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Extrêmes des processus branchants spatiaux

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À Mathieu et Kenza,
qui illuminent mes journées.
Et parfois mes nuits.

Le vent aussi dispersait certaines graines. En même temps que l'eau réapparut réapparaissaient les saules, les osiers, les prés, les jardins, les fleurs et une certaine raison de vivre.

Mais la transformation s'opérait si lentement qu'elle entrait dans l'habitude sans provoquer d'étonnement.

Jean Giono, *L'homme qui plantait des arbres.*

Remerciements

Contrairement à la thèse, qui peut sembler ne jamais avoir de fin, une habilitation donne l'impression de ne jamais vraiment commencer. Toutefois, de très nombreuses personnes m'ont aidé à mener ce projet à complétion. La recherche est une activité collective, et je suis heureux d'avoir ici un espace pour remercier celles et ceux qui m'ont apporté leur aide au cours des ces années.

Tout d'abord, il me faut remercier Bénédicte Haas, Grégory Miermont et Nina Gantert, qui ont accepté de relire le présent manuscrit. Je tiens à les remercier pour leur temps, et pour leur aide à l'amélioration de ce manuscrit. Mes plus sincères remerciements également à Julien Barral, Bernard Derrida et Thomas Duquesne qui ont accepté de compléter le jury. Je suis très heureux de pouvoir présenter devant eux les thématiques de recherche que j'ai développé depuis la fin de ma thèse.

Je souhaite également remercier les nombreuses et nombreux collègues et amis avec qui j'ai eu l'occasion d'échanger, débattre, apprendre, souvent autour d'un tableau ou d'un café, parfois dans des conditions plus exotiques. Ces interactions ont souvent été pour moi un moteur pour mes recherches, et toujours un moment d'humanité précieux. Je me compte extrêmement chanceux d'avoir pu profiter de tous ces échanges. Merci en particulier à mes coauteurs et coautrice, qui ont supporté la lenteur de mes réponses et corrigé mes nombreuses erreurs au cours de nombreuses séances de rédaction. J'aurais bien trop peur d'oublier de nombreuses personnes en commençant une liste de noms, mais mille mercis à toutes et tous.

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Pendant mon entrée dans le monde de la recherche, et ma découverte des processus de branchement, j'ai eu le bonheur d'être encadré par des mentors exceptionnels, qui n'ont pas hésité à perdre beaucoup de temps à répondre à mes questions naïves. Je pense bien sûr en premier lieu à mon directeur de thèse Zhan Shi, qui a toujours été à l'écoute, et auquel j'exprime ma plus grande reconnaissance. J'ai aussi une pensée émue pour Marc Yor, et les nombreux échanges que nous avons eu à cette période. Merci également pour leur accueil et leurs nombreux conseils à Edwin Perkins et Jean Bertoin.

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Pour conclure, je remercie ma famille pour leur soutien toutes ces années. Merci en particulier à mes parents Fabienne et Jean-François pour continuer à s'intéresser à la progression de mes recherches. Merci enfin à Yasmine, Mathieu et Kenza qui partagent mon quotidien, pour leurs sacrifices consentis et la joie qu'ils m'apportent chaque jour.

Résumé Abstract

Extrêmes des processus branchants spatiaux

Ce manuscrit est un document de synthèse et de présentation d'une sélection de mes travaux de recherches réalisés avant le 1er juillet 2022, dont on trouvera la liste complète ci-dessous. Les publications [M1-8] sont issues de ma thèse (juillet 2015), ou ont été en partie réalisées pendant celle-ci. Afin de présenter un manuscrit avec une unité thématique claire, les publications [M13, M14, M17, M20, M26, M28, M31, M34, M35] ne seront pas traitées ici dans ce document, avec toutes mes excuses vis-à-vis de mes co-auteurs.

Ce document a pour but de faire un état des lieux de la littérature consacrée à l'étude des valeurs extrêmes des processus branchants spatiaux, en particulier la convergence en loi du processus extrémal. Il consiste en un aperçu complet des techniques et résultats nécessaires pour démontrer la convergence en loi du processus extrémal d'une marche aléatoire branchante, suivi d'un bref chapitre d'application aux processus de branchement-sélection.

Mes contributions à ce domaine seront également présentées. Les résultats provenant de sources externes sont numérotés, tandis que ceux auxquels j'ai contribué sont énumérés par des lettres. Je tiens à m'excuser auprès des lecteurices non-anglophones, mais ce manuscrit étant probablement le dernier que je produis qui sera lu par plus de trois personnes, la plus grande partie de ce document sera rédigée en anglais pour en faciliter l'accessibilité.

Mots-clefs

Processus de branchement ; Marche aléatoire ; Marche aléatoire branchante ; Valeurs extrêmes ; Mouvement brownien ; Mouvement brownien branchant ; Processus de Lévy branchant ; Processus ponctuel.

Extrema of spatial branching processes

The present manuscript is a synthesis of a selection of my research up to July 1st 2022, of which a complete list is given below. Publications [M1-8] were produced during my Ph.D (before Jul. 2015). In order to keep thematic uniqueness in the manuscript, publications [M13, M14, M17, M20, M26, M28, M31, M34, M35] will not be treated in the present document, with my excuses to the co-authors of the concerned articles.

The objective of this manuscript is to give a self-contained introduction to the study of extreme values of spatial branching processes, in particular the convergence in law of the extremal process. It consists in a complete overview of the necessary techniques and results used to prove the convergence in distribution of the extremal process of a branching random walk, followed by a short application chapter to branching-selection particle systems.

In this text, my contributions to this domain will also be presented. Results coming from external sources are numbered, while the ones I collaborated to are enumerated by letters. I apologize to any non-French speaking reader, but this is probably the last mathematical document I'll be able to force my family to read, so its introduction will be written in French.

Keywords

Branching process; Random walk; Branching random walk; Extremal values; Brownian motion; Branching Brownian motion; Branching Lévy process; Point process.

Liste de mes publications et prépublications par ordre chronologique de prépublication. Ces publications sont disponibles sur ma page web, sur arXiv et sur Hal, identiques à des modifications mineures près aux versions publiées.

- [M1] B. Mallein. Maximal displacement in a branching random walk through interfaces. *Electron. J. Probab.* **20** (2015), no. 68, 40pp.
- [M2] B. Mallein. Maximal displacement of a branching random walk in time-inhomogeneous environment. *Stochastic Process. Appl.* **125** (2015), no. 10, 3958–4019.
- [M3] B. Mallein. Branching random walk with selection at critical rate. *Bernoulli* **23** (2017), no. 3, 1784–1821.
- [M4] B. Mallein. N-Branching random walk with α-stable spine. *Theor. Probab. Appl.* **62** (2018), no. 2, 295–318.
- [M5] B. Mallein. Maximal displacement of d-dimensional branching Brownian motion. *Electron. Commun. Probab.* **20** (2015), no. 76, 12 pp.
- [M6] X. Chen, T. Madaule and B. Mallein. On the trajectory of an individual chosen according to supercritical Gibbs measure in the branching random walk. *Stochastic Process. Appl.* **129** (2019), no. 10, 3821–3858.
- [M7] B. Mallein and P. Miłoś. Brownian motion and Random Walk above Quenched Random Wall. *Ann. Inst. Henri Poincaré Probab. Stat.*, **54** (2018), no. 4, 1877–1916.
- [M8] B. Mallein and P. Miłoś. Maximal displacement of a supercritical branching random walk in a time-inhomogeneous random environment. *Stochastic Process. Appl.* **129** (2019), no. 9, 3239–3260.
- [M9] B. Mallein. Necessary and sufficient conditions for the convergence of the consistent maximal displacement of the branching random walk. *Braz. J. Probab. Stat.* **33** (2019), no. 2, 356–373.
- [M10] A. Cortines and B. Mallein. A N-branching random walk with random selection. *ALEA Lat. Am. J. Probab. Math. Stat.* **14** (2017), no. 1, 117–137.
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- [M13] Y. Lin and B. Mallein. Second order behavior of the block counting process of beta coalescents. *Electron. Commun. Probab.*, **22** (2017), no. 61, 8pp.
- [M14] R. Biard, B. Mallein and L. Rabehasaina. Branching random walks with trapping zones. *Stochastic Process. Appl.* **128** (2018), 2341–2366.

[M15] B. Mallein and S. Ramassamy. Barak–Erdős graphs and the infinite-bin model. *Ann. Instit. Henri Poincaré Probab. Stat.* **57** (2021), no. 4, 1940–1967.

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[M29] D. Buraczewski, A. Iksanov and B. Mallein. On the derivative martingale in a branching random walk. *Ann. Probab.* **49** (2021), no. 3, 1164–1204.

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[M33] M. A. Belloum and B. Mallein. Anomalous spreading in reducible multitype branching Brownian motion. *Electron. J. Probab.* **26** (2021), no. 61, 39 pp.

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[M35] X. Chen and B. Mallein. Total number of births on the negative half-line of the binary branching Brownian motion in the boundary case. *Electron. Commun. Probab.* **27** (2022), no. 7, 11 pp.

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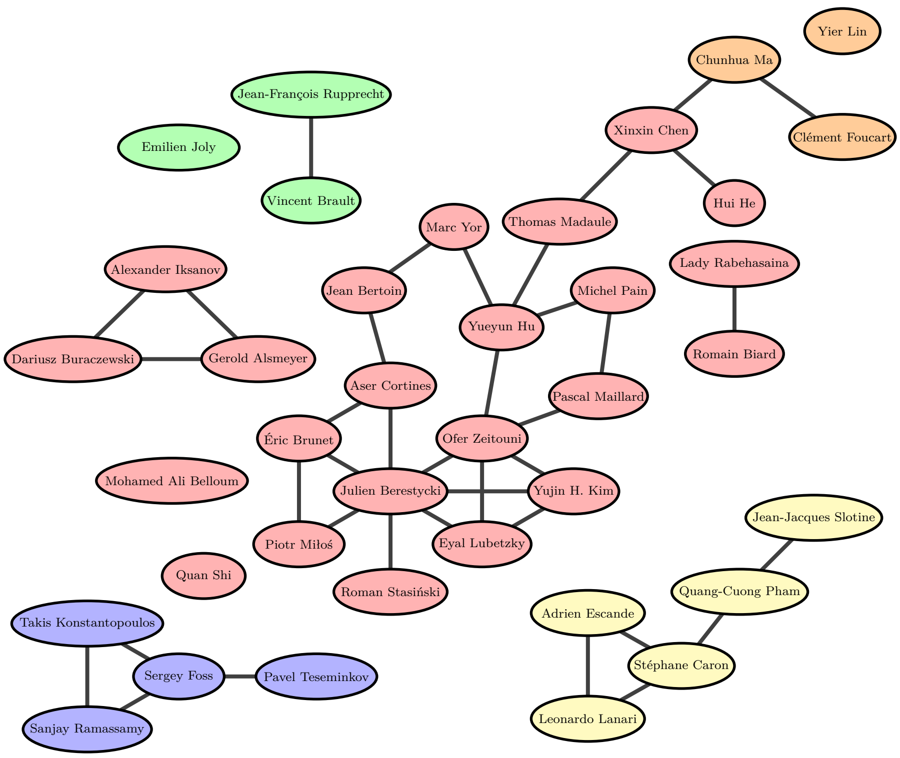


Graphe de collaborations (antérieures au 1er janvier 2022) de mes collaborateurs et collaboratrice directs, avec mes remerciements les plus sincères pour ces échanges nombreux et fructueux. Merci aussi à toutes celles et ceux ne figurant pas sur ce schéma qui ont à de nombreuses reprises échangé avec moi des idées, sans que cela ne se concrétise (encore) par une co-publication.

Table des matières

Introduction

Les processus de branchement sont des modèles mathématiques décrivant l'évolution stochastique d'une population au cours du temps. On peut retracer l'origine de ces processus aux travaux de Bienaymé [1] [41] ainsi que Galton et Watson [178]. À ce sujet, on pourra consulter l'article de Kendall [118] retraçant l'histoire de l'introduction des processus de branchement. Cet article inclut notamment une reproduction de l'article [41]. De façon générique, la théorie des processus de branchement permet d'étudier des objets qui se multiplient et créent des descendants, génération après génération, avec une influence importante du hasard sur les règles de cette évolution.

Au cours de leur histoire, les processus de branchement ont été utilisés pour étudier différents phénomènes et répondre à des questions reflétant les interrogations des mathématiciens de l'époque, ainsi que leurs a-priori et biais. Ainsi, les premiers articles introduisant les processus de branchement [41, 178] ont tous deux pour objet l'étude de la descendance mâle d'une famille, pour déterminer la probabilité de survie des noms de familles des "hommes de génie" et des aristocrates. L'histoire de la théorie des processus de branchement est liée aux mouvements malthusiens et eugénistes de la fin du 19e et du début du 20e siècle (on mentionnera ainsi Ronald A. Fisher, éditeur des *Annals of Eugenics* et titulaire de la chaire Galton d'Eugénisme de l'University College London). Cette origine infamante ne doit pas être effacée. Insistons toutefois sur le point suivant : on citera ici des articles en se référant aux modèles introduits et outils utilisés dans les dits articles, et non pour nous référer aux conclusions établies sur la base de ces résultats par les auteurs des dits articles [2] [3].

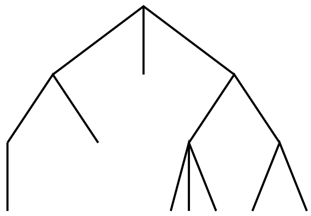
Les applications contemporaines de ces processus sont bien plus variées, et montrent la diversité des possibilités de modélisation offertes par les modèles de branchement (ainsi probablement que les biais existants dans la recherche contemporaine). On pourra par exemple mentionner les applications suivantes, sans volonté d'exhaustivité : la dynamique de propagation d'une épidémie [58], la répartition des neutrons dans un réacteur nucléaire [80, 104], les clients d'une file d'attente [117], la diffusion d'une rumeur [124], les états intermédiaires d'un jeu à somme nulle [144], les cascades de particules en physique des hautes énergie [151], la fixation d'un gène mutant dans une population [168], etc. De très

1. On se réfèrera à l'article de Heyde et Senteta [103] résumant les contributions de Bienaymé à la théorie des probabilités.

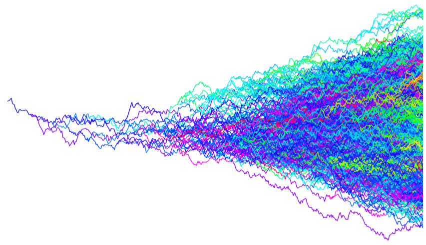
2. *La carte n'est pas le territoire :* le lien entre des résultats obtenus sur des modèles simplifiés avec un phénomène réel étudié doit toujours être fait avec prudence, et peut être réévaluée au cours du temps.

3. *Guillotine de Hume :* aucune conclusion obtenue sur l'évolution d'un modèle ne peut être utilisé pour induire un jugement moral ou politique sur le phénomène qu'il entend modéliser.

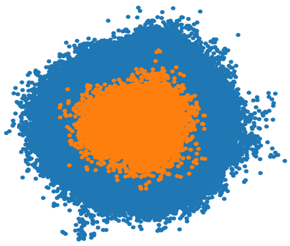
nombreux sujets de physique, de biologie, d'informatique ou d'économie peuvent être reliés à, ou modélisés par, des processus de branchement, et leur utilisation contemporaine reste importante dans ces domaines.



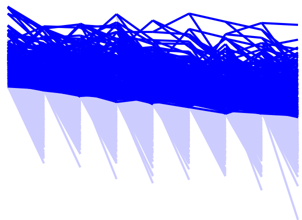
(a) Arbre de Galton-Watson



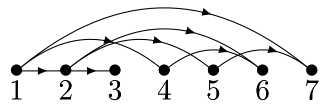
(b) Mouvement Brownien branchant



(c) Marche branchante dans le plan



(d) Marche branchante avec sélection



(e) Graphe d'Erős-Rényi dirigé

FIGURE 1 – Quelques membres de la grande (et grandissante) famille des processus branchants spatiaux et processus associés.

Il est courant, lorsqu'on étudie des processus de branchement, d'inclure une structure, qui associe à chaque individu du processus des traits (âge, taille, position, fitness, ...), qui pourront influencer la production de descendance directement (en augmentant/diminuant la vitesse de reproduction) ou indirectement (en transmettant à sa descendance des traits facilitant ou handicapant leur futur développement et reproduction). On pourra étudier l'effet de la variation de ces traits sur la dynamique de la population, ou l'effet de la dynamique de la population sur la fréquence des traits rencontrés.

On notera que la théorie des processus de branchement peut usuellement se séparer en trois catégories, selon que le branchement est *critique*, *sous-critique* ou *surcritique*, c'est à dire si un particule typique donne naissance en moyenne à 1, moins de 1 ou plus de 1 enfant. Dans les deux premiers cas, on montre de façon générique que la population s'éteindra naturellement en temps fini la plupart du temps, bien que le temps nécessaire à l'extinction tende à être bien plus long dans le cas critique. Au contraire, si le branchement est surcritique, la population a une probabilité positive de survivre et dans ce cas sa taille croît à un rythme exponentiel.

Bien entendu, il est rare qu'un objet naturel soit représentable par un processus de branchement purement critique, sous-critique ou sur-critique. La plupart du temps, il passe par ces différentes phases. Par exemple, une nouvelle maladie va initialement se répandre à un rythme exponentiel, comme dans un processus de branchement surcritique, jusqu'à contaminer une proportion suffisamment grande de la population (*heard immunity*, ou immunité de groupe). Par la suite, elle évoluera comme un processus critique ou sous-critique, jusqu'à l'extinction ou devenir endémique. On s'intéressera ici avant tout à l'étude des processus de branchement surcritiques, qui modélisent des populations dont la taille croît à vitesse exponentielle avec probabilité positive. Ces processus sont naturellement utilisés pour modéliser des populations invasives, des flambées épidémiques, des réactions autocatalytiques ou explosives, etc.

Plus précisément, l'objectif de ce manuscrit est de présenter l'état de l'art de la théorie des processus branchants spatiaux surcritiques, ainsi que quelques-uns de ses développements récents. Un processus de branchement spatial est un système de particules qui se déplacent sur la droite réelle et se reproduisent indépendamment les unes des autres. On associe à chaque particule dans le processus de branchement une position, qui est transmise à sa descendance, à une modification aléatoire près. La descendance d'une particule se distribue autour de la position de cette particule. Les questions d'intérêt associées à cette famille de modèles sont typiquement reliées aux positions et trajectoires des particules extrémales du système, qui réalisent le plus grand déplacement observé dans le processus. La non-linéarité de ces problèmes rend leur analyse complexe, et révèlent souvent une contribution non-triviale jointe des effets de branchement et de déplacement.

Parmi les processsus branchants spatiaux les plus simples on trouve d'une part la *marche aléatoire branchante*, et d'autre part son analogue en temps continu le *mouvement brownien branchant*. Dans une marche aléatoire branchante, chaque individu donne naissance à chaque génération à un *processus ponctuel* d'enfants autour de sa position, indépendamment des autres individus. Ce processus ponctuel représente une façon de tirer au hasard, de façon éventuellement corrélée, le nombre d'enfants et leur position. Dans le mouvement brownien branchant, chaque individu se déplace de façon indépendante selon un mouvement brownien. Les individus meurent en donnant naissance à des enfants au bout de temps exponentiels indépendants de leur déplacement.

Les processus branchants spatiaux sont utilisés dans de nombreux domaines spécifiques pour décrire l'évolution de populations soumis à un branchement et un déplacement aléatoire. Dans les applications possibles en mathématique, on notera en particulier les cartes aléatoires [38, 6], la gravité de Liouville [16, 142], la description des fronts de populations invasives [51, 129], les propriétés des équations de réaction-diffusion [145], la taille des composantes connexes d'un graphe aléatoire [88], parmi bien d'autres.

1 Organisation du manuscrit

Ce manuscrit est divisé en 5 chapitres. Les trois premiers sont conçus pour introduire les outils et notions nécessaires à la description du comportement asymptotique des particules réalisant un grand déplacement dans ces processus. Le quatrième chapitre donne cette description, notamment la convergence des processus extrémaux dans un grand nombre de processus branchants spatiaux, depuis les premiers résultats d'Aïdékon, Berestycki, Brunet et Shi [5] et d'Arguin, Bovier et Kistler [20] pour le mouvement Brownien branchant et de Madaule [136] pour la marche aléatoire branchante, jusqu'à des applications plus récentes de Bovier et Hartung [55, 56] aux processus inhomogènes en temps. Le cinquième chapitre est dédié à l'étude de quelques résultats reliés aux processus de branchement-sélection.

Définition des processus branchants spatiaux. Le premier chapitre est dédié à l'introduction des principaux objets et notations, particulièrement l'espace d'état sur lequel les processus branchants spatiaux sont construits. On donnera deux définitions différentes d'un processus de branchement spatial, d'abord comme un processus de Markov à valeurs dans l'espace des mesures ponctuelles satisfaisant la *propriété de branchement*, puis comme un arbre aléatoire décoré. Si la première définition permet de connaître l'ensemble des positions occupées par des particules à tout temps, la seconde est plus riche puisqu'elle contient également les informations généalogiques du processus. Cette deuxième définition permet alors de parler de façon non-ambiguë des relations généalogiques entre deux individus, et permet donc un contrôle plus détaillé du comportement des particules.

Dans le chapitre 1, on introduira tout d'abord la *marche aléatoire branchante*, analogue spatial du processus de Galton-Watson. Dans ce processus, à chaque génération, toutes les particules donnent naissance de façon indépendante à un processus ponctuel d'enfants, centré sur la position du parent. On montrera également les premières propriétés de ce processus, notamment qu'une population se reproduisant comme une marche aléatoire branchante envahit son environnement à un rythme linéaire.

On introduira ensuite la classe des *processus de Lévy branchants*, définis comme des systèmes de particules continus à droite $(Z_t, t \geq 0)$ tels que pour tout $h > 0$, $(Z_{hn}, n \in \mathbb{N})$ est une marche aléatoire branchante. Cette famille de processus, qui est l'analogue pour les processus de branchement des processus de Lévy pour les marches aléatoires, a été introduite par Bertoin et Mallein [M16], qui en ont fourni une classification sous une hypothèse de moments finis.

Enfin, on considérera un processus de Lévy branchant ayant des trajectoires continues : le *mouvement brownien branchant*. Ce processus fait partie des processus branchants spatiaux les plus étudiés, en partie à cause de sa connexion, par formule de Feynman-Kac, aux solutions des équations de réaction-diffusion de type Fisher–Kolmogorov–Petrovskii–Piskunov (abréviée en équations F-KPP). Cette connexion permet de relier des estimées probabilistes du mouvement brownien branchant à des quantités analytiques associées aux solutions d'une équation aux dérivées partielles. En utilisant un mélange de techniques probabilistes et analytiques, il devient souvent possible de simplifier des arguments, voire de montrer des résultats plus fins que pour les marches aléatoires branchantes.

Martingales additives et décomposition épinale. Comme pour de nombreux modèles étudiés en théorie des probabilités, l'étude des martingales associées aux processus de branchement permet d'obtenir de nombreuses informations sur leurs comportements asymptotiques. On introduira dans le chapitre 2 une famille à un paramètre de martingales positives $(W_t(\theta), t \geq 0)_{\theta \in I}$, appelées martingales exponentielles (ou parfois martingales de Biggins) de la marche aléatoire branchante. Ces martingales donnent une estimation du nombre de particules se déplaçant à une vitesse donnée [43]. Lyons [133] a obtenu une preuve simple des conditions nécessaires et suffisantes de l'uniforme intégrabilité de la marche aléatoire branchante basée sur une méthode de *décomposition épinale*. Cette méthode consiste à décrire explicitement la loi du processus biaisé par la martingale $(W_n(\theta), n \geq 0)$, puis à utiliser la propriété de branchement pour montrer l'absolue continuité de la loi biaisée par rapport à la loi originale.

L'ensemble des valeurs de θ telles que $(W_t(\theta))$ est uniformément intégrable est de façon typique un intervalle ouvert. Pour étudier plus en détails le nombre de particules se déplaçant à la vitesse la plus grande dans le processus, il est nécessaire d'introduire et d'étudier une nouvelle martingale. Cette martingale, appelée martingale dérivée, est une martingale signée non-uniformément intégrable, qui converge presque sûrement vers une valeur limite positive Z_∞.

La martingale dérivée joue un rôle crucial dans l'étude du comportement des particules extrêmes de la marche aléatoire branchante, et a par conséquent été l'objet de nombreuses recherches, pour étudier les conditions d'intégrabilité optimales garantissant sa convergence, sa vitesse de convergence, la queue-distribution de Z_∞, etc. Nous parlerons de quelques avancées notables dans l'étude de la vitesse de convergence de la martingale dérivée de la marche aléatoire branchante obtenue par Buraczewski, Iksanov et Mallein [M29], ainsi qu'une famille remarquable de martingales dérivées paramétrique du mouvement Brownien branchant multidimensionnel, obtenu par Stasiński, Berestycki et Mallein [M30].

Les points fixes de la transformée de smoothing. La transformée de smoothing est généralement décrite comme une identité en loi pour des variables aléatoires. Étant donné une suite aléatoire $(T_j, j \geq 1)$ positive et décroissant vers 0, on souhaite déterminer, si elles existent, les solutions de l'équation

$$X \stackrel{(d)}{=} \sum_{j=1}^{\infty} T_j X^{(j)}, \tag{1}$$

où les $(X^{(j)}, j \geq 1)$ sont des copies i.i.d. de X. En d'autre termes, X est égale en loi à la "moyenne pondérée" par la famille de poids $(T_j, j \geq 1)$ de copies indépendantes et identiquement distribuées de X (on notera toutefois qu'on n'impose pas en général la condition $\sum T_j = 1$). La transformée de smoothing peut être vue comme une généralisation de la notion de stabilité des variables aléatoires. En effet, on remarquera qu'une variable Y est α-stable si et seulement si

$$Y \stackrel{(d)}{=} \sum_{j=1}^{\infty} t_j Y^{(j)}$$

pour toute suite $(t_j, j \geq 1)$ telle que $\sum t_j^\alpha = 1$. Une variable aléatoire est un point fixe de la transformée de smoothing si cette équation est vraie pour un choix randomisé de la suite $(t_j, j \geq 1)$.

La caractérisation des points fixes de la transformée de smoothing est liée aux marches aléatoires branchantes, et en particulier à leurs martingales additives associées. En effet, on peut appliquer récursivement l'équation (1) aux variables $X^{(1)}, X^{(2)}$, etc. On observe ainsi que la variable X est égale en loi à la "moyenne pondérée" d'une famille de poids qu'on peut représenter comme l'exponentielle des positions à la génération 2 d'une marche aléatoire branchante. Grâce à cette connexion, on obtient une représentation explicite des solutions de l'équation de smoothing, sous la forme d'un produit d'une variable de loi stable par une fonction de la limite d'une martingale additive de la marche aléatoire branchante associée. Ce résultat a été obtenu par Alsmeyer, Biggins et Meiners [11] sous des conditions optimales d'intégrabilité.

On décrira dans le chapitre 3 les principales étapes menant à la caractérisation des points fixes de l'équation de smoothing. On notera que cette caractérisation passe en particulier par la caractérisation des fonctions harmoniques bornées des marches aléatoires [75]. On donnera ensuite une généralisation de ce résultat au cadre des martingales dérivées obtenue par Alsmeyer et Mallein [M27]. Enfin, on utilisera la caractérisation des points fixes de l'équation de smoothing pour déterminer les mesures ponctuelles invariantes pour les marches aléatoires branchantes, obtenue dans [M37]. Ces mesures jouent un rôle particulier dans l'étude des propriétés asymptotiques des marches aléatoires branchantes, notamment la positions des particules ayant réalisé un grand déplacement.

Plus grand déplacement et processus extrémal. L'un des sujets les plus étudiés des processus branchants spatiaux concernent les particules extrémales, c'est-à-dire les particules ayant réalisé les plus grands déplacements du processus. Ces questions sont centrales en particulier pour le mouvement Brownien branchant à cause de sa relation avec l'équation F-KPP, obtenue par McKean [145]. En particulier, le plus grand déplacement du mouvement Brownien branchant est lié à convergence vers la solution *travelling-wave* la plus lente de l'équation F-KPP partant d'une condition initiale simple. Afin de traduire ce résultat analytique vers le modèle probabiliste, il devient nécessaire de comprendre le comportement fin des trajectoires des particules réalisant un grand déplacement. On remarque ainsi [18, 19] que les particules réalisant un grand déplacement font soit partie de la même famille, soient proviennent de familles complètement différentes issues de l'origine du processus. Cela implique une description de la limite du processus extrémal comme un processus de Poisson ponctuel translaté et décoré : les leaders de chaque famille contribuant au processus extrémal se répartissent selon un processus de Poisson ponctuel d'intensité exponentielle, et le reste de la famille se répartit autour du leader selon un processus ponctuel indépendant.

L'étude du comportement asymptotique des particules extrêmes des marches aléatoires branchantes a nécessité l'introduction d'un grand nombre de techniques maintenant considérées comme classique dans l'étude de ces processus. On mentionnera ainsi les méthodes de concentration, basées sur des calculs de moments d'ordre 1 et 2 du nombre de particules satisfaisant une propriété particulière, les décompositions épinales pour un contrôle plus efficace de la trajectoire des particules extrémales, les méthodes de censure et de barrières absorbantes pour diminuer la variance des quantités considérées, etc.

On présentera dans le chapitre 4 les résultats classiques obtenus notamment par Bramson [57] et Lalley and Sellke [127] sur la convergence en loi du plus grand déplacement du mouvement brownien branchant. On décrira également les principales étapes menant à l'obtention par Aïdékon [4] de la convergence en loi du plus grand déplacement de la marche aléatoire branchante sous des hypothèses d'intégrabilité optimales. On mentionnera quelques-unes des nombreuses généralisation de ce résultat à des familles modifiées de processus de branchement, comme le plus grand déplacement d'un mouvement brownien branchant en dimension d [M40], ou d'un mouvement brownien branchant multitype réductible [M33]. On s'intéressera notamment aux effets des variations de l'environnement sur le comportement des particules extrémales, qu'elles soient brutales [M1], lentes à l'échelle de reproduction des individus [M2], ou bien aléatoire représentant des fluctuations aléatoires appliquées à l'ensemble de la population [M8].

On s'intéressera ensuite plus en détails à l'étude du processus extrémal des marches aléatoires branchantes. La caractérisation des mesures ponctuelles stables pour la propriété de branchement obtenue dans [M37] permet de simplifier certains arguments montrant cette convergence vers un processus de Poisson ponctuel décoré. Dans le cas du mouvement brownien branchant, il existe plusieurs définitions de la loi de la décoration apparaissant dans le processus extrémal [5, 20], tandis que celle-ci est essentiellement implicite dans le cas de la marche aléatoire branchante [M12]. Pour étudier cette convergence plus en détails, on pourra s'intéresser à la convergence de processus extrémaux enrichis d'information généalogiques [M12], trajectorielles [M6] ou directionnelles [M40] dans le cas d'un processus multidimensionnel.

Enfin, on s'intéressera à la convergence des processus extrémaux de processus branchant spatiaux généralisés, et du lien entre la loi de la limite de ce processus avec les trajectoires suivies par les particules y contribuant. On mentionnera en particulier le cas du mouvement brownien branchant inhomogène obtenue par Bovier and Hartung [56] et celui du mouvement brownien branchant multitype réductible étudié par Belloum et Mallein [M33].

Processus de branchement-sélection. Les processus de branchement-sélection sont une généralisation naturelle des processus de branchement. Dans un tel modèle, les particules se reproduisent de façon indépendante comme dans un processus de branchement spatial, mais un mécanisme de sélection limite leur capacité à se reproduire en fonction des propriétés des autres particules. Un tel modèle ne satisfait plus la propriété de branchement, ce qui rend son étude plus difficile. Toutefois, grâce à des méthodes de couplages, ces modèles peuvent être analysés de façon poussée.

Les processus de branchement-sélection permettent d'une part de modéliser des populations "réalistes" telles que le nombre d'individus ne croît pas exponentiellement, tout en évitant leur extinction presque sûre. D'autre part, c'est une modélisation naturelle des phénomènes de compétition entre individus, de limitation des ressources disponible et de sélection naturelle. Toutefois, les domaines d'application ne se limitent pas à la biologie, puisque les processus de branchement-sélection permettent notamment d'analyser la performance d'algorithmes de Monte-Carlo [84] ou de parallélisation [153].

Brunet et Derrida [59] ont introduit le modèle de la N-*marche aléatoire branchante*, dans lequel à chaque génération on sélectionne les N individus les plus à droite pour se reproduire, en supprimant tous les autres. Ils ont prédit que dans ce modèle, le nuage de particules dérive à une vitesse v_N satisfaisant $v_\infty - v_N \sim \frac{\chi}{(\log N)^2}$ lorsque $N \to \infty$. Ce comportement, qu'on appellera le *comportement de Brunet-Derrida*, a été démontré par Bérard et Gouéré [64], et étendu à de nombreux autres systèmes de branchement-sélection, comme le mouvement brownien branchant avec absorption [30], le *L-mouvement Brownien branchant* [154], ou des processus de branchement-sélection avec un taux de branchement [164] ou une loi de déplacement [177] inhomogène. On mentionnera également des modèles de marches branchantes avec sélection d'un nombre non-constant d'individus [M3], ou sortant de la classe d'universalité de Brunet-Derrida [M4], ou des marches branchantes avec déplacement à queue lourde [28, 158].

Une des questions cruciales de l'étude des processus de branchement-sélection est l'étude de la structure généalogique des individus. En effet, celle-ci peut être utilisée pour tester sur des populations réelles les traces d'épisodes de sélection importants et leur effet sur la vitesse d'adaptation des particules. Cette structure généalogique a également un effet important dans la performance d'algorithmes de Monte-Carlo basée sur des méthodes de branchement-sélection [84]. Berestycki, Berestycki et Schweinsberg [31] ont montré que la généalogie du mouvement Brownien branchant avec absorption quasi-critique converge vers le coalescent de Bolthausen-Sznitman [50], confirmant une conjecture de Brunet, Derrida, Mueller et Munier [61]. Cette conjecture a été obtenue grâce à l'introduction d'une famille exactement résoluble de processus de branchement-sélection. Cortines et Mallein ont étudiés des variations de cette famille exactement résoluble, choisissant les parents de la nouvelle génération de façon aléatoire [M10] ou introduisant une force de rappel diminuant l'avantage sélection d'une innovation [M21]. Dans le second cas, l'arbre généalogique de la population converge vers un processus Beta coalescent. La construction de modèles simples exhibant cette structure généalogique reste ouverte, malgré les progrès récents de Tourniaire [177].

On terminera ce chapitre par la présentation d'une application de l'étude des processus de branchement-sélection à l'étude de la longueur du plus long chemin croissant dans un graphe d'Erdős-Rényi. On déterminera ainsi les propriétés asymptotiques la fonction $p \mapsto C(p)$ obtenue par Newman [153] décrivant la proportion de sommets d'un graphe appartenant à ce plus long chemin croissant. L'analycité sur $(0, 1]$ de cette fonction a été démontrée dans [M19], tandis que le comportement au voisinage de 0 est analysé dans [M15] grâce à un couplage avec une marche aléatoire branchante avec sélection en temps continu. La construction d'une méthode de Monte-Carlo non-biaisée pour la simulation de C est obtenue dans [M39], tandis qu'on calcule dans [M41] la longueur du plus court chemin croissant entre 1 et n dans un graphe d'Erdős-Rényi peu dense.

2 Perspectives et futurs travaux

Malgré de très nombreux travaux récents, l'étude des valeurs extrêmes des processus branchants spatiaux et questions associées reste un domaine en plein expansion, avec une littérature féconde et grandissante. De nouveaux phénomènes, issus de la physique, la biologie, l'informatique... appellent à la création de processus modélisant ces phénomènes. Des avancées récentes sur sur différents modèles spécifiques appellent à la création d'une théorie unifiée, permettant de déterminer les propriétés universelles de ces familles de processus et d'en identifier les caractéristiques.

Une liste exhaustive des possibles perspectives de recherche dans ce domaine étant impossible, on se contentera d'indiquer ici quelques grandes lignes et problématiques spécifiques.

Connexions avec les équations de réaction-diffusion. Les liens entre le mouvement Brownien branchant et les équations de réaction-diffusion de type F-KPP ont d'abord été utilisés par McKean et Bramson. Toutefois, les récents développements d'Etheridge et Penington [89] ouvrent la possibilité de lier le mouvement Brownien branchant à des équations de réaction-diffusion ne faisant pas partie de la classe d'universalité F-KPP. Cela permettrait d'étudier des effets de type Allee, et la dualité entre les ondes pushed et pulled. Cette modélisation réclame toutefois la construction de résultats forts sur des modèles de propagation de votes sur des arbres aléatoires, qui nécessite de nombreux nouveaux développements.

Bouin et Calvez [51] ont introduit une équation F-KPP à diffusion variable pour modéliser l'invasion d'une population de crapauds-buffles en Australie. Un modèle associé proposé par Berestycki, Mouhot et Raoul [35] de mouvement Brownien branchant dont la variance dépend du temps présente des propriétés assez différentes du modèle de Bouin et Calvez. La construction d'un processus de branchement associé à cette équation reste donc un problème ouvert. De même, les résultats récent de Calvez et ses co-auteurs [66, 67] sur le comportement asymptotique des populations autopropulsées suggèrent de s'intéresser à des processus de populations dans lesquels les particules conservent une vitesse v (aléatoire) pour une durée aléatoire, avant de réévaluer leur direction de propagation (une dynamique *run-and-tumble*). La théorie des grandes déviations des marches aléatoires persistantes [90] permet de prédire des phénomènes exotiques pour le comportement des processus extrémaux, analogues à ceux rencontrés dans les marches aléatoires branchantes à déplacements à queue lourde par Bhattacharya et al. [40].

Processus de branchement multitypes. Les propriétés asymptotiques fines des processus de branchement multitypes réductibles révèlent une grande variété de phénomènes, dépendant des relations entre les différents types [M33]. D'autres modèles, comme le mouvement brownien branchant multitype peuvent être pensés comme des processus de branchement multitypes, tels que le type est donné par la norme de la position de la particule. Dans ce modèle, le "type" de la particule n'influence pas son déplacement, et change de plus en plus lentement à mesure que les particules s'éloignent de l'origine. On observe alors une "stabilisation" du type dans le processus extrémal [M40], et une famille de martingales dérivées dépendant du type [M30] apparaît.

Il est donc intéressant de tester les limites de ces familles de modèles. On pourra ainsi s'intéresser à un mouvement brownien branchant multitype dont le type n'influence pas le comportement de la particule, et ralentit au cours du temps. Si ce ralentissement est suffisamment lent, on s'attend à un phénomène de moyennisation dans le processus extrémal, et une martingale dérivée unique pour le processus, alors qu'on aurait une famille de martingales dérivées indexées par le type dans le cas contraire. Le comportement des mouvements browniens branchants irréductibles, bien que probablement plus proche de celui du mouvement brownien branchant à un seul type, n'est pas encore caractérisé également, malgré des travaux récents de Ren et Yang [161] sur l'existence d'une travelling wave critique de l'équation associée et [49, 48] sur le mouvement Brownien branchant "on/off".

