Random Multi-access Algorithms
A Mean Field analysis

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June 30, 2005

Abstract

In this paper, using mean field techniques, we present a performance analysis of random back-off algorithms, such as the exponential back-off algorithm, in the case of a finite number of saturated sources. We prove that when the number of sources grows large, the system is decoupled, i.e., source behaviors become mutually independent. These results are applied in the specific case of exponential back-off algorithms, where the transient and stationary distributions of the back-off processes are explicitly characterized. This work provides a theoretical justification of decoupling argument used by many authors, e.g. Bianchi [4], to analyze the performance of random algorithms in Wireless LANs.

1 Introduction

Random multi-access protocols, from the first version of Abramson's ALOHA algorithm [2] to the most recent protocols in IEEE802.xx standards [1], have concentrated a lot of research efforts. These efforts even increased recently due to the development of decentralized random access protocols for Wireless LANs and Ad-Hoc networks. However, theoretical results characterizing the stability and the performance of such protocols are quite rare, due to the extreme complexity arising because of the inherent interaction between sources.

In analyzing random access protocols, we can distinguish different kinds of models. First infinite population models where users are assumed to arrive arbitrarily and randomly in the system according to some point processes. In these models, each user generally has a single packet to transmit and leaves the system after successful transmission. In these scenarios, stability of protocols received a lot of attention: for example, the exponential backoff algorithm is unstable for all positive user arrival rate, this was first conjectured by Kelly [10] and proved by Aldous [3]. Other algorithms, see for example proposals by Hajek-van Loon [8], proved to stabilize the system for non-zero user arrival rates.

Other models consider a finite population: the number of users is fixed. In this class of models, we can further distinguish two cases: saturated sources where users always have a packet to send and unsaturated sources where packet arrivals are governed by some exogeneous arrival processes. The first case corresponds more or less to data traffic, where the congestion control mechanism ensures that buffers are never empty, whereas the second case is more appropriate in modelling streaming traffic. In case of unsaturated. In case of unsaturated sources, stability is again a major issue, please refer to the work by Håstad et al. [9] and references therein for a review of existing results and open issues.

In the present paper we consider a finite number of saturated sources. In this context, the stability analysis is simplified, because queues are not considered. The major issue is then to determine the performance of protocols, i.e., we would like to evaluate the throughput of the system and of each source, packet transmission delay, ... These performance parameters are largely unknown. This is due to the fact that the inherent interactions between sources have proven to be extremely complex to model and analyze. A very popular approach to circumvent this difficulty consists in decoupling the source behaviors, i.e., assuming that the (re)-transmission processes of the different sources are mutually independent. This assumption allows one to derive explicit estimates of the performance. This approach was for example applied by Bianchi [4] to analyze the IEEE 802.11 Decentralized Coordination Fucntion (DCF) algorithm and since then has been widely used to accurately predict the performance of similar protocols. Using mean field techniques, we prove that, for a wide range of random back-off algorithms, the decoupling assumption is asymptotically exact as the number of sources grows. In the specific case of exponential back-off algorithm (the DCF is based on this algorithm), the mean field analysis provides the transient and stationary distributions of the (re)-transmission processes.

The remainder of this paper is organized as follows. In Section 2 we introduce the finite-source model and present the basic principle for random back-off algorithms. In Section 3 the mean field results, that justify the decoupling approach, are presented. Section 4 conatins the proofs of the main results.

Notations. Let $\mathcal{S}$ be a separable, complete metric space, $\mathcal{P}(\mathcal{S})$ denotes the space of probability measures on $\mathcal{S}$. $\mathcal{L}(X)$ is the law of the $\mathcal{S}$-valued random variable X. $D(\mathbb{R}^+, \mathcal{S})$ the space of right-contiuous functions with left-handed limits, with the Skorohod topology [5].

2 Random back-off algorithms

We consider N users sharing a common access channel in a decentralized manner. Time is slotted and users are assumed to be synchronized. We consider saturated users only, which means that each user has always a packet to transmit. For data traffic, this assumption is more realistic than to assume that each user generates packets according to some predefined point process.

