but would also holds for many other functions χ. The estimate (5.9) would therefore follow from the following lemma.

Lemma 5.4. *Let $A\in S^{-\infty}(\mathbb{R}^{3n})$ be supported in $\{r>R,\ r'>R\}$. Then, the operator $\mathcal{A}(h)$ with kernel*

$$(2\pi h)^{-d}\iint e^{\frac{i}{h}(r-r')\rho+\frac{i}{h}(\theta-\theta')\cdot\eta}A(r,\theta,r',\theta',\rho,w(r)\eta)d\rho d\eta\zeta(r-r',\theta-\theta'),$$

satisfies

$$||\mathcal{A}(h)||_{L^2((R,\infty)\times\mathbb{R}^{n-1},w(r)^{1-n}drd\theta)\to L^q((R,\infty)\times\mathbb{R}^{n-1},w(r)^{1-n}drd\theta)} \lesssim h^{\frac{n}{q}-\frac{n}{2}},$$

for each $q\geq 2$.

Proof. It follows by interpolation between the case $q = 2$ and $q = \infty$ as in Lemma 2.3 and Lemma 2.4 of [1]. $\square$

This lemma implies clearly (5.9) since both a_1 and A_2, respectively in (5.15) and (5.16), belong to $S^{-\infty}(\mathbb{R}^{3n})$. $\square$

Proof of (5.10). If we denote by $h^2\mathcal{A}_2^{\mathcal{M}}(h)$ the pullback on the manifold of the operator with kernel (5.16), it follows from Lemma 5.4 and the uniform boundedness of $\widetilde{\Phi}(h)$ on $L^2(\mathcal{M}, dg)$ and $L^p(\mathcal{M}, dg)$ that

$$||[\widetilde{\Phi}(h), h^2\mathcal{A}_2^{\mathcal{M}}(h)]||_{L^2(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \lesssim h.$$

Therefore, to prove (5.10), it remains to show that

$$||[\widetilde{\Phi}(h), h\mathcal{A}_1^{\mathcal{M}}(h)]||_{L^2(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \lesssim h, \tag{5.17}$$

with $h\mathcal{A}_1^{\mathcal{M}}(h)$ the pullback on the manifold of the operator with kernel (5.15). In other words, we only have to consider commutators of operators with kernels of the form (2.1). This is the purpose of what follows.

We recall first a composition formula for properly supported differential operators. Let $B_1(h)$ and $B_2(h)$ be properly supported pseudo-differential operators on $\mathbb{R}^n$ defined by the Schwartz kernels

$$K_j(x, y, h) = (2\pi h)^{-n} \int e^{\frac{i}{h}(x-y)\cdot\xi} b_j(x, \xi) d\xi \chi_j(x - y), \qquad j = 1, 2, \tag{5.18}$$

where $\chi_j \in C_0^\infty(\mathbb{R}^n)$, $\chi_j \equiv 1$ near 0 and b_j symbols in a class that will be specified below and which guarantees the convergence of the integrals. The kernel $K(x_1, x_3, h)$ of $B_1(h)B_2(h)$ is

$$(2\pi h)^{-2n} \iiint e^{\frac{i}{h}(x_1-x_2)\cdot\xi_1 + \frac{i}{h}(x_2-x_3)\cdot\xi_2} b_1(x_1, \xi_1)\chi(x_1 - x_2) b_2(x_2, \xi_2)\chi_2(x_2 - x_3) d\xi_1 d\xi_2 dx_2,$$

that is, using the change of variables $\xi_1 = \xi_2 + \tau$, $x_2 = x_1 + t$,

$$K(x_1, x_3, h) = (2\pi h)^{-n} \int e^{\frac{i}{h}(x_1-x_3)\cdot\xi_2} b(x_1, x_3, \xi_2, h) d\xi_2,$$

with

$$b(x_1, x_3, \xi_2, h) = (2\pi h)^{-n} \iint e^{-\frac{i}{h}t\cdot\tau} b_1(x_1, \xi_2 + \tau)\chi_1(-t) b_2(x_1 + t, \xi_2)\chi_2(x_1 + t - x_3) dt d\tau.$$