D'autres familles de processus de branchement peuvent être considérées comme des processus branchants spatiaux multitypes. C'est par exemple le cas du mouvement brownien branchant à valeurs dans $\mathbb{R} \times I$, où I est un intervalle de $\mathbb{R}$, avec conditions réfléchissantes ou absorbantes au bord. Berestycki et Graham [29] ont ainsi montré l'existence de solutions de types travelling-waves pour le mouvement brownien branchant dans $\mathbb{R} \times \mathbb{R}_+$, avec condition absorbante sur la seconde coordonnée. Ce problème est lié à l'existence d'une famille de martingales dérivées (dépendant de l'ordonnée initiale du processus) pour le mouvement brownien branchant dans $\mathbb{R}^2$ avec absorption en $\mathbb{R} \times \{0\}$. Une étude plus détaillée du comportement asymptotique de ce processus permettrait d'étudier l'effet de la barrière absorbante sur la vitesse de propagation en fonction de la distance à cette barrière.

Processus de branchement-sélection. De très nombreuses questions restent ouvertes dans l'étude des processus de branchement sélection. Un problème naturel serait de proposer une méthode unifiée permettant de montrer la convergence vers le coalescent de Bolthausen-Sznitman des processus favorisant la reproduction des particules les plus à droite, qu'il s'agisse de la N-marche aléatoire branchante, de la L-marche aléatoire branchante ou de processus de branchements tels que la vitesse de reproduction dépend du rang. De plus, le comportement asymptotique fin de la vitesse du nuage de particules reste difficilement accessible, et le deuxième terme du développement asymptotique, qui devrait être universel d'après les prédictions de Brunet, Derrida, Mueller et Munier [60] reste inconnu.

Berestycki et Zhao [34] ont étudié un processus de branchement-sélection multidimensionnel dans lequel à chaque génération les N particules les plus loin de l'origine se reproduisent. Ils montrent que pendant un temps, les particules se répartissent sur un grand cercle centré en 0, mais qu'au fil des générations ce cercle se brise en segments, jusqu'à ce qu'il ne reste qu'un seul nuage de particules s'éloignant de l'origine dans une direction aléatoire. Ce résultat montre qu'il est en général difficile de conserver une population diverse dans les simulations de Monte-Carlo de processus branchant. Il serait intéressant de calculer le rythme minimal de croissance de la suite (N_n) pour qu'un processus de branchement-sélection multidimensionnel conserve des particules présentes dans plusieurs/toutes les directions. Grâce aux résultats de [M3], il semble que $\log N_n \approx n^{1/3}$ devrait être un rythme de croissance suffisant pour le nombre de particules à conserver. On pourra également étudier des procédures de branchement-sélection permettant de conserver des populations de particules survivant dans plusieurs directions sans diminuer la vitesse d'augmentation de la norme des particules.

Un autre modèle de branchement-sélection d'intérêt est la N-marche aléatoire branchante à vitesse d'amélioration prescrite. Étant donné $a > 0$ et $N \in \mathbb{N}$, dans ce système on garde à chaque étape les N particules les plus proches de la position an. Ce modèle devrait avoir un comportement similaire à celui des abeilles browniennes [33]. Le comportement asymptotique de l'arbre généalogique de ce processus est également un objet d'intérêt.

Processus de fragmentation et de croissance-fragmentation. Les processus de croissance-fragmentation ont pour objectif de décrire la dynamique d'une population de particules possédant une masse, dans laquelle à chaque événement de reproduction, la masse est distribuée entre les enfants. On peut de façon générale décrire ces processus comme l'exponentielle d'un processus de branchement, la masse étant une quantité positive et se distribuant généralement de façon multiplicative à chaque événement de branchement. Bertoin [37] a étendu cette classe de processus aux processus de croissance-fragmentation markoviens, dans lequel la taille d'une particule entre deux événements de reproduction peut évoluer selon n'importe quel processus markovien, toutefois la classe la plus naturelle de processus à étudier reste les processus autosimilaires.

Ces processus peuvent être construits comme un changement de temps de l'horloge locale de chaque particule dans un processus de Lévy branchant. Cette construction est analogue à la construction de Lamperti pour les processus autosimilaires comme un changement de temps de l'exponentielle d'un processus de Lévy. Il est alors naturel de s'intéresser à d'autres changements de temps naturels pour les processus de Lévy, comme la transformée de Lamperti pour les processus de branchement à états continus (CSBP). Ce type de processus pourrait être liés aux processus coalescents emboîtés de Foutel-Rodier et al. [97].

D'autres changements de temps plus exotiques pourraient également présenter un intérêt. On pourrait ainsi étudier le comportement asymptotique de l'enveloppe convexe de la trajectoire d'un mouvement brownien réfléchi à l'intérieur d'un disque grâce à un processus de fragmentation pur inhomogène en temps tel que la fragmentation d'un élément de taille x se produise à taux $e^{1/x}$. Ce processus permettrait ainsi de retrouver les résultats de De Bruyne et al. [62] prédisant un comportement en espérance de la longueur de cette enveloppe convexe se comportant en $2\pi - e^{-\mu t^{1/2}(1+o(1))}$ pour un certain $\mu > 0$.

CHAPTER 1

Construction of spatial branching processes

"Un voyage de mille lieues commence toujours par un premier pas"

Lao-Tseu – Tao Te King.

Summary.

In this chapter, we introduce the three main spatial branching processes of interest in this manuscript: the branching random walk, the branching Lévy process and the branching Brownian motion. We begin by introducing the space of point measure $\mathcal{P}(\mathbb{R})$, as well as the space of probability distributions on $\mathcal{P}(\mathbb{R})$ that we write $\mathfrak{P}(\mathbb{R})$. We introduce some notation such as the Laplace transform of a random point measure, that characterizes its distribution. We also introduce some probability distributions of interest, such as the randomly shifted decorated Poisson point processes, introduced by Subag and Zeitouni [176]. These distributions are used in Chapter 4 to describe the limiting distributions of the extremal processes of spatial branching processes.

Next, we introduce the branching random walk as a Markov chain on $\mathcal{P}(\mathbb{R})$ satisfying the branching property and invariance in distribution by translation: the point measure at time $n+1$ is obtained by replacing each atom at time n with an independent copy of the same point measure, shifted by the position of the atom. We then define the branching Lévy processes as càdlàg Markov processes on $\mathcal{P}(\mathbb{R})$ such that its discrete-time skeletons are Markov chains. The class of branching Lévy processes was characterized in [M16]. We also introduced branching-stable point measures, that were studied in [M22].

In a third section, we introduce a richer definition of branching random walks, such that not only the positions, but also the genealogical relationship of particles are measurable in the model. We extend this definition to branching Lévy processes and branching Brownian motions (which are branching Lévy processes with a.s. continuous trajectories). We end this section with the computation of the first two orders in the asymptotic expansion of the maximal displacement M_n of the branching random walk, and the convergence in distribution obtained by Bramson [57] for the maximum of the branching Brownian motion due to its connection with reaction diffusion equations.

A spatial branching process is a particle system evolving on a state space (usually the real line or $\mathbb{R}^d$) in which particles move and reproduce independently of one another. At each birth event, the newborn particles are positioned around their parent and added to the system. Then, they start evolving independently of one another and of the rest of the process. This fact, that particles evolve independently of one another after their birth, is the fundamental property of branching processes, which we refer to as the *branching property*.

A spatial branching process can be used to model a large variety of phenomena. It is for example a natural model for an invading species in a new environment without competition. For these populations, competition between individuals can be neglected as long as the local population remains small enough. For similar reasons, this model can also be used for the development of parasitic infection [25], the spread of epidemics [24], particles cascades in nuclear [104] and high energy [151] physics, and so on. A spatial branching process can also be used to model an environment exhibiting a random recursive branching structure, such as the lung or vascular system of a mammal [148], or a random graph [88]. It can also be adapted to the description of the value of a two-players game in min-max settings, with a given particle representing a state of the game, and its position the value of that position to the player [144].

Spatial branching processes are also used as toy-models in statistical physics for the study of complex phenomena, such as turbulence [159], spin glasses [53] or random polymer in large dimensions [76]. They are also used as a testing ground for the study of log-correlated random fields such as the Gaussian free field [182], and help the understanding of Liouville quantum gravity [142]. Log-correlated fields appear in a variety of other domains, such as the description of eigenvalues of some random matrices models [23] or the fluctuations of Riemann's zeta function on the critical line [17].

To describe a spatial branching process, one has to give at all times the location of particles. A natural way to encode this information is with the use of a point measure, in which each particle is represented by a Dirac mass at the position of that particle. Therefore a spatial branching process can be represented as a Markov process on the space of point measures, which is the angle we choose in Section 1.2. The notation and properties associated to random point measures are given in Section 1.1.

However, this representation does not give the necessary informations to recover the genealogical structure of the associated branching process. In other words, it does not allow to follow the trajectory of a particle, or to evaluate the age of the most recent common ancestor of two particles. In order to encode these more precise informations, a spatial branching process can be defined as a random function on a tree, encoded using the Ulam–Harris notation. We explicit this construction in Section 1.3.

1.1 Notation for point measures

A point measure on $\mathbb{R}$ is a Radon measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying

$$\forall A \in \mathcal{B}(\mathbb{R}),\ \mu(A) \in \mathbb{Z}_+ \cup \{\infty\}.$$

We consider the space of Radon point measures with a largest atom, defined as

$$\mathcal{P}(\mathbb{R}) := \{\mu \in \mathcal{M}(\mathbb{R}) : \forall x \in \mathbb{R}, \mu([x, \infty)) \in \mathbb{Z}_+\}. \tag{1.1}$$

The space $\mathcal{P}(\mathbb{R})$ can be canonically identified with the space of non-increasing sequences in $[-\infty, \infty)^{\mathbb{N}}$ that converge to $-\infty$ via the following bijection

$$\mu = \sum_{n=0}^{\infty} \mathbf{1}_{\{x_n \neq -\infty\}} \delta_{x_n} \quad \longleftrightarrow \quad \mathbf{x} = \left(\inf\{z > 0 : \mu([z,\infty)) \geq n\}\right). \tag{1.2}$$

For example, with this identification, the empty point measure is identified with the sequence $(-\infty, -\infty, \cdots)$, and the point measure $2\delta_a$ with the sequence $(a, a, -\infty, -\infty, \cdots)$. More generally, a point measure is identified with its sequence of atoms ranked in the non-increasing order, eventually completed with points at $-\infty$ if the measure is finite. We use here the point $\{-\infty\}$ as a cemetery state via the identification $\delta_{-\infty} = 0$.

For $\mu \in \mathcal{P}(\mathbb{R})$ and φ a measurable non-negative function, we define

$$\langle \mu, \varphi \rangle := \int_{\mathbb{R}} \varphi \mathrm{d}\mu = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{x_n > -\infty\}} \varphi(x_n),$$

writing $\mathbf{x}$ for the ranked sequence of atoms of μ.

We also define the translation operation on $\mathcal{P}(\mathbb{R})$ as follows. Given a point measure $\mu \in \mathcal{P}(\mathbb{R})$ identified with $\mathbf{x}$ and $a \in [-\infty, \infty)$, we define

$$\tau_a \mu = \tau_a \mathbf{x} = (x_n + a, n \in \mathbb{N}).$$

In other words, $\tau_a \mu$ the image measure of μ by the function $x \mapsto x + a$. We note in particular that $\tau_{-\infty}\mu$ is the empty measure for all $\mu \in \mathcal{P}(\mathbb{R})$.

We equip the space $\mathcal{P}(\mathbb{R})$ with the topology of vague convergence, i.e. $\mu_n \to \mu$ if and only if $\langle \mu_n, \varphi \rangle \to \langle \mu, \varphi \rangle$ as $n \to \infty$ for all continuous compactly-supported function φ. This topology turns $\mathcal{P}(\mathbb{R})$ into a separable completely metrizable space with a countable dense subset, which makes the definition of random variables on that space convenient.

1.1.1 Random point measures

A random point measure (or point process) is a random element of $\mathcal{P}(\mathbb{R})$. The space of probability distributions on $\mathcal{P}(\mathbb{R})$ is denoted $\mathfrak{P}(\mathbb{R})$. As a rule, in order to simplify the statements of the results, we will use the following typographic convention:

— the law of a random point measure in $\mathfrak{P}(\mathbb{R})$ is written as a cursive capital letter $(\mathcal{D}, \mathcal{E}, \cdots)$,

— a random point measure with the associated distribution is written as a straight capital letter $(D, E, \cdots)$,

— the (random) sequence of atoms of that random point measure is written in lowercase $(\mathbf{d} = (d_j, j \in \mathbb{N}), \mathbf{e} = (e_j, j \in \mathbb{N}), \cdots)$.

A particular class of point measures playing an important role in the rest of this presentation are the Poisson point processes, defined as follows.

Definition 1.1 (Poisson point process)**.** Given ϱ a Radon measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\varrho([0,\infty)) < \infty$, a Poisson point process with intensity ϱ is a random measure P on $\mathbb{R}$ satisfying the following properties:

1. for all $A \in \mathcal{B}(\mathbb{R})$ with $\varrho(A) < \infty$, we have $\mathbb{P}(P(A) = k) = e^{-\varrho(A)} \frac{\varrho(A)^k}{k!}$ (i.e. $P(A)$ is distributed as a Poisson random variable with parameter A);
2. if $A, B \in \mathcal{B}(\mathbb{R})$ with $A \cap B = \emptyset$, then $P(A)$ and $P(B)$ are independent.

The law $\mathcal{D}$ of a random point measure D is characterized by its (functional) Laplace transform, defined as

$$F_{\mathcal{D}} : \varphi \in \mathcal{T} \mapsto \int_{\mathcal{P}(\mathbb{R})} \exp\left(-\langle D, \varphi\rangle\right) \mathcal{D}(\mathrm{d}D) = \mathbb{E}\left(\exp\left(-\langle D, \varphi\rangle\right)\right), \tag{1.3}$$

where $\mathcal{T}$ is the space of non-negative measurable functions on $\mathbb{R}$. By approximations, it is a standard procedure to show that $\mathcal{D}$ can be characterized the restriction of its Laplace transform to the set $\mathcal{T}^*$ of positive Lipschitz non-decreasing bounded functions φ such that there exists $x \in \mathbb{R}$ satisfying $\varphi(x) = 0$.

Proposition 1.2 (Campbell's formulas)**.** *If P is a Poisson point process with intensity ϱ, then*

- *for all $f \in \mathrm{L}^1(\varrho)$, $\mathbb{E}\left(\langle P, f\rangle\right) = \int_{\mathbb{R}} f \mathrm{d}\varrho$;*
- *for all $f \in \mathrm{L}^1(\varrho) \cap \mathrm{L}^2(\varrho)$, $\mathbb{V}\mathrm{ar}\left(\langle P, f\rangle\right) = \int_{\mathbb{R}} f^2 \mathrm{d}\varrho$;*
- *for all $f \in \mathcal{T}$, $F_{\mathcal{P}}(f) = \mathbb{E}\left(\exp\left(-\langle P, f\rangle\right)\right) = \exp\left(-\int_{\mathbb{R}}\left(1 - e^{-f}\right)\mathrm{d}\varrho\right)$*

Observe that if P is a Poisson point process, its intensity measure can be recovered via the first Cambell's formula. More generally, given D a random point measure, if there exists a Radon measure ϱ such that

$$\mathbb{E}\left(\langle D, \varphi\rangle\right) = \int_{\mathbb{R}} \varphi \mathrm{d}\varrho$$

for all continuous compactly supported functions φ, we call ϱ the intensity measure of (the law of) D.

1.1.2 The branching convolution operator

We introduce the branching convolution operation $\circledast$ on $\mathfrak{P}(\mathbb{R})$, with is an analogue, in the space of random point measures, of the convolution equation for random variables.

Definition 1.3 (Branching convolution operation)**.** Given $\mathcal{D}$ and $\mathcal{E}$ two probability distributions on $\mathcal{P}(\mathbb{R})$, their convolution $\mathcal{D} \circledast \mathcal{E}$ is defined as the law of

$$\sum_{j=1}^{\infty} \tau_{d_j} E^{(j)},$$

where $\mathbf{d}$ is the ranked sequence of atoms of a point measure of law $\mathcal{D}$ and $(E^{(j)}, j \geq 1)$ are i.i.d. point measures with law $\mathcal{E}$.

We observe that $\mathcal{D} \circledast \mathcal{E}$ is the point measure obtained by replacing each atom of a point measure of law $\mathcal{D}$ by a point measure of law $\mathcal{E}$, shifted by the position of the atom, see Figure 1.1. In terms of branching processes, if $\mathcal{D}$ is the position of particles at a given time, and that each particle gives birth to a point process of descendants that spread according to the law $\mathcal{E}$ around the position of their parent, then $\mathcal{D} \circledast \mathcal{E}$ is the position of all the newborn particles after the branching event.

The branching convolution operation is an extension of the usual convolution. Indeed, considering two independent random variables X and Y with law μ and ν respectively, and writing $\mathcal{D}$ and $\mathcal{E}$ the laws of δ_X and δ_Y, we easily check that $\mathcal{D} \circledast \mathcal{E}$ is the law of δ_{X+Y}, a point measure with a single atom whose law is given by $\mu * \nu$. However, contrary to the usual convolution, the branching convolution is usually non-commutative.

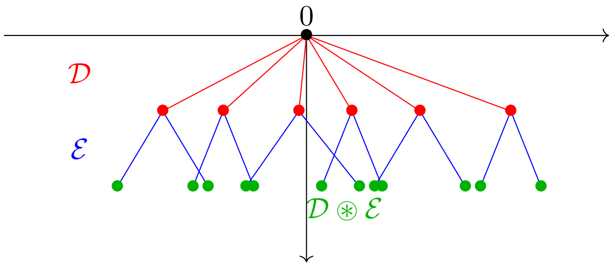


Figure 1.1 – Construction of the branching convolution equation, by making a step of a branching random walk with law $\mathcal{D}$ followed by a step of a branching random walk with law $\mathcal{E}$.

Remark 1.4. It is worth noting that $\mathcal{D} \circledast \mathcal{E}$ is not necessarily an element of $\mathfrak{P}(\mathbb{R})$. If we denote, for $\lambda > 0$, by $\mathcal{P}_\lambda$ the law of a Poisson point process with intensity $e^{-\lambda x}\mathrm{d}x$, it is then a simple exercise to verify that

- if $\lambda < \mu$, $\mathcal{P}_\lambda \circledast \mathcal{P}_\mu$ is a Poisson point process with (random) intensity $Ze^{-\mu x}$, where Z a positive stable random variable with index $\frac{\lambda}{\mu}$,
- if $\lambda \geq \mu$, then $\mathcal{P}_\lambda \circledast \mathcal{P}_\mu$ does not produce an element of $\mathfrak{P}$, as a variable F with law $\mathcal{P}_\lambda \circledast \mathcal{P}_\mu$ would satisfy $F((a,b)) = \infty$ a.s. for all $a < b$.

In order to guarantee for the branching convolution to be well-defined, it is sufficient to ask the point measures to satisfy an exponential integrability condition.

Property 1.5. *Let $\mathcal{D}, \mathcal{E} \in \mathfrak{P}(\mathbb{R})$, if there exists $\theta \geq 0$ such that*

$$\mathbb{E}\left(\langle D, \mathbf{e}_\theta\rangle\right) < \infty \text{ and } \mathbb{E}\left(\langle E, \mathbf{e}_\theta\rangle\right) < \infty,$$

where $\mathbf{e}_\theta : z \mapsto e^{\theta z}$, then $\mathcal{D} \circledast \mathcal{E} \in \mathfrak{P}(\mathbb{R})$, and moreover

$$\int_{\mathcal{P}(\mathbb{R})} \langle F, \mathbf{e}_\theta\rangle \mathcal{D} \circledast \mathcal{E}(\mathrm{d}F) = \mathbb{E}\left(\langle D, \mathbf{e}_\theta\rangle\right)\mathbb{E}\left(\langle E, \mathbf{e}_\theta\rangle\right).$$

The function $\theta \mapsto \mathbb{E}\left(\langle D, \mathbf{e}_\theta\rangle\right)$ is the Laplace transform of the intensity measure of the random point measure D. This property implies that if the laws $\mathcal{D}$ and $\mathcal{E}$ have intensity measures, then the intensity measure of $\mathcal{D} \circledast \mathcal{E}$ is the convolution of the intensity measures of $\mathcal{D}$ and $\mathcal{E}$.

However, note that the intensity measure does not characterize the law of a random point measure, contrary to its functional Laplace transform. To simplify computations associated with the branching convolution equation, it will be sometimes more convenient to work with the log-Laplace functional of random point measures.

Definition 1.6 (Log-Laplace functional)**.** Let $\mathcal{E}$ be a probability distribution on $\mathcal{P}(\mathbb{R})$. The log-Laplace functional of $\mathcal{E}$ is the functional defined, for all measurable non-negative function φ, by

$$\Psi_{\mathcal{E}}[\varphi] : z \in \mathbb{R} \mapsto -\log \mathbb{E}\left(e^{-\langle \tau_z E, \varphi\rangle}\right).$$

The distribution $\mathcal{E}$ is characterized by its log-Laplace functional $\Psi_{\mathcal{E}}$.

The following equation justifies the introduction of this functional for the study of spatial branching processes.

Property 1.7. *Let $\mathcal{D}$ and $\mathcal{E}$ be two probability distributions on $\mathcal{P}(\mathbb{R})$ such that $\mathcal{D} \circledast \mathcal{E}$ is well-defined, we have $\Psi_{\mathcal{D}\circledast\mathcal{E}} = \Psi_{\mathcal{D}} \circ \Psi_{\mathcal{E}}$.*

Proof. For all measurable non-negative function φ and $z \in \mathbb{R}$, by definition of the branching convolution operation, we have

$$\begin{aligned}\Psi_{\mathcal{D}\circledast\mathcal{E}}[\varphi](z) &= -\log \mathbb{E}\left(\exp\left(-\sum_{j=1}^{\infty} \langle \tau_{d_j+z} E^{(j)}, \varphi\rangle\right)\right)\\ &= -\log \mathbb{E}\left(\exp\left(-\sum_{j=1}^{\infty} \Psi_{\mathcal{E}}[\varphi](d_j+z)\right)\right)\\ &= -\log \mathbb{E}\left(\exp\left(-\langle \tau_z D, \Psi_{\mathcal{E}}[\varphi]\rangle\right)\right) = \Psi_{\mathcal{D}}\left[\Psi_{\mathcal{E}}[\varphi]\right](z),\end{aligned}$$

computing the expectation conditionally on D on a first step, and using the independence of the point processes $(E^{(j)}, j \geq 1)$. $\square$

1.1.3 Shifted decorated Poisson point processes

A class of point measures introduced by [176] plays a particular role in the study of extreme values of spatial branching processes: the shifted decorated Poisson point processes with exponential intensity. This class can be characterized in several different ways, we record in the present section some of its characteristics.

Definition 1.8. Let $\lambda > 0$, S a positive random variable and $\mathcal{D}$ a probability law on $\mathcal{P}(\mathbb{R})$. A shifted decorated Poisson point process with intensity $Se^{-\lambda x}\mathrm{d}x$ and decoration law $\mathcal{D}$ (shortened to $\mathrm{SDPPP}(Se^{-\lambda x}\mathrm{d}x, \mathcal{D})$) is a random point measure X with law

$$\mathcal{S} \circledast \mathcal{P}_\lambda \circledast \mathcal{D},$$

where $\mathcal{S}$ is the law of $\delta_{\lambda^{-1}\log S}$ and $\mathcal{P}_\lambda$ the law of a Poisson point process with intensity $e^{-\lambda x}\mathrm{d}x$. In other words, writing $(p_j, j \geq 1)$ the ranked sequence of atoms of a Poisson point process of law $\mathcal{P}_\lambda$, we can write

$$X = \sum_{j=1}^{\infty} \tau_{\lambda^{-1}\log S + p_j} D^{(j)},$$

where $D^{(j)}$ are i.i.d. copies of $\mathcal{D}$, as illustrated in Figure 1.2.

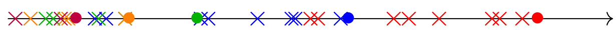


Figure 1.2 – Construction of shifted decorated Poisson point process via the superposition of i.i.d. random point measures, shifted by the positions of the atoms of a Poisson point process with exponential intensity.

The law of an $\mathrm{SDPPP}(Se^{-\lambda x}\mathrm{d}x, \mathcal{D})$ is well-defined as long as $\mathbb{E}\left(e^{\lambda d_1}\right) < \infty$, with d_1 the largest atom of a random point measure of law $\mathcal{D}$. We remark that without loss of generality, we can assume the law of the decoration is supported by

$$\mathcal{P}^*(\mathbb{R}) := \{D \in \mathcal{P}(\mathbb{R}) : d_1 = 0\}.$$

Indeed, given X an $\mathrm{SDPPP}(Se^{-\lambda x}\mathrm{d}x, \mathcal{D})$, we define the point measure distribution $\mathcal{D}^*$ by

$$\int_{\mathcal{P}(\mathbb{R})} F(D)\mathcal{D}^*(\mathrm{d}D) = \frac{\mathbb{E}\left(e^{\lambda d_1} F(\tau_{-d_1} D)\right)}{\mathbb{E}\left(e^{\lambda d_1}\right)},$$

and set $c = \mathbb{E}\left(e^{\lambda d_1}\right)$. Then, X can be defined as an SDPPP$(cSe^{-\lambda x}\mathrm{d}x, \mathcal{D}^*)$. As a result, without loss of generality we always assume that the decoration distribution is supported by $\mathcal{P}^*(\mathbb{R})$.

Remark 1.9. Even with the condition that $\mathcal{D}$ is supported by $\mathcal{P}^*(\mathbb{R})$, the description of an SDPPP by a triplet $(S, \lambda, \mathcal{D})$ is not necessarily unique. With the notation of Remark 1.4, we remark that for $\lambda < \mu$, a random point measure with law $\mathcal{P}_\lambda \circledast \mathcal{P}_\mu$ can be written as

— an SDPPP$(c_{\lambda,\mu}e^{-\lambda x}\mathrm{d}x, \mathcal{P}^*_\mu)$, i.e. with $S = c_{\lambda,\mu}$ a constant and $\mathcal{P}^*_\mu$ a Poisson point process with intensity $e^{-\mu x}\mathrm{d}x$ on $(-\infty, 0]$ conditioned on having an atom at 0;

— an SDPPP$(Ze^{-\mu x}\mathrm{d}x, \delta_{\delta_0})$, i.e. a decoration consisting of a single atom at 0 and Z a positive stable variable with index λ/μ.

More generally, it can be described as an SDPPP$(Z^\varrho e^{-\varrho x}\mathrm{d}x, \mathcal{P}^*_\mu)$ for all $\varrho \in [\lambda, \mu)$, with Z^ϱ a positive stable variable with index λ/ϱ.

By Property 1.7, the log-Laplace transform of the law $\mathcal{E}$ of an SDPPP$(Se^{-\lambda x}\mathrm{d}x, \mathcal{D})$ can be straightforwardly computed for all non-negative function φ as

$$\Psi_{\mathcal{E}}[\varphi] : z \in \mathbb{R} \mapsto -\log \mathbb{E}\left(\exp\left(-Se^{\lambda z}\int_{\mathbb{R}}\left(1 - e^{-\Psi_{\mathcal{D}}[\varphi](y)}\right)e^{-\lambda y}\mathrm{d}y\right)\right).$$

Subag and Zeitouni [176, Corollary 3] showed that this property is in fact characteristic of SDPPPs.

Theorem 1.10 (Characterization of SDPPP)**.** *Let $\lambda > 0$, and $\mathcal{E}$ a point measure distribution. The two following statements are equivalent:*

1. *There exists a random variable Z such that for all continuous non-negative compactly supported function φ, there exists $c_\varphi \in \mathbb{R}$ such that*
$$\forall z \in \mathbb{R}, \mathbb{E}\left(\exp\left(-\langle \tau_z E, \varphi\rangle\right)\right) = \mathbb{E}\left(\exp\left(-Ze^{-\lambda(z - c_\varphi)}\right)\right);$$
2. *There exists a positive random variable S and a decoration measure $\mathcal{D}$ such that E is an SDPPP$(Se^{-\lambda x}\mathrm{d}x, \mathcal{D})$.*

Although the law of an SDPPP is not uniquely characterized by a triplet $(S, \lambda, \mathcal{D})$, it is worth mentioning that if the value of λ is fixed, the laws of S and $\mathcal{D}$ such that $\mathcal{E}$ is an SDPPP$(Se^{-\lambda x}\mathrm{d}x, \mathcal{D})$ can be described using the law of $\mathcal{E}$.

Proposition 1.11. *If E is an SDPPP$(Se^{-\lambda x}\mathrm{d}x, \mathcal{D})$ with $\mathcal{D}$ supported by $\mathcal{P}^*(\mathbb{R})$, then*

$$\forall z \in \mathbb{R}, \mathbb{P}(e_1 \leq z) = \mathbb{E}\left(e^{-Se^{-\lambda z}}\right),$$

and for all $\varphi \in \mathcal{T}^*$,

$$\mathbb{E}\left(e^{-\langle D, \varphi\rangle}\right) = \lim_{z \to \infty} \mathbb{E}\left(e^{-\langle \tau_{-e_1} E, \varphi\rangle} \middle| e_1 \geq z\right),$$

where we recall that e_1 is the position of the largest atom of E.

Shifted decorated Poisson point processes play an important role in the study of extremal values of Poisson point processes, they are in fact the limiting distribution of the extremal processes of a large class of spatial branching processes, see Chapter 4. Moreover, they can be shown to be the unique class of solutions to the so-called branching convolution equation (see the results of [M37] and [74], that we discuss in more details in Chapter 3). In short, for a large class of random point measures $\mathcal{E}$, all fixed points of the branching convolution equation, i.e. solutions of

$$\mathcal{Z} = \mathcal{Z} \circledast \mathcal{E} \quad \text{or} \quad \mathcal{Z} = \mathcal{E} \circledast \mathcal{Z}, \tag{1.4}$$

are SDPPP with either prescribed random shift of prescribed decoration.

1.2 Branching processes as point-measure valued Markov processes

In this section, we present some branching processes constructed as point-measure valued Markov processes. We introduce the branching random walks in Section 1.2.1 as discrete-time Markov process on $\mathcal{P}(\mathbb{R})$ that satisfy the branching property and have shift-invariant increments. Its continuous-time analogue, satisfying the branching property and having stationary and independent increments is naturally called the branching Lévy process. This class is introduced in Section 1.2.2, presenting in particular the results of [M16] that characterize this class and give a representation of its law in terms of a characteristic triplet analogue to the Lévy-Khintchine triplet of Lévy processes. The continous-time branching Lévy process with continuous trajectories is the branching Brownian motion.

1.2.1 The branching random walk

A branching random walk is a particle system on the real line in which at every discrete time step, every particle creates an independent copy of a point process of offspring, shifted around its position. It can be thought off as a population model in which in every generation, each individual creates independently of any other individuals a random number of children, who then migrate away from the position of their parent. Using a point measure to describe the number and position of children of an individual allow us to consider a process in which the number of children and their displacement are correlated.

Definition 1.12. Let $\mathcal{Z}$ be a probability distribution on $\mathcal{P}(\mathbb{R})$ such that there exists $\theta > 0$ verifying

$$\int_{\mathcal{P}(\mathbb{R})} \langle Z, \mathbf{e}_\theta \rangle \mathcal{Z}(\mathrm{d}Z) = \mathbb{E}\left(\sum_{j=1}^{\infty} e^{\theta z_j} \right) < \infty, \tag{1.5}$$

where (z_j) is the ranked sequence of atoms of a point measure Z of law $\mathcal{Z}$.

A branching random walk with reproduction law $\mathcal{Z}$ is the Markov process $(Z_n, n \geq 0)$ on $\mathcal{P}(\mathbb{R})$ constructed recursively as follows

$$Z_0 = \delta_0 \quad \text{and} \quad Z_{n+1} = \sum_{k=1}^{\infty} \tau_{z_{n,k}} Y_{n+1,k}, \tag{1.6}$$

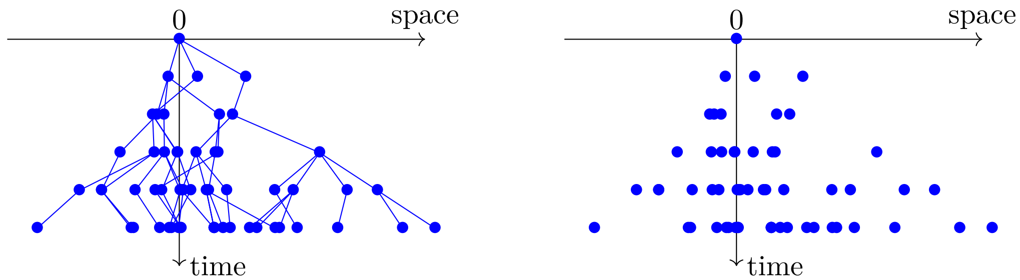
where $(z_{n,k}, k \geq 1)$ is the sequence of atoms of Z_n ranked in the decreasing order, and $(Y_{n,k}, (n,k) \in \mathbb{N}^2)$ are i.i.d. random point measures with law $\mathcal{Z}$, see Figure 1.3.

A condition for the well-definition of the branching random walk without assumption (1.5) was obtained by [15] for a branching random walk in which the number of children of an individual is independent of their displacement, which is a.s. non-positive. A general condition guaranteeing the well-definition of a branching random walk with reproduction law $\mathcal{Z}$ is however not known.

Equation (1.5) is reminiscent of the recursive equation defining Bienaymé–Galton–Watson processes. We recall that a BGW process with reproduction law ν is a Markov process $(Y_n, n \geq 0)$ on $\mathbb{Z}_+$ defined by

$$Y_0 = 1 \quad \text{and} \quad Y_{n+1} = \sum_{j=1}^{Y_n} \xi_{n+1,j}, \tag{1.7}$$

where $(\xi_{n,j}, n \in \mathbb{N}, j \in \mathbb{N})$ are i.i.d. random variables with law ν. If $(Z_n, n \geq 0)$ is a branching random walk such that $\langle Z_1, 1 \rangle \in \mathbb{Z}_+$ a.s, then writing ν the law of $\langle Z_1, 1 \rangle$, the



(a) Construction of the first five generations of a branching random walk.

(b) The same branching random walk, as encoded by random point measures.

Figure 1.3 – A branching random walk, with and without the genealogical informations between the generations of particles. The picture on the right side illustrate the information encoded in the branching random walk constructed as a Markov process in $\mathcal{P}(\mathbb{R})$.

process $(\langle Z_n, 1\rangle, n \geq 0)$ is a BGW process with reproduction law ν. In other words, the number of particles in a branching random walk is a discrete branching Markov process.

Using this observation, we can study the survival-extinction properties of branching random walks. A branching random walk is say to go extinct if at some finite time $n \in \mathbb{N}$ we have $Z_n = 0$, i.e. no particle remain on the real line. If the branching random walk never goes extinct, we say that it survives. We set

$$S := \{\forall n \in \mathbb{N}, \langle Z_n, 1\rangle > 1\}$$

the survival event of the branching random walk. By comparison with Galton-Watson processes (see [21] for a textbook introduction), the following result can be shown to hold.

Proposition 1.13. *Let $(Z_n, n \geq 0)$ be a branching random walk with reproduction law $\mathcal{Z}$. We define*

$$f : s \in [0,1] \mapsto \mathbb{E}\left(s^{\langle Z_n, 1\rangle}\right) \quad \text{with the convention } s^\infty = \mathbf{1}_{\{s=1\}}.$$

Then $1 - \mathbb{P}(S)$ is the smallest root on $[0,1]$ of the equation $f(s) = s$. In particular

— *if $\mathbb{E}(\langle Z_1, 1\rangle) \leq 1$, then $\mathbb{P}(S) = 0$;*

— *if $\mathbb{E}(\langle Z_1, 1\rangle) \geq 1$ then $\mathbb{P}(S) > 0$.*

In the rest of this manuscript, we will only consider branching random walks such that $\mathbb{E}\left(\langle Z_1, 1\rangle\right) > 1$, which are called supercritical branching random walks. We aim at describing the asymptotic properties of these processes on the survival event.

Let $(Z_n, n \geq 0)$ be a branching random walk with reproduction law $\mathcal{Z}$. With the notation of Section 1.1, writing $\mathcal{Z}_n$ the law of Z_n, the sequence $(\mathcal{Z}_n, n \geq 0)$ is defined by the following recursion equation:

$$\mathcal{Z}_0 = \delta_{\delta_0} \text{ and } \mathcal{Z}_{n+1} = \mathcal{Z} \circledast \mathcal{Z}_n = \mathcal{Z}_n \circledast \mathcal{Z}. \tag{1.8}$$

We denote by $\mathcal{Z}^{\circledast n}$ the law $\mathcal{Z}_n$ of Z_n, as it is the law of the nth iteration of the branching convolution operation. More generally, we observe that $\mathcal{Z}_{n+m} = \mathcal{Z}_n \circledast \mathcal{Z}_m$, i.e. in terms of the branching random walk

$$Z_{n+m} \overset{(d)}{=} \sum_{k=1}^{\infty} \tau_{z_{n,k}} Z_m^{(k)}, \tag{1.9}$$

with $(z_{n,k}, k \geq 1)$ the ranked sequence of atoms in Z_n, and $(Z_m^{(k)}, k \geq 1)$ i.i.d. copies of Z_m.

We refer to (1.9) as the branching property: the process at time $n+m$ is obtained by superposing the values at time m of independent copies of the same process starting from the position of each atom in the process at time n. We note that branching random walks are $\mathcal{P}(\mathbb{R})$-valued Markov processes satisfying the branching property with shift-invariant distribution, i.e. such that the law of the process started from z is the same as the law of the process stated from 0 shifted by z.

Remark 1.14. The integrability condition (1.5) in Definition 1.12 is not necessary in theory. Indeed, it is enough to assume that the reproduction law $\mathcal{Z}$ of the branching random walk is such that $\mathcal{Z}^{\circledast n} \in \mathfrak{P}(\mathbb{R})$ for all $n \in \mathbb{N}$. However, this assumptions is usually made, as in addition of guaranteeing the well-definition of the branching random walk, it allows for the use of the celebrated Many-to-one lemma (c.f. Lemma 1.15).

For all $\theta \geq 0$, we denote by

$$\mathfrak{P}_\theta(\mathbb{R}) = \left\{ \mathcal{D} \in \mathfrak{P}(\mathbb{R}) : \int \langle D, \mathbf{e}_\theta \rangle \mathcal{D}(\mathrm{d}D) < \infty \right\} \tag{1.10}$$

the set of point measure distributions such that $x \mapsto e^{\theta x}$ is integrable with respect to their intensity measure.

Given $(Z_n, n \geq 0)$ a branching random walk, we denote by $\kappa(\theta)$ the log-Laplace transform of (the intensity measure of) its reproduction law, defined as

$$\forall \theta \in \mathbb{R}, \kappa(\theta) := \log \mathbb{E}\left(\langle Z_1, \mathbf{e}_\theta \rangle\right) \in (-\infty, \infty].$$

The condition for the branching random walk to be supercritical can be expressed as $\kappa(0) > 0$ (without barring the possibility that $\kappa(0) = \infty$, or even that $\langle Z_1, 1 \rangle = \infty$ a.s.). With this definition and Property 1.5, it is a straightforward observation that for all $\theta \in \mathbb{R}$ such that $\kappa(\theta) < \infty$ and all $n \geq 0$, we have

$$\mathbb{E}\left(\langle Z_n, \mathbf{e}_\theta \rangle\right) = e^{n\kappa(\theta)}. \tag{1.11}$$

In particular, for all $n \in \mathbb{N}$, $\mathcal{Z}_n \in \mathfrak{P}_\theta$.