Each user runs a back-off timer. When a user is in stage $s \in \mathbb{N}$, her/his back-off timer is chosen according to a geometric random variable with mean W_s slots. The back-off timer is decremented at each slot where the channel is idle, until it reaches 0 in which case, the user attempts to use the channel. If the transmission is sucessful, the stage becomes $S(s)$ where $S : \mathbb{N} \to \mathbb{N}$ is a non-increasing function. If a collision occurs the stage becomes $C(s)$, where $C : \mathbb{N} \to \mathbb{N}$ is an increasing function. Then a new back-off timer is chosen. The assumption that the back-off timer is chosen according to a geometric r.v. is not crucial, and we believe the derived results hold for general distributions. The geometric distribution simplifies the analysis, since given that her/his stage is s, a user will attempt the channel with probability $p_s = 1/W_s$. Then, at time slot t, the state of the system is fully described by the attempt probabilties of all users, $p_i^N(t)$, $i = 1, \ldots, N$. The set of possible values for the attempt probabilities is denoted by $\mathcal{B}$. For the sake of simplicity we assume that $\mathcal{B}$ is at most denumerable. We denote by $p_0 = \sup \mathcal{B}$ the maximum attempt probability. We may re-define functions S and $C : \mathcal{B} \to \mathcal{B}$ expressing a successul and a collision respectively: Given that the attempt probability of a user is p and that she/he attempts to use the channel, the attempt probability becomes $S(p)$ (resp. $C(p)$) in case of successful transmission (resp. collision).

Finally, the evolution of the state of sthe sytem is described by the following recursion: for all i and t,

$$p_i^N(t{+}1) = p_i^N(t)1_{U_i(t)>p_i^N(t)} + \left(S(p_i^N(t)) \prod_{j\neq i} 1_{U_j(t)>p_j^N(t)} + C(p_i^N(t))(1 - \prod_{j\neq i} 1_{U_j(t)>p_j^N(t)}) \right) 1_{U_i(t)\leq p_i^N(t)}, \tag{1}$$

where the $U_i(t)$'s are i.i.d. r.v. uniformly distributed on $[0, 1]$. Having geometric backoffs implies that these r.v. are of course independent of the r.v. $U_i(s)$, $i = 1, \dots, N$, for all $s < t$.

Example 1 (Exponential backoff) *Most of MAC protocols use a so-called exponential backoff procedure. This is the case for the Ethernet or for the DCF function in wireless LANs. For these protocols the attempt probabilities are values of the set* $\mathcal{B} = \{p_0 \times 2^{-k}, k \in \mathbb{N}\}$ *and the functions* S *and* C *are defined by:*

$$S(p) = p_0 \quad \textit{and} \quad C(p) = \max(p/2, p_0 \times 2^K).$$

K *may be finite or not. Unless otherwise mentioned,* $K = \infty$.

We study the large N asymptotics. To this aim, we need an adequate renormalization. Note first that when the number of users grows large, the stationary probability a user attempts to use the channel must decrease and ultimately tend to zero, otherwise the global throughput will become negligible. Then, the mean time between two consecutive changes in the value of the backoff parameter is roughly inversely proportional to the stationary attempt probability. The above observations suggest the following renormalization:

$$q_i^N(t) \equiv N \times p_i^N[N \times t] \text{ where } [x] \text{ denotes the greatest integer in } x. \tag{2}$$

The set of possible values for $q_i^N(t)$ is $\mathcal{B}$. In case of Examples 1 and 2, it means that p_0 is replaced by p_0/N. In general, it means that the set of possible values for $p_i^N(t)$ becomes $\{p/N, p \in \mathcal{B}\}$. The process $q_i^N(.)$ may be seen as a $D(\mathbb{R}^+, \mathcal{B})$-valued process.

3 Convergence theorems

3.1 Transcient behavior

The following theorem constitutes the main result of the paper and states that when the number of sources become large, the backoff timers of all sources evolve independently from each other. In short, users behave independently when the number of users grows. The result also identifies the limit behavior of the distribution of the backoff timers.