Since χ_1 and χ_2 are compactly supported, we can clearly choose $\chi_3 \in C_0^\infty(\mathbb{R}^n)$ equal to 1 near 0 such that, for all $t, x_1, x_3 \in \mathbb{R}^n$,

$$\chi_1(-t)\chi_2(x_1 + t - x_3) = \chi_3(x_1 - x_3)\chi_1(-t)\chi_2(x_1 + t - x_3),$$

which shows that

$$K(x_1, x_3, h) = K(x_1, x_3, h)\chi_3(x_1 - x_3).$$

Assume now that the symbols $b_j(x, \xi)$ are of the form

$$b_j(x, \xi) = a_j(r, \theta, \rho, w(r)\eta), \qquad a_j \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n), \tag{5.19}$$

with $x = (r, \theta)$ and $\xi = (\rho, \eta)$. Writing $t = (t_r, t_\theta)$ and $\tau = (\tau_\rho, \tau_\eta)$, we then have

$$\begin{aligned} b_1(x_1, \xi_2 + \tau) b_2(x_1 + t, \xi_2) &= a_1(r_1, \theta_1, \rho_2 + \tau_\rho, w(r_1)(\eta_2 + \tau_\eta)) \times \\ &\qquad a_2(r_1 + t_r, \theta_1 + t_\theta, \rho_2, w(r_1 + t_r)\eta_2) \\ &= A_{12}(r_1, \theta_1, t, \tau, \rho_2, w(r_1)\eta_2) \end{aligned}$$

with

$$A_{12}(r, \theta, t, \tau, \rho, \eta) = a_1(r, \theta, \rho + \tau_\rho, \eta + w(r)\tau_\eta) a_2 \left(r + t_r, \theta + t_\theta, \rho, \frac{w(r + t_r)}{w(r)} \eta \right).$$

Setting finally

$$A(r_1, \theta_1, r_3, \theta_3, t, \tau, \rho, \eta) = A_{12}(r_1, \theta_1, t, \tau, \rho, \eta) \chi_1(-t) \chi_2((r_1, \theta_1) - t - (r_3, \theta_3)),$$

we have proved the main part of the following result.

Lemma 5.5. *Let $B_1(h)$, $B_2(h)$ be pseudo-differential operators with kernels of the form (5.18) and with symbols of the form (5.19). Then the kernel of $B_1(h)B_2(h)$ is of the form*

$$(2\pi h)^{-n} \iint e^{\frac{i}{h}(r_1 - r_3)\rho + \frac{i}{h}(\theta_1 - \theta_3)\cdot\eta} a(r_1, \theta_1, r_3, \theta_3, \rho, w(r_1)\eta, h) d\rho d\eta \chi_3 \left((r_1, \theta_1) - (r_3, \theta_3) \right), \quad (5.20)$$

with

$$a(r_1, \theta_1, r_3, \theta_3, \rho, \eta, h) = (2\pi h)^{-n} \iint e^{-\frac{i}{h} t \cdot \tau} A(r_1, \theta_1, r_3, \theta_3, t, \tau, \rho, \eta) dt d\tau.$$

Furthermore,

$$a(r_1, \theta_1, r_3, \theta_3, \rho, \eta, h) = a_1(r_1, \theta_1, \rho, \eta) a_2(r_1, \theta_1, \rho, \eta) + h \tilde{a}_h(r_1, \theta_1, r_3, \theta_3, \rho, \eta, h), \quad (5.21)$$

with $\tilde{a}_h$ bounded in $S^{-\infty}(\mathbb{R}^{3n})$.

Proof. It remains to prove (5.21) which is standard. We first insert a compactly supported cutoff equal to 1 close to $\tau = 0$ in the integral defining a. In the remaining integral, corresponding to $|\tau| \gtrsim 1$, we can use standard non stationary phase estimates to get integrability with respect to τ as well as arbitrary large powers of h, showing that it is $\mathcal{O}(h^\infty)$ in $S^{-\infty}(\mathbb{R}^{3n})$ since A and all its derivatives are of rapid decay with respect to (ρ, η). For the latter, we simply use that $w(r + t_r)/w(r)$ is bounded from above and below since $|t| \lesssim 1$. The 'main' integral, where $|\tau| \leq 1$, is then clearly in $S^{-\infty}(\mathbb{R}^{3n})$ by the decay of A again. Furthermore, thanks to the compact support with respect to (t, τ), we can use the stationary phase theorem and this gives (5.21). □