The Many-to-one lemma is an ubiquitous tool of the study of spatial branching processes, as it links the mean of $\langle Z_n, f \rangle$ to a random walk estimate. Its first appearance can be tracked back to the early work of Kahane and Peyrière [159, 116] on multiplicative cascades. A simple proof by recursion can be found in the lecture notes of Zhan Shi [172, Theorem 1.1].

Lemma 1.15. *Let $(Z_n, n \geq 0)$ be a branching random walk, and $\theta \in \mathbb{R}$ such that $\kappa(\theta) < \infty$. Let $(S_n, n \geq 0)$ be a random walk with step distribution satisfying*

$$\forall x \in \mathbb{R}, \quad \mathbb{P}(S_1 \leq x) = \mathbb{E}\left(\langle Z_1, \mathbf{e}_\theta \mathbf{1}_{(-\infty, x]} \rangle\right) e^{-\kappa(\theta)}.$$

For all measurable positive function f and $n \in \mathbb{N}$, we have

$$\mathbb{E}\left(\langle Z_n, f \rangle\right) = \mathbb{E}\left(e^{-\theta S_n + n\kappa(\theta)} f(S_n)\right).$$

Using this result, Biggins [42], building on earlier work from Hammersley [100] and Kingman [121] that considered branching random walks supported on $\mathbb{R}_+$, proved that the maximal position of a branching random walk has a well-defined speed as long as the intensity measure of the first generation decays exponentially at ∞.

Theorem 1.16 (Speed of the branching random walk, Biggins 1976)**.** *Let $\theta > 0$, we denote by $\mathcal{Z}$ a point measure distribution in $\mathfrak{P}_\theta(\mathbb{R})$ and by $(Z_n, n \geq 0)$ a branching random walk with reproduction law $\mathcal{Z}$. For all $n \in \mathbb{N}$, we set*

$$M_n := \max_{|u|=n} X(u) \quad \textit{and} \quad v := \inf_{\varphi>0} \frac{\kappa(\varphi)}{\varphi}.$$

We have $\lim_{n\to\infty} M_n/n = v$ a.s. on S.

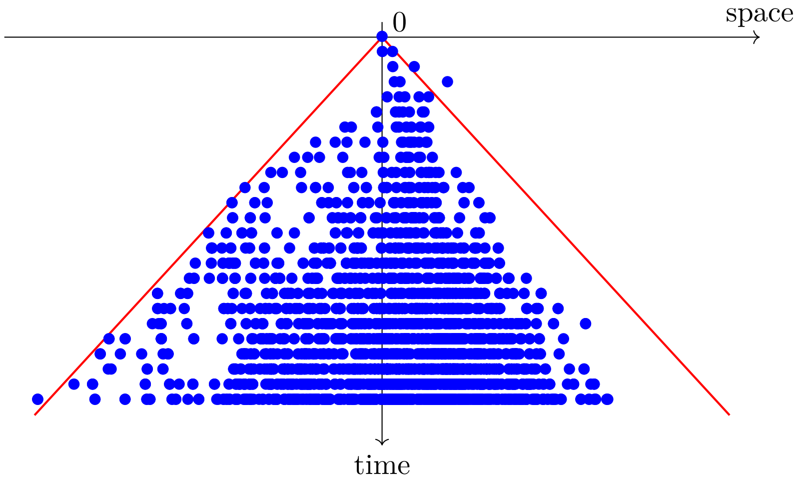


Figure 1.4 – First 30 generations of a branching random walk with symmetric exponential distribution. The linear speed of the particle system can be observed from its triangular shape.

In other words, the particle system represented by the branching random walk invades its environment at positive speed, as illustrated in Figure 1.4. However, martingales argument show that in typical settings, $M_n - nv$ converges to $-\infty$ almost surely. To describe more precisely the rate of growth of $nv - M_n$, it is crucial to consider the trajectory followed by particles, which will be made possible by the encoding introduced in Section 1.3.

1.2.2 The class of branching Lévy processes

Branching Lévy processes are the continuous-time counterparts of branching random walks, in the same way that Lévy processes are the continuous-time counterparts of random walks. Indeed, we recall that a Lévy process $(\xi_t, t \geq 0)$ can be described as a càdlàg (right continous with left limits at each point) Markov process such that for all $h > 0$, $(\xi_{nh}, n \geq 0)$ is a random walk. By analogy, we define a branching Lévy process $(Z_t, t \geq 0)$ as a càdlàg Markov process on $\mathcal{P}(\mathbb{R})$ such that for each $h > 0$, $(Z_{nh}, n \geq 0)$ is a branching random walk. We characterize in this section the branching Lévy processes satisfying an exponential integrability condition of the form (1.5), as well as their law at time 1 that we call *infinitely ramified point measures*, by analogy with infinitely divisible random variables.

The fine structure of Lévy processes and infinitely divisible distributions has been studied over the years, starting with the classical works of de Finetti, Itô, Khintchine, Kolmogorov and Lévy [81, 113, 119, 130, 131]. Over the years, their works uncovered a one-to-one correspondence between:

Infinitely divisible distributions: probability measures μ on $\mathbb{R}$ such that for all $n \in \mathbb{N}$, there exists a probability measure ν on $\mathbb{R}$ satisfying $\nu^{*n} = \mu$;

Lévy processes: càdlàg Markov processes with stationary and independent increments;

Lévy-Khintchine triplets: Triplets (σ^2, a, π) with $\sigma^2 \geq 0$, $a \in \mathbb{R}$ and π a measure on $\mathbb{R}\backslash\{0\}$ such that $\int (1 \wedge x^2)\pi(\mathrm{d}x) < \infty$.

More precisely, the following result holds.

Theorem 1.17 (Lévy, Khintchine)**.** *The three following statement hold.*

1. *For each infinitely divisible distribution μ, there exists a unique Lévy process X such that X_1 has law μ.*
2. *For any Lévy process X, there exists a unique Lévy-Khintchine triplet (σ^2, a, π) such that for all $t \geq 0$,*

$$\forall \xi \in \mathbb{R},\ \mathbb{E}\left(e^{i\xi X_t}\right) = e^{t\left(-\frac{\xi^2\sigma^2}{2} + ia + \int_{\mathbb{R}} (e^{i\xi z} - 1 + i\xi z \mathbf{1}_{\{|z|<1\}})\pi(\mathrm{d}z)\right)}.$$

3. *For any Lévy-Khintchine triplet (σ^2, a, π) there exists a unique infinitely divisible distribution μ such that*

$$\forall \xi \in \mathbb{R},\ \int_{\mathbb{R}} e^{i\xi x}\mu(\mathrm{d}x) = e^{-\frac{\xi^2\sigma^2}{2} + ia\xi + \int_{\mathbb{R}} (e^{i\xi z} - 1 + i\xi z \mathbf{1}_{\{|z|<1\}})\pi(\mathrm{d}z)}.$$

We refer to the Lévy-Khintchine triplet associated to a Lévy process or an infinitely divisible distribution as its characteristic triplet. In addition to this result, there exists an explicit construction of Lévy processes with a given Lévy-Khintchine exponent, called the Lévy-Itô formula.

Theorem 1.18 (Lévy, Itô)**.** *Let (σ^2, a, π) be a Lévy-Khintchine triplet. Let B be a standard Brownian motion and N an independent Poisson point process with intensity $\mathrm{d}t \otimes \pi(\mathrm{d}x)$, the process defined for $t \geq 0$ by*

$$X_t = \sigma B_t + at + \int_{[0,t]\times\mathbb{R}\backslash(-1,1)} xN(\mathrm{d}t\mathrm{d}x) + \int_{[0,t]\times(-1,1)} xN^c(\mathrm{d}t\mathrm{d}x)$$

is a Lévy process with characteristic triplet (σ^2, a, π). Here, the integral with respect to N^c represents the compensated Poisson integral.

An analogue of these characterization results of Lévy processes have been obtained by Bertoin and Mallein [M18] in the context of spatial branching processes. We can find correspondences between branching Lévy processes, infinitely ramified point measures and characteristic triplets, with a Lévy-Itô type construction of a branching Lévy process from its associated triplet. We first give a formal definition of those three classes of objects.

Definition 1.19 (Infinitely ramified point measure)**.** A law $\mathcal{Z} \in \mathfrak{P}(\mathbb{R})$ is called an infinitely ramified point measure if for all $n \in \mathbb{N}$, there exists $\mathcal{D} \in \mathfrak{P}(\mathbb{R})$ such that $\mathcal{Z} = \mathcal{D}^{\circledast n}$.

An infinitely ramified point measure can then be described as a probability distribution on $\mathcal{P}(\mathbb{R})$ that can be represented for all $n \in \mathbb{N}$ as the law of the value at time n of a branching random walk. They are the analogue of infinitely divisible distributions, that can be described for all $n \in \mathbb{N}$ as the law of the value at time n of a random walk.

Definition 1.20 (Branching Lévy process)**.** A branching Lévy process is a càdlàg Markov process $(Z_t, t \geq 0)$ on $\mathcal{P}(\mathbb{R})$ such that there exists $\theta \geq 0$ satisfying $\mathcal{Z}_t \in \mathfrak{P}_\theta(\mathbb{R})$ for all $t \geq 0$, and that for all $s < t$

$$\mathcal{Z}_{t+s} = \mathcal{Z}_t \circledast \mathcal{Z}_s, \quad \text{i.e. } Z_{t+s} \overset{(d)}{=} \sum_{k=1}^{\infty} \tau_{z_{t,k}} Z_s^{(k)}, \tag{1.12}$$

where $(z_{t,k}, k \in \mathbb{N})$ is the ranked sequence of atoms of Z_t, and $(Z_s^{(k)}, k \in \mathbb{N})$ are i.i.d. copies of Z_s.

We refer to (1.12) as the branching property for a spatial branching process. It expresses that a process at time $t+s$ can be described as the superposition, for each particle alive at time t at position z of a copy of the process at time s shifted by z. In other words, a branching Lévy processes is a spatial branching process, satisfying the branching property and such that the law of the process is invariant by translation. In other words, the process started from $z \in \mathbb{R}$ has same law as the shift by z of the law of a process started from 0.

We denote $\kappa : \theta \in \mathbb{R} \mapsto \log \mathbb{E}\left(\langle Z_1, \mathbf{e}_\theta \rangle\right) \in (-\infty, \infty]$, by analogy with the branching random walk. It is a straightforward consequence of the branching property that for all $t \geq 0$, $\mathbb{E}\left(\langle Z_t, \mathbf{e}_\theta \rangle\right) = e^{t\kappa(\theta)}$. Similarly to the Fourier transform of an infinitely divisible distribution, the log-Laplace transform of a branching Lévy process can be described by a characteristic triplet encoding the law of the process.

Definition 1.21 (Characteristic triplet)**.** A characteristic triplet of a branching Lévy process is a triplet (σ^2, a, Λ) with $\sigma^2 \geq 0$, $a \in \mathbb{R}$ and Λ a σ-finite measure on $\mathcal{P}(\mathbb{R}) \backslash \{\delta_0\}$ such that there exists $\theta \geq 0$ verifying

$$\int_{\mathcal{P}(\mathbb{R})} (1 \wedge x_1^2) \Lambda(\mathrm{d}\mathbf{x}) + \int_{\mathcal{P}(\mathbb{R})} \left(e^{\theta x_1} - 1 + \theta x_1 \mathbf{1}_{\{|x_1|<1\}} \right) \Lambda(\mathrm{d}\mathbf{x}) \\ + \int_{\mathcal{P}(\mathbb{R})} \left(\sum_{j=2}^{\infty} \mathbf{1}_{\{x_j > -\infty\}} e^{\theta x_j} \right) \Lambda(\mathrm{d}\mathbf{x}) < \infty. \tag{1.13}$$

The condition $\Lambda(\delta_0) = 0$ imposed on the branching Lévy measure Λ allows it to be uniquely defined, similar to the constraint $\pi(\{0\}) = 0$ imposed on the jump measure of a Lévy process. We underline that the condition (1.13) entails that $\Lambda(\{0\}) < \infty$.

The following result, obtained in [M16], gives a correspondence between branching Lévy processes, infinitely ramified point measures and characteristic triplets.

Theorem 1.A (Bertoin and Mallein (2019))**.** *Let $\theta \geq 0$, the three following statements hold.*

1. *For each infinitely ramified point measure $\mathcal{Z} \in \mathfrak{P}_\theta(\mathbb{R})$, there exists a branching Lévy process Z such that Z_1 has law $\mathcal{Z}$.*
2. *There exists a unique characteristic triplet (σ^2, a, Λ) associated to each branching Lévy process Z such that for all $t \geq 0$ and $z \in \mathbb{C}$ with $\kappa(\mathrm{Re}(z)) < \infty$, we have*

$$\mathbb{E}\left(\langle Z_t, \mathbf{e}_z \rangle\right) = e^{t\left(\frac{z\sigma^2}{2} + az + \int_{\mathcal{P}(\mathbb{R})} \left(\sum_{k=1}^{\infty} e^{zx_k} - 1 - zx_1 \mathbf{1}_{\{|x_1|<1\}} \Lambda(\mathrm{d}\mathbf{x})\right)\right)}. \tag{1.14}$$

3. *For any characteristic triplet (σ^2, a, Λ) there exists a unique branching Lévy process Z such that (1.14) holds.*

In other words, there exists a one-to-one correspondence between branching Lévy processes and characteristic triplets. Additionally, each infinitely ramified point measure can be constructed as the law at time 1 of a branching Lévy process, and has therefore at least one associated characteristic triplets. However, we were not able to prove the following conjecture.

Conjecture 1.22. *For each infinitely ramified point measure $\mathcal{Z} \in \mathfrak{P}_\theta(\mathbb{R})$, there exists a* unique *branching Lévy process Z such that Z_1 has law $\mathcal{Z}$.*

A simple way to prove the above conjecture would be to show that $(\sigma^2, a, \Lambda) \mapsto \mathcal{Z}_1$ is an injective map. In [M16], Bertoin and Mallein additionally proved a Lévy-Itô type construction of the branching Lévy process with a prescribed characteristic triplet. Informally, a branching Lévy process with triplet (σ^2, a, Λ) is a particle system in which each particle moves according to an independent Lévy process with parameters (σ^2, a, π) with π the image measure of Λ by the map $D \mapsto d_1$. With intensity $\Lambda(\mathrm{d}x)$ a particles makes a jump of size x_1 while giving birth to newborn offspring at distance $x_2, x_3, \cdots$ of its pre-jump position. The trajectory of a branching Lévy process such that $\Lambda(x_2 > -\infty) < \infty$ (i.e. such that only finitely many births occur per unit of time) is given in Figure 1.5, but in general the set of branching times of the process forms an a.s. everywhere dense set.

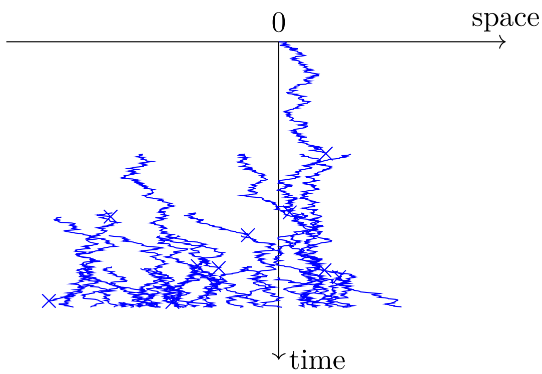


Figure 1.5 – Sample path of a branching Lévy process with finite birth intensity.

A branching Lévy process is a spatial branching process in which the times at which a birth event occurs can be an almost surely everywhere dense set. However, due to the integrability condition (1.13), most births occur at low enough level that the number of particles above a given position does not explode in finite time. As this construction is made easier with a genealogical structure for the branching Lévy process, we introduce it in details in Section 1.3.

Branching-stable point measures

Real-valued random variables are said to have a stable distribution if and only if for each $n \geq 2$, there exists $a(n)$ such that

$$X \stackrel{(d)}{=} \frac{X_1 + \cdots + X_n}{a(n)}, \tag{1.15}$$

where $(X_1, \ldots, X_n)$ are i.i.d. copies of X. Note that (barred the degenerate case $X = 0$ a.s.) we have for all n and m, $a(nm) = a(n)a(m)$, hence there exists an index α, called the index of stability of the distribution such that $a(n) = n^{1/\alpha}$. Observe that any stable distribution is infinitely-divisible, hence can be represented as the value at time 1 of a Lévy process $(X_t, t \geq 0)$. The stability property implies that for the Lévy process

$$(X_t, t \geq 0) \overset{(d)}{=} (\lambda^{-1/\alpha} X_{\lambda t}, t \geq 0)$$

for all $\lambda > 0$. This process is called a stable Lévy process.

Stable variables and stable Lévy processes often appear as scaling limit of a sequence of processes. In [M22], Bertoin, Cortines and Mallein characterized stable branching Lévy processes, satisfying a similar property to (1.15). A point measure distribution $\mathcal{Z}$ is said to be branching-stable if, writing $(Z_n, n \geq 0)$ a branching random walk with reproduction law $\mathcal{Z}$, for any $n \geq 2$ there exists $a(n)$ such that

$$Z \overset{(d)}{=} h_{a(n)}(Z_n), \tag{1.16}$$

where h_a is the dilation operator on $\mathcal{P}(\mathbb{R})$ that transforms the sequence of atoms $(z_k, k \geq 1)$ into $(az_k, k \geq 1)$. In other words, $h_a(Z)$ is the image measure of Z by the dilation $x \mapsto ax$.

Remark 1.23. The branching-stable point measure we defined here is a different notion from the one defined by Zanella and Zuyev in [181]. In that article, the branching operation is replaced by the superposition of point measures, and the dilation by the splitting of atoms (without motion) as Galton-Watson processes. This definition yields a different class of stable point measures.

The main result of [M22] is a characterization of branching-stable point measures.

Theorem 1.B (Bertoin, Cortines and Mallein (2019)). *The law of the random point measure Z is branching-stable with index α if and only if*

- *$\alpha > 2$ and either $Z = \delta_0$ a.s. or $Z = 0$ a.s.;*
- *$\alpha \in (0, 2]$ and $Z = \delta_X$, with X a stable random variable of index α;*
- *$\alpha < 0$ and $Z = Z_1$, with $(Z_t, t \geq 0)$ a branching Lévy process with parameters $(0, 0, \Lambda)$, where Λ is defined by*

$$\int_{\mathcal{P}(\mathbb{R})} F(Z)\Lambda(\mathrm{d}Z) = \int_0^\infty y^{\alpha-1} \int_{\mathcal{P}(\mathbb{R})} F(h_y(Z))\lambda(\mathrm{d}Z)\mathrm{d}y, \tag{1.17}$$

  with λ a finite point measure on $\mathcal{P}(\mathbb{R})$ satisfying

$$\lambda(x_1 \neq 0) = \lambda(x_2 \neq 1) = 0 \quad \textit{and} \quad \int_{\mathcal{P}(\mathbb{R})} \sum_{k=2}^\infty x_k^{-\alpha} \lambda(\mathrm{d}\mathbf{x}).$$

This results allows us similarly to characterize stable branching Lévy processes. Barring the case of the degenerate Lévy process consisting of a single particle staying at 0 forever, α-stable branching Lévy processes can be split into two categories:

- if $\alpha \in (0, 2]$, then Z consists of a single particle moving as an α-stable Lévy process, without giving birth to offspring,
- if $\alpha < 0$, then Z is a particle system in which particles do not move, but a particle at position z creates at rate $y^{\alpha-1}\lambda(\mathbb{R})\mathrm{d}y$ newborn particles at position $z + yx_2, z + yx_3, \cdots$, with $(x_n, n \geq 1)$ the ranked sequence of atoms sampled according to the law $\lambda/\lambda(\mathbb{R})$.

In other words, in stable branching processes, either particles move without branching if $\alpha > 0$, or they branch without moving if $\alpha < 0$. In the later case, the branching rate and position of offspring are chosen in a consistent fashion with the scaling property. In [39], stable branching Lévy processes are shown to be the scaling limits of branching random walks, with some well-chosen space-time scaling.

One particular stable branching Lévy process of interest is the Cauchy-Yule, process, which has index -1. In this process, a particle at position x creates at rate $\mathrm{d}t\mathrm{d}y$ a particle at position $x+y$ at time t. It can be thought of as a branching random walk on $[0,\infty)^2$ in which a particle at position (t,x) creates a Poisson point process of offspring with unit intensity in the quadrant $(t,\infty)\times(x,\infty)$. The first coordinate of the process gives it the time at which the particle is born in the branching Lévy process, and the second coordinate its position.

Branching Brownian motion

A branching Brownian motion is a branching Lévy process with a.s. continuous trajectories. As a result, its associated characteristic triplet is (σ^2, a, Λ) with Λ having support in $\{n\delta_0, n \in \mathbb{Z}_+\backslash\{1\}\}$. In other words, in this process each particle move according to an independent Brownian motion, at each particle splits at rate $\Lambda(\{k\delta_0\})$ into k daughter particles, that start from the position of their mother independent copies of the branching Brownian motion. We refer in more details to this process in Section 1.3.4.

1.3 Genealogical structure for branching processes

To give a more precise description of the properties of particles in spatial branching processes, such as the trajectory followed by particles, or the genealogical relationship between two particles, we have to encode not only the positions of particles at all times, but also the genealogical structure of the process. This encoding will allow us to give an alternative description of the branching random walk and the branching Lévy process, and to describe the Lévy-Itô construction for a branching Lévy process with prescribed characteristic triplet. We will also introduce in more details the branching Brownian motions as well as their connections with reaction-diffusion equations.

1.3.1 The Ulam–Harris–Neveu notation for trees

The Ulam–Harris–Neveu notation for trees is a structure used to encode plane rooted trees, such as genealogical structures. Each node of the plane tree is associated with a unique label in the set of finite sequences of integers

$$\mathbb{U} := \bigcup_{n\geq 0} \mathbb{N}^n, \tag{1.18}$$

with the usual convention $\mathbb{N}^0 = \{\emptyset\}$, i.e. $\emptyset$ is the sequence of length 0. The labelling is made as follows: the root vertex has label $\emptyset$, and its k neighbours are given labels $1, 2, \cdots, k$ according to their ordering from left to right in the place. Recursively, the jth leftmost child of a vertex labelled $u = (u(1), \ldots, u(n))$ is labelled $(u(1), \ldots, u(n), j)$. In other words, the label $u = (u(1), \ldots, u(n))$ is associated to the $u(n)$th child of the $u(n-1)$th child of the ... of the $u(1)$th child of the root.

We introduce some notation for the manipulation of these labels. For all $u \in \mathbb{U}$, we denote by

— $|u|$ the length of u, i.e. the integer n such that $u \in \mathbb{N}^n$;

— for $k \leq u$, $u(k)$ is the kth element of the sequence u, i.e. $u = (u(1), \dots u(|u|))$ as long as $u \neq \emptyset$;

— for $k \leq u$, $u_k = (u(1), \dots, u(k))$ is the sequence consisting of the first k values of u;

— for $v \in \mathbb{U}$, $u.v = (u(1), \dots, u(|u|), v(1), \dots, v(|v|))$ is the concatenation of u and v.

Let $\mathbb{U}^* = \mathbb{U} \backslash \{\emptyset\}$ be the set of non-empty finite sequences. For all $u = (u(1), \dots, u(n)) \in \mathbb{U}^*$, we denote by $\pi u = (u(1), \dots, u(n-1)) = u_{|u|-1}$ the sequence obtained by erasing its last value.

In terms of a genealogical tree, $|u|$ corresponds to the generation to which u belongs (in other words its distance from the root), πu is the parent of u, u_k is the ancestor of u that was alive at generation k, and for $j \in \mathbb{N}$ $u.j$ is the jth child of u. The set $\mathbb{U}^*$ can be seen as the set of possibles labels for non-root vertices of a plane tree, or sometimes be used to represent the genealogy of a population starting from more than one ancestor, i.e. a plane rooted forest.

With this notation, a plane rooted tree can be defined as a subset of $\mathbb{U}$ satisfying some properties.

Definition 1.24 (Plane rooted tree)**.** A plane rooted tree is a subset T of $\mathbb{U}$ satisfying the three following properties

Root: $\emptyset \in T$;

Parent: for all $u \in T$ with $u \neq \emptyset$, we have $\pi u \in T$;

Children enumeration: for all $u \in T$ and $j \geq 2$, if $u.j \in T$ then $u.(j-1) \in T$.

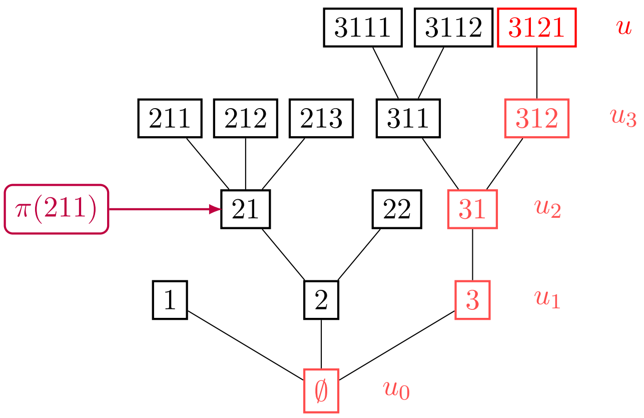


Figure 1.6 – Illustration of the Ulam-Harris notation for a plane rooted tree of height 4.

A Bienaymé-Galton-Watson tree is the genalogical tree of a population in which individuals create offspring independently of one another, according to a common reproduction law ν on $\mathbb{Z}_+$. It can be explicitely constructed from the Ulam–Harris–Neveu notation in the following fashion.

Definition 1.25 (Bienaymé-Galton-Watson tree)**.** Let ν be a probability distribution on $\mathbb{Z}_+$, a Bienaymé-Galton-Watson tree with reproduction law ν is a random plane rooted tree defined as

$$\mathcal{T} := \{u \in \mathbb{U} : \forall 0 \leq k < |u|,\ u(k+1) \leq \xi_{u_k}\},$$

where $(\xi_u, u \in \mathbb{U})$ is a family of i.i.d. random variables of law ν.

In the above definition, it can be observed that the random variable ξ_u represents the number of children of the particle labelled u, if this particle belongs to the tree. In particular, the process $(\#\{u \in \mathcal{T} : |u| = n\}, n \geq 0)$ is a BGW process. In fact, the tree $\mathcal{T}$ gives the detailed genealogical structure of this branching process as for any individual u, the label of its parent, its number of siblings or its offspring are measurable functions of the tree $\mathcal{T}$.

Using Proposition 1.13, we observe that a BGW process may survive with positive probability, provided that $\mathbb{E}(\xi_\emptyset) > 1$. More precisely, we denote by $S := \{\#\mathcal{T} = \infty\}$ the survival event for the process.

1.3.2 Genealogical structure of the branching random walk

We introduce here an alternative definition of branching random walks as random maps $\mathbb{U} \to \mathbb{R} \cup \{-\infty\}$. Similarly to BGW trees versus BGW associated processes, this definition allows a more precise description of the genealogy of particles in the branching random walk.

Definition 1.26 (Branching random walk)**.** Let $\mathcal{Z}$ be a probability distribution on $\mathfrak{P}(\mathbb{R})$. A branching random walk with reproduction law $\mathcal{Z}$ is a random map $X : \mathbb{U} \to \mathbb{R} \cup \{-\infty\}$ defined for all $u \in \mathbb{U}$ by

$$X(\emptyset) := 0 \quad \text{and} \quad X(u) := X(\pi u) + z^{\pi u}_{u(|u|)} = \sum_{j=0}^{|u|-1} z^{(u_j)}_{u(j+1)},$$

where $(Z^{(u)}, u \in \mathbb{U})$ are i.i.d. random point measures with law $\mathcal{Z}$, and $(z^{(u)}_k, k \geq 1)$ is the ranked sequence of atoms of $Z^{(u)}$.

With this construction, the quantity $X(u)$ represents the position of the particle labelled u in the system. Observe that the point process of the positions at time n can be obtained as $Z_n = \sum_{|u|=n} \delta_{X(u)}$, when this quantity is well-defined. Here and in the rest of this manuscript, a sum over the set $\{|u| = n\}$ represents a sum over $\{u \in \mathbb{N}^n : X(u) \neq -\infty\}$.

Remark 1.27. This definition of the branching random walk removes the requirement for $\mathcal{Z}$ to belong in $\cup_{\theta>0}\mathfrak{P}_\theta(\mathbb{R})$, i.e. to satisfy an exponential integrability condition. Indeed, this definition does not require that at all times, the set of positions of particles is locally finite or admits a largest element.

Similarly to the definition of the BGW tree above, the point measure $Z^{(u)}$ give the number and the relative position of the children of the particle u with respect to the position of their parent $X(u)$, provided that $X(u) \neq -\infty$. The description of a branching random walk as a random map $\mathbb{U} \to \mathbb{R} \cup \{-\infty\}$ gives additional information of the genealogical relationship, and the trajectories followed by a particle and the positions of its descendants in the branching random walk. More precisely, for all $u \in \mathcal{U}$, we call the trajectory followed by the particle u the function $k \in [\![0, |u|]\!] \mapsto X(u_k)$, i.e. the sequence of positions of the ancestors of particle u.

Given a branching random walk X, we may define $\mathcal{T} := \{u \in \mathbb{U} : X(u) > -\infty\}$ the set of particles in the branching random walk. We note that $\mathcal{T}$ is a BGW tree, and we define S the survival set of the branching random walk as

$$S := \{\#\mathcal{T} = \infty\} = \{\forall n \in \mathbb{N}, \exists |u| = n : X(u) > -\infty\} \quad \text{a.s.}$$

The many-to-one lemma can be extended to consider functions of the trajectory of particles in a branching random walk, provided that it verifies an integrability condition. We refer again to [172, Theorem 1.1] for a proof of this result.

Lemma 1.28 (Trajectorial many-to-one lemma)**.** *Let X be a branching random walk and $\theta \in \mathbb{R}$ such that $\kappa(\theta) = \log \mathbb{E}\left(\sum_{|u|=1} e^{\theta X(u)}\right) < \infty$. We denote by $(S_n, n \geq 0)$ a random walk with step distribution given by*

$$\mathbb{P}(S_1 \leq x) = \mathbb{E}\left(\sum_{|u|=1} e^{\theta X(u) - \kappa(\theta)} \mathbf{1}_{\{X(u) \leq x\}}\right).$$

For all $n \geq 1$ and measurable non-negative function f, we have

$$\mathbb{E}\left(\sum_{|u|=n} f(X(u_j), 0 \leq j \leq n)\right) = \mathbb{E}\left(e^{-\theta S_n + n\kappa(\theta)} f(S_j, 0 \leq j \leq n)\right).$$

This version of the many-to-one lemma an be used to control the trajectory of extremal particles in the branching random walk. This control can be used to obtain the second order in the asymptotic behaviour of the maximal displacement $M_n = \max_{|u|=n} X(u)$. More precisely, we recall that $\kappa : \theta \mapsto \log \mathbb{E}\left(\sum_{|u|=1} e^{\theta X(u)}\right)$ is a $\mathcal{C}^\infty$ and convex function over the interior of its domain of definition. Assuming that $\varphi \mapsto \frac{\kappa(\varphi)}{\varphi}$ attains its minimum at point θ^*, and that κ is well-defined in a neighbourhood of θ^*, we will have

$$\theta^* \kappa'(\theta^*) = \kappa(\theta^*), \qquad \text{hence } v = \kappa'(\theta^*). \tag{1.19}$$

Under these assumptions, Addario-Berry and Reed [3] proved that $M_n - nv + \frac{3}{2\theta^*} \log n$ is tight. Hu and Shi [105] proved that while $\frac{M_n - nv}{\log n}$ converges to $\frac{-3}{2\theta^*}$ in probability, this quantity exhibits almost sure fluctuations with $\limsup_{n\to\infty} \frac{M_n - nv}{\log n} = \frac{-1}{2\theta^*}$ a.s.

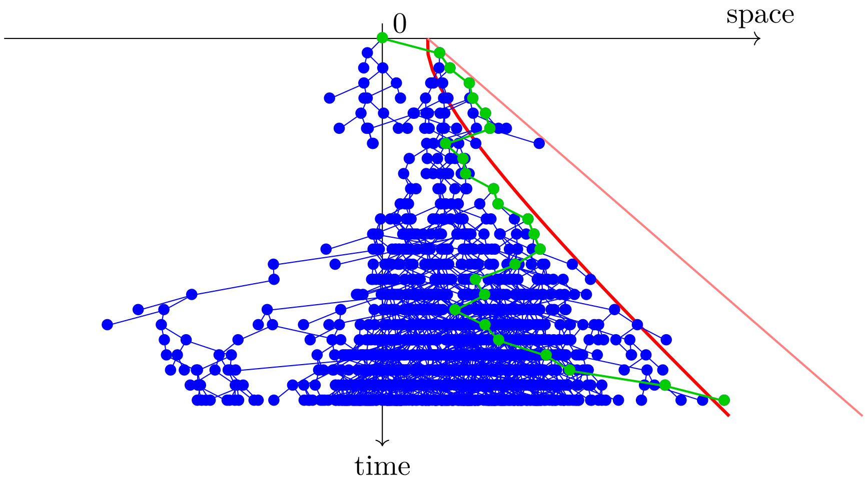


Figure 1.7 – Logarithmic correction for the speed of the branching random walk. Observe that the trajectory yielding to the rightmost occupied position (in green) looks like an excursion below the curve $k \mapsto kv + c$ for some $c > 0$. This observation yields the simple proof for the convergence of the logarithmic correction in [8], and is formalized in [73].

As a first step towards this result, using the many-to-one lemma we remark that for all $y \geq 0$,

$$\mathbb{P}\left(\exists u \in \mathbb{U} : X(u) \geq nv + y\right) \leq \mathbb{E}\left(\sum_{u \in \mathbb{U}} \mathbf{1}_{\{X(u) \geq nv+y, X(u_j) < jv+y, j<n\}}\right)$$
$$\leq \sum_{n=1}^{\infty} \mathbb{E}\left(e^{-\theta^* S_n + n\kappa(\theta^*)} \mathbf{1}_{\{S_n \geq nv+y, S_j \leq jv+y, j<n\}}\right).$$

Therefore, as $\sum_{n=1}^{\infty} \mathbb{P}\left(S_n \geq nv + y, S_j \leq jv + y, j < n\right) = 1$, we have

$$\mathbb{P}\left(\exists u \in \mathbb{U} : X(u) \geq nv + y\right) \leq e^{-\theta^* y}, \tag{1.20}$$

or in other words, $\sup_{u \in \mathbb{U}} X(u) - |u|v < \infty$ a.s. This observation has been used by Aïdékon and Shi in [8] to obtain a simple proof for the computation of the logarithmic second order in the asymptotic behaviour of the maximal displacement of the branching random walk.

Theorem 1.29 (Addario–Berry and Reed, Hu and Shi, 2009)**.** *Let X be a branching random walk such that (1.19) hold and that*

$$\mathbb{E}\left(\sum_{|u|=1} X(u)^2 e^{\theta^* X(u)}\right) + \mathbb{E}\left(\left(\sum_{|u|=1} e^{\theta^* X(u)}\right)^2\right) < \infty.$$

We have $\lim_{n\to\infty} \frac{M_n - nv}{\log n} = -\frac{3}{2}$ in probability.

1.3.3 Trajectorial construction of branching Lévy processes

Similarly to the branching random walk, the branching Lévy process can be constructed as a process together with its genealogy. However, in this process, the set of branching times typically is an everywhere dense set, so there is no natural way to define the first, second... child of an individual, thus to associate an Ulam-Harris index to each individual. Therefore, to describe the genealogy of branching Lévy processes, we have to change our point of view from the one we had for branching random walks. Instead of seeing particles as dying at each generation being replaced by its offspring, we identify the parent particle with its rightmost child, seeing the process as a system of persisting particles, that move while giving birth to offspring at integer times.

For branching Lévy processes, we can describe the particle system in the following fashion: an initial particle moves according to a certain Lévy process until an independent exponential time at which it dies. During its lifetime, this particle creates offspring at a Poisson rate, which can correlate with the jump measure of its displacement. Each newborn particle immediately starts an independent copy of the branching Lévy process with its birth time and position. We will now turn to a more precise description of this process in terms of the characteristic triplet (σ^2, a, Λ) of the branching Lévy process satisfying (1.13).

Let $(B_t, t \geq 0)$ be a Brownian motion and N an independent Poisson point process on $\mathbb{R}_+ \times \mathcal{P}(\mathbb{R})$ with intensity $\mathrm{d}t\Lambda(\mathrm{d}X)$. We define the trajectory followed by the root particle $\emptyset$ for all $t \geq 0$ by

$$X_t(\emptyset) := \sigma B_t + at + \int_{[0,t]\times\mathcal{P}(\mathbb{R})} x_1 \mathbf{1}_{\{|x_1|\geq 1\}} N(\mathrm{d}s\mathrm{d}\mathbf{x}) + \int_{[0,t]\times\mathcal{P}(\mathbb{R})} x_1 \mathbf{1}_{\{|x_1|<1\}} N^c(\mathrm{d}s\mathrm{d}\mathbf{x}),$$

where N^c represents the compensated Poisson integral. The trajectory followed by $\emptyset$ is a Lévy process with characteristic triplet (σ^2, a, π) with π the image measure of Λ by

$\mathbf{x} \mapsto x_1$. The trajectory of this process will jump $-\infty$ at rate $\pi(\{-\infty\}) = \Lambda(\{0\})$, which is interpreted as the death of that root particle.

Additionally, the particle $\emptyset$ give birth to offspring in the following fashion. For each atom $(s, \mathbf{x})$ of N, it creates newborn children in positions $X_{t-}(\emptyset) + x_2, X_{t-}(\emptyset) + x_3, \cdots$ at time s. Hence the point measure N encodes both the jumps of the measure $\emptyset$ and the offspring production. The set of children of the root can be labelled by $\mathbb{N}$, by enumeration. Several enumerations have been proposed in the literature, although an adaptation of the scheme proposed by Shi and Watson in [171, Section 4] is particularly efficient as it associates to each newborn particle a label in an adapted fashion to the natural time filtration.

In this scheme, for all $k \in \mathbb{N}$, the atoms $(s, \mathbf{x})$ of N such that $|x_2| \in [k, k+1)$ are ranked in an increasing order of their first coordinate. This ranking is possible as

$$\Lambda(\{\mathbf{x} \in \mathcal{P}(\mathbb{R}) : |x_2| \leq z\}) < \infty \text{ for all } z \geq 0,$$

hence the number of atoms of N with $|x_2| \in [k, k+1)$ and $s \leq t$ is a.s. finite. Then, an integer triplet (k, p, j) is associated to each child particle of the original ancestor $\emptyset$ as follows. For the pth smallest atom $(s, \mathbf{x})$ of N with $|x_2| \in [k, k+1)$, the particle born at time s and position $X_{s-}(\emptyset) + x_j$ is associated to (k, p, j). Then, fixing a bijection $\alpha : \mathbb{N}^3 \to \mathbb{N}$, the label of that particle is $\alpha((k, p, j))$.