Theorem 1 *We assume that the initial values* $q_i^N(0)$, $i = 1, \dots, N$, *are exchangeable and chaotic. This means the empirical measure* μ_0^N *of the* $q_i^N(0)$ *converges weakly to* *a deterministic limit Q_0. In practice the assumption that the $q_i^N(0)$ are i.i.d. suffices. There exists a probability measure Q on $D(\mathbb{R}^+, \mathcal{B})$ such that for all subset $I \subset \mathbb{N}$ of finite cardinal $|I|$,*

$$\lim_{N\to\infty} \mathcal{L}\left((q_i^N(.))_{i\in I}\right) = Q^{\otimes |I|} \quad \textit{weakly in } \mathcal{P}(D(\mathbb{R}^+, \mathcal{B})^{|I|}). \tag{3}$$

To establish this result, we use the method developed by Sznitman [11] and further investigated by Graham [7]. The empirical measure on path space, with samples in $\mathcal{P}(D(\mathbb{R}^+, \mathcal{B}))$, is defined as

$$\mu^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{q_i^N}, \tag{4}$$

where q_i^N stands for the process $q_i^N(\cdot)$. From Sznitman [11] Proposition 2.2, we know that (3) is equivalent to:

$$\lim_{N\to\infty} \mathcal{L}(\mu^N) = \delta_Q \quad \text{weakly in } \mathcal{P}(\mathcal{P}(D(\mathbb{R}^+, \mathcal{B}))). \tag{5}$$

Moreover, (5) allows us to consider functionals on $D(\mathbb{R}^+, \mathcal{B})$ such as the the indicator of the event that a sample path backs off more than b in a finite it time interval $[0, T]$. Let X denote a canonical trajectory in $D(\mathbb{R}^+, \mathcal{B})$. Let ϕ denote this functional on X. Then $\langle \mu^N, \phi\rangle$ is the proportion of the particles that back off more than b over the time interval b. If we studied the weaker convergence of $\mu_t^N \to Q(t)$ then we could not obtain sample path information like this.

For a large class of protocols such as the exponential backoff protocol, we are also able to characterize the evolution of the distribution of the backoff of a source when the number of sources grows large, i.e., to evaluate the marginals of Q.

Theorem 2 *In case of exponential backoff, the marginals of Q satisfy the following set of differential equations: define by $Q_k(t) \equiv Q(t)(\{2^{-k}p_0\})$,*

$$\frac{dQ_k}{dt}(t) = 2^{1-k}p_0\left(Q_{k-1}(t)\big(1 - \exp(-\sum_{i=0}^{\infty} 2^{-i}p_0 Q_i(t))\big) - \frac{Q_k(t)}{2}\right), \quad \textit{for all } k \geq 1, \tag{6}$$

$$\frac{dQ_0}{dt}(t) = \sum_{k=0}^{\infty} 2^{-k}p_0 Q_k(t)\exp(-\sum_{i=0}^{\infty} 2^{-i}p_0 Q_i(t)) - p_0 Q_0(t). \tag{7}$$

3.2 Stationary regimes

We now investigate the system behavior in equilibrium. We consider here exponential backoff protocols only. We first prove that when the number of sources N is fixed, the backoff process $(q_i^N(t))_i$ is ergodic. The theorem below is proved assuming $p_0 < \ln 2$, which is consistent with usual MAC protocols (in case of the DCF, $p_0 = 1/32$). The results actually hold for any p_0, but we lack of space for a full proof.