As a direct consequence of Lemma 5.5, we obtain that $[B_1(h), B_2(h)] = hB(h)$, where $B(h)$ has a kernel as in Lemma 5.4. Therefore, using Lemma 5.4, we have

$$||[\widetilde{\Phi}(h), \mathcal{A}_1^{\mathcal{M}}(h)]||_{L^2(\mathcal{M}, dg) \to L^p(\mathcal{M}, dg)} \lesssim h \times h^{\frac{n}{p} - \frac{n}{2}} \lesssim 1,$$

which proves (5.17) and completes the proof of (5.10). □

A Proof of Proposition 3.1

We first define special families of partitions of Ω. Given $n_0 \in \mathbb{N}$ and $k \geq -n_0$ an integer, we denote by $\mathcal{P}(k)$ a countable partition of Ω, i.e.

$$\mathcal{P}(k) = (\mathcal{P}_l(k))_{l\in\mathbb{N}}, \qquad \Omega = \sqcup_{l\in\mathbb{N}}\mathcal{P}_l(k).$$

In the sequel, the sets $\mathcal{P}_l(k)$ will always be measurable and bounded.

Given a family of partitions $\mathcal{P} := (\mathcal{P}(k))_{k\geq -n_0}$, we shall say that

- $\mathcal{P}$ is non increasing if: for all $k \geq 1-n_0$ and all $l \in \mathbb{N}$, there exists $l' \in \mathbb{N}$ such that
$$\mathcal{P}_l(k) \subset \mathcal{P}_{l'}(k-1), \tag{A.22}$$
- $\mathcal{P}$ is locally finite if: for all compact subset $K \subset \Omega$ there exists a compact subset $K' \subset \Omega$ such that, for all $k \geq -n_0$,
$$\bigsqcup_{\substack{l\in\mathbb{N}\\ \mathcal{P}_l(k)\cap K\neq\emptyset}} \mathcal{P}_l(k) \subset K' \tag{A.23}$$
- $\mathcal{P}$ is of vanishing diameter if: there exists a sequence $\epsilon_k \to 0$ such that, for all $k \geq -n_0$ and all $l \in \mathbb{N}$ there exists $x_{k,l} \in \Omega$ such that
$$\mathcal{P}_l(k) \subset \{x \in \Omega \mid |x - x_{k,l}| \leq \epsilon_k\}.$$

The following useful remarks are easy to check.

Remarks. 1. If $\mathcal{P}$ is non increasing, in (A.22), l' is uniquely defined by l and k.
2. If $\mathcal{P}$ is non increasing, then it is locally finite if and only if for all compact subset K there exists another compact subset K' such that (A.23) holds for $k = -n_0$.
3. If $\mathcal{P}$ is non increasing, it follows by a simple induction that if $\mathcal{P}_{l_1}(k_1) \cap \mathcal{P}_{l_2}(k_2) \neq \emptyset$ for some $k_2 \geq k_1 \geq -n_0$ and $l_1, l_2 \in \mathbb{N}$, then $\mathcal{P}_{l_2}(k_2) \subset \mathcal{P}_{l_1}(k_1)$.

Definition A.1. *A family of partitions $(\mathcal{P}(k))_{k\geq -n_0}$ is admissible if it is non increasing, locally finite and of vanishing diameter.*

The proof of Proposition 3.1 is based on a suitable choice of admissible partitions which we now describe.

Construction of a family of admissible partitions. For $m = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$, we set

$$\square_m = [m_1, m_1+1) \times \cdots \times [m_{n-1}, m_{n-1}+1)$$

and for $\tau > 0$, we set $\tau\square_m = \{\tau\theta \ ; \ \theta \in \square_m\}$ so that $\sqcup_{m\in\mathbb{Z}^{n-1}}\tau\square_m = \mathbb{R}^{n-1}$ is a decomposition of $\mathbb{R}^{n-1}$ into cubes of side τ. Setting $k_+ = \max(0,k)$, we can define, for all $k \in \mathbb{Z}$,

$$\mathcal{P}_{(i,m)}(k) = 2^{-k_+}(i, i+1] \times 2^{-k}w([2^{-k_+}i])\square_m \tag{A.24}$$

for all $i \in \mathbb{N} \cap [2^{k_+}r_{\mathcal{K}}, \infty)$ and $m \in \mathbb{Z}^{n-1}$. Here $[2^{-k_+}i]$ denotes the integer part of $2^{-k_+}i$.