More generally, we denote by $(B^{(u)}, N^{(u)}, u \in \mathbb{U})$ i.i.d. copies of the pair (B, N). For each $u \in \mathbb{U}$, we assume that the trajectory $(X_t(u), t \geq b_u)$ followed by particle u is defined from its birth time b_u to its death time, using the Brownian motion $B^{(u)}$ and the point process $N^{(u)}$ as

$$X_t(u) = X_{b_u}(0) + \sigma B^{(u)}_{t-b_u} + a(t - b_u) + \int_{[0,t-b_u] \times \mathcal{P}(\mathbb{R})} x_1 \mathbf{1}_{\{|x_1| \geq 1\}} N^{(u)}(\mathrm{d}s\mathrm{d}\mathbf{x}) \\ + \int_{[0,t-b_u] \times \mathcal{P}(\mathbb{R})} x_1 \mathbf{1}_{\{|x_1| < 1\}} N^{(u),c}(\mathrm{d}s\mathrm{d}\mathbf{x}).$$

The offspring of the particle u is encoded in the point process $N^{(u)}$, and labelled by $\mathbb{N}$ using the above scheme. Assuming that the jth children of particle u is associated to the kth element of the atom $(s, \mathbf{x})$ of $N^{(u)}$, then particle uj is born at time $b_u + s$ in position $X_{b_u+s} = X_{b_u+s-}(u) + x_k$.

We extend definition of the trajectory by setting $X_s(u) = X_s(\pi u)$ if $s < b_u$. In this way, the càdlàg process $(X_t(u), t \geq 0)$ represents the trajectory followed by the ancestors of particle u up to time b_u, then its own displacement. We denote by $\mathcal{D}([0, \infty))$ the set of càdlàg functions $[0, \infty) \to \mathbb{R} \cup \{-\infty\}$, corresponding to the trajectories of particles.

Definition 1.30 (Branching Lévy process)**.** A branching Lévy process with characteristic triplet (σ^2, a, Λ) is constructed as the random map $X : \mathbb{U} \to \mathcal{D}([0, \infty))$ defined above. For all $t \geq 0$, we denote by

$$\mathcal{N}_t = \{u \in \mathbb{U} : b_u \leq t, X_t(u) > -\infty\}$$

the set of particles alive at time t in the process, and by

$$Z_t = \sum_{u \in \mathcal{N}_t} \delta_{X_t(u)}$$

the point measure of the position of particles in this process.

This trajectorial construction of branching Lévy processes is introduced in [M16] to provide an explicit construction of the branching Lévy process with characteristic triplet (σ^2, a, Λ). The following result allows the identification of the two definitions of branching Lévy processes defined above.

Theorem 1.C (Bertoin and Mallein (2019))**.** *Let (σ^2, a, Λ) be a characteristic triplet satisfying (1.13) and $X : \mathbb{U} \to \mathcal{D}([0,\infty))$ a branching Lévy process as defined in Definition 1.30. The process Z is a branching Lévy process with characteristic (σ^2, a, Λ).*

Remark that conversely, by Theorem 1.A, the law of a branching Lévy process Z as defined in Definition 1.20 is unique associated to a characteristic triplet (σ^2, a, Λ). Therefore, given a branching Lévy process Z, there exists a unique law for the genealogical structure of this process.

1.3.4 Branching Brownian motion and the F-KPP reaction-diffusion equation

The branching Brownian motion is a branching Lévy process in which particles have continuous trajectories almost surely. It can therefore be described informally as the following particle system. It starts from a single particle at position 0 at time 0. Particles move in this system as independent Brownian motions with diffusion σ and drift a. Additionally, after an independent exponential time of parameter β, the particles split into a random number of children of law ν. Children then start from the position of their parent independent copies of the branching Brownian motion.

Definition 1.31 (Branching Brownian motion)**.** Let ν be a probability measure on $\mathbb{N}$ with $m = \sum_{k\in\mathbb{N}} k\nu(k) < \infty$, $\beta > 0$, $\sigma^2 \geq 0$ and $a \in \mathbb{R}$. A branching Brownian motion with diffusion coefficient σ^2, drift a, branching rate β and reproduction law ν is a branching Lévy process with characteristic triplet (σ^2, a, Λ), setting $\Lambda = \sum_{k=0}^{\infty} \beta\nu(k)\delta_{k\delta_0}$.

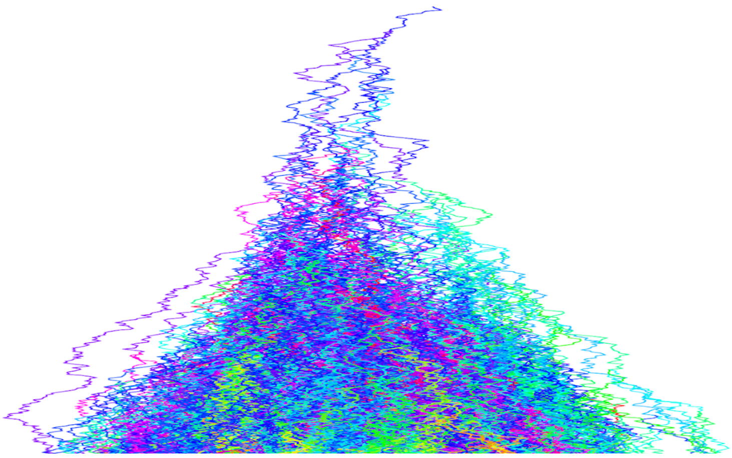


Figure 1.8 – Sample path of a binary branching Brownian motion. The colour of each particle is inherited from parent to child, with a random mutation. We remark a triangular shape similar to the one observed for branching random walks.

Up to choosing a correct time and space units, as well as a reference frame, one can assume, without loss of generality that particles move as standard Brownian motions and branch into children at unit rate, i.e. $\sigma^2 = 1$, $a = 0$ and $\beta = 1$. Unless otherwise explicitly stated, the branching Brownian motions we consider will have this choice of parameters. A particular branching Brownian motion of interest is the binary branching Brownian motion, with reproduction law $\nu = \delta_2$. In this process, every particle move as an independent standard Brownian motion, and split at rate 1 into two children, that start independent copies of the process. A realization of that process is drawn in Figure 1.8.

Similarly to branching Lévy processes, we denote by $\mathcal{N}_t$ the subset of $\mathbb{U}$ consisting of the label of particles alive at time t, and for $u \in \mathcal{N}_t$ and $s \leq t$, by $X_s(u)$ the position at time s of either particle u or its ancestor that was alive at time s.

The branching Brownian motion has been very well-studied over the years, in particular due to its connection with the Fisher–Kolmogorov-Petrovskii-Piskunov reaction-diffusion equation

$$\partial_t u = \frac{1}{2}\Delta u + f(u) \tag{1.21}$$

with f a positive concave continuous function $[0,1] \to [0,1]$ with $f(0) = f(1) = 0$. This partial differential equation was independently introduced by Fisher [94] and Kolmogorov, Petrovskii and Piskunov [123] as a model for the propagation of advantageous gene in a spatial population.

The F-KPP equation is linked with the law of the branching Brownian motion by a duality relationship (see [173, 107, 145]) for early appearances of this relationship): the Laplace transform of the branching Brownian motion is the solution of the F-KPP equation with prescribed initial condition. More precisely, given φ a non-negative measurable function, we set

$$u_\varphi(t,x) = \mathbb{E}\left(1 - \exp\left(-\sum_{u \in \mathcal{N}_t} \varphi(X_t(u) + x)\right)\right) = \mathbb{E}\left(1 - e^{-\langle \tau_x Z_t, \varphi \rangle}\right),$$

with $Z_t = \sum_{u \in \mathcal{N}_t} \delta_{X_t(u)}$ the point process associated to the branching Brownian motion. Then, the function u_φ is the unique solution of the equation

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u + f(u) \\ u(0,x) = 1 - e^{-\varphi(x)}, \end{cases} \tag{1.22}$$

where $f(z) = 1 - z - \sum_{k=0}^\infty \nu(k)(1-z)^k$.

In particular, by straightforward approximation, the tail distribution function of the maximal displacement M_t is a solution of the F-KPP equation with Heaviside initial condition. It was proved that this solution stabilises as a travelling-wave function, that invades the real line at positive speed [123, 145]. This result was used to prove the convergence in law of maximal displacement of the branching Brownian motion by Bramson [57].

Theorem 1.32 (Bramson, 1978)**.** *Let X be a branching Brownian motion with reproduction law ν such that $\sum_{k=0}^\infty k^2\nu(k) < \infty$. Then, writing $m = \sum_{k=0}^\infty k\nu(k) - 1$, we have*

$$\lim_{t\to\infty} M_t - \left(\sqrt{2m}t - \tfrac{3}{2\sqrt{2m}} \log t\right) = G \quad \textit{in law.}$$

A simple probabilistic proof for the tightness of $M_t - \sqrt{2m}t + \frac{3}{2\sqrt{2m}} \log t$ is given by Roberts [162]. The law of the random variable G obtained above was described by Lalley and Sellke [127] using an additive martingale of the branching Brownian motion that we describe in the next section.

1.4 Extended family of spatial branching models

Spatial branching processes form an ever-expanding family of models beyond the branching random walk, branching Lévy process and branching Brownian motion that we introduced. Over the years, a number of generalizations of the branching random walk models have been introduced, to explore the tools used to study them and their limits, or to model specific situations. In the rest of the section, we aim at presenting a small sample of the variety of models which can be proposed over the years.

Branching Markov process A branching Markov process is a generalized class of spatial branching processes, in which particles satisfy the branching property, but not the invariance by translation of the law of the process. In these systems, the reproduction law of particles depend on their position. A general study of the properties of branching Markov processes has been undertaken by Ikeda, Nagasawa and Watanabe [106, 107, 108]. The branching Markov process class covers a large variety of models, among which branching diffusions [160], catalytic branching processes [95, 10], branching processes with absorption [30], branching processes in random environment [128], growth-fragmentation processes [37] among many others.

Multitype branching processes Multitype branching processes represent a natural generalization of branching processes. Each particle in that system carries a type, which influences its reproduction law or its displacement. It allows us to model the effect of mutations on the evolution of a population [M33], or the evolution of particles whose development go through different phases [104, 48]. If spatial branching processes with recurrent type structure appear to share many properties with single-type systems [161], a larger variety of behaviour can be observed in models with transient type structure [27], or in multidimensional branching processes [M40], that can be though off as multitype branching processes with type evolution influenced by the position of particles.

Branching random walk in time-inhomogeneous environment Branching processes in time-inhomogeneous environment are used to observe the impact of modifying the environment on the evolution of the process. Among these models we can identify branching Brownian motions with inhomogeneous branching rate [32, 164], branching Brownian motions with time-inhomogeneous environment [92, 93, 143] and branching random walks in random environment [M8]. The asymptotic behaviour of these processes often appear non-standard, and remain an active field of study.

Branching-selection particle systems A branching-selection particle system is a process evolving under the repeated application of the two following steps:

branching step each particle creates newborn offspring around its position as in a spatial branching process;

selection step among the children, some are chosen to reproduce in the next generation, with a procedure depending on the position of particles.

The selection procedure usually depend on the position of the children, often interpreted as an expression of the genotypic makeup of that particle. From the phenotype generated by this genotype one associate a fitness level to each particle, and particles with highest fitness are selected to reproduce in the next generation. As such, this process models simple examples of natural selection. Some branching-selection particle systems are studied in Chapter 5.

Many selection procedures have been introduced over the years. When particles further to the right are assumed to have a higher fitness, one can consider the selection of the (fixed) N rightmost individuals at each step [59], all particles within distance L from the rightmost particle [154] or the absorption of particles by a boundary [30]. On the contrary, if particles have higher fitness close to a certain point, one can consider a branching process with selection of the N closest particles at each step [33], defining the Brownian bees model.

CHAPTER 2

Additive martingales and the spine decomposition

"La distance n'y fait rien; il n'y a que le premier pas qui coûte."

Marie Anne de Vichy-Chamrond, marquise du Deffand – Letter to d'Alembert, July 7, 1763.

Summary.

We introduce in this chapter the additive martingales of the branching random walk. This family of martingales, indexed by a parameter $\theta \in \mathbb{R}_+$ presents a phase transition at some critical parameter θ^*. A more detailed study of this phase transition reveals the important role played by the so-called derivative martingale, which, while non-uniformly integrable, converges almost surely to a non-negative limit. We show that the additive and the derivative martingales allow of the exact computation of the asymptotic growth rate of particles moving at a given speed, and show an example of application of this result to the measure of occupancy in a nested Bernoulli sieve [M38]. After introducing the *spine decomposition*, we give necessary and sufficient conditions, in terms of its characteristic triplet, for the non-degeneracy of the limit of additive martingales in branching Lévy processes, as obtained in [M18] and [M24]. These articles are adaptations of the results proved by Lyons [133] and Alsmeyer and Iksanov [13] for branching random walks.

We then take a closer look at the derivative martingale, recalling the necessary and sufficient conditions due to Aïdékon and Chen [4, 72] for its convergence to a non-degenerate limit, as well as its rate of convergence obtained in [M29]. We then take interest to the derivative martingale of the branching Lévy process. Using tight estimates on the finiteness of perpetuities for Lévy processes conditioned to stay positive, we obtain a necessary and sufficient condition for the non-degeneracy of the limit similar to the one obtained for branching random walks [M36]. Finally, we present a multidimensional extension of the convergence of the derivative martingale obtained in [M30], showing that the derivative martingale of a d-dimensional branching Brownian motion converges simultaneously in almost all directions almost surely.

As a general rule, martingales are a powerful tool for the study of stochastic processes. In branching random walks settings, a family of martingales of interests are the exponential martingales, defined as

$$W_n(\theta) = \sum_{|u|=n} e^{\theta X(u) - n\kappa(\theta)}.$$

These martingales can be seen as partition functions for spin glasses or polymer in random environment, in which to a configuration u is associated the energy $X(u)$, so that the configuration u is occupied with probability proportional to $e^{\theta X(u)}$, with θ the inverse temperature. A freezing phase transition occurs in the branching random walk: at high temperature $\theta < \theta_*$, $W_n(\theta)$ converges to a positive limit while $W_n(\theta) \to 0$ at low temperature $\theta \geq \theta_*$.

These positive martingales can also be used to construct change of measures, linking the large deviations of a process with the almost sure behaviour of an other one. In the case of spatial branching processes, a *spine decomposition* property then hold : the law of a branching process biased by an additive martingale can be described as the law of an immortal particle (the spine), which gives birth to other particles that evolve as in the original branching process. Spine decomposition methods are a natural generalization of the Many-to-one lemma, which can be used to compute the asymptotic behaviour of moments of the spatial branching process.

2.1 Additive martingales of the branching random walk

The additive martingales of the branching random walk are a family of martingales which can be used to study this process. Precisely, given X a branching random walk and $\theta \in \mathbb{R}$ such that $\kappa(\theta) = \log \mathbb{E}\left(\sum_{|u|=1} e^{\theta X(u)}\right) < \infty$, the additive martingale with parameter θ is the stochastic process defined for $n \in \mathbb{N}$ by

$$W_n(\theta) = \sum_{|u|=n} e^{\theta X(u) - n\kappa(\theta)}. \tag{2.1}$$

Using the branching property, simple computations show that $(W_n(\theta), n \geq 0)$ is a non-negative martingale. In particular, $W_n(\theta)$ converges almost surely, as $n \to \infty$ to a non-negative limit $W_\infty(\theta)$.

Using the branching property of the branching random walk at time 1, we remark immediately that the law of $W_\infty(\theta)$ is a solution of the following equation in distribution

$$W_\infty(\theta) \overset{(d)}{=} \sum_{|u|=1} e^{\theta X(u) - \kappa(\theta)} W^{(u)}, \tag{2.2}$$

where $W^{(u)}$ are i.i.d. copies of $W_\infty(\theta)$. In particular, writing $q_\theta = \mathbb{P}(W_\infty(\theta) = 0)$, the above equation yields

$$q_\theta = \mathbb{E}\left(\prod_{|u|=1} q_\theta\right) =: f(q_\theta),$$

with $f : s \in [0,1] \mapsto \mathbb{E}\left(s^{\sum_{|u|=1} \mathbf{1}_{\{X(u) > -\infty\}}}\right)$, using the convention $s^\infty = \mathbf{1}_{\{s=1\}}$. We remark that f, as the generating function of the Galton-Watson process of the number of particles in the branching random walk, is a strictly convex and increasing function with $f(1) = 1$. Therefore there are at most 2 solution to the equation $f(x) = x$, which are 1 and the extinction probability of the branching random walk. As a consequence, for all $\theta \in \mathbb{R}$ with $\kappa(\theta) < \infty$, the following dichotomy holds

$$W_\infty(\theta) = 0 \text{ a.s.} \quad \text{or} \quad W_\infty(\theta) > 0 \text{ a.s. on } S,$$

with $S = \{\#\{u \in \mathbb{U} : X(u) > -\infty\} = \infty\}$ the survival event.

Among others, Biggins [43, 44] took interest in the characterization of the asymptotic behaviour of these additive martingales. In particular, he obtained the following integral criterion for the uniform integrability of $(W_n(\theta))$, guaranteeing that $W_\infty(\theta) > 0$ a.s. on S. For all $\theta > 0$ such that $\kappa(\theta) < \infty$, assuming that

$$\mathbb{E}\left(\sum_{|u|=1} |X(u)| e^{\theta X(u)}\right) < \infty,$$

we denote by

$$\kappa'(\theta) := \mathbb{E}\left(\sum_{|u|=1} X(u) e^{\theta X(u) - \kappa(\theta)}\right) = -i \left.\frac{\mathrm{d}}{\mathrm{d}\xi}\kappa(\theta + i\xi)\right|_{\xi=0}. \tag{2.3}$$

We remark that this definition does not require for κ to be finite in any point but point θ, however if there exists $\delta > 0$ such that $\kappa(\theta - \delta) + \kappa(\theta + \delta) < \infty$, then κ is $\mathcal{C}^\infty$ and convex on $[\theta - \delta, \theta + \delta]$, and $\kappa'(\theta)$ corresponds to the derivative of κ at point θ.

Theorem 2.1 (Biggins 1977)**.** *Let $\theta \in \mathbb{R}$ such that $\mathbb{E}\left(\sum_{|u|=1} |X(u)| e^{\theta X(u)}\right) < \infty$, then*

$$\mathbb{P}(W_\infty(\theta) > 0) > 0 \iff \mathbb{E}(W_\infty(\theta)) = 1 \iff \begin{cases} \theta\kappa'(\theta) < \kappa(\theta) < \infty \\ \mathbb{E}\left(W_1(\theta)\log_+(W_1(\theta))\right) < \infty, \end{cases}$$

where $\log_+(x) = \log(\max(x, 1))$.

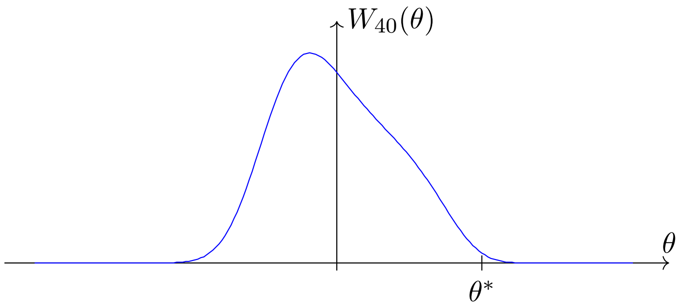


Figure 2.1 – Value at generation 40 of the additive martingale of a branching random walk. The limit is degenerate outside of an interval, defined as $\{\theta : \theta\kappa'(\theta) - \kappa(\theta) < 0\}$ in which it converges to a positive limit.

A simple proof for Theorem 2.1 was obtained by Lyons [133]. This proof is based on the application of spinal decomposition methods pioneered in [134] for the study of the convergence of the martingale in a Bienaymé-Galton-Watson process, that we describe in Section 2.2. Using perpetuity estimates, Alsmeyer and Iksanov [13] obtained a necessary and sufficient condition for the uniform integrability of $(W_n(\theta))$ which does not require the integrability condition $\mathbb{E}\left(\sum_{|u|=1} |X(u)| e^{\theta X(u)}\right) < \infty$. This result is stated further down as Theorem 2.8.

The rate of convergence of the additive martingale towards its limit was studied by Iksanov, Kolesko and Meiners [111, 112]. This rate of convergence is exponential in the domain of non-degeneracy of the additive martingale, but gets significantly smaller as $\theta \to \theta_*$, as can be seen in Figure 2.1.

The additive martingale is a very useful tool in the study of the asymptotic properties of branching random walks. In the same way that the martingale of a Galton-Watson process encodes the speed at which the population size grow in the process, the additive martingale allows us to estimate the growth of the number of particles that are at time n in a neighbourhood of position $n\kappa'(\theta)$. This result was proved by Biggins in [44].

Theorem 2.2 (Biggins 1992)**.** *Let $0 < \theta_* < \theta^*$ such that $\kappa(\theta_*) + \kappa(\theta^*) < \infty$. We assume there exists $\gamma > 1$ such that*

$$\forall \theta \in (\theta_*, \theta^*), \quad \mathbb{E}\left(W_1(\theta)^\gamma\right) < \infty \quad \text{and} \quad \theta^*\kappa'(\theta^*) - \kappa(\theta^*) \leq 0.$$

For $\theta \in (\theta_*, \theta^*)$, we write $\kappa''(\theta) = \mathbb{E}\left(\sum_{|u|=1}(X(u) - \kappa'(\theta))^2 e^{\theta X(u) - \kappa(\theta)}\right)$. Let f be a measurable function $\mathbb{R} \to \mathbb{R}$.

- *For all $\delta > 0$, $\lim_{n\to\infty} \sup_{\theta \in [\theta_*+\delta, \theta^*-\delta]} |W_n(\theta) - W_\infty(\theta)| = 0$ a.s.*
- *If f is direct Riemann-integrable then*

$$\lim_{n\to\infty} n^{1/2} \sum_{|u|=n} e^{\theta X(u) - n\kappa(\theta)} f(X(u) - n\kappa'(\theta)) = \frac{W_\infty(\theta)}{\sqrt{2\pi\kappa''(\theta)}} \int_{\mathbb{R}} f(x)\mathrm{d}x \quad a.s.$$

- *If f is continuous and bounded, then*

$$\lim_{n\to\infty} \sum_{|u|=n} e^{\theta X(u) - n\kappa(\theta)} f\left(\tfrac{X(u) - n\kappa'(\theta)}{n^{1/2}}\right) = W_\infty(\theta)\mathbb{E}(f(N)) \quad a.s.$$

  with N a Gaussian variable with variance $\kappa''(\theta)$.

Loosely speaking, the above theorem expresses that for any reasonable definition of $\approx$, if the number of particles u at generation n satisfying $X(u) \approx n\kappa'(\theta)$ is growing exponentially, we have

$$\lim_{n\to\infty} \frac{\#\{X(u) \approx n\kappa'(\theta)\}}{\mathbb{E}\left(\#\{X(u) \approx n\kappa'(\theta)\}\right)} = W_\infty(\theta) \quad \text{a.s.}$$

It gives a uniform control on the number of particles around any point in the bulk of the branching random walk, i.e. in each area where there is an exponential growth of the number of particles as n grows. Pain [155] provided a trajectorial version of the theorem of Biggins, proving that under the appropriate integrability conditions, for all continuous bounded function f, we have

$$\lim_{n\to\infty} \sum_{|u|=n} e^{\theta X(u) - n\kappa(\theta)} f\left(\tfrac{X(u_{\lfloor nt \rfloor}) - nt\kappa'(\theta)}{n^{1/2}}, t \in [0,1]\right) = W_\infty(\theta)\mathbb{E}\left(f\left(\sigma_\theta B_t, t \in [0,1]\right)\right) \quad \text{a.s.} \tag{2.4}$$

where $\sigma_\theta^2 = \kappa''(\theta)$.

The last two points of Theorem 2.2 are proved in this form in [M38, Section 3] as straightforward consequences of the results of Biggins [44]. In that article, they are used to study the asymptotic behaviour of a random urn scheme constructed on the branching random walk.

Nested occupancy scheme in random environment

The convergence of additive martingales can be used to study the asymptotic behaviour of the nested occupancy scheme in random environment, that we now describe. Let $(X(u), u \in \mathbb{U})$ be a branching random walk satisfying

$$\sum_{|u|=1} e^{X(u)} = 1 \quad \text{a.s.} \quad \text{and} \quad \mathbb{E}\left(\#\{|u|=1\}\right) > 1. \tag{2.5}$$

Observe that under these conditions, the branching random walk is supercritical, does not get extinct a.s. and for all $u \in \mathbb{U}$, $X(u) \leq 0$. We additionally impose a non-lattice assumption, i.e. that for all $a > 0$ and $b \in \mathbb{R}$, we have

$$\mathbb{P}\left(\exists |u| = 1 : X(u) \notin a\mathbb{Z} + b\right) > 0.$$

The nested occupancy scheme is a process of urns and bins defined on the the branching random walk as follows. We define a Markov chain $(R_n, n \geq 0)$ constructed as $R_0 = \emptyset$ and

$$\forall u \in \mathbb{U}, j \in \mathbb{N}, \quad \mathbb{P}(R_{n+1} = uj | R_n = u) = \frac{e^{X(uj)}}{\sum_{i=1}^{\infty} e^{X(ui)}} = e^{X(uj)-X(u)}.$$

The process $(R_n, n \geq 0)$ can be thought of as a ball falling down the tree $\mathbb{U}$ in such a way for each vertex u in which it goes through, the ball will go to its jth child with probability proportional to $e^{X(uj)}$. Another way to visualize this model is to split the interval $[0, 1]$ into bins labelled $1, 2, \ldots$, the bin labelled j having width proportional to $e^{X(j)}$. Then, each bin labelled u is recursively subdivided into smaller bins labelled uj which all have width proportional to $e^{X(uj)}$. Then the process $(R_n, n \geq 0)$ corresponds to the sequence of labels of bins at each generation of a uniformly randomly chosen point on $[0, 1]$.

Remark 2.3. The nested occupancy scheme can be constructed for an arbitrary supercritical branching random walk Y satisfying $\mathbb{E}(\sum_{|u|=1} e^{Y(u)}) < \infty$ and that almost surely does not get extinct. The process (R_n) is defined such that $\mathbb{P}(R_{n+1} = uj | R_n = u)$ is proportional to $e^{Y(uj)}$. However, in this situation, the definition of the process is identical with the branching random walk defined by

$$X(u) = Y(u) - \sum_{k<|u|} \log \sum_{j=1}^{\infty} e^{Y(u_k j)},$$

and the branching random walk X will satisfy (2.5).

The nested occupancy scheme in random environment consists in the consideration of an infinite series $(R^{(j)}, j \in \mathbb{N})$ of i.i.d. copies of R. For all $n, N \in \mathbb{N}$, we take interest in

$$K_N^{(n)}(k) = \sum_{|u|=n} \mathbf{1}_{\{\#\{j \leq N : R_n^{(j)} = u\} = k\}}.$$

In other words, $K_N^{(n)}(k)$ is the number of vertices of the nth generation of $\mathbb{U}$ through which k of the first N balls fell. In [M38], Iksanov and Mallein studied the asymptotic behaviour of $K_N^{(n)}(1)$ as n and N grow to ∞ with $\log N/n \to a$, expanding on earlier work by Bertoin [36].

Before stating the main result, we introduce a few notation. We write $\underline{\theta} = \inf\{\theta \in \mathbb{R} : \kappa(\theta) < \infty\}$, the function κ is decreasing and convex on $(\underline{\theta}, \infty)$. We assume there exists

$\theta^* > 0$ such that $\theta^*\kappa'(\theta^*) - \kappa(\theta^*) = 0$, which the the minimizer of the function $\theta \mapsto \frac{\kappa(\theta)}{\theta}$. We set

$$a_* := \begin{cases} -\kappa(2)/2 & \text{if } \theta^* > 2, \\ -\kappa(\theta^*)/\theta^* = v & \text{otherwise,} \end{cases} \quad a_c = -\kappa'(1) \quad \text{and} \quad \overline{a} = - \lim_{\theta \to \max(\underline{\theta}, 0)} \kappa'(\theta).$$

Under the conditions (2.5), for all $\theta \in (\underline{\theta}, \theta^*)$ the martingale $(W_n(\theta), n \geq 0)$ is uniformly integrable, and we denote by $W(\theta)$ its almost sure limit.

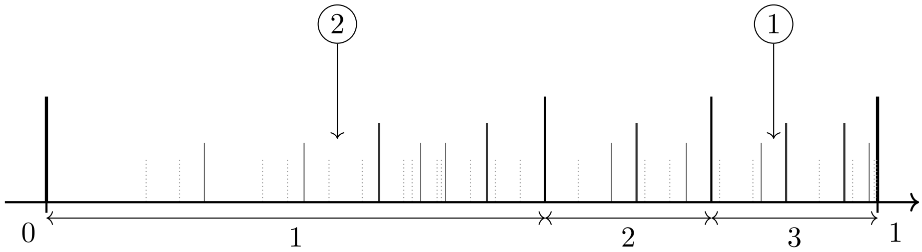


Figure 2.2 – First few levels of a nested occupancy scheme in random environment constructed over the interval $[0, 1]$. Each interval is recursively subdivided at random according to a scaled point measure. The first ball in this occupancy scheme falls into boxes 3, 31 and 312, the second one into boxes 1, 11 and 113.

Theorem 2.A (Iksanov and Mallein (2022))**.** *Let $a \in (0, \overline{a})$ and $b \in \mathbb{N}$, we let $n, N \to \infty$ in such a way that $\log N = an + bn^{1/2}(1 + o(1))$. We fix $\theta > 0$ such that $a = -\kappa(\theta)$. The following result then holds.*

1. *If $a < a_*$, then $K_N^{(n)}(1) = n$ a.s. for n large enough.*
2. *If $a_\star < a < a_c$, then*
   (a) if $a < -\kappa'(2)$ then $K_N^{(n)}(1) = n - \frac{W(2)}{2} N^2 e^{\kappa(2)n}(1 + o(1))$ a.s. as $n \to \infty$,
   (b) if $a > \kappa'(2)$, then $K_N^{(n)}(1) = n - \frac{W(\theta)\Gamma(2-\theta)}{\theta(\theta-1)(2\pi\kappa''(\theta)n)^{1/2}} e^{-b^2/2\kappa''(\theta)} N^2 e^{\kappa(2)n}(1 + o(1))$ a.s. as $n \to \infty$.
3. *If $a_c < a < \overline{a}$, then $K_N^{(n)}(1) = \frac{\Gamma(1-\theta)}{\theta(2\pi\kappa''(\theta)n)^{1/2}} e^{-b^2/2\kappa''(\theta)} W(\theta) N^\theta e^{\kappa(\theta)n}(1 + o(1))$*

The explicit asymptotic behaviour of $K_n^{(n)}(1)$ at the critical points a_c and $-\kappa'(2)$ (when $\theta^* > 2$) is also obtained. The nested occupancy scheme in random environment can be used to describe the way polluting elements can accumulate in a lung. In a current joint project, we are working on an application of the nested occupancy scheme to the modelling of induce fibrosis on mice lungs.

2.2 The spine decomposition: a change of measure

In [133], Lyons introduced the spine decomposition for the branching random walk as an alternative description of the law of the branching random walk biased by its additive martingale. This construction can be used to give a simple proof of the uniform integrability, or degeneracy, of the additive martingales. This method is an extension of the spine decomposition obtained by Lyons, Pemantle and Peres [134] for Galton-Watson processes.

Let $(X(u), u \in \mathbb{U})$ be a branching random walk and $\theta \geq 0$ such that $\kappa(\theta) < \infty$. We denote by $\mathbb{P}$ the law of X, and we introduce the size-biased law $\widehat{\mathbb{P}}$ defined, for $n \in \mathbb{N}$ by

$$\widehat{\mathbb{P}}\big|_{\mathcal{F}_n} = W_n(\theta) \cdot \mathbb{P}\big|_{\mathcal{F}_n}, \tag{2.6}$$

where $\mathcal{F}_n = \sigma(X(u), |u| \leq n)$ the natural filtration of the branching random walk.

The spine decomposition consists in an alternative description of the law $\widehat{\mathbb{P}}$ constructed as a branching random walk with spine, that we now define. Writing $\mathcal{Z}$ the reproduction law of X, we introduce the law $\overline{\mathcal{Z}}$ on $\mathcal{P}(\mathbb{R}) \times \mathbb{N}$ defined by

$$\int_{\mathcal{P}(\mathbb{R})\times\mathbb{N}} f(Z,k)\overline{\mathcal{Z}}(\mathrm{d}Z\mathrm{d}k) = \int_{\mathcal{P}(\mathbb{R})} \sum_{k=1}^{\infty} e^{\theta z_k - \kappa(\theta)} f(Z,k)\mathcal{Z}(\mathrm{d}Z). \tag{2.7}$$

In other words, to construct a pair (Z, ξ) of law $\overline{\mathcal{Z}}$, we first define a random point measure Z with law given by $\langle Z, \mathbf{e}_\theta\rangle e^{-\kappa(\theta)}\mathcal{Z}$, then, conditionally on Z, we fix $\xi = k$ with probability proportional to $e^{\theta z_k}$.

Definition 2.4 (Branching random walk with spine)**.** The branching random walk with spine with reproduction law $\mathcal{Z}$ is the random pair (X, ξ), with X a map $\mathbb{U} \to \mathbb{R}$ and $\xi \in \mathbb{N}^n$ an infinite ray in $\mathbb{U}$ called the spine of the process, defined as follows. Let $(Z^{(u)}, u \in \mathbb{U})$ i.i.d. random point measures with law $\mathcal{Z}$ and $(\overline{Z}_n, \xi(n+1), n \geq 0)$ i.i.d. vectors with law $\overline{\mathcal{Z}}$. We denote by $\xi = (\xi(1), \xi(2), \ldots)$, and set

$$\forall u \in \mathbb{U},\ \overline{Z}^{(u)} = \begin{cases} \overline{Z}_n & \text{if } u = \xi_n := (\xi(1), \ldots, \xi(n)), \\ Z^{(u)} & \text{otherwise.} \end{cases}$$

Finally, the map X is defined by

$$\forall u \in \mathbb{U},\ X(u) = \sum_{j=0}^{|u|-1} \overline{z}^{(u_j)}_{u(j+1)}.$$

We write $\overline{\mathbb{P}}$ the law of the branching random walk with spine.

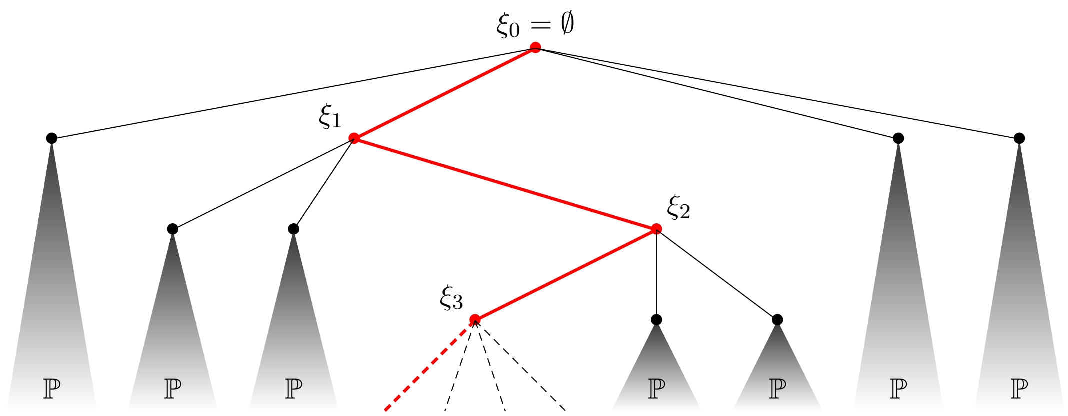


Figure 2.3 – Construction of a branching random walk with spine ξ. The spine particle reproduces and chooses its child that will be part of the spine according to the law $\overline{\mathcal{Z}}$, all other children then reproduce as in a (regular) branching random walk.

We can rephrase the above construction in the following fashion represented in Figure 2.3. A branching random walk with spine is a branching particle with a distinguished particle at each generation (the spine). The process starts with a single particle $\emptyset$, which is the spine particle ξ_0. This particle simultaneously produces offspring and chooses one of its children to be the spine particle at time 1 according to the law $\overline{\mathcal{Z}}$. Then, at each generation n, every particle reproduces independently. Non-spine particles create offspring around their position according to random point measures of law $\mathcal{Z}$, while the spine particle jointly creates offspring and choose one of its children to be the next spine particle according to the law $\overline{\mathcal{Z}}$.

For all $n \in \mathbb{N}$, we denote by $\mathcal{F}_n = \sigma((X(u), |u| \leq n))$ the filtration of the branching random walk (without knowledge of the value of the spine) and by $\overline{\mathcal{F}}_n = \mathcal{F}_n \vee \sigma(\xi_n)$ the natural filtration of the branching random walk with spine. Another filtration of interest is $\mathcal{G}_n = \sigma((\overline{Z}_j, \xi(j-1)), j \leq n) = \sigma(\xi_n, X(\xi_k j), k < n, j \in \mathbb{N})$, which is the filtration associated to the spine and its offspring. Observe that conditionally on $\mathcal{G}_n$ the children of the spine create independent copies of the branching random walk with law $\mathbb{P}$ shifted by their position.

The spine decomposition theorem corresponds in the identification of the law of the size-biased branching random walk and the law of the branching random walk with spine. This result was proved by Lyons [133], with a prototype version of this process for the branching Brownian motion appearing in [71].

Theorem 2.5 (Lyons, 1997)**.** *Let X be a branching random walk, we denote by $\widehat{\mathbb{P}}$ the size-biased law of X and $\overline{\mathbb{P}}$ the law of the branching random walk with spine (X, ξ). For all $n \in \mathbb{N}$, we have*

$$\widehat{\mathbb{P}}\big|_{\mathcal{F}_n} = \overline{\mathbb{P}}\big|_{\mathcal{F}_n}, \tag{2.8}$$

and moreover, for all $|u| = n$, we have

$$\overline{\mathbb{P}}\left(\xi_n = u | \mathcal{F}_n\right) = \frac{e^{\theta X(u)}}{\sum_{|v|=n} e^{X(v)}} \quad a.s. \tag{2.9}$$

This result generalizes the spine decomposition introduced by Lyons, Pemantle and Peres [134] for the BGW process to study the convergence of the martingale of the Galton-Watson process. The spine decomposition was also extended by Kurtz, Lyons Pemantle and Peres [126] to multitype Galton-Watson processes with a finite number of types, and by Biggins and Kyprianou [46] to general branching Markov processes. Indeed, studying the size-biased law of the process can be used to prove the uniform integrability of the biasing martingale. More precisely, the following result, which can be found in [86, Theorem 4.3.5], is used to prove the uniform integrability of additive martingales of branching processes.