Theorem 3 *In case of exponential backoff, the Markov chain $(q_i^N(t))_i$ is positive recurrent. Furthermore the set of laws of $q_i^N(0)$ in equilibrium is tight, i.e., $\mathcal{L}_{\mathrm{st}}(q_i^N(0))$ is tight.*

Proof. It can be easily proved that the considered Markov chain is irreducible. Consider an arbitrary source, say source 1. Its backoff is less than that in a fictive system, where all other sources attempt to use the channel with highest probability p_0/N. In this fictive system, $\log(Nq_1^N(t)/p_0)$ evolves as a Markov chain with transition matrix $P = (p_{ij})$ defined by: $p_{00} = 1 - p_0(1-a)/N$, $p_{01} = p_0 a/N$ and $p_{i0} = p_0 2^{-i}(1-a)/N$, $p_{ii} = 1 - p_0^2 - i/N$, $p_{ii+1} = p_0 2^{-i} a/N$. $a = 1-(1-p_0/N)^{N-1} < 1/2$ (since $p_0 < \ln 2$). The analysis of this Markov chain is straightforward, it is positive recurrent and we can also prove that in this fictive system, the mean value of $q_1^N(0)$ in stationary regime is $p_0(1-2a)/(1-a)$. We deduce that $(q_i^N(t))$ in the actual system is also ergodic and that $E_{\mathrm{st}}[q_1^N(0)] \le p_0(1-2a)/(1-a)$, which provides the desired tightness. □

It can be easily proved that the dynamic system decribed by differential equations (6)-(7) admits a unique equilibrium $q_{\mathrm{st}} = (q_k)_k$ defined by:

$$\forall k, \quad q_k = (2(1-e^{-S}))^k q_0, \quad q_0 = Se^{-S} \quad \text{and} \quad S = \sum_k 2^{-k} q_k.$$

The stability of this equilibrium point is obtained remarking that the gradient matrix of the dynamical system has eigen values with strictly negative real part at the equilibrium point. Due to lack of scapce, we let the study of the global stability of (6)-(7) for future work. In the following we will assume that it is globally stable. We are now able to characterize the system behavior in equilibrium when the number of sources grows large.

Theorem 4 *In equilibrium, for all subset $I \subset \mathbb{N}$ of finite cardinal $|I|$,*

$$\lim_{N\to\infty} \mathcal{L}_{\mathrm{st}}\left((q_i^N(.))_{i\in I}\right) = q_{\mathrm{st}}^{\otimes |I|} \quad \textit{weakly in } \mathcal{P}(D(\mathbb{R}^+, \mathcal{B})^{|I|}), \tag{8}$$

This theorem is proved in Section 4. It states that in equilibrium, the behavior of sources are independent. It then implies that the decoupling approach used by Bianchi is asymptotically correct when the number of sources grows large.

4 Proof of Theorems 1 and 2

4.1 Proof of Theorems 1 and 2

Step 1. The sequence $\mathcal{L}(\mu^N)$ is tight in $\mathcal{P}(\mathcal{P}(D(\mathbb{R}^+, \mathcal{B})))$. Thanks again to Sznitman [11] Proposition 2.2, we only have to prove that $\mathcal{L}(q_1^N(.))$ is tight in $\mathcal{P}(D(\mathbb{R}^+, \mathcal{B}))$. The jumps of $q_1^N(.)$ are included in those of a Poisson process of intensity p_0. the jump sizes are bounded (by 1). We conclude by the tightness criterion in Ethier-Kurtz [6] p 128.
Step 2. We can mimic the Step 2 in [7]. We show that any accumulation point of $\mathcal{L}(\mu^N)$ satisfies a certain martingale problem. For $f \in L^\infty(\mathcal{B})$, the bounded and forcibly measureable functions of $\mathcal{B} \to \mathbb{R}$, define $f^s(q) = f(S(q)) - f(q)$ and $f^c(q) = f(C(q)) - f(q)$ (s and c stand for *success* and *collision*, respectively). Now, for $f \in L^\infty(\mathcal{B})$,

$$\begin{aligned} f(q_i^N(t)) - f(q_i^N(0)) &= \sum_{k=0}^{[Nt]-1} (f(q_i^N(\frac{(k+1)}{N}) - f(q_i^N(\frac{k}{N})) \\ &= \sum_{k=0}^{[Nt]-1} \mathcal{G}^{f,i,N}(k) + \mu^{f,i,N}(t), \end{aligned}$$