For notational convenience, we then relabel $(\mathcal{P}_{(i,m)}(k))_{(i,m)\in\mathbb{N}\cap[2^{k_+}r_{\mathcal{K}},\infty)\times\mathbb{Z}^{n-1}}$ as $(\mathcal{P}_l(k))_{l\in\mathbb{N}}$.

Let us notice that, for $k\in\mathbb{Z}$ and $l\in\mathbb{N}$, we have

$$\nu(\mathcal{P}_l(k)) = 2^{-k(n-1)}\int_{2^{-k_+}i}^{2^{-k_+}(i+1)}\left(\frac{w([2^{-k_+}i])}{w(r)}\right)^{n-1} dr.$$

Thus, using (1.10), there exists $C_2\geq 1$ such that

$$C_2^{1-n}2^{-k(n-1)} \leq \nu(\mathcal{P}_l(k)) \leq C_2^{n-1}2^{-k(n-1)} \qquad \text{if } k\leq 0, \tag{A.25}$$
$$C_2^{1-n}2^{-kn} \leq \nu(\mathcal{P}_l(k)) \leq C_2^{n-1}2^{-kn} \qquad \text{if } k\geq 1. \tag{A.26}$$

Lemma A.2. *For all $n_0\in\mathbb{N}$, $(\mathcal{P}(k))_{k\geq -n_0}\equiv\big((\mathcal{P}_l(k))_{l\in\mathbb{N}}\big)_{k\geq -n_0}$ (defined by (A.24)) is an admissible family of partitions of Ω. Furthermore, there exists $C_3>1$ independent of n_0 such that, for all $k\geq 1-n_0$ and all $l\in\mathbb{N}$,*

$$C_3^{-1}\leq\frac{\nu(\mathcal{P}_l(k))}{\nu(\mathcal{P}_{l'}(k-1))}\leq C_3, \tag{A.27}$$

with $l'=l'(k,l)$ the unique integer satisfying (A.22).

Let us already point out that our family of admissible partitions has been designed in order to have (A.27) which will be crucial in the proof of Lemma A.4 below.

Proof. For each $k\geq -n_0$, $\mathcal{P}(k)=(\mathcal{P}_l(k))_{l\in\mathbb{N}}$ is obviously a partition of Ω. Since w is bounded, the family $\mathcal{P}=(\mathcal{P}(k))_{k\geq -n_0}$ is of vanishing diameter ($\epsilon_k\approx 2^{-k}$). Let us prove that $\mathcal{P}$ is non increasing. If $k\leq 0$, we have

$$(i,i+1]\times 2^{-k}w(i)\Box_m\subset(i',i'+1]\times 2^{1-k}w(i')\Box_{m'}$$

provided $i=i'$ and $\Box_m\subset 2\Box_{m'}$, which clearly holds for some $m'\in\mathbb{Z}^{n-1}$. Thus (A.22) holds if $k\leq 0$. If $k\geq 1$, we remark that if

$$2^{-k}(i,i+1]\subset 2^{1-k}(i',i'+1] \tag{A.28}$$

then $[2^{-k}i]=[2^{1-k}i']$. This follows easily from the fact that $2^{-k}(i,i+1)\cap\mathbb{N}=2^{1-k}(i',i'+1)\cap\mathbb{N}=\emptyset$. Thus

$$2^{-k}(i,i+1]\times 2^{-k}w([2^{-k}i])\Box_m\subset 2^{1-k}(i',i'+1]\times 2^{1-k}w([2^{1-k}i'])\Box_{m'}$$

with i' such that (A.28) holds and m' such that $\Box_m\subset 2\Box_{m'}$. Therefore $\mathcal{P}$ is non increasing. Using Remark 2, it is then easy to check that $\mathcal{P}$ is locally finite and hence admissible. The estimate (A.27) follows from (A.25) and (A.26). $\Box$

We now recall a basic result which is a version of Lebesgue's Lemma.

Lemma A.3. *Let $u\in L^1(\Omega,d\nu)$ and $\mathcal{P}$ be an admissible family of partitions of Ω. Assume that $A\subset\Omega$ is a measurable subset such that there exists $C>0$ satisfying: for all $k\geq -n_0$ and all $l\in\mathbb{N}$*

$$\mathcal{P}_l(k)\cap A\neq\emptyset \quad\Rightarrow\quad \frac{1}{\nu(\mathcal{P}_l(k))}\int_{\mathcal{P}_l(k)}|u(x)|\ d\nu(x)\leq C. \tag{A.29}$$

Then $|u|\leq C$ almost everywhere on A.