Proposition 2.6 (Radon-Nikodým derivative theorem)**.** *Let $(W_n, n \geq 0)$ be a $(\mathcal{F}_n)$-positive martingale under the law $\mathbb{P}$. We define the biased law $\widehat{\mathbb{P}}$ by setting, for all $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$, $\widehat{\mathbb{P}}(A) = \mathbb{E}\left(W_n \mathbf{1}_A\right)$. Let $W_\infty = \limsup_{n\to\infty} W_n$, for any $A \in \mathcal{F}_\infty$ we have*

$$\widehat{\mathbb{P}}(A) = \mathbb{E}\left(W_\infty \mathbf{1}_A\right) + \widehat{\mathbb{P}}(A \cap \{X = \infty\}).$$

Using that in branching random walk settings, $\mathbb{P}\left(W_\infty(\theta) = 0\right) \in \{\mathbb{P}(S^c), 1\}$, we remark that the additive martingale of the branching random walk is either uniformly integrable or converges to 0 $\mathbb{P}$-a.s. As a result, Proposition 2.6 implies the following dichotomy.

Corollary 2.7. *Let X be a branching random walk with $\kappa(\theta) < \infty$. One of the two following alternatives holds:*

$$\limsup_{n\to\infty} W_n(\theta) < \infty \quad \widehat{\mathbb{P}}\text{-}a.s. \quad \iff \quad \mathbb{E}\left(W_\infty(\theta)\right) = 1;$$
$$\widehat{\mathbb{P}}\left(\liminf_{n\to\infty} W_n(\theta) = \infty\right) > 0 \quad \iff \quad W_\infty(\theta) = 0 \quad \mathbb{P}\text{-}a.s.$$

Thanks to this dichotomy, Lyons provided a simple proof of Theorem 2.1, using the following method. Denoting by

$$\mathcal{G} := \sigma\left(\xi, X(\xi_n j), n \geq 0, j \in \mathbb{N}\right) = \sigma\left(\overline{Z}_n, \xi(n+1), n \geq 0\right)$$

the filtration associated to the trajectory of the spine and the position of its offspring, we remark that

$$\widehat{\mathbb{E}}\left(W_n(\theta)|\mathcal{G}\right) = \sum_{k=0}^{n-1}\sum_{j=1}^{\infty} e^{\theta X(\xi_k j)-(k+1)\kappa(\theta)}.$$

This sum can be represented as a perpetuity $\sum_{k=0}^{n-1} B_k \prod_{j=0}^{k-1} A_j$, by setting

$$A_k = e^{\theta(X(\xi_{k+1})-X(\xi_k))-\kappa(\theta)} \quad \text{and} \quad B_k = \sum_{j=1}^{\infty} e^{\theta(X(\xi_k j)-X(\xi_k))-\kappa(\theta)}.$$

We refer to [63, 110] for an introduction to perpetuities. Using [110, Theorem 2.1.1], barred degenerate cases such that $A_k \in \{0,1\}$ a.s. we have

$$\sum_{k=0}^{\infty} B_k \prod_{j=0}^{k-1} A_j < \infty \quad \text{a.s.} \quad \Longleftrightarrow \quad \sup_{k\in\mathbb{N}} B_k \prod_{j=0}^{k-1} A_j < \infty \quad \text{a.s.}$$

and integral tests are available to determine whether the perpetuity is a.s. finite or infinite with positive probability. This integral test leads to the following generalization of Biggins' martingale convergence theorem, obtained by Alsmeyer and Iksanov [13].

Theorem 2.8 (Alsmeyer and Iksanov, 2009)**.** *Let $\theta > 0$, the martingale $(W_n(\theta), n \geq 0)$ is uniformly integrable if and only if the following condition holds*

$$\lim_{n\to\infty} \theta X(\xi_n) - n\kappa(\theta) = -\infty \quad \widehat{\mathbb{P}}\text{-}a.s. \quad \text{and} \quad \mathbb{E}\left(W_1(\theta) J(\log_+ W_1(\theta))\right) < \infty, \tag{2.10}$$

where J is defined for $x \geq 0$ by $J(x) = \frac{x}{\mathbb{E}(\min(X(\xi_1)_-, x))}$.

We remark that this theorem holds without the necessity for $\kappa'(\theta)$ to be finite. However, assuming that $\kappa'(\theta)$ is well-defined and finite, (2.10) is equivalent to

$$\theta\kappa'(\theta) - \kappa(\theta) < 0 \quad \text{and} \quad \mathbb{E}\left(W_1(\theta)\log_+(W_1(\theta))\right) < \infty,$$

which are the conditions obtained by Biggins [43] and Lyons [133]. The result of Alsmeyer and Iksanov additionally covers situations in which the spine of the branching random walk moves as a non-integrable random walk such that $\theta X(\xi_n) - n\kappa(\theta)$ drifts to $-\infty$.

2.2.1 Spine decomposition for branching Lévy processes

The spine decomposition technique can be extended to branching Lévy processes. This result was first introduced in [171], where a genealogical construction of the branching Lévy process is introduced. The spine decomposition for the branching Brownian motion is stated in full generality and proved in [M18, Lemma 2.3]. We first introduce branching Lévy processes with spine.

Let (σ^2, a, Λ) be the characteristic triplet of a branching Lévy process X. Let $\theta > 0$ such that

$$\int_{\mathcal{P}(\mathbb{R})} (1 \wedge x_1^2)\Lambda(\mathrm{d}\mathbf{x}) < \infty \quad \text{and}$$

$$\int_{\mathcal{P}(\mathbb{R})} \left(\sum_{j=1}^{\infty} \mathbf{1}_{\{x_j > -\infty\}} e^{\theta x_j} - 1 - \theta x_1 \mathbf{1}_{\{|x_1|<1\}}\right) \Lambda(\mathrm{d}\mathbf{x}) < \infty. \tag{2.11}$$

The additive martingale of the branching Lévy process X is the process defined for $t \geq 0$ by

$$W_t(\theta) := \sum_{u \in \mathcal{N}_t} e^{\theta X_t(u) - t\kappa(\theta)}. \tag{2.12}$$

We denote by $\overline{\mathbb{P}}$ the size-biased law of the branching Lévy process defined by

$$\overline{\mathbb{P}}\big|_{\mathcal{F}_t} = W_t(\theta) \cdot \mathbb{P}\big|_{\mathcal{F}_t},$$

where $\mathcal{F}_t := \sigma\left(\mathcal{N}_s, s \leq t, (X_s(u), s \leq t), u \in \mathcal{N}_t\right)$ is the natural time filtration of the branching Lévy process.

We introduce the tilted measure as a σ-finite measure on $\mathcal{P}(\mathbb{R}) \times \mathbb{N}$ defined by

$$\int_{\mathbb{P}(\mathbb{R}) \times \mathbb{N}} f(\mathbf{x}, k) \widehat{\Lambda}(\mathrm{d}\mathbf{x}\mathrm{d}k) = \int_{\mathcal{P}(\mathbb{R})} \sum_{k=1}^{\infty} e^{\theta x_k} f(\mathbf{x}, k) \Lambda(\mathrm{d}\mathbf{x})$$

for all measurable positive function f. Then, let $\widehat{\mathcal{N}}$ be a Poisson point process on $\mathbb{R}_+ \times \mathcal{P}(\mathbb{R}) \times \mathbb{N}$ with intensity $\mathrm{d}t\widehat{\Lambda}(\mathrm{d}\mathbf{x}, \mathrm{d}k)$ and $\widehat{B}$ an independent Brownian motion. We use these processes to construct the spine trajectory

$$\begin{aligned}\widehat{S}_t := \sigma \widehat{B}_t + \widehat{a}t + \int_{[0,t] \times \mathcal{P}(\mathbb{R}) \times \mathbb{N}} x_k \mathbf{1}_{\{|x_k| < 1\}} \mathrm{d}\widehat{\mathcal{N}}^c(s, \mathbf{x}, k) \\ + \int_{[0,t] \times \mathcal{P}(\mathbb{R}) \times \mathbb{N}} x_k \mathbf{1}_{\{|x_k| \geq 1\}} \mathrm{d}\widehat{\mathcal{N}}(s, \mathbf{x}, k),\end{aligned}$$

where we have set

$$\widehat{a} = a + \theta\sigma^2 + \int_{\mathcal{P}(\mathbb{R})} \sum_{k \geq 1} x_k e^{\theta x_k} \mathbf{1}_{\{|x_k| < 1\}} - x_1 \mathbf{1}_{\{|x_1| < 1\}} \Lambda(\mathrm{d}\mathbf{x}).$$

The branching Lévy process with spine is constructed in the following fashion. The process starts with the spine particle $\emptyset$ at time 0 and position 0. The spine particle moves according to the process $\widehat{S}$ while giving birth to offspring according to the point measure $\widehat{\mathcal{N}}$ until time

$$T := \inf\{t > 0 : \exists (t, \mathbf{x}, k) \in \widehat{\mathcal{N}} \text{ with } k \geq 2\}.$$

At that time T, the spine becomes the particle labelled u born at position $\widehat{S}_{T-} + x_k$, with $(T, \mathbf{x}, k)$ the corresponding atom of $\widehat{\mathcal{N}}$. This particle again moves and reproduce according to the process $\widehat{S}$ and the point measure $\widehat{\mathcal{N}}$ until its jump to one of the children of u in a similar fashion. Non-spine particles reproduce and displace as in the usual branching Lévy process. The branching Lévy process with spine is then given as the pair $(\widehat{X}, \xi)$, with $\widehat{X}$ the random map $\mathbb{U} \to \mathcal{D}([0, \infty))$ giving the trajectories followed by each particle and their ancestors, and $\xi : [0, \infty) \to \mathbb{U}$ the random process such that for all time $t \geq 0$, ξ_t is the label of the spine particle. In particular, we remark that the trajectory followed by the spine particle is given, for all $t \geq 0$ by $\widehat{S}_t = \widehat{X}_t(\xi_t)$.

The spine decomposition for the branching Lévy processes is an identification of the law of the branching Lévy process with spine with the law of the size-biased branching Lévy process. This result can be found in [M18], but earlier versions (assuming more restrictive conditions on the support of Λ) were previously available, for example in [171].

Proposition 2.B (Bertoin and Mallein (2018))**.** *Let X be a branching Lévy process with characteristic triplet (σ^2, a, Λ), we denote by $\overline{\mathbb{P}}$ the size-biased law of X and $\widehat{\mathbb{P}}$ the law of the branching Lévy process with spine (X, ξ). For all $t \geq 0$, we have*

$$\widehat{\mathbb{P}}\big|_{\mathcal{F}_t} = \overline{\mathbb{P}}\big|_{\mathcal{F}_t}, \tag{2.13}$$

and moreover, for all $u \in \mathcal{N}_t$, we have

$$\overline{\mathbb{P}}(\xi_t = u | \mathcal{F}_t) = \frac{e^{\theta X_t(u)}}{\sum_{v \in \mathcal{N}_t} e^{X_t(v)}} \quad a.s. \tag{2.14}$$

Similarly to the branching random walk, the convergence of additive martingales of branching Lévy process can be studied using the spine decomposition and Proposition 2.6. A version of Biggins' convergence theorem applied to the branching Lévy processes is obtained by Bertoin and Mallein in [M18].

Theorem 2.C (Bertoin and Mallein (2018))**.** *Let X be a branching Lévy process with characteristics (σ^2, a, Λ) and $\theta > 0$ such that (2.11) holds. We recall that*

$$\kappa(\theta) = \frac{\sigma^2\theta^2}{2} + \theta a + \int_{\mathcal{P}(\mathbb{R})} \left(\sum_{j=1}^{\infty} e^{\theta x_j} - 1 - \theta x_1 \mathbf{1}_{\{|x_1|<1\}} \right) \Lambda(\mathrm{d}\mathbf{x}),$$

and we assume that

$$\kappa'(\theta) = \sigma^2\theta + a + \int_{\mathcal{P}(\mathbb{R})} \left(\sum_{j=1}^{\infty} x_j e^{\theta x_j} - x_1 \mathbf{1}_{\{|x_1|<1\}} \right) \Lambda(\mathrm{d}\mathbf{x})$$

is well-defined and finite. We have

$$\mathbb{P}(W_\infty(\theta) > 0) \iff \mathbb{E}(W_\infty(\theta)) = 1 \iff \begin{cases} \theta\kappa'(\theta) - \kappa(\theta) < 0 \\ \int_{\mathcal{P}(\mathbb{R})} \langle Z, \mathbf{e}_\theta \rangle \log_+(\langle Z, \mathbf{e}_\theta \rangle - 1)\Lambda(\mathrm{d}Z). \end{cases}$$

A general necessary and sufficient condition for the uniform integrability of the additive martingale of branching Lévy processes has been obtained by Iksanov and Mallein in [M24]. The theorem can be stated as follows.

Theorem 2.D (Iksanov and Mallein (2019))**.** *Let X be a branching Lévy process with characteristics (σ^2, a, Λ) and $\theta > 0$ such that (2.11) holds. Let $(\widehat{X}, \xi)$ the branching Lévy with spine of same characteristics. The martingale $(W_t(\theta), t \geq 0)$ is uniformly integrable if and only if*

$$\lim_{t \to \infty} \theta X_t(\xi_t) - t\kappa(\theta) = -\infty \ \widehat{\mathbb{P}}\text{-}a.s. \ \ and \qquad \int_{\mathcal{P}(\mathbb{R})} \sum_{k \geq 1} e^{\theta x_k} J(\log \sum_{j \neq k} e^{\theta x_j}) \Lambda(\mathrm{d}\mathbf{x}) < \infty,$$

with $J : y \mapsto \frac{y}{A(y)} \mathbf{1}_{\{y \geq 1\}}$ and

$$A : y \mapsto 1 + \int_{\mathcal{P}(\mathbb{R})} \sum_{k \geq 1} e^{\theta x_k} ((-x_k) \wedge y - 1)_+ \Lambda(\mathrm{d}\mathbf{x}).$$

Moreover, for $p \in (1, 2]$, the martingale $W_t(\theta)$ converges to $W_\infty(\theta)$ in L^p if and only if

$$\kappa(p\theta) < p\kappa(\theta) \quad and \quad \int_{\mathcal{P}(\mathbb{R})} \sum_{k=1}^{\infty} \left(\sum_{j \neq k} e^{\theta x_j} \right)^{p-1} \mathbf{1}_{\{\sum_{j \neq k} e^{\theta x_j} > e\}} \Lambda(\mathrm{d}\mathbf{x}) < \infty.$$

It is worth mentioning that although the process $(W_{nh}(\theta), n \geq 0)$ can be seen as the additive martingale of a branching random walk, Theorems 2.C and 2.D were not obtained by using the associated results for branching random walks. Indeed, the necessary and sufficient conditions for the uniform integrability of additive martingales of the branching random walk can be expressed using the reproduction law $\mathcal{Z}$ of the process at time 1. However, there is no simple map linking the law of the branching Lévy process at time 1 with its characteristic triplet. As a result, the analysis using spine decomposition techniques is necessary to prove the convergence of the additive martingales.

Convergence of the additive martingale for branching Brownian motion

As a special case, we consider necessary and sufficient conditions for the convergence of the additive martingales in a branching Brownian motion. Let $(X_t(u), u \in \mathcal{N}_t)_{t\geq 0}$ be a branching Brownian motion with reproduction law ν (setting, without loss of generality, diffusion coefficient $\sigma^2 = 1$, drift $a = 0$ and branching rate $\beta = 1$). We recall that for this process, the log-Laplace transform is given, for all $\theta \in \mathbb{R}$, by

$$\kappa(\theta) = \log \mathbb{E}\left(\sum_{u\in\mathcal{N}_1} e^{\theta X_t(u)}\right) = \log(m-1) + \frac{\theta^2}{2},$$

where $m = \sum_{k=0}^{\infty} k\nu(k) < \infty$ is the mean number of offspring at each branching time. The additive martingale of the branching Brownian motion of parameter θ is then defined as the following stochastic process

$$W_t(\theta) := \sum_{u\in\mathcal{N}_t} e^{\theta X_t(u) - t\left(\frac{\theta^2}{2} + \log(m-1)\right)}.$$

Study of the convergence of additive martingales of the branching Brownian motion was started by McKean [145]. Necessary and sufficient conditions for the uniform integrability of these martingales can be obtained using the same spine decomposition methods as in Lyons [133], or be seen as a consequence of Theorem 2.C.

Theorem 2.9. *Given a branching Brownian motion with reproduction law ν of mean m, the martingale $(W_t(\theta), t \geq 0)$ is uniformly integrable if and only if $\theta < \sqrt{2(m-1)}$ and $\sum_{k=1}^{\infty} \nu(k) k \log k < \infty$.*

The $L \log L$ integrability condition appearing to guarantee the non-degeneracy of the additive martingales of the branching Brownian motion is reminiscent of the similar integrability condition for the non-degeneracy of the martingale associated to the BGW process.

The uniformity of these additive martingales is deeply related to the existence of travelling-wave solutions to the associated F-KPP reaction-diffusion equation. Indeed, for all $\theta \in \mathbb{R}$, the function

$$u^{(\theta)}(t,x) = \mathbb{E}\left(1 - \exp\left(-\sum_{u\in\mathcal{N}_t} e^{\theta(X_t(u)+x)}\right)\right)$$

is the unique solution to (1.22) with initial condition $u^\theta(0,x) = 1 - e^{-e^{\theta x}}$. The convergence of $W_t(\theta)$ to a non-degenerate limit is then equivalent to

$$\lim_{t\to\infty} u^{(\theta)}(t, x - v_\theta t) = w^{(\theta)}(-x) = 1 - \mathbb{E}\left(e^{-e^{\theta x} W_\infty(\theta)}\right) \quad \text{uniformly on compacts,}$$

with $v_\theta = \frac{\theta}{2} + \frac{m-1}{\theta}$ and $w^{(\theta)}$ is a so-called *travelling-wave* solution to the F-KPP equation, satisfying

$$\frac{1}{2} w_\theta'' - v_\theta w_\theta' + w_\theta(1-w_\theta) = 0, \text{ with } \lim_{x\to\infty} w_\theta(x) = 0 \text{ and } \lim_{x\to-\infty} w_\theta(x) = 1. \tag{2.15}$$

Indeed, it can be straightforwardly observed that $(t,x) \mapsto w^{(\theta)}(x - v_\theta t)$ is a solution to the F-KPP equation[1].

1. Among others, this fact can be obtained using the equality in distribution (2.2) for the additive martingales of the branching Brownian motion.

McKean [145] proved that for all $0 < \theta < \sqrt{2(m-1)}$, there exists a unique (up to translation) travelling wave solution to (2.15) with speed v_θ, that can be represented as

$$w^{(\theta)}(x) = \mathbb{E}\left(1 - \exp\left(-e^{-\theta x} W_\infty(\theta)\right)\right). \tag{2.16}$$

Moreover, for any solution u of the F-KPP equation satisfying $u(0,x) \sim_{x\to\infty} ce^{-\theta x}$ for some $\theta < \sqrt{2(m-1)}$, there will exists $\varphi \in \mathbb{R}$ such that

$$\lim_{t\to\infty} u(t, x + v_\theta t) = w^{(\theta)}(x + \varphi) \quad \text{uniformly on compact sets.}$$

The question of the asymptotic behaviour of solutions to the F-KPP equations with an initial condition with lighter right tails (for example bounded initial conditions) motivated the introduction of the derivative martingale of branching random walks by Lalley and Sellke [127], that we describe in the next section.

2.3 The derivative martingale

As noted in Theorem 2.2, the additive martingale limit $W_\infty(\theta)$ of the branching random walk can be used to give an accurate estimate of the number of particles in a branching random walk moving at speed $\kappa'(\theta)$. However, this estimate is precise as long as the martingale $(W_n(\theta))$ remains uniformly integrable, in particular if $\theta\kappa'(\theta) < \kappa(\theta)$.

We recall that the maximal speed of particles in the branching random walk is given by $v = \inf_{\theta>0} \frac{\kappa(\theta)}{\theta}$. Assuming that this minimum is attained at a point θ^* such that κ is well-defined in a neighbourhood of θ^*, we observe immediately that

$$\theta^*\kappa'(\theta^*) - \kappa(\theta^*) = 0. \tag{2.17}$$

As a result, the martingale $(W_n(\theta^*))$ is non-integrable, and cannot be use to estimate the number of particles going at maximal speed in the process.

This observation underlines the necessity of exhibiting a different martingale to study the asymptotic behaviour of extremal particles in the branching random walk. This martingale is the so-called derivative martingale, defined as follows.

Definition 2.10. Let X be a branching random walk with log-Laplace transform κ. We assume that there exists $\theta^* > 0$ such that Equation (2.17) holds (with $\kappa'(\theta^*)$ defined by (2.3)). The parameter θ^* is called the critical parameter, and the derivative martingale of the branching random walk is defined for $n \geq 0$ by

$$Z_n := \sum_{|u|=n} (n\kappa(\theta^*) - \theta^* X(u)) e^{\theta^* X(u) - n\kappa(\theta^*)}. \tag{2.18}$$

Using (2.17) and the many-to-one lemma, it is a straightforward application of the branching property to observe that $(Z_n, n \geq 0)$ is a (signed) martingale. The name *derivative martingale* comes from the fact that Z_n can be rewritten as $\frac{-1}{\theta^*} \left.\frac{\mathrm{d}W_n(\theta)}{\mathrm{d}\theta}\right|_{\theta=\theta^*}$.

Our definition of the derivative martingale of the branching random walk does not match the usual literature, in which the derivative martingale is usually defined by Z_n/θ^*. However, the current definition allows for a simpler description of the main results, in particular due to the reduction of branching random walks to the boundary case, that we now describe.

Remark 2.11. Usually, when studying the asymptotic behaviour of quantities associated to the derivative martingales, authors study the process $Y : \mathbb{U} \to \mathbb{R}$ defined by

$$Y(u) = |u|\kappa(\theta^*) - \theta^* X(u).$$

We call Y a *branching random walk in the boundary case*, which is characterized by the equations

$$\mathbb{E}\left(\sum_{|u|=1} e^{-Y(u)}\right) = 1 \quad \text{and} \quad \mathbb{E}\left(\sum_{|u|=1} Y(u)e^{-Y(u)}\right) = 0, \tag{2.19}$$

with the second condition implicitly assuming that $\sum_{|u|=1} Y(u)e^{-Y(u)}$ is integrable. Note that contrary to the branching random walks we consider so far, the branching random walk Y puts finite mass on intervals of the form $(-\infty, a]$. However, it is usually more convenient to study this process rather than $-Y$ as almost surely $Y(u) > 0$ for all $|u|$ large enough. With this notation, the derivative martingale can be rewritten

$$Z_n = \sum_{|u|=n} Y(u)e^{-Y(u)}, \ n \in \mathbb{N}.$$

The convergence of the derivative martingale of the branching random walk was first obtained by Biggins and Kyprianou [46]. Aïdékon [4] proved the almost sure convergence of Z_n to a non-degenerate limit under broad sufficient conditions, that Chen [72] proved to be necessary.

Theorem 2.12 (Aïdékon 2013, Chen 2015)**.** *Let X be a branching random walk with critical parameter θ^*. We assume that*

$$\sigma^2 := (\theta^*)^2\kappa''(\theta^*) = \mathbb{E}\left(\sum_{|u|=1} (\kappa(\theta^*) - \theta^* X(u))^2 e^{\theta^* X(u) - \kappa(\theta^*)}\right) < \infty. \tag{2.20}$$

The martingale $(Z_n, n \geq 0)$ converges a.s. to a non-degenerate non-negative limit Z_∞ if and only if

$$\mathbb{E}\left(W_1(\theta^*)\left(\log W_1(\theta^*)\right)^2\right) + \mathbb{E}\left(\widetilde{W}_1(\theta^*) \log_+ \widetilde{W}_1(\theta^*)\right) < \infty, \tag{2.21}$$

where $\widetilde{W}_1(\theta^*) = \sum_{|u|=1}(\kappa(\theta^*) - \theta^* X(u))_+ e^{\theta^* X(u) - \kappa(\theta^*)}$.

The convergence of the derivative martingale to a non-degenerate limit plays an important role in the study of extremal particles in the branching random walk. Therefore, we refer to the conditions of application of Theorem 2.12 as assumption $\mathcal{A}$: there exists $\theta^* > 0$ such that

$$\begin{gathered} \theta^*\kappa'(\theta^*) - \kappa(\theta^*) = 0, \quad \sigma^2 = \kappa''(\theta^*) \in (0, \infty) \quad \text{and} \\ \mathbb{E}\left(W_1(\theta^*)\left(\log W_1(\theta^*)\right)^2\right) + \mathbb{E}\left(\widetilde{W}_1(\theta^*) \log_+ \widetilde{W}_1(\theta^*)\right) < \infty. \end{gathered} \tag{$\mathcal{A}$}$$

Boutaud and Maillard [52] obtained convergence of an analogue of the derivative martingale for branching random walks that do not satisfy Assumption $\mathcal{A}$. Specifically, they considered branching random walks that do not satisfy (2.20), but such that the random walk associated to the branching random walk is in the domain of attraction of an α-stable random variable.

The existence of the limit of the derivative martingale is deeply linked with the properties of the renewal function of the descending ladder heights of the random walk associated to the branching random walk with critical parameter θ^*. More precisely, let us set $(S_n, n \geq 0)$ the random walk with step distribution satisfying

$$\mathbb{E}(f(S_1)) = \mathbb{E}\left(f(\kappa(\theta^*) - \theta^* X(u))e^{\theta^* X(u) - \kappa(\theta^*)}\right).$$

We remark that the fact that θ^* is the critical parameter implies that $\mathbb{E}(S_1) = 0$, so the random walk S is centred. Additionally, condition (2.20) is equivalent to saying that $\sigma^2 = \mathbb{V}\mathrm{ar}(S_1) < \infty$.

The renewal function of the descending ladder heights of the random walk S is then defined as

$$U(x) := \mu^{-1} \sum_{k \geq 0} \mathbb{P}(-S_{\tau_k} < x), \quad \text{with } \mu = -\mathbb{E}(S_{\tau_1}),$$

with τ_k defined recursively by $\tau_0 = 0$ and $\tau_{k+1} = \inf\{j > \tau_k : S_j \leq S_{\tau_k}\}$ the $(k+1)$st time at which the random walk S attains its running minimum. By the duality lemma, it can be observed that

$$\forall x \in \mathbb{R},\ U(x) = x - \mathbb{E}_x(S_\tau),$$

where $\tau = \inf\{n \in \mathbb{N} : S_n \leq 0\}$ and $\mathbb{P}_x$ satisfies $\mathbb{P}_x(S_0 = x) = 1$.

It is well-known that under assumption (2.20), the function U satisfies $U(x) \sim x$ as $x \to \infty$ and $U(x) = \mathbb{E}(U(x + S_1))$ for all $x > 0$. Note that U is the unique such function satisfying $U(x) = 0$ for $x < 0$, i.e. U is the unique (up to a multiplicative constant) harmonic function of the killed random walk with at most linear growth (see [M27] or the forthcoming Proposition 3.B).

The existence of the limit of the derivative martingale is then associated with the uniform integrability of the martingale

$$D_n^{(\alpha)} := \sum_{|u|=n} U(Y(u) + \alpha) \mathbf{1}_{\{\min_{k \leq n} Y(u_k) + \alpha \geq 0\}} e^{-Y(u)},$$

with Y the branching random walk in the boundary case associated to X. The martingales $D^{(\alpha)}$ are often called the truncated derivative martingales, for self-explanatory reasons.

More precisely, it can be show that for any (and for all) $\alpha > 0$, the martingale $D^{(\alpha)}$ is uniformly integrable if and only if condition (2.21) is satisfied. We then note that

$$-\inf_{u \in \mathbb{U}} Y(u) = \theta^* \sup_{u \in \mathbb{U}} (X(u) - v|u|) < \infty \quad \text{a.s.},$$

by (1.20). Therefore, almost surely there exists $\alpha > 0$ so that no particle u goes above $v|u| + \alpha$. As there exists $c > 0$ such that $x \leq U(x) \leq x + c$ for all $x \geq 0$, we conclude that almost surely, for all α large enough and $n \in \mathbb{N}$, we have

$$D_n^{(\alpha)} \leq Z_n \leq D_n^{(\alpha)} + cW_n.$$

Hence, letting $n \to \infty$ we conclude that $\lim_{\alpha \to \infty} D_\infty^{(\alpha)} = \lim_{n \to \infty} Z_n$ a.s., proving the existence and the non-degeneracy of the derivative martingale.

The limit of the derivative martingale, under assumptions (2.20) and (2.21) gives an accurate estimation of the number of particles growing at speed v in the branching random walks. For example, we can cite the following extension of Theorem 2.1 to the derivative martingale, proved by Madaule [135].

Theorem 2.13 (Madaule 2016)**.** *Under assumption $\mathcal{A}$, for all continuous bounded function f, we have*

$$\lim_{n\to\infty} \sum_{|u|=n} (n\kappa(\theta^*) - \theta^* X(u)) e^{\theta^* X(u) - n\kappa(\theta)} f\left(\frac{n\kappa(\theta^*) - \theta^* X(u)}{\sigma\sqrt{n}}\right) = Z_\infty \mathbb{E}\left(f(M)\right),$$

with M a random variable with the Rayleigh distribution.

Madaule [135] also proved a trajectorial version of Theorem 2.13, in the same spirit as (2.4), that for any continuous bounded function f, under appropriate integrability conditions

$$\lim_{n\to\infty} \sum_{|u|=n} Y(u) e^{-Y(u)} f\left(\frac{Y(u_{\lfloor nt \rfloor}, t \in [0,1])}{\sigma\sqrt{n}}\right) = Z_\infty \mathbb{E}\left(f(M_t, t \in [0,1])\right), \tag{2.22}$$

where Y is a branching random walk in the boundary case, $(M_t, t \in [0,1])$ is a Brownian meander and one can rewrite $\sigma^2 = \mathbb{E}\left(\sum_{|u|=1} Y(u)^2 e^{-Y(u)}\right)$. Pain [155] extended this result to estimating the number of particles ending in the interval $[vn - a_n, vn]$. This number is well-approached by a deterministic asymptotic rate, depending on a_n, multiplied by Z_∞ as long as $\log n \ll a_n \leq n^{1/2}$.

The relationship between the additives and the derivative martingales have been the subject of a large literature. In particular, Madaule [135] proved, under some integrability conditions that

$$\lim_{\theta\to\theta^*} \frac{W_\infty(\theta)}{\theta - \theta^*} = -2Z_\infty. \tag{2.23}$$

This result shows an absence of uniformity in the convergence of the family of martingales $(W_n(\theta), \theta \in [0, \theta^*])$, contrasting with the result of Theorem 2.1 which proves the uniform convergence on compact subsets of $(0, \theta^*)$ of that family.

The derivative martingale can also be used as a measure of the decay rate of the additive martingale $(W_n(\theta^*), n \geq 0)$ with critical parameter. These results are known in the literature as the Seneta-Heyde norming for the branching random walk, after the pioneering work of Seneta [170] and Heyde [102] on the martingale of BGW processes. Partial answers to this question were provided by Biggins and Kyprianou [46, 47] and Hu and Shi [105]. A proof of this result under optimal integrability conditions were obtained by Aïdékon and Shi [9].

Theorem 2.14 (Aïdékon and Shi 2014)**.** *Under assumption $\mathcal{A}$, we have*

$$\lim_{n\to\infty} n^{1/2} W_n(\theta^*) = \sqrt{\frac{2}{\pi\sigma^2}} Z_\infty \text{ in probability.}$$

Using the branching property of the branching random walk (and that $W_n(\theta^*) \to 0$ a.s. as $n \to \infty$), it can be checked that, the limit Z_∞ of the derivative martingale is a non-degenerate solution of the equation in distribution

$$Z_\infty \overset{(d)}{=} \sum_{|u|=1} e^{\theta^* X(u) - \kappa(\theta^*)} Z_\infty^{(u)}, \tag{2.24}$$

with $(Z_\infty^{(u)}, |u| = 1)$ i.i.d. copies of Z_∞. This equality in distribution extends (2.2) to branching random walks in the boundary case.

In [M29], Buraczewski, Iksanov and Mallein took interest in the rate of convergence of Z_n to Z_∞, i.e. in the asymptotic behaviour of $Z_\infty - Z_n$. By (2.24), it can be noted that

$$Z_\infty - Z_n \overset{(d)}{=} \sum_{|u|=n} e^{\theta^* X(u) - n\kappa(\theta^*)} \left(Z_\infty^{(u)} - (n\kappa(\theta^*) - \theta^* X(u)) \right),$$

so $Z_\infty - Z_n$ is a weighted sum of independent random variables. It should therefore be no surprise that a stable distribution appears in the fluctuations of that process.

The rate of convergence of the derivative martingale were studied by Maillard and Pain [141] for the branching Brownian motion under an $L(\log L)^3$-type integrability condition, which is believed to be optimal. This study was motivated by the observations of Mueller and Munier [150] on numerical simulations for this process. We refer to Section 2.3.2 for more details on these results.

The first result of [M29] is a general estimate of the tail of the random variable Z_∞.

Proposition 2.E (Buraczewski, Iksanov and Mallein (2021))**.** *Under assumption $\mathcal{A}$, the derivative martingale of the branching random walk satisfies*

$$\mathbb{E}\left(Z_\infty \mathbf{1}_{\{Z_\infty \leq x\}} \right) \sim \log x, \quad \text{as } x \to \infty. \tag{2.25}$$

In particular, we note that Z_∞ is in the domain of attraction of the spectrally positive stable law of exponent 1.

A more precise estimate on the tail of Z_∞, which is used to prove the convergence in law of the rescaled fluctuations of $Z_\infty - Z_n$ holds under some restrictive integrability condition, that we now introduce. With Y the branching random walk in the boundary case associated to X, define

$$W_1^+ = \sum_{|u|=1} e^{-Y(u)} \mathbf{1}_{\{Y(u)\geq 0\}}, \quad W_1^- = \sum_{|u|=1} e^{-Y(u)} \mathbf{1}_{\{Y(u)<0\}}, \quad \widetilde{W}_1 = \sum_{|u|=1} Y(u)_+ e^{-Y(u)}.$$

and $Y_{\min} = \min_{|u|=1} Y(u)$. We introduce the integrability conditions

$$\mathbb{E}\left(W_1^+ \left(\log_+ W_1^+ \right)^3 \right) + \mathbb{E}\left(\widetilde{W}_1 \left(\log_+ \widetilde{W}_1 \right)^2 \right) + \mathbb{E}\left(\sum_{|u|=n} (Y(u)_-)^3 e^{-Y(u)} \right) < \infty, \tag{2.26}$$

and

$$\exists C > 0 : \mathbb{E}\left(W_1^- \left(\log W_1^- \right)^3 \mathbf{1}_{\left\{\sum_{|u|=1} (1+Y(u)-Y_{\min}) e^{Y_{\min}-Y(u)} \mathbf{1}_{\{Y(u)<0\}} > C\right\}} \right) < \infty. \tag{2.27}$$

The following result then holds.

Theorem 2.F (Buraczewski, Iksanov and Mallein (2021))**.** *Let X be a branching random walk satisfying assumption $\mathcal{A}$. We assume that X is non-lattice, i.e.*

$$\mathbb{P}(X(u) \in a\mathbb{Z} + b) < 1 \quad \text{for all } a > 0,\ b \in \mathbb{R}.$$

There exists $c > 0$ such that

$$\mathbb{E}\left(Z_\infty \mathbf{1}_{\{Z_\infty \leq x\}} \right) = \log x + c + o(1) \quad \text{as } x \to \infty \tag{2.28}$$

if and only if (2.26) and (2.27) hold.

In particular, if (2.26) and (2.27) hold, then $\mathbb{P}(Z_\infty > x) \sim \frac{1}{x}$ as $x \to \infty$.

This result is mainly based on a tight estimate of the Laplace transform of Z_∞ and the recursive equation (2.24). Indeed, we observe that setting $\varphi : \lambda \in [0, \infty) \mapsto \mathbb{E}\left(e^{-\lambda Z_\infty}\right)$, the equation in distribution satisfied by Z_∞ implies that

$$\varphi(\lambda) := \mathbb{E}\left(\prod_{|u|=1} \varphi\left(\lambda e^{-Y(u)}\right)\right).$$

Therefore, using Jensen inequality, we obtain that the function

$$D : x \in \mathbb{R} \mapsto e^x \left(1 - \varphi(e^{-x})\right)$$

is a sub-harmonic function of the random walk (S_n) with step distribution satisfying $\mathbb{E}(f(S_1)) = \mathbb{E}(f(Y(u))e^{-Y(u)})$, obtained by the many-to-one lemma.

In order to give estimates on the asymptotic behaviour of $D(x)$ as $x \to \infty$, which will, by Tauberian theorem, translate in an estimate on the tail of Z_∞, we characterize the set of sub-harmonic functions of the killed random walk with at most linear growth. More precisely, let $g : (0, \infty) \to [0, \infty)$ be a càdlàg function and $h : (-\infty, 0] \to \mathbb{R}$ a right-continuous bounded function. We consider the right-continuous functions f satisfying

$$\begin{cases} f(x) = \mathbb{E}(f(x + S_1)) - g(x) & \text{if } x > 0, \\ f(x) = h(x) & \text{if } x < 0, \\ \sup_{x>0} f(x)/(1 + |x|) < \infty. \end{cases} \tag{2.29}$$

We observe that if f is a solution of this equation, then $\mathbb{E}(f(x + S_1)) \geq f(x)$ for all $x \geq 0$, hence f is a sub-harmonic function for the random walk S. These functions can be characterized in the following fashion.

Theorem 2.G (Buraczewski, Iksanov and Mallein (2021)). *If there exists a function f satisfying (2.29), then for all $x > 0$, we have*

$$\mathbb{E}_x\left(\sum_{k=0}^{\tau-1} g(S_k)\right) < \infty, \tag{2.30}$$

where $\tau = \inf\{k \geq 1 : S_k < 0\}$ and $\mathbb{P}_x(S_0 = x) = 1$.

Conversely, if (2.30) holds and g is direct Riemann integrable on $\mathbb{R}_+$, then for any solution f of (2.29), there exists $c \in \mathbb{R}$ such that

$$\forall x > 0, \quad f(x) = cU(x) + \mathbb{E}_x\left(h(S_\tau)\right) - \mathbb{E}_x\left(\sum_{k=0}^{\tau-1} g(S_k)\right), \tag{2.31}$$

where $U(x) = \lim_{y\to\infty} y\mathbb{P}_x(\exists k < \tau : S_k > y)$ is the renewal function associated with the negative ladder heights of the random walk S.

This result is essentially obtained by using the optional stopping theorem at the random time $\tau \wedge \inf\{n \geq 0 : S_n > y\}$ to the martingale $(f(S_n) - \sum_{k=0}^{n-1} g(S_k), n \geq 0)$, and an associated characterization of the harmonic functions of the killed random walk obtained by Alsmeyer and Mallein [M27], that we will discuss in the next chapter.

Thanks to Theorem 2.G and the observation that D satisfies (2.29) for some functions g and h, we obtain immediately its decomposition (2.31), from which we deduce that $D(x) \sim x$, with the constant c being identified using the branching property (2.24). This immediately implies Proposition 2.E.

Additionally, (2.31) allows us to obtain an integral test to verify if $D(x) = x+c+o(1)$, based on verifying that

$$\lim_{x\to\infty} \mathbb{E}_x\left(\sum_{k=0}^{\tau-1} g(S_k)\right) \text{ exists } \iff \int_0^\infty yg(y)\mathrm{d}y < \infty.$$

It allows us to describe the necessary and sufficient conditions obtained in Theorem 2.F. Finally, using the tight estimate on the tail of Z_∞, it is a standard computation to deduce the following central limit theorem for the convergence of Z_n.