where

$$M^{f,i,N}(t) = \sum_{k=0}^{[Nt]-1} f^s(q_i^N(\frac{k}{N}))\left(1_{\{NU_i(k)\le q_i^N(\frac{k}{N})\}}\prod_{j\neq i}1_{\{NU_j(k)>q_j^N(\frac{k}{N})\}} - \frac{q_i^N(\frac{k}{N})}{N}\prod_{j\neq i}(1-\frac{q_j^N(\frac{k}{N})}{N})\right)$$
$$+f^c(q_i^N(\frac{k}{N}))\left(1_{\{NU_i(k)\le q_i^N(\frac{k}{N})\}}(1-\prod_{j\neq i}1_{\{NU_j(k)>q_j^N(\frac{k}{N})\}}) - \frac{q_i^N(\frac{k}{N})}{N}(1-\prod_{j\neq i}(1-\frac{q_j^N(\frac{k}{N})}{N})\right.$$

and where

$$\mathcal{G}^{i,N}f(k) = \left[f^s(q_i^N(\frac{k}{N}))\frac{q_i^N(\frac{k}{N})}{N}\prod_{j\neq i}(1-\frac{q_j^N(\frac{k}{N})}{N}) + f^c(q_i^N(\frac{k}{N}))\frac{q_i^N(\frac{k}{N})}{N}\left(1-\prod_{j\neq i}(1-\frac{q_j^N(\frac{k}{N})}{N})\right)\right],$$

The proofs of the two following lemmas are given at the end of this section.

Lemma 1 *For the martingale $M^{f,i,N}(t)$ defined at (9), the Doob-Meyer Brackets $\langle M^{f,i,N}, M^{f,j,N}\rangle ==$ $t||f||_\infty\mathcal{O}(1/N)$ as $N\to\infty$ uniformly in $i\neq j$.*

Lemma 2 *The martingale $M^{f,i,N}(t)$ defined at (9) satisfies*

$$M^{f,i,N}(t) = f(q_i^N(t)) - f(q_i^N(0)) - \int_0^t \mathcal{G}f(\mu_v^N, q_i^N(v))dv + \epsilon^{f,i,N}(t) \quad (10)$$

where

$$\mathcal{G}f(\mu,q) = q\cdot(f^s(q)-f^c(q))\exp(-\langle Id,\mu\rangle) + q\cdot f^c(q) \quad (11)$$

and where $\epsilon_t^{f,i,N} = t||f||_\infty\mathcal{O}(1/N)$ uniformly in i.

If indeed μ^N does converge to Q then (10) will converge to a solution of a martingale problem. Recall that X denotes a canonical trajectory in $D(\mathbb{R}^+,\mathcal{B})$. A probability Q in $\mathcal{P}(D(\mathbb{R}^+,\mathcal{B}))$ solves the non-linear martingale problem if

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{G}f(Q(s),X_s)ds \quad (12)$$

is a Q-martingale. It solves the martingale problem starting at q if $Q(0)=Q_0$.

Let Π^∞ be an accumulation point of $\mathcal{L}(\mu^N)$. Let $R\in\mathcal{P}(D(\mathbb{R}^+,\mathcal{B}))$ belong to the support of Π^∞. Recall as in [7] that the projection map $X\to X_t$ is R-a.s. continuous for all t except perhaps in at most a countable subset D_R of R_+. Further it is shown easy ?? to show that $D=\{t\in R_+ : \Pi^\infty(\{R : t\in D_R) > 0\}$ is at most denumberable.

Lemma 3 *R satisfies the non-linear martingale problem (12)*

The continuity of $X\to X_0$ implies $R_0=Q_0$, Π^∞-a.s.. The proof is postponed to the end of the section.