Proof. For all $v \in L^1(\Omega, d\nu)$, we set

$$(\mathcal{E}_k v)(x) = \sum_{l\in\mathbb{N}} \frac{1}{\nu(\mathcal{P}_l(k))} \int_{\mathcal{P}_l(k)} v(y) d\nu(y) \chi_{\mathcal{P}_l(k)}(x),$$

$\chi_{\mathcal{P}_l(k)}$ being the characteristic function of $\mathcal{P}_l(k)$. We first remark that

$$\lim_{k\to\infty} \mathcal{E}_k v = v \qquad \text{in} \quad L^1(\Omega, d\nu). \tag{A.30}$$

Indeed, since $||\mathcal{E}_k v||_{L^1(\Omega,d\nu)} \leq ||v||_{L^1(\Omega,d\nu)}$ for all v, we may assume that v is continuous and compactly supported. Then, denoting by K the support of v, we have for all $k \geq -n_0$

$$||\mathcal{E}_k v - v||_{L^1(\Omega,\mathrm{d}\nu)} \leq \sum_{\substack{l\in\mathbb{N} \\ \mathcal{P}_l(k)\cap K\neq\emptyset}} \nu(\mathcal{P}_l(k)) \sup_{x,y\in\mathcal{P}_l(k)} |v(y) - v(x)|.$$

Using the local finiteness of $\mathcal{P}$ and the fact that it is of vanishing diameter, there exists a compact subset K' such that

$$||\mathcal{E}_k v - v||_{L^1(\Omega,\mathrm{d}\nu)} \leq \nu(K') \sup_{\substack{x,y\in K' \\ |x-y|\leq 2\epsilon_k}} |v(y) - v(x)| \to 0, \qquad k \to \infty,$$

and (A.30) follows. In particular, $\chi_A \mathcal{E}_k |u| \to \chi_A |u|$ in $L^1(\Omega, d\nu)$ so there exists a subsequence $\chi_A \mathcal{E}_{k_j} |u|$ converging almost everywhere to $\chi_A |u|$. Using (A.29) we have

$$0 \leq \left(\chi_A \mathcal{E}_{k_j} |u|\right)(x) \leq C, \qquad x \in \Omega,$$

and the result follows. □

The next lemma contains half of Proposition 3.1. It is based on the classical stopping time argument of the usual Calderón-Zygmund covering lemma.

Lemma A.4. *For all $u \in L^1(\Omega, d\nu)$ and all $\lambda > 0$, we can find $n_0 \in \mathbb{N}$, an admissible family $\mathcal{P}$ of partitions of Ω, a set $\mathcal{I} \subset \{(k,l) \in \mathbb{Z} \times \mathbb{N} \mid k \geq -n_0\}$ and functions $(w_{k,l})_{(k,l)\in\mathcal{I}}$ and v satisfying*

$$u = v + \sum_{\mathcal{I}} w_{k,l}, \tag{A.31}$$

$$|v(x)| \leq C_3 \lambda, \qquad a.e., \tag{A.32}$$

$$\int_\Omega w_{k,l} \, d\nu = 0 \quad \textit{and} \quad \text{supp } w_{k,l} \in \mathcal{P}_l(k), \tag{A.33}$$

$$\sum_{\mathcal{I}} \nu(\mathcal{P}_l(k)) \leq \lambda^{-1} ||u||_{L^1(\Omega,d\nu)}, \tag{A.34}$$

$$||v||_{L^1(\Omega,d\nu)} + \sum_{\mathcal{I}} ||w_{k,l}||_{L^1(\Omega,d\nu)} \leq 3||u||_{L^1(\Omega,d\nu)}. \tag{A.35}$$

The constant C_3 in (A.32) is the one chosen in (A.27).