Theorem 2.H (Buraczewski, Iksanov and Mallein (2021)). *Let X be a non-lattice branching random walk satisfying assumption $\mathcal{A}$ and equations (2.26) and (2.27). We have*

$$\lim_{n\to\infty} n^{1/2}(Z_\infty - Z_n + (2^{-1}\log n)W_n(\theta_*)) = Z_\infty L \quad \textit{in law,}$$

where L has a 1-stable distribution with the generating triple $\left(\sqrt{\frac{2(c+1-\gamma)}{\pi\sigma^2}}, \sqrt{\frac{\pi}{2\sigma^2}}, 1\right)$ and is independent of Z_∞, where γ is the Euler-Mascheroni constant and c the same constant as in (2.28). In particular, the law of L is spectrally positive.

The derivative martingale of the branching random walk is an essential tool in the study of the asymptotic behaviour of extremal particles in the process. Theorem 2.13 expresses that this martingale gives an accurate estimate for the number of particles within distance $n^{1/2}$ from position nv at time n. We will show in Chapter 4 that Z_∞ also influences the number of particles reaching the highest positions in this process.

We mention below an extension of Theorem 2.12 to branching Lévy processes, i.e. determining necessary and sufficient conditions for the non-degeneracy of the derivative martingale of branching Lévy processes. We then give a quick overview of some results and applications of the derivative martingale to branching Brownian motions, in particular a multidimensional extension of this convergence.

2.3.1 The derivative martingale of the branching Lévy process

Similarly to the branching random walk, we take interest in the derivative martingale for the branching Lévy process. We recall that if X is a branching Lévy process with characteristics (σ^2, a, Λ), for all $\theta > 0$ we have

$$\kappa(\theta) = \log \mathbb{E}\left(\sum_{u\in\mathcal{N}_1} e^{\theta X_t(u)}\right) = \frac{\sigma^2\theta^2}{2} + a\theta + \int_{\mathcal{P}(\mathbb{R})} \left(\sum_{j=1}^n e^{\theta x_j} - 1 - \theta x_1 \mathbf{1}_{\{|x_1|<1\}}\right) \Lambda(\mathrm{d}\mathbf{x}).$$

We assume in this section the existence of a parameter θ^* such that $\theta^*\kappa'(\theta^*) - \kappa(\theta^*) = 0$.

In the same way as for the branching random walk, we can define a branching Lévy process in the boundary case as a process satisfying the above equation with $\theta^* = 1$ and $\kappa(\theta^*) = 0$. In other words, $\kappa(1) = \kappa'(1) = 0$. These conditions can be explicitly restated in terms of the characteristic triplet of the branching Lévy process.

Definition 2.15. A *branching Lévy process in the boundary case* is a process Y such that $-Y$ is a branching Lévy process with characteristics (σ^2, a, Λ) satisfying

$$\sigma^2 = 2\int_{\mathcal{P}(\mathbb{R})} \left(\sum_{j=1}^\infty \mathbf{1}_{\{x_j > -\infty\}}(1 - x_j)e^{x_j}\right) - 1\ \Lambda(\mathrm{d}\mathbf{x}) = 0,$$

$$a = \int_{\mathcal{P}(\mathbb{R})} \left(\sum_{j=1}^\infty \mathbf{1}_{\{x_j > -\infty\}}(2 - x_j)e^{x_j}\right) - 2 - x_1\mathbf{1}_{\{|x_1|<1\}}\Lambda(\mathrm{d}\mathbf{x}) = 0.$$

We call $(\sigma^2, -a, \overline{\Lambda})$ the characteristics of the branching Lévy process in the boundary case Y, with $\overline{\Lambda}$ the image of Λ by the application $\mathbf{x} \mapsto -\mathbf{x}$.

Remark that similarly to branching random walks, it is traditional to consider the process Y (with a possible accumulation of positions of particles at ∞ at any finite time) rather than the branching Lévy process $-Y$ when working with branching Lévy processes in the boundary case. Indeed, under these conditions particles in $-Y$ will typically go to $-\infty$, so working with Y allows to work with particles with a typically positive position. Up to a slight abuse of notation, we consider in this section that the set $\mathcal{P}(\mathbb{R})$ will be the set of Radon point measures with a *smallest* atom, which is the support set of the characteristic Λ of a branching Lévy process in the boundary case.

The study of the convergence of the derivative martingale of a branching Lévy process in the boundary case, defined by

$$Z_t := \sum_{u \in \mathcal{N}_t} Y_t(u) e^{-Y_t(u)}$$

was started by Shi and Watson [171], in which the predictable genealogy of particles and the spine decomposition for branching Lévy processes are introduced. A necessary and sufficient condition was obtained by Mallein and Shi [M36], which gives optimal condition on the characteristic measure Λ under the assumption $\kappa''(1) < \infty$. This result should be thought of as an analogue, in branching Lévy processes settings, of Theorem 2.12.

Theorem 2.I (Mallein and Shi (2022)). *Let Y be a branching random walk in the boundary case with characteristics (σ^2, a, Λ). We assume that*

$$\int_{\mathcal{P}(\mathbb{R})} \sum_{j=1}^{\infty} \mathbf{1}_{\{x_j < \infty\}} x_j^2 e^{-x_j} \ \Lambda(\mathrm{d}\mathbf{x}) < \infty. \tag{2.32}$$

The martingale $(Z_t, t \geq 0)$ converges a.s. to a non-degenerate non-negative limit Z_∞ if and only if

$$\int_{\mathcal{P}(\mathbb{R})} Y(\mathbf{x}) \log_+ (Y(\mathbf{x}) - 1)^2 + \widetilde{Y}(\mathbf{x}) \log_+ (\widetilde{Y}(\mathbf{x}) - 1)^2 \Lambda(\mathrm{d}\mathbf{x}), \tag{2.33}$$

where $Y(\mathbf{x}) = \sum_{j=1}^{\infty} \mathbf{1}_{\{x_j < \infty\}} e^{-x_j}$ and $\widetilde{Y}(\mathbf{x}) = \sum_{j=1}^{\infty} \mathbf{1}_{\{x_j \in [0,\infty)\}} x_j e^{-x_j}$.

Similarly to Theorem 2.12, this result is based on the connection of the derivative martingale with the following non-negative martingale

$$D_t^\alpha := \sum_{u \in \mathcal{N}_t} U(Y_t(u) + \alpha) \mathbf{1}_{\{\inf_{s \leq t} Y_s(u) \geq -\alpha\}} e^{-Y_t(u)},$$

with U the renewal function associated to the (weakly) decreasing ladder heights of the Lévy process ξ satisfying for all measurable bounded function f,

$$\forall t \geq 0, \ \mathbb{E}(f(\xi_t)) = \mathbb{E}\left(\sum_{u \in \mathcal{N}_t} f(Y_t(u)) e^{-Y_t(u)} \right).$$

Using that U is asymptotically linear under assumption (2.32), we deduce that Z_t converges a.s. to a non-degenerate limit if and only if D_t^α is uniformly integrable for some (and thus all) $\alpha > 0$.

Using the spine decomposition to describe the law of the branching Lévy process biased by the martingale D^α, Mallein and Shi proved that the uniform integrability of D^α is equivalent to

$$\int_0^\infty f_\Lambda(\xi_s)\mathrm{d}s < \infty \quad \mathbb{P}^\uparrow\text{-a.s.}$$

where $\mathbb{P}^\uparrow$ is the law of the Doob's h-transform of ξ by U, which can be thought off as the law of ξ conditioned on never becoming negative, and f_Λ is a numeric decreasing positive function on $\mathbb{R}_+$, that depend on Λ. Theorem 2.I is then a consequence of the following integral test for the finiteness of perpetuities of Lévy processes conditioned to stay positive.

Proposition 2.J (Mallein and Shi (2021))**.** *Let ξ be a centred Lévy process with finite variance, and denote by $\mathbb{P}^\uparrow$ the law of this process conditioned to stay positive. For all measurable bounded function $f : [0,\infty) \to [0,\infty)$ eventually non-increasing, we have*

$$\int_0^\infty f(\xi_s)\mathrm{d}s < \infty \;\; \mathbb{P}^\uparrow\text{-a.s.} \quad \Longleftrightarrow \quad \int_0^\infty y f(y)\mathrm{d}y < \infty$$
$$\int_0^\infty f(\xi_s)\mathrm{d}s = \infty \;\; \mathbb{P}^\uparrow\text{-a.s.} \quad \Longleftrightarrow \quad \int_0^\infty y f(y)\mathrm{d}y = \infty$$

This result is based on the work of Baguley, Döring and Kyprianou [22] providing a general result for the 0-1 law of perpetuities of Markov processes. Using Proposition 2.J, Theorem 2.I is then obtained by connecting the condition $\int y f(y)\mathrm{d}y$ with the integrability condition (2.33).

2.3.2 The derivative martingale of the branching Brownian motion

The study of the derivative martingale in branching Brownian motion has a much longer history than the one of branching random walks and branching Lévy processes, due to the link between the branching Brownian motion and the F-KPP reaction diffusion equation obtained by McKean [145]. In fact, in that article, McKean uses the limit of additive martingales of the branching Brownian motion with critical parameter $\sqrt{2(m-1)}$ to describe the travelling-wave solutions to the F-KPP equation with the formula (2.16). He later observe that the martingale $(W_t(\sqrt{2(m-1)}))$ being non-uniformly integrable, this formula could not hold for that parameter [146].

To overcome this problem, Lalley and Sellke [127] introduced and prove the almost sure convergence of the derivative martingale of the branching Brownian motion, defined as

$$Z_t := \sum_{u \in \mathcal{N}_t} (\sqrt{2(m-1)}t - X_t(u)) e^{\sqrt{2(m-1)}(X_t(u) - \sqrt{2(m-1)}t)},$$

to a non-degenerate, non-negative limit Z_∞. They use the limit Z_∞ to describe the travelling-wave equation of the F-KPP equation with minimal speed

$$w(x) := 1 - \mathbb{E}\left(\exp\left(e^{-\theta x} Z_\infty\right)\right).$$

The function $u(t,x) = w(x - \sqrt{2(m-1)}t)$ is a solution to F-KPP, and there is no non-degenerate solution to this equation of the form $(t,x) \mapsto \varphi(x - vt)$ with $v < \sqrt{2(m-1)}$.

In particular, using the convergence obtained by Bramson [57] for the solution of the F-KPP equation with Heavyside initial condition, the result of Lalley and Sellke readily implies that for all $y \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{P}\left(M_t \leq \sqrt{2(m-1)}t - \frac{3}{2\sqrt{2(m-1)}} \log t + y \right) = \mathbb{E}\left(\exp\left(-c_\star Z_\infty e^{-\sqrt{2(m-1)}y}\right)\right). \tag{2.34}$$

In other words, $M_t - m_t$ converges in law to $G + \log Z_\infty$, with G a Gumbel random variable independent of the branching Brownian motion. This result is an other example of the deep connections existing between the derivative martingale and the extremal values of the branching Brownian motion.

The optimal integrability conditions on the reproduction law ν of the branching Brownian motion, guaranteeing that the limit of the derivative martingale of the branching Brownian motion is non-degenerate were obtained by Yang and Ren [180]. It can be seen as an analogue of Theorem 2.12, or a particular case of Theorem 2.I.

Theorem 2.16 (Yang and Ren 2011). *Let $(Z_t, t \geq 0)$ be the derivative martingale associated to a (supercritical) branching Brownian motion with reproduction law ν. The derivative martingale (Z_t) converges a.s. to a non-degenerate limit Z_∞ if and only if $\sum_{k=1}^\infty \nu(k) k (\log k)^2 < \infty$.*

The speed of convergence of the derivative martingale towards its limit was studied by Maillard and Pain [141], who proved the following functional central limit theorem for the derivative martingale of the branching Brownian motion.

Theorem 2.17 (Maillard and Pain 2019). *Let $(Z_t, t \geq 0)$ be the derivative martingale of a (supercritical) branching Brownian motion with a reproduction law ν satisfying*

$$\sum_{k=1}^\infty \nu(k) k (\log k)^3 < \infty.$$

We have $\lim_{t\to\infty} \left(\sqrt{t}\,(Z_\infty - Z_{at}) + \frac{\log t}{\sqrt{2\pi a t}} Z_\infty, a \geq 1 \right) = \left(S_{Z_\infty/\sqrt{a}}, a \geq 1 \right)$, in finite dimensional distributions, where S is a spectrally positive 1-stable Lévy process.

In particular, the convergence of the 1-dimensional marginal with $a = 1$ is the analogue of Theorem 2.H in the context of branching Brownian motion. As a result, it can be deduced that the $L \log L^3$ integrability condition is optimal for Theorem 2.17 to hold. This theorem is in particular

Maillard [139] obtained precise estimates on the tail of the derivative martingale of the branching Brownian motion, which are analogue to Theorem 2.F. More precisely, the following results hold.

Proposition 2.18 (Maillard 2012). *Let Z_∞ be the limit of the derivative martingale of a branching Brownian motion with reproduction law ν. We have*

$$\mathbb{P}(Z_\infty > x) \sim_{x\to\infty} \frac{1}{x} \iff \sum_{k=1}^\infty k (\log k)^2 \nu(k) < \infty \quad \text{and}$$

$$\mathbb{E}(Z_\infty \mathbf{1}_{\{Z_\infty \leq x\}}) = \log x + c + o(1) \text{ as } x \to \infty \iff \sum_{k=1}^\infty k (\log k)^3 \nu(k) < \infty.$$

Multidimensional branching Brownian motion Before closing this chapter on additive martingales of spatial branching processes, we mention the problems related to the extension of the convergence of additive martingales of the branching Brownian motion to multidimensional settings. In some sense, this question is already present when considering additive martingales with complex parameters which was studied by various authors, see e.g. [44, 122, 112] in branching random walk settings, or [138, 101] for questions related to the convergence of martingales of the branching Brownian motion with complex parameters.

In particular, we remark that Biggins [44] obtained the convergence of additive martingales in multidimensional branching random walks uniformly in compact subsets of the set of parameters for which the martingales are uniformly convergent. However, uniform convergence in all directions of the derivative martingale cannot hold in general settings, due to the existence of exceptional directions in which this martingale diverges to $-\infty$, due to the presence of anomalously high particles, that was obtained in [M5].

More precisely, let $d \geq 2$, we consider a standard binary branching Brownian motion $(X_t(u), u \in \mathcal{N}_t)_{t\geq 0}$ in dimension d. It is worth noting that for each $\varphi \in \mathbb{S}^{d-1}$, the process $(X_t(u).\varphi, u \in \mathcal{N}_t)_{t\geq 0}$ is a unidimensional branching Brownian motion. Therefore, for each $\varphi \in \mathbb{S}^{d-1}$, the martingale

$$Z_t(\varphi) := \sum_{u\in\mathcal{N}_t} (\sqrt{2}t - X_t(u).\varphi)e^{\sqrt{2}(X_t(u).\varphi-\sqrt{2}t)}$$

converges almost surely to a non-degenerate limit written $Z_\infty(\varphi)$. However, using results of Mallein [M5], it can be straightforwardly proved that at least in dimension $d \geq 3$,

$$\liminf_{t\to\infty} \inf_{\varphi\in\mathbb{S}^{d-1}} Z_t(\varphi) = -\infty \quad \text{a.s.}$$

so the convergence of Z_t to Z_∞ cannot hold uniformly in φ.

Recalling that the derivative martingale encodes information on the asymptotic behaviour of extremal particles, if one wishes to describe particles in the multidimensional branching Brownian motions which went further from the origin, it becomes necessary to obtain some simultaneous convergence result for the martingale $(Z_t(\varphi), \varphi \in \mathbb{S}^{d-1})_{t\geq 0}$. Stasiński, Berestycki and Mallein in [M30] proved that almost surely, $Z_t(\varphi)$ converges to $Z_\infty(\varphi)$ as $t \to \infty$ in almost every direction.

Theorem 2.K (Stasiński, Berestycki and Mallein (2021))**.** *There exists almost surely a Borelian subset A of $\mathbb{S}^{d-1}$ of full Lebesgue measure such that*

$$\forall \varphi \in A,\ \lim_{t\to\infty} Z_t(\varphi) = Z_\infty(\varphi).$$

Moreover, for all continuous bounded function f, we have

$$\lim_{t\to\infty} \int_{\mathbb{S}^{d-1}} f(\varphi)Z_t(\varphi)\sigma(\mathrm{d}\varphi) = \int_A f(\varphi)Z_\infty(\varphi)\sigma(\mathrm{d}\varphi) \quad \textit{a.s.}$$

In other words, this theorem proves the almost sure convergence of $Z_t(\varphi)\sigma(\mathrm{d}\varphi)$ as a random measure, and that the limiting distribution has density with respect to the Lebesgue measure equal, up to a subset of null measure, to the limit of the derivative martingale in each fixed direction. Although this result was only proved for binary branching Brownian motions, minor modifications of the argument would allow to prove it for any branching Brownian motion whose reproduction law satisfies an $L(\log L)^2$ integrability condition.

It remains currently an open problem to describe the regularity of the function Z_∞ on the sphere $\mathbb{S}^{d-1}$. It is believed that this function should become more irregular in larger dimensions. The question of characterizing the random variable Z_∞ up to multiplications by a constant by an analogue to the equality in law (2.24) remains open as well.

CHAPTER 3

Fixed point of the smoothing transform and branching stable point measures

"On peut braver les lois humaines, mais non résister aux lois naturelles."

Jules Verne – Vingt Mille Lieues sous les mers (1870).

Summary.

We take interest in this chapter in the identification and the expression of the fixed points of the so-called smoothing transform. A function f is said to be a fixed point of the smoothing transform associated to the decreasing null sequence $(T_n, n \geq 1)$ if

$$\forall t > 0,\ \ f(t) = \mathbb{E}\left(\prod_{n=1}^{\infty} f(tT_j) \right).$$

We aim at describing the set of solutions to this equation, in particular its links with additive martingales of the branching random walk. This identification was obtained by Alsmeyer, Biggins and Meiners [11]. We present here a simple proof for this result, also working in the boundary case, obtained by Alsmeyer and Mallein [M27].

Using this characterization of the fixed points of the smoothing transform, we are able to describe random point measures that are stable for the branching convolution equation, i.e. satisfying (1.4) under some condition. This result, obtained by Maillard and Mallein [M37] proves that if the extremal process of the branching random walk converges, its limit has to be a shifted decorated Poisson point process with prescribed intensity.

The smoothing transform is a regularization transform defined as a mapping on the space of non-negative measurable functions bounded by 1. More precisely, given a non-increasing null sequence $(T_j, j \geq 1)$ of non-negative random variables, the associated smoothing transform is defined by

$$\mathcal{F}_T : f \mapsto \left(t \mapsto \mathbb{E}\left(\prod_{j=1}^{\infty} f(tT_j) \right) \right).$$

A fixed point of the smoothing transform is then defined as a measurable function f satisfying

$$\forall t \geq 0, \ f(t) = \mathbb{E}\left(\prod_{j=1}^{\infty} f(tT_j) \right). \tag{3.1}$$

The problem of identifying fixed points of the smoothing transform belonging to a certain class of functions, such as Laplace transforms of non-negative random variables or right-continuous non-increasing functions have been the subject of a large literature, among which can be mentioned [116, 85, 132, 109, 47] and the references therein. An overview of the literature, and optimal results on the existence of fixed points of the smoothing transform with f taking values in $[0,1]$ can be found in [11].

We note that if f is the Laplace transform of a non-negative random variable X and a fixed point of the smoothing transform, the smoothing transform equation (3.1) can be rewritten as

$$R \overset{(d)}{=} \sum_{j=1}^{\infty} T_j R^{(j)}, \tag{3.2}$$

where $(R^{(j)}, j \geq 1)$ are i.i.d. copies of R that are further independent of $(T_j, j \geq 1)$. This equation is sometimes called the stochastic fixed point equation. Similarly, if f is a decreasing right-continuous function with $f(0) = 1$, there exists a non-negative random variable Y such that for all $x \geq 0$, $\mathbb{P}(Y \geq x) = f(x)$. Then Y is a fixed point of the equation

$$S \overset{(d)}{=} \min_{j \in \mathbb{N}: T_j > 0} \frac{S^{(j)}}{T_j}, \tag{3.3}$$

with $(S^{(j)}, j \geq 1)$ i.i.d. copies of S that are further independent of $(T_j, j \geq 1)$.

Remark 3.1. Various generalization of the smoothing transforms have been considered: real-valued random variables satisfying (3.2) in [14], complex-valued random variables in [147], matrix-valued solutions of the smoothing transform [79]. However, in this chapter we will only focus on non-negative solutions to (3.2).

The work of Kahane and Peyrière [116] on fixed points of the smoothing transform is related to Mandelbrot's cascades and the study of turbulence models. In view of (3.2), Durrett and Liggett [85] took interest in these fixed points interpreted as analogue, in branching random walk settings, of an invariant distribution for particle systems on the real line. This observation is made rigorous by Maillard and Mallein [M37] (see forthcoming Section 3.3) Moreover, equation (3.3) motivates the study of the smoothing transform when considering the asymptotic behaviour of extremal particles in a branching random walk, as the limit in distribution of the centred maximum has to satisfy an equation in distribution similar to $-\log S$.

Fixed points of the smoothing transforms were also introduced to study the asymptotic of objects exhibiting a recursive structure, such as recursive algorithm. For example, Rösler

[167, 165] characterized the limit X of the normalized number of comparison made by the Quicksort algorithm as solution to the distributional equation

$$R \overset{(d)}{=} UR^{(1)} + (1-U)R^{(2)} + g(U),$$

where $R^{(1)}, R^{(2)}$ are two independent copies of R, U an independent uniform random variable and $g : t \mapsto 1 + 2t\log(t) + 2(1-t)\log(1-t)$. Then, writing φ the characteristic function of R, it is a straightforward computation to observe that

$$\forall t \geq 0,\ \ \varphi(t) = \mathbb{E}\left(\varphi(Ut)\varphi((1-U)t)e^{itg(U)}\right).$$

Hence the law of R is related to a fixed point of the smoothing transform, we refer to Alsmeyer and Dyszewski [12] for more details on this relationship.

The smoothing transform, and in particular its iterations, are deeply related with branching random walks in the following way. We denote by X a branching random walk with reproduction law given by

$$\sum_{j=1}^{\infty} \mathbf{1}_{\{T_j>0\}} \delta_{\log T_j}.$$

For all suitable function f, we observe that

$$\mathcal{F}_T(f) : t \mapsto \mathbb{E}\left(\prod_{|u|=1} \mathbb{E}\left(te^{X(u)}\right)\right),$$

hence, by a trivial recursion, for any $n \in \mathbb{N}$,

$$\mathcal{F}_T^n(f) : t \mapsto \mathbb{E}\left(\prod_{|u|=n} f\left(te^{X(u)}\right)\right).$$

In particular, a fixed point of the smoothing transform will satisfy, for all $n \in \mathbb{N}$

$$\forall t > 0,\ \ f(t) = \mathbb{E}\left(\prod_{|u|=n} f\left(te^{X(u)}\right)\right). \tag{3.4}$$

Existence, uniqueness and representation of the fixed points of the smoothing transform mostly under the following standard assumptions. First, it is assumed that

$$\mathbb{E}\left(\sum_{j=1}^{\infty} \mathbf{1}_{\{T_j>0\}}\right) > 1, \tag{3.5}$$

which is equivalent to the supercriticality of the associated branching random walk. It is a straightforward consequence of (3.4) that the only solution to (3.1) is the constant $f \equiv 1$ if (3.5) does not hold. Additionally, we remark that if f is a non-negative solution to the fixed point equation, for all $t \geq 0$,

$$f(t) = \mathbb{E}\left(\prod_{|u|=n} f\left(te^{X(u)}\right)\right) \geq \mathbb{P}(X \text{ gets extinct by time } n),$$

using that $\prod_{\emptyset} = 1$. Therefore, any non-negative solution to the fixed point equation is bounded from below by the extinction probability of the branching random walk.

It is then assumed that there exists $\alpha > 0$ such that

$$\mathbb{E}\left(\sum_{j=1}^{\infty} T_j^\alpha\right) = 1 \quad \text{and} \quad \mathbb{E}\left(\sum_{j=1}^{\infty} T_j^\alpha \log T_j\right) \leq 0. \tag{3.6}$$

This second condition implies that $\lim_{n\to\infty} \max_{|u|=n} X(u) = -\infty$ in the associated branching random walk. Hence, by (3.4), the function f is characterized by its asymptotic behaviour as $t \to 0$. We mention that Liu [132] obtained under weaker conditions than (3.6) the existence of non-negative solutions to (3.2), via a truncation argument.

Under condition (3.6), we remark that $\mathbb{E}\left(\sum_{j=1}^{\infty} T_j^\alpha \log T_j\right) = 0$ is equivalent to saying that $\kappa(\alpha) = \kappa'(\alpha) = 0$ for the associated branching random walk, therefore α is the critical parameter. We call this case the *boundary case* of the smoothing transform, while we call *regular case* the situation such that

$$\mathbb{E}\left(\sum_{j=1}^{\infty} T_j^\alpha \log T_j\right) < 0.$$

Construction of fixed points of the smoothing transform in the regular case is done in the next section. This construction is then extended to the boundary case in Section 3.2

3.1 Fixed points of the smoothing transform in the regular case

We consider in this section a smoothing transform in the regular case, i.e. such that

$$\mathbb{E}\left(\sum_{j=1}^{\infty} T_j^\alpha\right) = 1 \quad \text{and} \quad \mathbb{E}\left(\sum_{j=1}^{\infty} T_j^\alpha \log T_j\right) < 0. \tag{3.7}$$

Under these conditions, by Theorem 2.1, the martingale

$$W_n = W_n(\alpha) = \sum_{|u|=n} e^{\alpha X(u)}$$

converges almost surely to a limit W_∞, and this limit is non-degenerate if and only if

$$\mathbb{E}(W_1 \log_+ W_1) = \mathbb{E}\left(\left(\sum_{j=1}^{\infty} T_j^\alpha\right) \log_+ \left(\sum_{j=1}^{\infty} T_j^\alpha\right)\right) < \infty. \tag{3.8}$$

We denote by r the geometric span of the random sequence $(T_j, j \geq 1)$, defined as the largest number larger than 1 such that

$$\mathbb{P}(T_j \in \{r^n : n \in \mathbb{Z}\} \text{ for all } j \geq 1) = 1, \tag{3.9}$$

If no such real number exist, we set $r = 1$. We remark that $(T_j, j \geq 1)$ has geometric span $r > 1$ if and only the branching random walk X takes value in $(\log r)\mathbb{Z}$. We then introduce, for $r > 1$

$$\mathcal{H}_r := \left\{ h : [0,\infty) \to (0,\infty) : \begin{array}{l} \forall t > 0, h(rt) = h(t) \text{ and} \\ t \mapsto h(t)t^\alpha \text{ is non-decreasing} \end{array} \right\}$$

the set of geometrically r-periodic functions such that $t \mapsto t^\alpha h(t)$ is non-decreasing. By convention, $\mathcal{H}_1$ is taken as the set of constant functions. The following result was proved by Alsmeyer, Biggins and Meiners [11].

Theorem 3.2 (Alsmeyer, Biggins, Meiners 2012)**.** *Let $(T_j, j \geq 1)$ be a non-increasing null sequence satisfying (3.5), we assume there exists $\alpha > 0$ such that (3.7) holds. Then, if (3.8) holds and r is the geometric span of $(T_j, j \geq 1)$ the non-increasing fixed points of the smoothing transform (3.1) are the functions f satisfying*

$$\forall t \geq 0, f(t) = \mathbb{E}\left(\exp\left(-h(t)t^\alpha W_\infty\right)\right), \tag{3.10}$$

with h a left-continuous function in $\mathcal{H}_r$.

A simple proof of Theorem 3.2 was obtained by Alsmeyer and Mallein [M27] under the additional condition

$$\mathbb{E}\left(\sum_{j=1}^{\infty} T_j^\alpha \log T_j\right) > -\infty. \tag{3.11}$$

This proof scheme can be extended to work under the boundary case, and allows the identification of the function h as an harmonic bounded function of the Markov chain $(e^{S_n}, n \geq 1)$, with $(S_n, n \geq 1)$ the random walk defined by

$$\mathbb{E}\left(g(S_1)\right) = \mathbb{E}\left(\sum_{|u|=1} e^{\alpha X(u)} g(X(u))\right). \tag{3.12}$$

Remark 3.3. We note that if f is a function of the form given by (3.10), it is a fixed point of the smoothing transform, at it can be immediately seen using the equality in distribution (2.2) satisfied by W_∞.

We give below the main steps of this proof. The first step is to obtain an a priori estimate on the asymptotic behaviour of $f(t)$ as $t \to 0$ for any non-trivial solution to (3.1). This property, called the *tameness* of f, is written as

$$\limsup_{t \to 0} \frac{-\log f(t)}{t^\alpha} < \infty. \tag{3.13}$$

The proof that any non-trivial fixed point of the smoothing transform is tame follows a straightforward proof by contradiction. Assuming that there exists a non-trivial non-tame solution to (3.1), we extract two decreasing null sequences (t_n) and (δ_n) such that

$$\forall n \in \mathbb{N}, \inf_{s \in [t_n \delta_n, t_n]} \frac{-\log f(s)}{s^\alpha} \geq n.$$

Then, using recursively that f is a fixed point of the smoothing transform, we obtain that for all $n \in \mathbb{N}$ and $t > 0$,

$$f(t) \leq \mathbb{E}\left(\prod_{u \in L_n} f(te^{X(u)})\right),$$

where we have set

$$L_n := \left\{u \in \mathbb{U} : te^{X(u)} \in [t_n\delta_n, t_n] \text{ and } te^{X(u_k)} > t_n \text{ for all } k < |u|\right\}.$$

This defines a stopping line, which is a notion introduced by Jagers [114] (see also Chauvin [70] for an analogous notion for branching Brownian motion), that generalizes the notion of stopping time for spatial branching processes. Using the spatial branching property, we obtain

$$f(t) \leq \mathbb{E}\left(e^{-n\sum_{u \in L_n} e^{\alpha X(u)}}\right).$$

Using that (L_n) is a sequence of cutting lines as well as the results of Biggins and Kyprianou [46], by (3.11) we have $\sum_{u \in L_n} e^{\alpha X(u)} \to W_\infty$ a.s. as $n \to \infty$. As a result, we obtain that $f(t) = \mathbb{P}(W_\infty = 0)$ for all $t > 0$, which contradicts that f is non-trivial.

The second step of the proof of Theorem 3.2 is based on harmonic analysis: using that f is a fixed point of the smoothing transform, the process $(\prod_{|u|=n} f(te^{X(u)}), n \geq 0)$ is a bounded martingale, therefore converges a.s. and in L^1 to

$$M(t) := \lim_{n\to\infty} \prod_{|u|=n} f(te^{X(u)}).$$

This limit is sometimes called the *disintegration* of the fixed point f. Using the branching property of the branching random walk, we remark that M satisfies the following functional equation in distribution:

$$(M(t), t \geq 0) \stackrel{(d)}{=} \left(\prod_{|u|=1} M^u(te^{X(u)}), t \geq 0 \right),$$

with $(M^u, |u| = 1)$ i.i.d. copies of M. Using the tameness of f and this equality in distribution, we observe that the function

$$G : x \in \mathbb{R} \mapsto e^{\alpha x} \mathbb{E}\left(-\log M(e^x)\right)$$

is an harmonic bounded function of the random walk S defined in (3.12). By Choquet and Deny's theorem [75], we deduce that $t \mapsto G(\log t) \in \mathcal{H}_r$.

Finally, the third step of the proof consists in the identification of the disintegration, which allows the reconstruction of f. We have

$$\begin{aligned}
\log M(t) &= \lim_{n\to\infty} \mathbb{E}\left(\log M(t) | \mathcal{F}_n\right) \\
&= -\lim_{n\to\infty} \sum_{|u|=n} t^\alpha e^{\alpha X(u)} G(X(u) + \log t) = W_\infty G(-\log t),
\end{aligned}$$

hence using that $f(t) = \mathbb{E}(e^{\log M(t)})$, the proof of (3.10) is complete.

3.2 Fixed points of the smoothing transform in the boundary case

We now turn, in this section, to the study of the smoothing transform in the boundary case, i.e. assuming that

$$\mathbb{E}\left(\sum_{j=1}^\infty T_j^\alpha \right) = 1 \quad \text{and} \quad \mathbb{E}\left(\sum_{j=1}^\infty T_j^\alpha \log(T_j) \right) = 0. \tag{3.14}$$

We add the finite variance hypothesis

$$\mathbb{E}\left(\sum_{j=1}^\infty T_j^\alpha \log(T_j)^2 \right) < \infty. \tag{3.15}$$

Under these conditions, by Theorem 2.12, the martingale

$$Z_n = \sum_{|u|=n} X(u) e^{\alpha X(u)}$$

converges almost surely to a limit Z_∞, and this limit is non-degenerate if and only if

$$\begin{aligned} &\mathbb{E}\left(\left(\sum_{j=1}^\infty T_j^\alpha\right) \log_+ \left(\sum_{j=1}^\infty T_j^\alpha\right)^2\right) < \infty \\ \text{and} \quad &\mathbb{E}\left(\left(\sum_{j=1}^\infty T_j^\alpha \log_+(T_j)\right) \log_+ \left(\sum_{j=1}^\infty T_j^\alpha \log_+(T_j)\right)\right) < \infty. \end{aligned} \tag{3.16}$$

The following result, giving an analogue of Theorem 3.2 in the boundary case, has been proved by Alsmeyer and Mallein in [M27].

Theorem 3.A (Alsmeyer and Mallein (2022)). *Let $(T_j, j \geq 1)$ be a random non-increasing null sequence satisfying (3.5), we assume there exists $\alpha > 0$ such that (2.19) holds. Then, under assumptions (3.15) and (3.16), writing r the geometric span of $(T_j, j \geq 1)$, the non-increasing fixed points of the smoothing transform (3.1) are the functions f satisfying*

$$\forall t \geq 0, f(t) = \mathbb{E}\left(\exp\left(-h(t)t^\alpha Z_\infty\right)\right), \tag{3.17}$$

with h a left-continuous function in $\mathcal{H}_r$.

The proof of this result follows a similar three-step structure as for the characterization of fixed points of the smoothing transform in the regular case. We first prove the tameness of any non-increasing solution f satisfying (3.1), i.e. that they satisfy

$$\limsup_{t \to 0} \frac{-\log f(t)}{t^\alpha \log t} < \infty. \tag{3.18}$$

This tameness assumption allows us to define, for $a > 1$, the function

$$\overline{G}^a : x \in \mathbb{R} \mapsto e^{\alpha x} \mathbb{E}\left(-\log M^a(e^x)\right),$$

with M^a the stopped disintegration martingale of f defined by

$$M^a(t) = \lim_{n \to \infty} \prod_{|u|=n} \left(\mathbf{1}_{\{\max_{k<n} te^{X(u_k)} > a\}} + \mathbf{1}_{\{\max_{k<n} te^{X(u_k)} < a\}} f(te^{X(u)})\right).$$

By the branching property of the branching random walk, and the tameness assumption, we observe that $\overline{G}$ is an harmonic function of the random walk S killed at level $\log a$ and has at most linear growth at $-\infty$. We mention that in the boundary case, with assumption (3.15), the random walk S is centred with finite variance. To complete the identification of the function h associated to the solution f of the smoothing transform, we need an analogue of the Choquet and Deny's theorem for harmonic functions of the killed random walk with at most linear growth. We use the following result, proved in [M27], which extends a similar result of Spitzer [175, Thm. E3, p. 332] proved for centred random walks on the integer lattice.

Proposition 3.B (Alsmeyer and Mallein (2022)). *Given S a non-trivial centred random walk with finite variance and lattice span $d \geq 0$, let G be a right-continuous function satisfying*

$$\forall x > 0, \; G(x) = \mathbb{E}(G(x + S_1)), \quad \sup_{x<0} G(x) < \infty \quad \textit{and} \quad \limsup_{x \to \infty} \frac{G(x)}{x+1} < \infty. \tag{3.19}$$

There exists a function κ, d-periodic if $d > 0$ and constant if $d = 0$ such that

$$G(x) = \kappa(x)U(x) + \mathbb{E}_x(G(S_\tau)) \quad \text{for all } x \in \mathbb{R},$$

where $\tau = \inf\{n \geq 0 : S_n \leq 0\}$ and $U(x) = x - \mathbb{E}_x(S_\tau)$ is the renewal function of the weakly descending ladder heights of the random walk S.

This proposition can be obtained using the optional stopping theorem as well as classical estimates on the overshoot distribution of the random walk S. For all $y \in \mathbb{R}$, we set $T_y = \inf\{n \geq 0 : S_n > y\}$, if G satisfies (3.19), then for all $n \in \mathbb{N}$

$$G(x) = \mathbb{E}_x\left(G(S_\tau \mathbf{1}_{\{\tau < T_y \wedge n\}}\right) + \mathbb{E}_x\left(G(S_{T_y})\mathbf{1}_{\{T_y \leq \tau \wedge n\}}\right) + \mathbb{E}_x\left(G(S_n)\mathbf{1}_{\{n\tau \wedge T_y\}}\right).$$

Letting $n \to \infty$ and the monotone and the dominated convergence theorems, we obtain

$$G(x) = \mathbb{E}_x\left(G(S_\tau)\mathbf{1}_{\{\tau < T_y\}}\right) + \mathbb{E}_x\left(G(S_{T_y})\mathbf{1}_{\{T_y < \tau\}}\right).$$

Then, using that $\lim_{y\to\infty} y\mathbb{P}_x(S_{T_y} > \tau) = U(x)$, we conclude that for each $x \in \mathbb{R}$, $G(x) = \mathbb{E}_x\left(G(S_\tau)\mathbf{1}_{\{\tau < T_y\}}\right) + \kappa(x)U(x)$. The function κ being an harmonic bounded function of the random walk S, we then deduce by Choquet and Deny's theorem that it is d-periodic.

Coming back to the fixed points of the smoothing transform and using Proposition 3.B, we conclude that for all $a > 1$, $h : t \mapsto \frac{G^a(-\log t)}{U^-(-\log(at))} \in \mathcal{H}_r$, where U^- is the renewal function of the weakly descending ladder heights of $-S$. Then, using the identification of the limit in the same way as in the regular case, we deduce that for all $a > 1$,

$$-\log M^a(t) = h(t)D^a_\infty,$$

with D^a_∞ the truncated derivative martingale of X truncated at level $-\log a$. Finally, letting $a \to \infty$ we have $M^a(t) \to M(t)$, from which we deduce

$$M(t) = \exp\left(-h(t)Z_\infty\right),$$

which completes the proof of Theorem 3.A by uniform integrability of M.