Step 3. We now show the solution to (12) is unique so $R=Q$. We can mimic Theorem 3.3 in [7] to show the solution to the martingale problem (12) is unique. We remark $\mathcal{G}f(\mu,q)=\int_{\mathcal{B}}(f(y)-f(x))J(\mu,x,dy)$ where

$$J(\mu,x,dy) = x\exp(-\langle Id,\mu\rangle)(\delta_{S(x)}(dy)-\delta_{C(x)}(dy)) + x\delta_{C(x)}(dy).$$

Next, $||J(\mu, x, \cdot)|| \leq x \leq p_0$ and

$$\begin{aligned}
||J(\alpha, x, \cdot) - J(\beta, x, \cdot)|| &\leq x|\exp(-\langle Id, \alpha\rangle) - \exp(-\langle Id, \beta\rangle)| \\
&\leq x|1 - \exp(\langle Id, \alpha\rangle - \langle Id, \beta\rangle)| \\
&\leq xC|\langle Id, \alpha\rangle - \langle Id, \beta\rangle \\
&\leq p_0 C||\alpha - \beta||
\end{aligned}$$

where $||\cdot||$ denotes the total variation norm. As in Theorem 3.3 in [7] we use Proposition 2.3 in [7] to establish the solution to the martingale problem (12) is unique.

Step 4. We have now proved convergence because any subsequence converges to the same limit Q. We can now identify this limit for the protocols considered in Examples 1 and 2. If Q satisfies the martingale problem then $(Q(t))_{t\geq 0}$ solves the non-linear Komogorov equation derived by taking the expected value:

$$\langle f, Q(t)\rangle - \langle f, Q(0)\rangle = \int_0^t \langle \mathcal{G}f(Q(s), q); Q(s)\rangle ds. \tag{13}$$

Applying (13) to $f = 1_{p_0 2^{-k}}$ for all k, we get the set differential equations (6)-(7) in case of exponential backoff.

Proof of Lemma 1.

$M^{f,i,N}(t)$ is a martingale by the Dynkin formula.

Next, we define the following variables : $S^N_{i,k} = 1_{\{NU_i(k)\leq q_i^N(\frac{k}{N})\}} \prod_{j\neq i} 1_{\{NU_j(k)>q_j^N(\frac{k}{N})\}}$ and $C^N_{i,k} = 1_{\{NU_i(k)\leq q_i^N(\frac{k}{N})\}}\left(1 - \prod_{j\neq i} 1_{\{NU_j(k)>q_j^N(\frac{k}{N})\}}\right)$.

$S^N_{i,k} = 1$ if the user i is the only user accessing the channel at time k and $C_{i,k} = 1$ if a collision occurs for user i at time k. Both these events have a probability less than p_0/N.

Note in particular for $i \neq j$ that :

$$S^N_{i,k}C^N_{i,k} = 0, \quad S^N_{i,k}S^N_{j,k} = 0, \quad S^N_{i,k}C^N_{j,k} = 0 \text{ and } P(C^N_{i,k}C^N_{j,k} = 1) = \mathcal{O}(1/N^2), \tag{14}$$

where the last equation stands uniformly in i, j.

In the sequel, $E_{U(k)}(.)$ will denote $E(.|(p_i^N(k))_i)$. $E_{U(k)}(S^N_{i,k})$ and $E_{U(k)}(S^N_{i,k})$ are bounded by p_0/N. With this notation, we can rewrite as Equation 9 as :

$$M^{f,i,N}(t) = \sum_{k=0}^{[Nt]-1} f^s(q_i^N(\frac{k}{N}))\left(S^N_{i,k} - E_{U(k)}S^N_{i,k}\right) + f^c(q_i^N(\frac{k}{N}))(C^N_{i,k} - E_{U(k)}C^N_{i,k}) \tag{15}$$

To prove Lemma 1, we first show $EM^{f,1,N}(t)M^{f,2,N}(t)$ tends to 0. Since $(M^{f,i,N}(t))$ is a martingale this product is equal to :

$$\begin{aligned}
EM^{f,1,N}(t)M^{f,2,N}(t) = \sum_{k=0}^{[Nt]-1} & Ef^s(q_1^N(\frac{k}{N}))\left(S^N_{1,k} - E_{U(k)}S^N_{1,k}\right) f^s(q_2^N(\frac{k}{N}))\left(S^N_{2,k} - E_{U(k)}S^N_{2,k}\right) \\
&+Ef^c(q_1^N(\frac{k}{N}))\left(C^N_{1,k} - E_{U(k)}C^N_{1,k}\right) f^c(q_2^N(\frac{k}{N}))\left(C^N_{2,k} - E_{U(k)}C^N_{2,k}\right) \\
&+2Ef^c(q_1^N(\frac{k}{N}))\left(C^N_{1,k} - E_{U(k)}C^N_{1,k}\right) f^s(q_2^N(\frac{k}{N}))\left(S^N_{2,k} - E_{U(k)}S^N_{2,k}\right).
\end{aligned}$$