Proof. We first choose $n_0 \in \mathbb{N}$ such that $C_2^{1-n} 2^{n_0(n-1)} > \lambda^{-1} ||u||_{L^1(\Omega,d\nu)}$, using the same constant C_2 as in (A.25), and then consider the admissible family of partitions $\mathcal{P} = ((\mathcal{P}_l(k))_{l\in\mathbb{N}})_{k\geq -n_0}$ defined by (A.24). By (A.25), we have

$$\nu(\mathcal{P}_l(-n_0)) > \lambda^{-1} ||u||_{L^1(\Omega,d\nu)}, \tag{A.36}$$

for all $l \in \mathbb{N}$. Next, we define $I_{1-n_0} \subset \mathbb{N}$ and $B_{1-n_0} \subset \Omega$ by

$$I_{1-n_0} = \left\{ l \in \mathbb{N} \mid \int_{\mathcal{P}_l(1-n_0)} |u| \; d\nu \geq \lambda \nu(\mathcal{P}_l(1-n_0)) \right\},$$
$$B_{1-n_0} = \sqcup_{l \in I_{1-n_0}} \mathcal{P}_l(1-n_0).$$

By induction, we then construct I_k and B_k, for $k \geq 2-n_0$, by

$$I_k = \left\{ l \in \mathbb{N} \;\Big|\; \int_{\mathcal{P}_l(k)} |u| \; d\nu \geq \lambda \nu(\mathcal{P}_l(k)) \;\text{ and }\; \mathcal{P}_l(k) \cap B_{k-1} = \emptyset \right\}$$
$$B_k = B_{k-1} \sqcup \bigsqcup_{l \in I_k} \mathcal{P}_l(k).$$

Let us set $B = \cup_{k \geq 1-n_0} B_k$, $A = \Omega \setminus B$ and $\mathcal{I} = \cup_{k \geq 1-n_0} \{k\} \times I_k$. We can then define

$$v(x) = \begin{cases} u(x) & x \in A, \\ \\ \sum_{(k,l) \in \mathcal{I}} \frac{1}{\nu(\mathcal{P}_l(k))} \int_{\mathcal{P}_l(k)} u \; d\nu \; \chi_{\mathcal{P}_l(k)}(x), & x \in B, \end{cases} \tag{A.37}$$

and, for each $(k,l) \in \mathcal{I}$,

$$w_{k,l} = (u - v)\chi_{\mathcal{P}_l(k)}. \tag{A.38}$$

Let us now check the properties (A.31) to (A.35). First, by construction, we have

$$B = \sqcup_{(k,l) \in \mathcal{I}} \mathcal{P}_l(k) \tag{A.39}$$

and this implies (A.31). To prove (A.32), we start by observing that for all $k \geq 1-n_0$ and all $l \in \mathbb{N}$

$$\mathcal{P}_l(k) \cap A \neq \emptyset \quad \Rightarrow \quad \frac{1}{\nu(\mathcal{P}_l(k))} \int_{\mathcal{P}_l(k)} |u| \; d\nu < \lambda. \tag{A.40}$$

Indeed, assume that $\mathcal{P}_l(k) \cap A \neq \emptyset$. Then $l \notin I_k$ otherwise $\mathcal{P}_l(k) \subset B_k \subset B$. Furthermore, we have $\mathcal{P}_l(k) \cap B_k = \emptyset$, otherwise $\mathcal{P}_l(k)$ should meet B_{k-1} and we could find $k' \leq k-1$ and $l' \in I_{k'}$ such that $\mathcal{P}_l(k) \cap \mathcal{P}_{l'}(k') \neq \emptyset$ in which case we would have $\mathcal{P}_l(k) \subset \mathcal{P}_{l'}(k')$ (since $\mathcal{P}$ is non decreasing) and then $\mathcal{P}_l(k) \subset B$, which is excluded. Thus, if $\mathcal{P}_l(k) \cap A \neq \emptyset$, then the right hand side of (A.40) holds by definition of I_k, since $l \notin I_k$ and, if $k \geq 2-n_0$, since $\mathcal{P}_l(k) \cap B_{k-1} = \emptyset$ (since $B_{k-1} \subset B_k$). Therefore, (A.40) (which is also true for $k = -n_0$ by (A.36)) and Lemma A.3 show that $|u| \leq \lambda$ almost everywhere on A. Let us now prove that $|v| \leq C_3\lambda$ almost everywhere on B. Using (A.27), it is enough to show that, for all $(k,l) \in \mathcal{I}$ (ie $l \in I_k$),

$$\int_{\mathcal{P}_{l'}(k-1)} |u| \; d\nu < \lambda \nu(\mathcal{P}_{l'}(k-1)). \tag{A.41}$$