3.3 Application to the identification of the fixed points of the branching convolution equation

Using the identification of the fixed points of the smoothing transform, we are able to identified the fixed points of the branching convolution equation defined in Chapter 1. More precisely, two family of point measures distributions can be thought of as fixed point of the branching convolution by $\mathcal{Z}$, the invariant measures of the branching random walk, satisfying

$$\mathcal{E} = \mathcal{E} \circledast \mathcal{Z}, \tag{3.20}$$

and the extremal point measures of the branching random walk, which solve

$$\mathcal{E} = \mathcal{Z} \circledast \mathcal{E}. \tag{3.21}$$

The systematic study of extremal measures was carried out in [M37] using the characterization of fixed points of the smoothing transform. Maillard [140] previously studied random point measures satisfying a general superposability property. Solutions to (3.20) were obtained by Kabluchko in [115]. The invariant measures of the branching Brownian motion in the boundary case was obtained by Chen, Garban and Shekhar in [74].

Let X be a branching random walk with reproduction law $\mathcal{Z}$ and E i.i.d. point measures with law $\mathcal{E}$. In terms of random processes, if $\mathcal{E}$ is an invariant measure of the branching random walk, then for all $n \in \mathbb{N}$,

$$E \overset{(d)}{=} \sum_{e \in E} \sum_{|u|=n} \delta_{e+X^e(u)},$$

where $(X^e, e \in E)$ are i.i.d. branching random walks independent of E. In other words, if $\overline{X}$ is a branching random walk starting from the distribution $\mathcal{E}$, then at all positive times $(X(u), |u| = n)$ has the same law as E (justifying its name invariant measure).

Similarly, if $(E^u, u \in \mathbb{U})$ are i.i.d. point measure with law $\mathcal{E}$ independent of X, if $\mathcal{E}$ is an extremal point measure of the branching random walk, then for all $n \in \mathbb{N}$,

$$E \overset{(d)}{=} \sum_{|u|=n} \sum_{e \in E^u} \delta_{X(u)+e}. \tag{3.22}$$

In other words, the point measure E can be factorized as independent copies of E shifted by the position at time n of the branching random walk X.

Observe that if there exists a sub-linear sequence (a_n) such that the point measure associated to X, shifted by a_n, converges in distribution, then using the branching property it would holds that

$$\lim_{m \to \infty} \sum_{|u|=m} \delta_{X(u)-a_m} = \lim_{m \to \infty} \sum_{|u|=n} \sum_{|v|=m} \delta_{X(u.v)-X(u)-a_{n+m}} \quad \text{in distribution}$$

Thus, the limit $\overline{\mathcal{E}}$ of the extremal point measure of the branching random walk would satisfy (3.21). This is why we refer to solution of (3.22) as extremal measure.

We recall from Section 1.1 that the law of a random point measure is characterized by its log-Laplace functional (see Definition 1.6). We denote by Ψ the log-Laplace functional of the point measure $\mathcal{E}$. By Property 1.7, Equation 3.20 can be rewritten

$$\Psi[\varphi] = \Psi\left[\Psi_{\mathcal{Z}}[\varphi]\right],$$

with $\Psi_{\mathcal{Z}}[\varphi] : z \mapsto \mathbb{E}\left(\exp\left(-\sum_{|u|=1} \varphi(z + X(u))\right)\right)$. Similarly (3.21) can be rewritten

$$\Psi[\varphi] = \Psi_{\mathcal{Z}}\left[\Psi[\varphi]\right] = \mathbb{E}\left(\exp\left(-\sum_{|u|=1} \Psi[\varphi](z + X(u))\right)\right).$$

In other words, if $\mathcal{E}$ satisfies (3.21), for all non-negative function φ the function

$$f_\varphi : z \in [0, \infty) \mapsto \Psi[\varphi](-\log z)$$

is a fixed point of the smoothing transform with weight sequence $(e^{X(u)}, |u| = 1)$.

With this observation, Maillard and Mallein [M37] characterized the extremal point measures of the branching random walk in the regular, and in the boundary case. More precisely, they prove the following result, under assumptions translating the conditions for characterization of fixed points of the smoothing transform in Theorems 3.2 and 3.A. We assume there exists $\alpha > 0$ such that

$$\mathbb{E}\left(\sum_{|u|=1} \mathbf{1}_{\{X(u) > -\infty\}}\right) > 1 \quad \text{and} \quad \mathbb{E}\left(\sum_{|u|=1} e^{\alpha X(u)}\right) = 1. \tag{3.23}$$

Then, we assume that the smoothing transform satisfied by f_φ is either in the regular case

$$\mathbb{E}\left(\sum_{|u|=1} X(u)e^{\alpha X(u)}\right) \in (-\infty, 0) \quad \text{and} \quad \mathbb{E}(W_1(\alpha)\log_+(W_1(\alpha))) < \infty, \tag{3.24}$$

or in the boundary case and satisfy assumption $\mathcal{A}$, i.e.

$$\begin{aligned} \mathbb{E}\left(\sum_{|u|=1} X(u)e^{\alpha X(u)}\right) = 0, \quad \mathbb{E}\left(\sum_{|u|=1} X(u)^2 e^{\alpha X(u)}\right) \\ \text{and} \quad \mathbb{E}(W_1(\alpha)\log_+(W_1(\alpha))^2) + \mathbb{E}\left(\widetilde{W}_1 \log_+(\widetilde{W}_1)\right) < \infty, \end{aligned} \tag{3.25}$$

where $\widetilde{W}_1 = \sum_{|u|=1} X(u)_+ e^{\alpha X(u)}$. Finally, we assume the branching random walk to be non-lattice, i.e.

$$\forall a > 0,\ \forall b \in \mathbb{R},\ \mathbb{P}\left(\forall |u| = 1, X(u) \in a\mathbb{Z} + b\right) < 1. \tag{3.26}$$

Under these three conditions, extremal point measures of the branching random walk are characterized as follows.

Theorem 3.C (Maillard and Mallein (2022))**.** *Let X a branching random walk satisfying (3.23), (3.24) (respectively (3.25)) and (3.26), a random point measure E satisfies (3.22) if and only if there exits a probability distribution $\mathcal{D}$ on $\mathfrak{P}(\mathbb{R})$ such that*

$$E \text{ is a SDPPP}(Se^{-\alpha x}\mathrm{d}x, \mathcal{D}),$$

where $S = W_\infty$ is the limit of the additive martingale $(W_n(\alpha))$ (resp. $S = Z_\infty$ is the limit of the derivative martingale $Z_n(\alpha)$).

We observe that under the assumption of Theorem 3.C, if φ is a positive non-decreasing function, then f_φ is a non-increasing fixed point of the smoothing transform, hence there exists $c > 0$ such that

$$\forall z > 0, f_\varphi(z) = \mathbb{E}\left(\exp\left(-cz^\alpha S\right)\right)$$

by Theorem 3.2 or 3.A depending on whether we are in the regular or the boundary case. As a result, we obtain that for a large class of functions φ, the log-Laplace transform of any random point measure E satisfying (3.22) will verify

$$F_{\mathcal{E}}[\varphi](x) = \mathbb{E}\left(\exp(-ce^{\alpha x}S)\right).$$

Then, using the characterization of shifted decorated Poisson point processes of Subag and Zeitouni [176] (see Theorem 1.10), we conclude that $\mathcal{E}$ is an SDPPP with intensity $Se^{\alpha x}\mathrm{d}x$.

It is worth observing that in terms of point measure distribution, writing $\mathcal{S}$ the law of the random point measure with a single atom $\delta_{-\log S}$ and $\mathcal{P}_\alpha$ the law of a Poisson point process with intensity $e^{\alpha x}\mathrm{d}x$, then a point measure distribution $\mathcal{E}$ satisfies (3.22) if and only if

$$\mathcal{E} = \mathcal{S} \circledast \mathcal{P}_\alpha \circledast \mathcal{D},$$

with $\mathcal{D}$ a random point measure called the *decoration* of the SDPPP $\mathcal{E}$

We observe that no information can be obtained on the law of the decoration from the single equation (3.22). Indeed, if $\mathcal{E}$ satisfies this equation, then so does $\mathcal{E} \circledast \mathcal{F}$ for any random point measure $\mathcal{F}$ such that the branching convolution equation is well-defined.

Invariant point measures of the branching random walk, satisfying (3.20), are less well-known. In [115, Theorem 2.1], Kabluchko proved that under some restrictive integrability conditions such invariant measures can be represented as SDPPP$(Se^{-\alpha x}\mathrm{d}x, \mathcal{D}^\alpha)$, with S any non-negative random variable and $\mathcal{D}^\alpha$ the random point measure obtained as

$$\mathcal{D}^\alpha := \lim_{n\to\infty} \mathcal{P}_\alpha \circledast \mathcal{Z}^{\circledast n},$$

i.e. as the limiting distribution of a branching random walk with initial distribution given by a Poisson point process with intensity $e^{-\alpha x}\mathrm{d}x$. It is worth noting that this limit exist only if α satisfies

$$\mathbb{E}\left(\sum_{|u|=1} X(u)e^{\alpha X(u)}\right) > 0 \quad \text{and} \quad \mathbb{E}\left(W_1(\alpha)\log_+(W_1(\alpha))\right) < \infty. \tag{3.27}$$

In [74], Chen, Garban and Shekhar characterized the invariant measures of the binary branching Brownian motion with drift $-\sqrt{2}$, giving an example of invariant measure of a branching process in the boundary case. Specifically, their main theorem can be stated as as follows.

Theorem 3.4 (Chen, Garban and Shekhar [74])**.** *Let X be a binary standard branching Brownian motion with drift $-\sqrt{2}$. For all $t \geq 0$, we denote by $\mathcal{B}_t$ the law of $\sum_{u\in\mathcal{N}_t} \delta_{X_t(u)}$. If $\mathcal{E}$ is a random point measure on $\mathfrak{P}$ such that for all $t \geq 0$, we have $\mathcal{E} \circledast \mathcal{B}_t = \mathcal{E}$, then there exists a non-negative random variable S such that $\mathcal{E}$ is a SDPPP$(Se^{-\alpha x}\mathrm{d}x, \mathcal{D}^*)$, where $\mathcal{D}^*$ is defined by*

$$\int e^{-\langle D,\varphi\rangle}\mathcal{D}^*(\mathrm{d}D) = \lim_{t\to\infty} \mathbb{E}\left(e^{-\sum_{u\in\mathcal{N}_t}\varphi(X_t(u)-M_t)}\middle| M_t \geq 0\right). \tag{3.28}$$

The existence of the measure $\mathcal{D}^*$ defined by (3.28) was obtained (with a slightly modified expression) by Arguin, Bovier and Kiesler [20], when studying the convergence in distribution of the extremal point measure of the branching Brownian motion. The present expression was obtained for branching Brownian motions by Cortines, Hartung and Louidor [77]. The law $\mathcal{D}^*$ is often called the decoration law of the branching Brownian motion, for reasons that will become clear in the next section.

We conjecture that, under suitable conditions, a version of Theorem 3.4 should hold for most branching random walks.

Conjecture 3.5. *Let X a branching random walk satisfying (3.23), (3.25) of (3.27), and (3.26), a random point measure E satisfies (3.20) if and only if there exits a non-negative random variable S such that*

$$E \text{ is a SDPPP}(Se^{-\alpha x}\mathrm{d}x, \mathcal{D}^*),$$

where $\mathcal{D}^*$ is defined by

$$\int e^{-\langle D,\varphi\rangle}\mathcal{D}^*(\mathrm{d}D) = \lim_{t\to\infty} \mathbb{E}\left(e^{-\sum_{|u|=n}\varphi(X(u)-M_n)}\middle| M_n \geq 0\right). \tag{3.29}$$

The random point measures defined by (3.29) are called the decoration point measure of the branching random walk if X satisfies (3.25). If the branching random walk verifies (3.27), the condition $\{M_n \geq 0\}$ becomes a large deviation event with exponentially small probability of occurrence. In this case, the well definition of the decoration defined by

(3.29) is an open question, as well as the optimal integrability conditions for the existence of this limit.

It is worth noting that fixed points of the branching convolution equation, on the left or on the right, are SDPPPs. However, equation (3.20) gives information on the decoration law $\mathcal{D}$ of $\mathcal{E}$, while equation (3.21) gives information on its shift S. Combining Theorem 3.C and Conjecture 3.5 would give rise to the following result.

Conjecture 3.6. *Let $\mathcal{Z}_1$, $\mathcal{Z}_2$ be two random point measures that satisfy (3.23) and (3.26). If $\mathcal{Z}_1$ verifies (3.24) or (3.25) and $\mathcal{Z}_2$ verifies (3.25) or (3.27), then the unique random point measure $\mathcal{E}$ (up to shifts by a constant) satisfying*

$$\mathcal{E} = \mathcal{Z}_1 \circledast \mathcal{E} = \mathcal{E} \circledast \mathcal{Z}_2$$

is the SDPPP$(Se^{-\alpha x}\mathrm{d}x, \mathcal{D}^*)$, with S the limit of the additive (or the derivative) martingale associated to $\mathcal{Z}_1$ and $\mathcal{D}^*$ the decoration point measure associated to $\mathcal{Z}_2$.

Using this conjecture, one can predict the asymptotic behaviour of extremal particles in a large number of extremal processes, by applying the branching property at time 1, to determine the random shift, and at time $n-1$, to describe the extremal distribution. However, there is not yet a general convergence result, that would allow to deduce the convergence in distribution of extremal processes through a tightness argument.

CHAPTER 4

Extreme values of spatial branching processes

"Et le chemin est long du projet à la chose."

Molière – Le Tartuffe, 1664.

Summary.

We treat in this chapter the central focus of this manuscript: the convergence of the extremes of spatial branching processes at large times. We present here some of the contemporary literature dealing with the description of the law of the maximal displacement at large time in this process, as well as the joint distribution of all particle within distance 1 from the rightmost particle. The form of the limiting object reveals important insight for the behaviour of the branching process, in particular the strategy followed by particles reaching these unusually large position.

We first focus on the convergence in distribution of the maximal displacement of spatial branching processes. We recall that Bramson [57] and Lalley and Sellke [127] obtained the convergence in distribution of the maximal displacement centred around its mean through a mixture of probabilistic and analytic methods. These analytic tools are no longer available when dealing with the branching random walk. Aïdékon [4] proved the convergence in distribution of the centred maximal displacement of a branching random walk satisfying assumption $\mathcal{A}$ using tight estimates on the right tail of this process. We then present some extension of this result, among which [M1, M2, M8, M9].

We then consider the convergence of extremal processes, showing that the results of Subag and Zeitouni [176] can be used to provide a simple proof for the convergence in distribution of the extremal process of the branching Brownian motion towards an SDPPP, obtained by Aïdékon and al. [5] and Arguin et al. [20] with descriptions of the decoration distribution. The convergence in distribution of the extremal process of the branching random walk was obtained by Madaule [136]. We introduce some generalization of this results from [M6], [M12] and [M40] in which convergence of enriched extremal processes are obtained. We conclude with an extension to time-inhomogeneous branching Brownian motions, mentioning among others [M23], and to a multitype reducible model obtained in [M33].

The maximal displacement of the branching random walk X at time n is the random variable defined as

$$M_n := \max_{|u|=n} X(u).$$

It represent the rightmost occupied position in the branching random walk at time n, and is a good proxy for the speed of invasion of the particle system. We take interest in the asymptotic behaviour of (M_n) as $n \to \infty$. In typical situations, this random sequence converges in distribution, when centred around its mean, to a mixture of a Gumbel random variable and $\log Z_\infty$. This convergence can be thought off as a generalization of the convergence of the maximum of independent random variables to log-correlated settings, as the Gumbel law is a max-stable distribution. The additive contribution from $\log Z_\infty$ is a mark of the branching property of the process, and that a large population at a finite time will produce a large population further down the line.

To describe the joint convergence in distribution of the particles realizing an anomalously large displacement in the branching random walk, we also take interest in the extremal process of the branching random walk, defined as the point measure

$$E_n := \sum_{|u|=n} \delta_{X(u)-a_n},$$

for a suitable sequence (a_n) such that $M_n - a_n$ converges in distribution. The limit in distribution of E_n are usually SDPPPs with exponential intensities, shifted by $\log Z_\infty$.

We focus in a first time on the convergence of the maximal displacement of spatial branching processes, and in particular the branching random walk, studied over the years by a large group of authors including Biggins [42], Bramson [57], Addario-Berry and Reed [3], Hu and Shi [105], Aïdékon [4] among many others. We treat in a second time the convergence of the extremal processes in similar settings, following the steps of Aïdékon, Berestycki, Brunet and Shi [5], Arguin, Bovier and Kistler [20] and Madaule [136].

4.1 Convergence in law of the maximal displacement

We focus in this section on the convergence in distribution of the maximal displacement of a spatial branching process centred by its median. We consider in a first time the case of the branching Brownian motion, which can be obtained through analytic methods, thanks to its connection with the F-KPP equation observed by McKean [145]. Using the interplay between the stochastic model and the associated PDE equation, it becomes possible to describe quite explicitly the asymptotic behaviour of large particles in that process.

In a second time, we extend these convergence results to the branching random walk. Due to the lack of analyticity of the sequence $(\mathbb{P}(M_n \leq x), x \geq 0)_{n\in\mathbb{N}}$, the methods used in that context are more focus on the detailed study of the probabilistic model, and a precise understanding of the behaviour of particles in this system. These methods are often robust enough to be extended to models not satisfying exactly the branching property, such as time-inhomogeneous branching processes [92] or the Gaussian free field [137].

4.1.1 Maximal displacement of the branching Brownian motion

We recall that the tail distribution function of the maximal displacement of the branching Brownian motion, defined by $u(t, x) = \mathbb{P}(M_t \geq x)$ is a solution to the F-KPP equation

$$\partial_t u = \frac{1}{2}\Delta u + u(1-u)$$

started from the initial condition $u(0, x) = \mathbf{1}_{\{x \leq 0\}}$. Kolmogorov, Petroviskii and Piskunov [123] proved the existence of a function $(m_t, t \geq 0)$ satisfying $\lim_{t\to\infty} \frac{m_t}{t} = \sqrt{2}$ such that $x \mapsto u(t, x + m_t)$ converges uniformly on compact sets to a limit w, which is a travelling-wave solution to F-KPP with speed $\sqrt{2}$.

In [57], Bramson proved that $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$ is one such function. In terms of branching Brownian motions, it proves that

$$M_t - m_t \text{ converges in distribution to a limit } X,$$

with the law of X being defined by $\mathbb{P}(X \geq x) = w(x)$. Then, Lalley and Sellke [127] obtained a formula for the travelling-wave solution, namely that there exists $c > 0$ such that

$$\forall x \in \mathbb{R}, \quad w(x) = 1 - \mathbb{E}(e^{-ce^{\sqrt{2}x} Z_\infty}). \tag{4.1}$$

This representation of the travelling-wave solutions of the F-KPP equation, together with Bramson's convergence in law yields the convergence in law of the maximal displacement, as observed in (2.34). The following result, due to Lalley and Sellke [127] identifies the limit of the derivative martingale in the Gumbel mixture.

Theorem 4.1 (Lalley and Sellke, 1987)**.** *Writing M_t for the maximal displacement of the branching Brownian motion, we have*

$$\lim_{t\to\infty} M_t - m_t = \frac{\sqrt{2}}{2} \log Z_\infty + G, \text{ in law,}$$

with G a Gumbel random variable, independent of Z_∞. More precisely, for all $y > 0$, there exists $c_\star > 0$ such that

$$\lim_{s\to\infty} \lim_{t\to\infty} \mathbb{P}(M_t \leq m_t + y | \mathcal{F}_s) = e^{-c_\star Z_\infty e^{-\sqrt{2}y}}$$

Although this result was initially proved for the binary branching Brownian motion, a similar result holds as soon as the reproduction law satisfies an $L(\log L)^2$ integrability condition (necessary for the well-definition of the limit of the derivative martingale). We remark that

$$M_{t+s} = \max_{u \in \mathcal{N}_t} \max_{v \in \mathcal{N}_{t+s} : v \text{ descendant of } u} X_{t+s}(v),$$

therefore

$$1 - u(t + s, x) = \mathbb{E}\left(\prod_{u \in \mathcal{N}_t} (1 - u(s, X_t(u) + x)) \right).$$

Then, using Bramson's result and letting $s \to \infty$, this equation yields

$$1 - w(x) = \mathbb{E}\left(\prod_{u \in \mathcal{N}_t} (1 - w(X_t(u) - \sqrt{2}t + x)) \right).$$

In other words, $1 - w$ is a fixed point of the smoothing transform (in the boundary case), which in view of Theorem 3.A yields (4.1), hence Theorem 4.1.

Before discussing the analogous result in branching random walk settings, we mention a multidimensional extension of Theorem 4.1.

Multidimensional branching Brownian motion The branching Brownian motion in dimension d is a spatial branching process in which particles move according to i.i.d. Brownian motions in dimension d, while branching into two offspring at rate 1. In this process, we denote by

$$R_t := \max_{u \in \mathcal{N}_t} \|X_t(u)\|$$

the maximal displacement of this multidimensional branching Brownian motion. It is worth observing that contrary to the dimension 1, the tail distribution function of R_t is not the solution of an F-KPP-type partial differential equation. As a result, it is not directly possible to use the method of Lalley and Sellke to study the asymptotic behaviour of R_t as $t \to \infty$.

Using first and second moment methods to bound the tail distribution function of R_t at large times, Mallein [M5] proved that $(R_t - r_t, t \geq 0)$ is tight, where $r_t = \sqrt{2}t + \frac{d-4}{2\sqrt{2}} \log t$. In other words, with high probability the farthest particle away from the origin in dimension d is $\frac{d-1}{2\sqrt{2}} \log t$ farther than it would be in dimension 1. This larger logarithmic correction can be heuristically explained noting that a sphere of radius ct can be covered by $O(t^{(d-1)/2})$ spherical caps of height 1, corresponding to $O(t^{(d-1)/2})$ possible directions for the multidimensional branching Brownian motion to realize its maximum.

Using a precise estimate for the right-tail asymptotic of $\mathbb{P}(R_t \geq r_t + y)$ uniform in $y \in [A, \delta \log t - A]$, Kim, Lubetzki and Zeitouni [120] proved the convergence in law of the variable $R_t - r_t$. This asymptotic is obtained using that far away from 0, the norm of the Brownian motion behaves similarly to a one-dimensional Brownian motion, hence the tail distribution of R_t can be compared to the tail distribution of M_t.

Theorem 4.2 (Kim, Lubetzki and Zeitouni, 2021)**.** *There exists a positive non-degenerate random variable Z such that*

$$\lim_{t \to \infty} \mathbb{P}(R_t \leq r_t + y) = \mathbb{E}\left(\exp\left(-Ze^{-\sqrt{2}y}\right)\right),$$

i.e. $R_t - r_t$ converges in law to a Gumble distribution shifted by $\frac{\sqrt{2}}{2} \log Z$.

This result is formally similar to the one obtained by Lalley and Sellke for the unidimensional branching Brownian motion with the following caveat: contrarily to Z_∞, which is the limit of the derivative martingale of the branching Brownian motion, the random variable Z is not measurable with respect to the branching Brownian motion, and is obtained as a limit in distribution. This result was later strengthen in [M40], using the radial derivative martingale of the branching Brownian motion identified in Theorem 2.K.

Theorem 4.A (Berestycki, Kim, Lubetzki, Mallein and Zeitouni (2022))**.** *We denote by $Z = \int_{\mathbb{S}^{d-1}} Z_\infty(\varphi)\sigma(\mathrm{d}\varphi)$, there exists $c_\star^d > 0$ such that*

$$\lim_{s \to \infty} \lim_{t \to \infty} \mathbb{P}(R_t \leq r_t + y | \mathcal{F}_s) = e^{-c_\star^d Z e^{-\sqrt{2}y}}.$$

The techniques used to prove this result rely on the observation that with high probability, particles that will make a large displacement in the process at time t move in a fixed direction after some finite time. Thanks to this observation, it becomes possible to link tail estimates of largest displacement in the multidimensional branching Brownian motion with unidimensional moderate deviations estimates for the maximal displacement.

4.1.2 Maximal displacement of the branching random walk

The convergence in law of the maximal displacement of the branching random walk was obtained by Aïdékon [4], who proved a discrete-time analogue to Theorem 4.1. His proof rely on a precise estimation of a moderately large deviation event for the branching random walk. More precisely, writing M_n the minimal displacement at time n of a branching random walk in the boundary case, Aïdékon proved that there exists $c_\star > 0$ such that

$$\lim_{A\to\infty} \limsup_{n\to\infty} \sup_{y\in[A,\frac{3}{2}\log n - A]} \left| \mathbb{P}\left(M_n \leq \tfrac{3}{2}\log n - y \right) - c_\star y e^{-y} \right| = 0. \tag{4.2}$$

This estimate is obtained using the precise first and second moment computations of the number of particles in the branching random walk satisfying specific properties, which, along with concentration inequalities, allows for the obtention of this uniform equivalent.

Using the tight estimate (4.2) on the probability for a branching random walk to make a moderate deviation event and the branching property of the branching random walk, it becomes possible to justify the following heuristic computation as n goes to ∞

$$\begin{aligned}\mathbb{P}(M_{n+k} \geq \tfrac{3}{2}\log n - y) &= \mathbb{E}\left(\prod_{|u|=k} \mathbb{P}(\overline{M}_n \geq \tfrac{3}{2}\log n - (y + X(u))|\mathcal{F}_k) \right)\\ &\approx \mathbb{E}\left(\prod_{|u|=k} \left(1 - c_\star(y+X(u))e^{-y-X(u)}\right) \right) \approx \mathbb{E}\left(\exp\left(-c_\star(W_k + Z_k)\right)\right),\end{aligned}$$

where $\overline{M}_n$ is an independent copy of M_n. As a result, using Theorems 2.2 and 2.12, the following result then holds.

Theorem 4.3 (Aïdékon (2013)). *Let X be a non-lattice branching random walk satisfying assumption $\mathcal{A}$. We set*

$$v = \inf_{\theta>0} \frac{\kappa(\theta)}{\theta} = \frac{\kappa(\theta^*)}{\theta^*} \quad \textit{and} \quad m_n = nv - \frac{3}{2\theta^*}\log n.$$

Then with $M_n = \max_{|u|=n} X(u)$, the variable $M_n - m_n$ converges in law to a Gumbel distribution shifted by $\frac{1}{\theta^*}\log Z_\infty$. More precisely,

$$\lim_{k\to\infty} \lim_{n\to\infty} \mathbb{P}\left(M_n \leq m_n + y|\mathcal{F}_k\right) = \exp\left(-c_\star e^{-\theta^* y} Z_\infty\right).$$

This result, similarly to the one obtained by Bramson for branching Brownian motions, give a precise estimation of the position of the rightmost particle at a large time. We mention that [M11] gives a simple proof for the tightness of $(M_n - m_n, n \geq 1)$, under the same, optimal, integrability assumptions. This result is used by Addario-Berry and Ford [1] to compute the asymptotic height of random recursive tree, and by Pain and Sénizergues for the height of weighted recursive trees and other branching-type structures [156, 157].

Theorem 4.3 can, in some sense, be compared to the analysis of the law of the largest element in a set of i.i.d. random variables. We recall that writing $\overline{M}_n$ the maximum of 2^n i.i.d. Gaussian random variables with variance n, it is well-known that

$$\lim_{n\to\infty} \overline{M}_n - \sqrt{2}n - \frac{1}{2\sqrt{2}}\log n = G \quad \text{in law},$$

with G a Gumbel random variable. We refer to the book of Bovier [54, Chapter 1–4] for more details

Hence, compared to i.i.d. settings, the maximal displacement in spatial branching processes present several distinct features. First, the logarithmic correction is slightly larger, due to the presence of correlation between particles alive at generation n. Additionally, the limiting distribution of the maximal displacement is obtained as a mixture of a Gumbel random variable and the logarithm of a martingale of the process. This shift by the logarithm of a martingale expresses the dependence of the law of the maximal displacement to the first steps of the process. Indeed, if the initial particle makes an anomalously large displacement to the right, it will move in the same way the rest of the process to the right, therefore have a macroscopic effect on the maximal displacement after a large time. This effect is absent when considering i.i.d. random variables.

Before turning in the next section to the study of extremal processes, we mention below several time-inhomogeneous extension for the position of the maximal displacement in a time-inhomogeneous branching process. By modifying the correlation structure of particles, the asymptotic behaviours of the maximal displacement in these time-inhomogeneous processes differ from the one observed in the time-homogeneous branching processes.

Time-inhomogeneous branching random walks

A time-inhomogeneous branching random walk is a spatial branching process defined in such a way that the reproduction law depend on the generation. Given $n \in \mathbb{N}$, we fix $(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1})$ a family of point measures distributions. The time-inhomogeneous branching random walk $X^{(n)}$ is the branching process such that for all $k \leq n-1$, we have

$$\left(\sum_{j \in \mathbb{N}} \delta_{X^{(n)}(uj) - X^{(n)}(u)}, |u| = k \right) \text{ are i.i.d. point measures of law } \mathcal{L}_k.$$

In other words, a particle alive at generation k gives birth to offspring that are positioned around their parent according to the law $\mathcal{L}_k$. We refer to $(\mathcal{L}_k)$ as the environment of the process.

Time-inhomogeneous branching random walks were introduced by Fang and Zeitouni [92, 93], who studied in particular the maximal displacement in a branching random walk with an interface, such that particles reproduce according to the law $\mathcal{L}_1$ for the $n/2$ first units of time, and according to the law $\mathcal{L}_2$ for the $n/2$ next ones. This time-inhomogeneous process models the effect of a brutal modification of the environment on the evolution of an invading species.

Fang and Zeitouni [92] observed that in this model, the maximal displacement M_n behaves as $an - b \log n + O_{\mathbb{P}}(1)$, and that the parameters a and b depend on the order of the law $\mathcal{L}_1$ and $\mathcal{L}_2$. Mallein [M1] studied a generalization of this problem, and gave a formula for the parameters a and b such that

$$M_n = an - b \log n + O_{\mathbb{P}}(1),$$

in a branching random walk with a finite number of interfaces, which hold under quite general integrability assumptions analogous to assumption $\mathcal{A}$.

Another class of time-inhomogeneous branching random walks are defined using a continuous family $(\mathcal{L}_t, t \in [0, 1])$, such that particles alive at generation k reproduce according to the law $\mathcal{L}_{k/n}$. This class of processes models the evolution of a population under gradual shift of its environment. One of the simplest such models is the branching

Brownian motion with time-inhomogeneous variance profile $(\sigma_s, s \in [0,1])$. Bovier and Hartung [55, 56] proved the convergence in distribution of the maximal displacement in this time-inhomogeneous branching Brownian motion in the weak correlation regime.

Theorem 4.4 (Bovier and Hartung (2015)). *Let M_t be the maximal displacement at time t of a time-inhomogeneous branching Brownian motion with time-inhomogeneous variance profile σ. We write $\tau^2 := \int_0^1 \sigma_r^2 \mathrm{d}r$. If $\forall s \in (0,1)$ $\int_0^s \sigma_r^2 \mathrm{d}r < s\tau^2$, then*

$$\lim_{t\to\infty} M_t - \sqrt{2\tau^2}t + \frac{1}{2\sqrt{2\tau^2}} \log t = G - \frac{1}{2\sqrt{2\tau^2}} \log W_\infty(\sqrt{2\sigma_0^2/\tau^2}) \text{ in law,}$$

with $W_\infty(\beta)$ the limit of the additive martingale at parameter β of a time-homogeneous branching Brownian motion and G an independent Gumbel random variable.

At the other end of the correlation spectrum lies the branching Brownian motion with decreasing variance. In this process, the motion of particles becomes more difficult as time grows, hence a particle reaching a far position has to have stayed close to the boundary of this process for a long time, and particles that realize a large displacement are usually very correlated. As a result, the maximal displacement in this process lies much lower than for i.i.d. Gaussian variables with similar variance. This result was obtained by Maillard and Zeitouni [143].

Theorem 4.5 (Maillard and Zeitouni (2016)). *Let M_t be the maximal displacement at time t of a time-inhomogeneous branching Brownian motion with time-inhomogeneous variance σ. We assume that σ is $\mathcal{C}^2$, decreasing and $\sigma_1 > 0$, and we set*

$$m_t = \sqrt{2}t \int_0^1 \sigma_s \mathrm{d}s + \frac{\alpha_1}{2^{1/3}} t^{1/3} \int_0^1 \sigma(s)^{1/3} |\sigma'(s)|^{2/3} \mathrm{d}s - \sigma_1 \log t,$$

where $\alpha_1 \approx -2.3381$ is the largest zero of the Airy Ai *function of the first kind. The sequence $(M_t - m_t, t \geq 0)$ is tight.*

The $t^{1/3}$ main correction term in the branching Brownian motion with decreasing variance holds for any branching Brownian motion in the large disorder regime, as has been proved in [M2] for general branching random walks.

Theorem 4.B (Mallein (2015)). *Let M_t be the maximal displacement at time t of a time-inhomogeneous branching Brownian motion with time-inhomogeneous variance σ. We assume that σ is $\mathcal{C}^2$ and that there exists $t_0 \in (0,1)$ such that $\int_0^{t_0} \sigma_s^2 \mathrm{d}s < t_0 \int_0^1 \sigma_s^2 \mathrm{d}s$. There exists $v > 0$ and $\alpha > 0$ such that*

$$\lim_{t\to\infty} \frac{M_t - vt}{t^{1/3}} = -\alpha \quad \text{in probability.}$$

More precisely, writing $(a_t, t \geq 0)$ the solution of the optimization problem

$$\max\{\int_0^1 b_s \mathrm{d}s : b \in \mathcal{C}^1, \int_0^t \frac{b_s^2}{2\sigma_s^2} \mathrm{d}s \leq t\},$$

we have $v = \int_0^1 a_s \mathrm{d}s$ and $\alpha = -\frac{\alpha_1}{2^{1/3}} \int_0^1 \sigma_s^{2/3} \frac{(2a_s\sigma_s' - a_s'\sigma_s)^{2/3}}{a_s} \mathrm{d}s$.

Note that the first two orders asymptotic of the maximal displacement obtained in [M2] coincide with the ones obtained by Maillard and Zeitouni as $a_s = \sqrt{2}\sigma_s$ for all $s \in [0,1]$ if σ is decreasing.

Finally, a third class of time-inhomogeneous branching processes concern branching random walks in random environment. In this model, which explores the effect of seasonal fluctuations on the behaviour of an invading population, at each generation the reproduction law of all the particles is chosen at random, according to some distribution of $\mathfrak{P}$. In this setting, Kriechbaum [125] proved the tightness of the maximal displacement around its quenched mean m_n. Additionally, Mallein and Miłoś [M7] computed constants v and $\gamma > 0$ such that

$$\frac{m_n - nv}{\log n} \to -\gamma \quad \text{in probability,}$$

while observing almost sure fluctuations at the logarithmic level for the median of $M_n - nv$.

Consistent maximal displacement It is worth noting that the $n^{1/3}$ second order term in the asymptotic behaviour of time-inhomogeneous branching processes can be understood as a penalty for requiring particles to stay as close as possible to the boundary of the process. The existence of this correction was previously observed by Fang and Zeitouni [91], who computed the asymptotic behaviour of the so-called *consistent maximal displacement* of the branching random walk, defined as

$$L_n := \min_{|u|=n} \max_{k \leq n} kv - X(u_k). \tag{4.3}$$

They proved that $L_n/n^{1/3}$ converges a.s. to an explicit well-defined limit. Mallein [M9] obtained a necessary and sufficient condition for this almost sure convergence to hold (see also Roberts [163] for a similar result on branching Brownian motion).

Theorem 4.C (Mallein (2019))**.** *Let X be a branching random walk such that there exists θ^* verifying $\theta^*\kappa'(\theta^*) - \kappa(\theta^*) = 0$ with $\sigma^2 = \kappa''(\theta^*) \in (0, \infty)$. Then*

$$\lim_{n\to\infty} \frac{L_n}{n^{1/3}} = \left(\frac{3\pi^2\sigma^2}{2}\right)^{1/3} \text{ a.s. on the survival event}$$

if and only if $\lim_{x\to\infty} x^2 \mathbb{E}\left(\sum_{|u|=1} e^{-X(u)} \mathbf{1}_{\left\{\sum_{|v|=1} e^{-X(v)} \geq e^x\right\}}\right) = 0$.

In particular, $\frac{L_n}{n^{1/3}}$ converges almost surely under the assumptions of Theorem 2.12. The proof of this result is based on Mogul'skii estimates on the probability for random walks to stay in a domain of width of order $n^{1/3}$, following the method used by Aïdékon and Jaffuel [7] to study the critical line above which a particle can survive in the branching random walk.

4.2 Convergence of extremal processes

In order to refine the convergence in distribution results obtained in the previous section, one can consider the extremal process of the spatial branching processes, i.e. the relative positions of all particles within distance $O(1)$ from the position of the rightmost individual. We note that if the extremal process of a spatial branching process converges, it has to satisfy the branching property, which can be rephrased as a branching fixed point equation (3.20). As a result, the extremal process of a spatial branching process satisfying mild conditions is a shifted decorated Poisson point process with exponential intensity.

We will first discuss the convergence of the extremal process of the branching Brownian motion, using again the convergence of solutions of the F-KPP equation to travelling waves, then Theorem 3.C to identify the law of the limit. We then illustrate several extensions of this result, to multidimensional, multitype and time-inhomogeneous branching Brownian motions, to illustrate the variety of results one might find in the extremal process of spatial branching processes.

We then turn to the branching random walk setting, in which the literature is less developed. If the convergence in law of the extremal process of a branching random walk seen from its tip is now well-known, few extensions of this result are currently available. The study of this problem remains an active field.

4.2.1 Extremal process of the branching Brownian motion

Let $(X_t(u), u \in \mathcal{N}_t)$ be a standard branching Brownian motion. We denote by

$$E_t := \sum_{u \in \mathcal{N}_t} \delta_{X_t(u) - m_t}$$

its extremal process. We take interest in the convergence in distribution of $\mathcal{E}_t$ as $t \to \infty$, for the topology of weak convergence. This topology is equivalent to the joint convergence in distribution of the position of the k rightmost atoms for all $k \in \mathbb{N}$, thus informs us on the repartition of particles in the branching Brownian motion close to its tip.

Given f a continuous compactly supported function, we define for $x \in \mathbb{R}$ and $t \geq 0$

$$\Phi_t(x) = \mathbb{E}\left(\exp\left(-\langle \tau_x E_t, f \rangle\right)\right) = \mathbb{E}\left(\exp\left(-\sum_{u \in \mathcal{N}_t} f(X_t(u) - m_t + x)\right)\right).$$

We have $\Phi_t(x) = 1 - u(t, m_t - x)$ by McKean's representation, where u is the solution to the F-KPP equation with (compact) initial condition $1 - e^{-f}$. By uniform convergence on compact sets to a travelling-wave solution to the equation with speed $\sqrt{2}$ of $u(t, m_t - x)$ and Lalley-Sellke's representation of this travelling wave solution, we immediately deduce that there exists $c_f > 0$ such that

$$\lim_{t \to \infty} \Phi_t(x) = \mathbb{E}\left(\exp\left(-c_f e^{\sqrt{2}x} Z_\infty\right)\right). \tag{4.4}$$

Hence, we conclude that $(E_t, t \geq 0)$ converges in law to a random point measure satisfying the assumptions of Theorem 1.10. We therefore obtain the first part of the following result, first proved by Arguin, Bovier and Kistler [20] and Aïdékon, Berestycki, Brunet and Shi [5].