Using (14), it appears easily that each term of the sum is bounded by $||f||_\infty \mathcal{O}(1/N^2)$ and the lemma follows.

Proof of Lemma 2.

$$\begin{aligned}\int_0^t \mathcal{G}f(\mu_s^N, q_i^N(s))ds &= \sum_{k=0}^{[Nt]-1}\int_{[k/N]}^{(k+1)/N} \mathcal{G}f(\mu_s^N, q_i^N(s))ds + \int_{[Nt]/N}^{t} \mathcal{G}f(\mu_s^N, q_i^N(v))ds \\ &= \sum_{k=0}^{[Nt]-1}\int_{k}^{k+1} \mathcal{G}f(\mu_{w/N}^N, q_i^N(\frac{w}{N}))dw + \frac{1}{N}\int_{[Nt]}^{Nt} \mathcal{G}f(\mu_{w/N}^N, q_i^N(\frac{w}{N}))dw\end{aligned}$$

Hence

$$\begin{aligned}|\epsilon_t^{f,i,N}| &\leq \sum_{k=0}^{[Nt]-1} |\mathcal{G}^{i,N}f(k) - \frac{1}{N}\int_k^{k+1} \mathcal{G}f(\mu_{w/N}^N, q_i^N(\frac{w}{N}))dw| + \frac{1}{N}|\int_{[Nt]}^{Nt} \mathcal{G}f(\mu_{w/N}^N, q_i^N(\frac{w}{N}))dw| \\ &\leq \sum_{k=0}^{[Nt]-1} |\mathcal{G}^{i,N}f(k) - \frac{1}{N}\mathcal{G}f(\mu_{k/N}^N, q_i^N(\frac{k}{N}))| + \frac{1}{N}|\mathcal{G}f(\mu_{k/N}^N, q_i^N(\frac{k}{N}))|.\end{aligned}$$

The last term of this inequality is obviously less than $3p_0||f||_\infty/N$ (since $q_i^N(t) \leq p_0/N$). To finish the proof, it remains to bound the terms appearing in the sum and, giving a closer look at $\mathcal{G}$, it suffices to bound $\exp(-1/N\sum_{i=1}^N q_i^N(k/N)) - \prod_{i\neq j}(1 - q_i^N(k/N)/N)$ by $\mathcal{O}(1/N)$ uniformly in i, j. We have :

$$|e^{-\frac{1}{N}\sum_{i=1}^N q_i^N(k/N)} - \prod_{i\neq j}(1 - \frac{q_i^N(k/N)}{N})| \leq e^{-p_0}|1 - e^{\frac{q_i^N(k/N)}{N} + \sum_{j\neq i}|\ln(1-\frac{q_j^N(k/N)}{N}) + \frac{q_j^N(k/N)}{N}|}|.$$

Using for example the inequality for $x \in (0,1)$, $|\ln(1-x)+x| \leq \frac{x^2}{2(1-x)^2}$, we easily deduce the required bound.

Proof of Lemma 3.

We mimic the proof in Step 2 of Theorem 3.4 of [7]. For the sake of completeness, we detail the proof.

Take $0 \leq s_1 < s_2 < \cdots s_k \leq s < t$ outside D and $g \in L^\infty(\mathcal{B}^k)$. Take $f \in L^\infty(\mathcal{B})$. The map

$$\begin{aligned}&G : R \in \mathcal{P}(D(\mathbb{R}^+, \mathcal{B})) \\ &\quad \to \langle\left(f(X_t) - f(X_s) - \int_s^t \mathcal{G}f(R_u, X_u)du\right) g(X_{s_1}, \ldots, X_{s_k}); R\rangle\end{aligned}$$

is Π^∞-a.s. continuous.