If $k = 1-n_0$, this follows from (A.36). If $k \geq 2-n_0$, we first remark that $l' \notin I_{k-1}$ otherwise l could not belong to I_k since we would have $\mathcal{P}_l(k) \subset \mathcal{P}_{l'}(k-1) \subset B_{k-1}$. Therefore, by definition of I_{k-1}, either (A.41) holds or $\mathcal{P}_{l'}(k-1) \cap B_{k-2} \neq \emptyset$. The latter is excluded, otherwise Remark 3 (before Definition A.1) would imply that $\mathcal{P}_l(k) \subset \mathcal{P}_{l'}(k-1) \subset B_{k-2} \subset B_{k-1}$ which would prevent

l to belong to I_k. This completes the proof of (A.32). The property (A.33) is a straightforward consequence of (A.37) and (A.38) . The estimate (A.34) is a direct consequence of the definition of the sets I_k, $k \geq 1 - n_0$, and of the fact that the sets $\mathcal{P}_l(k))$ are disjoint if $(l,k) \in \mathcal{I}$. Finally, (A.37) shows that $\int |v| d\nu \leq \int |u| d\nu$ and (A.38) that $\int |w_{k,l}| d\nu \leq \int_{\mathcal{P}_l(k)} |u| d\nu + \int_{\mathcal{P}_l(k)} |v| d\nu$, and these inequalities clearly imply (A.35). $\square$

Proof of Proposition 3.1. We relabel the family $(\mathcal{P}_l(k))_{(k,l)\in\mathcal{I}}$ obtained in Lemma A.4 as $(\mathcal{P}_{l_j}(k_j))_{j\in\mathbb{N}}$ and define accordingly the functions $\tilde{u} = v$ and $u_j = w_{k_j,l_j}$. Using Lemma A.4, (3.1), (3.2), (3.3) and (3.5) are consequences of (A.31), (A.35), (A.32) and (A.33) respectively. It remains to show that the sets $\mathcal{P}_{l_j}(k_j)$ are contained in balls or cylinders of the form (3.4) which satisfy (3.6). Let $j \in \mathbb{N}$ and consider $\mathcal{P}_{l_j}(k_j)$, which is of the form (A.24) for some $i \in \mathbb{N}$ and $m \in \mathbb{Z}^{n-1}$. Since any cube of side 2 in $\mathbb{R}^{n-1}$ is contained in a euclidean ball of radius $(n-1)^{1/2}$, we have

$$\mathcal{P}_{l_j}(k_j) \subset \left\{ |r - 2^{-k_{j+}} i| \leq 2^{-k_{j+}} \text{ and } \frac{|\theta - 2^{-k_j} w([2^{-k_{j+}} i]) m|}{w(2^{-k_{j+}} i)} \leq 2^{-k_j} (n-1)^{1/2} \frac{w([2^{-k_{j+}} i])}{w(2^{-k_{j+}} i)} \right\}.$$

Therefore, if we set

$$r_j = 2^{-k_{j+}} i, \qquad \theta_j = 2^{-k_j} w([2^{-k_{j+}} i]) m$$

and use the fact that $w([2^{-k_{j+}} i])/w(2^{-k_{j+}} i) \leq C$, by (1.10), we have $\mathcal{P}_{l_j}(k_j) \subset \mathcal{B}(r_j, \theta_j, t_j)$ with

$$t_j = 2^{-k_{j+}} + C 2^{-k_j} (n-1)^{1/2}$$

if this quantity is ≤ 1. Otherwise, we have $\mathcal{P}_{l_j}(k_j) \subset \mathcal{C}(r_j, \theta_j, t_j)$ with

$$t_j = \max(1, C 2^{-k_j} (n-1)^{1/2}).$$

Since $\nu(\mathcal{C}(r_j, \theta_j, t_j)) \approx t_j^{n-1}$ if $t_j > 1$ and $\nu(\mathcal{B}(r_j, \theta_j, t_j)) \approx t_j^n$ if $t_j \leq 1$, which follow easily from (1.37) and (1.38), and since $t_j \approx 2^{-k_j}$ in all cases, the estimates (A.25) and (A.26) show that $\nu(Q_j) \lesssim \nu(\mathcal{P}_{l_j}(k_j))$. Thus (3.6) follows from (A.34). $\square$

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