Let Π^N be the law of μ^N, following [7], we write :

$$\begin{aligned}\langle G^2, \Pi^N\rangle &= E(G(\mu^N)^2) \\ &= E(G(\frac{1}{N}\sum_{i=1}^N \delta_{q_i^N})^2) \\ &= E\left(\frac{1}{N}\sum_{i=1}^N \left(f(q_i^N(t)) - f(q_i^N(s)) - \int_s^t \mathcal{G}f(\mu_u^N, q_i^N(u))du\right) g(q_i^N(s_1), \ldots, q_i^N(s_k))\right)^2 \\ &= E\left(\frac{1}{N}\sum_{i=1}^N \left(M^{f,i,N}(t) - M^{f,i,N}(s) - (\epsilon^{f,i,N}(t) - \epsilon^{f,i,N}(s))\right) g(q_i^N(s_1), \ldots, q_i^N(s_k))\right)^2 \quad (16)\end{aligned}$$

Let $g^{i,N} = g(q_i^N(s_1), \dots, q_i^N(s_k))$ and for a process Y let $Y_{s,t} = Y_t - Y_s$. Using exchangeability, from Equation (16) we obtain :

$$\begin{aligned}\langle G^2, \Pi^N \rangle &= E\big(\frac{1}{N}\sum_{i=1}^{N}\left(M_{s,t}^{f,i,N} - \epsilon_{s,t}^{f,i,N}\right) g(q_i^N(s_1), \dots, q_i^N(s_k))\big)^2 \\ &= \frac{1}{N}E\big((M_{s,t}^{f,1,N} - \epsilon_{s,t}^{f,1,N})g^{1,N}\big)^2 + \frac{N-1}{N}E(M_{s,t}^{f,1,N} - \epsilon_{s,t}^{f,1,N})g^{1,N}(M_{s,t}^{f,2,N} - \epsilon_{s,t}^{f,2,N})g^{2,N}.\end{aligned}$$

Lemmas 1 and 2 imply that $\langle G^2, \Pi^N\rangle$ tends to 0. From Fatou lemma, $\langle G^2, \Pi^\infty\rangle \leq \lim_N \langle G^2, \Pi^N\rangle = 0$ and thus Π^∞-a.s, $G(R) = 0$. Since this holds for arbitrary $0 \leq s_1 < s_2 < \cdots s_k \leq s < t$ outside D and $g \in C_b(\mathcal{B}^k)$ it follows that R satisfies the non-linear martingale problem (12).

4.2 Proof of Theorem 4

Let $(q_i^N(0))_i$ with law the invariant law of the system with N users. By symmetry, $(q_i^N(0))_i$ is exchangeable. We define $\Pi^N = 1/N\sum_{i=1}^N q_i^N$. By Theorem 3, Π_0^N is tight, consider an accumulation point Π_0^∞. We cannot apply directly Theorem 1 since we do not know wheter the subsequence of Π_0^N converges weakly toward a deterministic limit.

We now circumvent this difficulty. As in step 1 in the proof of Theorem 1, we deduce from Sznitman [11] Proposition 2.2, that Π^N is tight in $\mathcal{P}(\mathcal{P}(D(\mathbb{R}^+, \mathcal{B})))$. Let R in $\mathcal{P}(D(\mathbb{R}^+, \mathcal{B}))$ in the support of an accumulation point of Π^N. We can prove simlarly that Lemma 3 still holds for R.

By Step 3 of Theorem 1, the solution of the martingale problem is unique and R solves it with initial condition R_0. The global stability of (6)-(7) implies that $\lim_{t\to+\infty} R_t = q_{st}$. However, by stationarity, Π_t^N and Π_0^N are equal, we deduce immediately that the support Π_0^N is reduced to q_{st} and $R_0 = q_{st}$.

Theorem 4 is then a consequence of Theorem 1.

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