Theorem 4.6 (Aïdékon et al., Arguin et al. (2013)). *Let X be a branching Brownian motion such that $\sum_{k=1}^\infty k(\log k)^2 \nu(k) < \infty$. The extremal process E_t converges in law as $t \to \infty$ to a decorated Poisson point process with intensity $c_\star \sqrt{2(m-1)} e^{-\sqrt{2(m-1)}x} \mathrm{d}x$ and decoration measure $\mathcal{D}$, introduced in (3.28), which can be defined by*

$$\mathcal{D} = \lim_{t \to \infty} \mathbb{P}\left(\sum_{u \in \mathcal{N}_t} \delta_{X_t(u) - M_t} \middle| M_t > \sqrt{2}t\right) = \lim_{t \to \infty} \mathbb{P}\left(\sum_{u \in \mathcal{N}_t} \delta_{X_t(u) - M_t} \mathbf{1}_{\{u \in G_t\}} \in \cdot\right),$$

where G_t is the set of particles alive at time t that share a common ancestor with the rightmost particle at time t that was alive after time $t/2$.

Remark that in some sense, the law of the decoration $\mathcal{D}$ is encoded by the constant c_f appearing in (4.4), i.e. by the function which associates to the initial condition of the F-KPP equation the shift of the the limiting travelling wave front. However, the connection between this constant and a construction of the law $\mathcal{D}$ is far from obvious. Two constructions of this decoration distribution are available, as the extremal process of the branching Brownian motion conditioned on an anomalously large displacement [20], or as the extremal process of a branching Brownian motion with a spine moving in a complex potential [5].

Using the branching property of the branching Brownian motion, we remark that, writing E_∞ for the limiting distribution of the extremal process, the following equation in distribution holds for all $t \geq 0$:

$$E_\infty \overset{(d)}{=} \sum_{u \in \mathcal{N}_t} \tau_{X_t(u) - \sqrt{2(m-1)}t} E^u,$$

with $(E^u, u \in \mathcal{N}_t)$ i.i.d. copies of E_∞. Using the superposition property of the Poisson point process, this equality implies that all particles belonging to the same decoration at a large time T have a most recent common ancestor larger that t for all $t > 0$. Conversely, particles in different decorations have a most recent common ancestor with an age of order 1. This heuristic observation, made precise in [M12], allows us to describe the limiting extremal process as follows. The atoms of the Poisson point process with exponential intensity correspond to *leaders*, particles realizing the largest displacement at time t among their relatives. The decoration point process then represents the relative positions of the close family of these leaders.

Multidimensional branching Brownian motion A version of Theorem 4.6 was proved in [M40] for the multidimensional binary branching Brownian motion. We recall that in this situation, the largest particle in the process is, with high probability, at distance of order $r_t = \sqrt{2}t + \frac{d-4}{2\sqrt{2}} \log t$ of the origin. To give a more precise description of the extremal process of this model, we define the extremal process on $\mathbb{R} \times \mathbb{S}^{d-1}$

$$\overline{E}_t := \sum_{u \in \mathcal{N}_t} \delta_{\|X_t(u)\| - r_t, \frac{X_t(u)}{\|X_t(u)\|}},$$

which encodes both the radial and the angular positions of particles that made a large displacement in that process.

Theorem 4.D (Berestycki, Kim, Lubetzky, Mallein and Zeitouni (2022))**.** *The extremal process $\overline{E}_t$ converges in law as $t \to \infty$ for the topology of the weak convergence to a decorated Poisson point process with intensity $c_\star^d Z_\infty(\varphi)\sqrt{2}e^{-\sqrt{2}x}\mathrm{d}x\sigma(\mathrm{d}\varphi)$ with decoration $\mathcal{D} \otimes \delta_0$.*

In other words, in the multidimensional branching Brownian motion, the leaders choose the direction φ with probability proportional to $Z_\infty(\varphi)$, and all their relatives share the same direction. Indeed, as the angular direction of a particle at distance t from the origin remains almost fixed for times of order 1, no relative has enough time to move in a different direction from its leader. The decoration distribution appearing in the multidimensional branching Brownian motion is identical to the one obtained in dimension 1.

Using that $\|x + y\| = \|x\| + y\frac{x}{\|x\|} + O(\|y\|^2/\|x\|)$ and the branching property, one can prove that the motion of particles far away from the origin can be coupled with the motion of particles in a unidimensional branching Brownian motion [120, Section 5]. The proof of Theorem 4.D then relies on proving an analogue to (4.4) using connections with the unidimensional branching Brownian motion.

4.2.2 Decoration(s) of the branching Brownian motion

The decoration distribution $\mathcal{D}$ of the branching Brownian motion plays an important role in the description of the extremal process. To simplify the exposition, we restrict ourselves to *binary* branching Brownian motions. The properties of $\mathcal{D}$ have been studied notably by Cortines, Hartung and Louidor [77, 78], notably the behaviour of the intensity of this random point measure. However, a larger class of decoration distributions can be constructed, defined for all $a > \sqrt{2}$ by

$$\mathcal{D}^a := \lim_{t\to\infty} \mathbb{P}\left(\sum_{u\in\mathcal{N}_t} \delta_{X_t(u)-M_t} \in \cdot \middle| M_t \geq at \right). \tag{4.5}$$

These decorations distribution were introduced by Bovier and Hartung [55] when investigating the asymptotic behaviour of the extremal process of a time-inhomogeneous branching Brownian motion. They appear as the decoration point measures of the extremal processes of several variants of the branching Brownian motion.

An alternative description of these decorations was given in [M23], as random point measures conditioned on an event of positive probability. More precisely, given B a standard Brownian motion, $(\tau_j, j \in \mathbb{N})$ the atoms of a Poisson point process with intensity 2 and X^j i.i.d. branching Brownian motions, we define

$$\overline{D}^a = \delta_0 + \sum_{k=1}^\infty \sum_{u\in\mathcal{N}^k_{\tau_k}} \delta_{B_{\tau_k}-a\tau_k+X_{\tau_k}(u)}.$$

As long as $a > \sqrt{2}$, $\overline{D}^a$ is a well-defined random point measure, which can be described as the positions of particle alive at time 0 in the following process: a spine particle moves backwards in time according to a Brownian motion with drift $-a$, while giving birth to particles that will start independent branching Brownian motions (forward in time) at rate 2. The law $\mathcal{D}^a$ can the be constructed as follows.

Theorem 4.E (Berestycki, Brunet, Cortines, Mallein (2018))**.** *For all $a > \sqrt{2}$, we have*

$$\mathcal{D}^a := \mathbb{P}(\overline{D}^a \in \cdot | \overline{D}^a((0,\infty)) = 0).$$

In particular, the family of laws $(\mathcal{D}^a, a > \sqrt{2})$ is continuous in distribution. Moreover,

$$\mathbb{P}(\overline{D}^a((0,\infty)) = 0) = \sqrt{2\pi}a \lim_{t\to\infty} t^{1/2} e^{(\frac{a^2}{2}-1)t} \mathbb{P}(M_t \geq at).$$

This alternative description of the family of point measure distributions $\mathcal{D}^a$ allows in particular an explicit description of the constant appearing in the large deviations of the maximal displacement of the branching Brownian motion. Derrida, Meerson and Sasorov [83] conjectured that this constant satisfies $\mathbb{P}(\overline{D}^a((0,\infty)) = 0) \sim c(a-\sqrt{2})$ as $a \to \sqrt{2}$ for some explicit constant c. We also conjecture that this family can be used to represent the decoration $\mathcal{D}$ appearing in Theorem 4.6.

Conjecture 4.7. *We have $\lim_{a\to\sqrt{2}} \mathcal{D}^a = \mathcal{D}$ for the topology of weak convergence.*

Time-inhomogeneous branching Brownian motion. In [56], Bovier and Hartung proved that in the weak correlation regime, the centred extremal process of the time-inhomogeneous branching Brownian motion converges in law to a decorated Poisson point process.

Theorem 4.8 (Bovier and Hartung (2015)). *Let M_t be the maximal displacement at time t of a time-inhomogeneous branching Brownian motion with time-inhomogeneous variance profile σ satisfying the assumptions of Theorem 4.4. We set*

$$E_t := \sum_{u \in \mathcal{N}_t} \delta_{X_t(u) - \sqrt{2\tau^2}t + \frac{1}{2\sqrt{2\tau^2}} \log t},$$

where we recall that $\tau^2 = \int_0^1 \sigma_s^2 \mathrm{d}s$. The process E_t converges in law, for the topology of weak convergence to a decorated Poisson point process with intensity $cW_\infty(\sqrt{2\sigma_0^2/\tau^2})e^{-\sqrt{2\tau^2}x}\mathrm{d}x$, and decoration point measure obtained by dilating the point measure $\mathcal{D}^{\sqrt{2\sigma_1^2/\tau^2}}$ by $\sigma(1)$.

The form of the extremal point measure obtained here is consistent with Theorem 3.C and with Conjecture 3.6. In the strong correlation regime, the convergence in distribution of the extremal point measure is currently unknown. However, due to Conjecture 3.6, it becomes possible to make the following prediction.

Conjecture 4.9. *Let $(X_t(u), u \in \mathcal{N}_t)$ be a branching Brownian motion with time-inhomogeneous variance σ. We assume that σ is $\mathcal{C}^2$ and decreasing, and write*

$$E_t = \sum_{u \in \mathcal{N}_t} \delta_{X_t(u) - m_t},$$

with m_t the quantity defined in Theorem 4.5. The extremal process converges in law to a decorated Poisson point process with intensity $cZ_\infty e^{-\sqrt{2}/\sigma_0 x}\mathrm{d}x$, and with a decoration point measure that can be described as a decorated Poisson point process with intensity $e^{-\sqrt{2}/\sigma_1 x}\mathrm{d}x$ and decoration measure given by the dilatation of $\mathcal{D}$ by σ_1.

We remark that this conjecture is consistent with [55, Theorem 1.2], which describes the extremal process of a branching variance with piecewise constant non-increasing variance.

Multitype branching Brownian motion. In [M33], Belloum and Mallein study the asymptotic behaviour of the extremal process of two-type reducible branching Brownian motion. In this process, particles of type 1 move according to independent Brownian motions with variance σ^2 and branch at rate β into two children. Additionally, particles of type 1 create at rate 1 particles of type 2, which behave as independent standard binary branching Brownian motions. Biggins [45] computed the asymptotic behaviour of the speed in such processes, observing in particular that the speed of the two-type particle system is larger that the speed of either particles of type 1 or particles of type 2 alone if

$$\beta + \sigma^2 > 2 \quad \text{and} \quad (2\beta - 1)\sigma^2 < \beta. \tag{4.6}$$

These conditions in particular ensure that particles of type 1 branch at a higher rate ($\beta > 1$) but move with a lower speed ($\sigma^2 < 1$) than type 2 particles. In this case, the following result then holds.

Theorem 4.F (Belloum and Mallein (2021)). *We denote by*

$$v = \frac{\sigma^2 - \beta}{2\sqrt{(1-\sigma^2)(\beta - 1)}} \quad \text{and} \quad \theta = \sqrt{2\frac{\beta - 1}{1 - \sigma}},$$

and set $E_t := \sum_{u \in \mathcal{N}_t} \delta_{X_t(u) - vt}$. This process converges in law for the topology of the weak convergence to a decorated Poisson point process with intensity $cW(\sigma\theta)e^{-\theta x}\mathrm{d}x$ and decoration distribution $\mathcal{D}^\theta$.

This result is notably analogous to Theorem 4.4, obtained by Bovier and Hartung for the variable speed branching Brownian motion. Indeed, in this regime, particles that contribute to the extremal process at time t remain of type 1 for a time $pt + O(\sqrt{t})$ before changing to type 2. This strategy allows to use the first half of the time to produce a lot of offspring, of which one will reach a high position at time t.

When parameters β, σ^2 do not satisfy (4.6), similar results to Theorem 4.F are known to hold. In particular, if $\beta = \sigma^2 = 1$, Belloum [27] proved that the extremal process of this system is the same as the limiting extremal process of a single type branching Brownian motion, shifted by $\sqrt{2}\log t$. It yields to the following conjecture.

Conjecture 4.10. *Let $(X_t(u), u \in \cup_{k\geq 1}\mathcal{N}_t^k)$ be a cascading family of branching Brownian motions, in which particles of type k (in the set $\mathcal{N}_t^k$) behave as in a standard branching Brownian motion, and additionally give birth to particles of type $k+1$ at rate α. For all $k \in \mathbb{N}$, the extremal process*

$$E_t^k := \sum_{u\in\mathcal{N}_t^k} \delta_{X_t(u)-\sqrt{2}t+\frac{3-2k}{2\sqrt{2}}\log t}$$

converges in law as $t \to \infty$ for the topology of weak convergence towards a decorated Poisson point process with intensity $c_{k,\alpha}e^{-\sqrt{2}x}\mathrm{d}x$ and decoration measure $\mathcal{D}$.

4.2.3 The extremal process of the branching random walk

Let X be a non-lattice branching random walk satisfying assumption $\mathcal{A}$. We recall the notation

$$v = \inf_{\theta>0}\frac{\kappa(\theta)}{\theta} = \frac{\kappa(\theta^*)}{\theta^*} \quad \text{and} \quad m_n = nv - \frac{3}{2\theta^*}\log n.$$

Under these conditions, we recall that Aïdékon [4] proved the convergence in distribution of $M_n - m_n$ towards a shifted Gumbel distribution. Under the same conditions, Madaule [136] proved the convergence in distribution for the extremal process

$$E_n := \sum_{|u|=n} X(u) - m_n.$$

Theorem 4.11 (Madaule (2017))**.** *Given X a non-lattice branching random walk satisfying assumption $\mathcal{A}$, the point measure E_n converges in law to a decorated Poisson point process with intensity $c_\star\theta^* Z_\infty e^{-\theta^* x}$.*

The proof used by Madaule to obtain the convergence in distribution of the extremal process relies on a tightness argument, and an identification of the limit by its superposition properties (due to Maillard [140]). As a result, the distribution of the decoration does not has a simple an expression as in branching Brownian motions settings. In [M12], an expression of this distribution is obtained as

$$\mathcal{D} = \lim_{n\to\infty}\sum_{|u|=n} \delta_{X(u)-M_n}\mathbf{1}_{\{u\in G_n\}},$$

with G_n the set of particles alive at time n that share a common ancestor with the rightmost particle alive at time n alive after generation $n/2$. The proof is based on the following straightforward extension of the convergence obtained by Madaule.

Theorem 4.G (Mallein (2018)). *Given X a non-lattice branching random walk satisfying assumption $\mathcal{A}$, the random point measure $\overline{E}_n := \sum_{|u|=n} \delta_{u,X(u)-m_n}$ converges in law to a decorated Poisson point process with intensity $c_\star \theta^* Z_\infty(\mathrm{d}u)e^{-\theta^* x}$, with Z_∞ the random measure on $\mathbb{N}^{\mathbb{N}}$ such that for all $u \in \mathbb{U}$,*

$$Z_\infty(\{v : v_{|u|} = u\}) = \lim_{n\to\infty} \sum_{|w|=n} (nv - X(w))e^{-\theta^*(X(w)-nw)}\mathbf{1}_{\{w_{|u|}=u\}} \quad a.s.$$

Note that the enriched point measure $\overline{E}_n$ allows to study the joint convergence in distribution of the positions of faraway particles and the genealogical relationships of the different clusters of particles. It remains an open problem to provide a convergence result for the extremal process of spatial branching processes recording both the genealogical relationships of leaders belonging to different cluster, as well as those within each cluster. This result can be used to obtain the convergence in distribution of the so-called overlap distribution of the branching random walk at low temperature.

Enriching the information in the extremal process allows us to obtain more precise information on the behaviour of particles reaching a large position in the branching random walk. For example, Chen, Madaule and Mallein [M6] obtained the convergence in distribution of the trajectories yielding to the family of particles in the branching random walk.

Theorem 4.H (Chen, Madaule and Mallein (2015)). *Let X be a non-lattice branching random walk satisfying assumption $\mathcal{A}$, we denote by*

$$\widetilde{E}_n := \sum_{|u|=n} \delta_{X(u)-m_n, H_n(u)},$$

with $H_n(u) = \left(\frac{X(u_{\lfloor nt \rfloor})-ntv}{\sqrt{n}}, t \in [0,1]\right)$ the scaled trajectory followed by particle u. We have

$$\lim_{n\to\infty} \widetilde{E}_n = \sum_{k=1}^{\infty} \sum_{d \in D^{(k)}} \delta_{\xi_k + d, \sigma \mathbf{e}_k},$$

where $(\xi_k, k \geq 1)$ are the atoms of a Poisson point process with intensity $cZ_\infty e^{-\theta_* x}\mathrm{d}x$, $(D^{(k)})$ are i.i.d. random point measures with law $\mathcal{D}$ and $(\mathbf{e}_k, k \geq 1)$ are i.i.d. standard Brownian excursions

In other words, the trajectories followed by the leaders of each cluster of particles are given by i.i.d. Brownian excursion above the boundary of the branching random walk, and all particles in a same cluster follow the same trajectory as its leader. This result generalizes the observation of Chen [73], that the scaled trajectory leading to the maximal displacement in the branching random walk converges to a Brownian excursion.

We strongly expect that the decoration appearing in the branching random walk can be described, similarly to the branching Brownian motion setting, as the extremal process on the branching random walk conditioned on a moderate deviation event. More precisely, we expect the following to hold.

Conjecture 4.12. *Given X a non-lattice branching random walk satisfying assumption $\mathcal{A}$, we have*

$$\mathcal{D} = \lim_{n\to\infty} \mathbb{P}\left(\sum_{|u|=n} \delta_{X(u)-M_n} \in \cdot \middle| M_n \geq nv \right).$$

Similarly, we expect that similar results to the one proved for branching Brownian motions to hold in a multitype reducible branching random walk or in a time-inhomogeneous branching random walk.

CHAPTER 5

Branching-selection particle systems

"La raison du plus fort est encore la meilleure :
Nous l'allons montrer tout à l'heure."

Jean de la Fontaine – Le loup et l'agneau, 1668.

Summary.

We give in this chapter some result related to the evolution of branching-selection particle systems. These particle systems model the evolution of a population with limited resources that individuals compete for. A prototypical example of such a system is the N-branching random walk. In this model, at each generation the N particles occupying the rightmost positions (treated as their fitness in this model) reproduce in the next generation, while all others die. Brunet, Derrida and their co-authors made prediction on the asymptotic behaviour of the cloud of particle, as well as the genealogical relationships between particles over time. Bérard and Gouéré obtained the asymptotic behaviour of the speed of the N-branching random walk as N grows to ∞. This result was generalized in [M4], while the case of branching random walk with a varying population size is considered in [M3].

We then focus on an exactly solvable model, the exponential model introduced by Brunet, Derrida, Mueller and Munier [61]. Cortines and Mallein studied various generalization of that process, which we describe below. In particular, a solvable model for branching Ornstein-Uhlenbeck process can be showed to fall outside the universality class of the Brunet-Derrida processes.

In a final section, we present an application of the computation of the speed of branching-selection particle systems to the study of the longest path in a directed Erdős-Rényi random graph. More precisely, we show that the growth rate of the longest increasing path in a sparse Erdős-Rényi graph with vertex set $\{1, \ldots, n\}$ can be computed via a coupling with a continuous-time N-branching random walk [M15]. We also give the asymptotic properties of the growth rate $C(p)$ of the Erdős-Rényi graph in the dense phase. This function is proved to be analytic on $(0, 1]$ with an asymptotic expansion at 1 consisting only of integers [M19].

A branching-selection particle system is a particle system in which each individual moves and reproduces as in a spatial branching process, while undergoing a selection procedure. This selection has the effect of giving a reproductive advantage to some of the particles, based on their position. These models can often be based on a *fitness* function f, which associate to a particle located at x a value $f(x)$ representing its degree of adaptation to the current environment. A large class of phenomena observed in natural selection can be modelled by branching-selection particle systems.

In the present chapter, we focus on the so-called N-branching random walk, a particle system in which every particle alive at each generation reproduces as in the branching random walk, but among the children only the N rightmost are selected to reproduce in the next generation. We discuss in particular the rate of adaptation of the process, i.e. the speed at which particles move to areas of higher fitness. We take a particular interest in the exponential model, an exactly solvable branching-selection system. Exact computations on this model are used to exhibit phenomena of interest and make prediction on the asymptotic behaviour of generic branching-selection particle systems.

We present in Section 5.2 an example of application of the study of the speed of adaptation of the N-branching random walk. We take interest in the length of the longest increasing path in an Erdős-Rényi random graph. The length of this path in a large sparse graph is connected to the speed of a continuous-time N-branching random walk in which every particle gives birth at rate 1 to a child one unit of space to its right, such that at every branching event, the leftmost particle gets killed.

5.1 The N-branching random walk

The N-branching random walk is a simple branching-selection particle system in which at each generation, the N rightmost particles are allowed to reproduce and give birth to the next generation. More precisely, for all $n \in \mathbb{N}$, we write $X_n^N(1), \ldots X_n^N(N)$ for the positions of all particles alive in the nth generation, ranked in the decreasing order. Then, setting $(L_n(1), \ldots, L_n(N))$ the point measures describing the relative positions of the children of each of these particles, we define $X_{n+1}^N(1), \ldots X_{n+1}^N(N)$ as the N largest values, ranked in the decreasing order, of

$$\left\{ X_n^N(j) + \ell, \ell \in L_n(j), 1 \leq j \leq N \right\}.$$

The genealogical relationships of particles as recorded using ordered partitions of integers. For $k \leq n$, we write $\Pi_{k,n}^N$ for the partition $(\pi_1, \ldots, \pi_N)$ of $\{1, \ldots, N\}$ such that $i \in \pi_j$ if and only if the particle at position $X_n^N(i)$ is a descendant of the particle at position $X_k^N(j)$. This family of partitions satisfy a flow property, for all $p \leq q \leq r$

$$\Pi_{p,r}^N = \mathrm{Coag}(\Pi_{q,r}^N, \Pi_{p,q}^N),$$

with $\mathrm{Coag}(\Pi, \Pi') = \left(\cup_{i \in \pi'_j} \pi_i, 1 \leq j \leq n \right)$.

The N-branching random walk was introduced by Brunet and Derrida [59] as a model connected to a noisy versions of the F-KPP equation. In this article, they conjectured that the speed v_N of this cloud of particle, defined as

$$v_N = \lim_{n \to \infty} \frac{X_n^N(1)}{n} = \lim_{n \to \infty} \frac{X_n^N(N)}{n} \quad \text{a.s.}$$

will satisfy, under typical integrability conditions

$$v_\infty - v_N \sim_{N \to \infty} \frac{\chi}{(\log N)^2}, \tag{5.1}$$

with χ a positive constant depending only on the reproduction law of the branching-selection process, and v_∞ the speed of the associated branching random walk without selection. We refer to this slow convergence of the speed of the branching particle system with selection to the speed of the branching particle system without selection as the Brunet-Derrida behaviour of branching-selection particle systems. This result hints at the *algorithmic hardness* [2] of the exploration of a branching process to identify a particle at time n with a large displacement. The number of sites that need to be visited in order to detect a particle in the neighbourhood of nv_N at time n is indeed of order nN.

Through the introduction of more precise analytic approximations of the N-branching random walk and the study of an exactly solvable model [60, 61], Brunet, Derrida, Mueller and Munier refined the predictions for the asymptotic behaviour of the N-branching random walk. They predicted that

$$v_\infty - v_N = \frac{\chi}{(\log N + 3\log\log N + O(1))^2}, \tag{5.2}$$

and that its genealogical structure, on a time-scale $(\log N)^3$, converges to the Bolthausen-Sznitman coalescent [50].

Bérard and Gouéré [64] proved the Brunet-Derrida behaviour of a N-branching random walk in which every particle gives birth to two children at each generation. Their proof is based on the following general coupling argument, which informally states that a branching-selection particle system with more particles further to its right will displace faster to the right.

Lemma 5.1 (Bérard and Gouéré (2010)). *Let $n \geq p$ and $(x_1, \ldots x_n), (y_1, \ldots, y_p)$ such that*

$$x_1 \geq y_1, x_2 \geq y_2, \ldots x_p \geq y_p.$$

Then, we can construct on the same probability space the positions $(\overline{X}_k)$ of the children of particles located at positions $(x_1, \ldots, x_n)$ and $(\overline{Y}_k)$ of the particles located at positions $(y_1, \ldots y_p)$ such that

$$\forall j \in \mathbb{N}, \overline{X}_j \geq \overline{Y}_j.$$

This lemma implies that in any branching-selection particle system in which the rightmost particles are selected at each generation, the partial order

$$x \preccurlyeq y \iff x_1 \leq y_1, \ldots x_{|x|} \leq y_{|x|}$$

can be preserved as long as there are at all times more particles in the larger particle system than in the smaller particle system. This coupling allows us to compare different selection procedures, as long as the total number of particles remain of the same order of magnitude.

In particular, the Brunet-Derrida behaviour of the branching random walk can be obtained by comparison with a branching random walk killed by a linear, close to critical boundary. Gantert, Hu and Shi [98] precisely computed the survival probability in such a branching process with quasi-critical killing (see also a similar proof by Bérard and Gouéré [65]).

Theorem 5.2 (Gantert, Hu, Shi (2011)). *Let X be a non-lattice branching random walk satisfying assumption $\mathcal{A}$. For $\varepsilon > 0$, we set*

$$\varrho(\varepsilon) = \mathbb{P}\left(\forall n \in \mathbb{N}, \exists u \in \mathbb{U} : |u| = n \text{ and } X(uk) \geq n(v - \varepsilon)\right).$$

We have $\log \varrho(\varepsilon) \sim -\sqrt{\frac{\pi^2}{2\varepsilon}\theta^*\kappa''(\theta^*)}$ *as $\varepsilon \to 0$.*

Using this result, Bérard and Gouéré proved that the N binary branching random walk with selection satisfy the Brunet-Derrida behaviour. Indeed, note that choosing $\varepsilon \approx (\log N)^2$, the probability for one particle to stay above the line of slope $v - \varepsilon$ will be close to $\frac{1}{N}$. Moreover, conditionally on surviving, this particle will take a time of order $(\log N)^3$ to create N new particles. Chaining this argument gives the following result, which was generalised [M4] to arbitrary N-branching random walks satisfying $\mathcal{A}$.

Theorem 5.3 (Bérard and Gouéré (2010), Mallein (2018))**.** *Writing v_N for the speed of a non-lattice N-branching random walk satisfying assumption $\mathcal{A}$, and v the speed of the associated branching random walk without selection, we have*

$$v - v_N \sim \frac{\pi^2}{2(\log N)^2}\theta^* \kappa''(\theta^*) \text{ as } N \to \infty.$$

One can consider N-branching random walks that do not satisfy the Brunet-Derrida behaviour. For example, in [M4] we studied N-branching random walks such that the spine associated to the critical parameter are in the domain of attraction of an α-stable random variable. In this case, one has

$$v - v_N \sim \frac{L(\log N)}{(\log N)^\alpha} \text{ as } N \to \infty,$$

where L is a slowly varying random variable. In other words, these random walks do not satisfy the Brunet-Derrida behaviour. Bérard and Maillard [28] studied the asymptotic behaviour of N-branching random walks with polynomial tails, in which the asymptotic behaviour of the cloud of particles as well as the genealogical structure are dominated by the large jumps. Penington, Roberts and Talyigás [158] took interest in N-branching random walks with stretch exponential tails, which also falls in the "one big jump" class of random variables.

An other way to give a description of the cloud of particles drifting at speed v_N in the N-branching random walk is to prove its scaled convergence as $N \to \infty$, to a limiting profile [87, 82]. For example, the evolution of the density u^N of an N-branching Brownian motion can be shown to converge, up to a proper scaling, to the solution (u, γ) to the following free boundary problem

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x) & \text{if } x > \gamma_t \\ u(t,x) = 0 & \text{if } x \leq \gamma_t \\ \int_{\gamma_t}^{\infty} u(t,x) = 1. \end{cases}$$

The first equation in this free boundary problem illustrates that while not under selections, particle are diffusing while creating offspring freely, while the second one reproduces the strong selection procedure at the left boundary of the system. The third equation defines γ_t as the position of the leftmost particle sustainable by the environment. Generalizations of this free boundary problem can be obtained for various branching-selection systems, as in [33].

The exponential model The exactly solvable N-branching random walk, introduced by Brunet, Derrida, Mueller and Munier [61] is a branching-selection particle system in which every particle gives birth to a Poisson point process with intensity $e^{-x}\mathrm{d}x$. Then the N rightmost particles are selected to form the next generation. For all $n \in \mathbb{N}$, we set

$$X_n^N(\mathrm{eq}) = \log\left(\sum_{j=1}^{N} e^{X_n^N(j)}\right). \tag{5.3}$$

Using the superposition property of the Poisson point process, we remark that the set of children of the nth generation are positioned as a Poisson point process with intensity $e^{-(x-X_n^N(\mathrm{eq}))}\mathrm{d}x$ (in other words, the set of children of the process is similar to the children of one particle positioned at $X_n(\mathrm{eq})$). As a result,

$$\left(\left(X_{n+1}^N(j) - X_n^N(\mathrm{eq}), j \leq N\right), n \in \mathbb{N}\right)$$

are i.i.d. random variables distributed as the largest N atoms in a Poisson point process. The asymptotic behaviour of the cloud of particles becomes explicit, as $\left(X_n^N(\mathrm{eq}), n \geq 1\right)$ is a random walk with finite mean, yielding by law of large numbers

$$v_N = \mathbb{E}\left(X_1^N(\mathrm{eq}) - X_0^N(\mathrm{eq})\right) = \ln\ln N + \frac{\ln\ln N + 1}{\ln N}(1 + o(1)) \text{ as } N \to \infty.$$

By comparing this result to N-branching random walks, Brunet, Derrida, Mueller and Munier were able to state the conjecture (5.2).

The genealogical structure of this process becomes exactly solvable as well. Indeed, using again the superposability property of Poisson point processes, we observe that independently for each individual i alive at time $n+1$, the probability that i is a child of the parent j is given by

$$\frac{e^{X_n^N(j)}}{\sum_{k=1}^N e^{X_n^N(k)}} = e^{X_n^N(j) - X_n^N(\mathrm{eq})}.$$

In particular, the law of the parent of a particle does not depend on its position, only on the relative positions of the particles at the previous generation. Remarking that the N rightmost particles in a Poisson point process with exponential intensity can be represented as a random shift of N i.i.d. exponential random variables, we deduce that the genealogical structure of the exponential model is given by a Cannings model [68, 69]. As a result, we deduce (see e.g. [149]) that the genealogical tree of the exponential model converges in law to the Bolthausen-Sznitman coalescent on the time scale $\log N$.

In [M10] and [M21], Cortines and Mallein considered modifications of the exponential model to construct branching-selection particles system that do not satisfy the Brunet-Derrida behaviour, and more importantly that do not exhibit a dichotomy between Bolthausen-Sznitman and Kingman's genealogies. It is well-known that in neutral population models, a one-parameter family of neutral models, the Beta coalescents, appear as an intermediate family between the Bolthausen-Sznitman coalescent in which a small number of ancestors dominate the genalogy, and the Kinman's coalescent in which small genealogical events appear on a longer time scale, see e.g. Schweinsberg [169]. However, in branching-selection models no known intermediary model, which would represent a moderate pressure of selection, was known.

In [M10], Cortines and Mallein replaced the selection of the N rightmost particles with a random selection procedure that advantages particles to the right. More precisely, for a fixed $\beta > 1$, the N particles of the $(n+1)$st generation are selected from the offspring of particles in the nth generation at random without replacement, such that a particle at position x is selected with probability proportional to $e^{\beta x}$.

Theorem 5.A (Cortines and Mallein (2017)). *In the (N, β)-branching random walk, we have*

$$v_N \sim \log\log N \textit{ as } N \to \infty,$$

and the genealogy of the process converges on the $\log N$ scale to a Bolthausen-Sznitman coalescent.

In other words, the random selection procedure does not modify the main macroscopic properties of the branching-selection procedure. Schertzer and Wences [179] studied further modifications of the random selection procedure and obtained an interplay between the Bolthausen-Sznitman coalescent and a discrete Poisson-Dirichlet coalescent.

In [M21], Cortines and Mallein turned to an exponential branching-selection particle system evolving according to the following procedure. Given $a < 1$, in this system a particle at position x gives birth to a Poisson point process with exponential intensity shifted by ax. In this model, the selection advantage of an individual is partially diminished by the multiplication by a. We refer to this process as the (N, a) branching Ornstein-Uhlenbeck process, as the multiplication to the positions by a can be identified by the application of a linear force calling back particles to 0.

The effect of a partially inherited fitness by a child can be observed in populations with sexual reproduction [166], in which an individual with an important selective advantage reproduces with average members of the population. For example, the height of the children of tall parents is usually larger than in the overall population, but smaller than their parent.

In the (N, a) branching Ornstein-Uhlenbeck process, the formula for the position of the center of mass of the cloud has to be modified to

$$X_n^N(\mathrm{eq}) = \log\left(\sum_{j=1}^N e^{aX_n^N(j)}\right).$$

There, we have again that $\left(\left(X_{n+1}^N(j) - X_n^N(\mathrm{eq}), j \leq N\right), n \in \mathbb{N}\right)$ are i.i.d. copies of the N rightmost atoms in a Poisson point process with exponential intensity, using again the superposability properties of Poisson point processes. In particular, observe that

$$X_{n+1}^N(\mathrm{eq}) = aX_n^N(\mathrm{eq}) + Y_{n+1},$$

with $(Y_n, n \geq 1)$ i.i.d. random variables with finite moments.

As a result, as long as $|a| < 1$, $X_n^N(\mathrm{eq})$ converges in distribution to a well-defined random variable as $n \to \infty$. Hence the cloud of particles remains asymptotically stable. Similarly, the genealogy can be expressed explicitly using that the probability that the individual i at the $n+1$st generation is a child of the parent j is given by

$$\frac{e^{aX_n^N(j)}}{\sum_{k=1}^N e^{aX_n^N(k)}}.$$

This dynamics again define a Canning's model, which yields the following result.

Theorem 5.B (Cortines and Mallein (2018)). *In the (N, a)-branching Ornstein-Uhlenbeck process, if $|a| < 1$ we have*

$$\lim_{N\to\infty} \lim_{n\to\infty} \mathbb{E}(X_n^N(1)) - \log N - \gamma = \lim_{N\to\infty} \lim_{n\to\infty} \mathbb{E}(X_n^N(N)) = -\frac{\log(1-a)}{1-a},$$

where γ is the Euler-Mascheroni constant.

Moreover, writing (Π_n^N) for the genealogical process, as $N \to \infty$, we have

1. *if $0 < a < 1/2$ then $(\Pi^N_{\lfloor tN \rfloor}, t \geq 0)$ converges in law to the Kingman's coalescent;*
2. *if $a = 1/2$ then $(\Pi^N_{\lfloor tN/logN \rfloor}, t \geq 0)$ converges in law to the Kingman's coalescent;*
3. *if $1/2 < a < 1$ then $(\Pi^N_{\lfloor tN^{(1-a)/a} \rfloor}, t \geq 0)$ converges in law to the Beta$(2 - a^{-1}, a^{-1})$ coalescent;*

4. *if* $a = 1$ *then* $\Pi^N_{\lfloor t \log N \rfloor}, t \geq 0)$ *converges in law to the Bolthausen-Sznitman coalescent;*
5. *if* $a > 1$ *then* Π^N *converges in law to a discrete-time coalescent.*

The (N, a)-branching Ornstein-Uhlenbeck process is the first proposed model of natural selection with a genealogical structure different from the Bolthausen-Sznitman or the Kingman's coalescents. Since then, several newer, and more biologically relevant, models were introduced, which could exhibit a genealogical structure given by the Bolthausen-Sznitman coalescent. Tourniaire [177] introduced a branching Brownian motion with absorption in which the branching mechanism is larger close to the absorbing barrier, whose population size converges to a stable CSBP whose genealogy is given by the Beta coalescent. Roberts and Schweinsberg [164] proposed a branching Brownian motion in which particles at position x has an effective (birth rate minus death rate) branching of βx.

The results obtained on the (N, a)-branching Ornstein-Uhlenbeck process yield to the following conjecture on the asymptotic behaviour of branching Ornstein-Uhlenbeck processes with selection, present in [M21]

Conjecture 5.4. *Let* Π^N *be the genealogy of a branching Ornstein-Uhlenbeck process with selection of the* N *rightmost particles, such that particles at position* x *have a drift of* $-\frac{\gamma x}{(\log N)^2}$. *There exists* $a_\gamma \in [0, 1]$ *such that* $(\Pi^N_{\lfloor t N^{a_\gamma} \ell(N) \rfloor})$ *converges in distribution to the Beta*$(1 - a_\gamma, 1 + a_\gamma)$ *coalescent (or the Kingman's coalescent if* $a_\gamma = 1$*). Moreover,* $\gamma \mapsto a_\gamma$ *is non-decreasing, and there exists* $0 < \gamma_\star < \infty$ *such that*

$$a_\gamma < 1 \iff \gamma \leq \gamma_\star.$$

5.2 Length of the longest path in a directed Erdős-Rényi graph

In this section, we present an application of the computation of the speed of a N-branching random walk to the computation of the length of the longest path in a directed acyclic Erdős-Rényi graph, which we refer to as the Barak-Erdős graph [26]. This random graph was introduced by Barak and Erdős, who studied its number of strongly independent vertices, and can be constructed as follows. Given $p \in [0, 1]$ and $n \in \mathbb{N}$, the set of vertices is given by $\{1, \ldots, n\}$, and for each $1 \leq i < j \leq n$, a directed edge from i to j is present with probability p, independently of any other edge.

The Barak-Erdős graph can be used to model community food webs in ecology [152], or as the task graph for parallel processing in computer sciences [99]. In this case, an edge is present between i and j if completing task i is necessary to start task j. A commonly studied feature of this random graph is the length of its longest path L_n, as $L_n + 1$ is exactly the number of steps that a massively parallelized system would need to complete all n tasks. Using the subadditivity of the sequence $(L_{n+1}, n \geq 0)$, Newman [153] obtained the almost sure asymptotic behaviour for the maximal displacement.

Theorem 5.5 (Newman (1992))**.** *There exists a continuous increasing function* C *such that for all* $p \in [0, 1]$*, writing* L_n *for the length of the longest path in a Barak-Erdős graph with parameters* n *and* p*, we have*

$$\lim_{n \to \infty} \frac{L_n}{n} = C(p) \quad \textit{in probability.}$$

Moreover, $C(0) = 0$*,* $C'(0) = e$ *and* $C(1) = 1$*.*

To study more precisely the properties of the function C, Foss and Konstantopoulos [96] introduced a coupling between the Barak-Erdős graph and a long-memory Markov chain called the infinite-bin model. This process can be described as a balls-and-bins scheme, in which a bin is associated to each integer $k \in \mathbb{Z}$. In the initial configuration, an infinite number of balls is added to the bins, in such a way that one can define the rightmost occupied bin, that we call the front of the configuration. Then, at each step $n \geq 0$, a new ball is added to the right of the ξ_nth rightmost ball, with $(\xi_n, n \geq 1)$ a sequence of i.i.d. integer-valued random variables. For all $n \in \mathbb{N}$, we denote by F_n the position of the front at the nth step.

Proposition 5.6 (Foss and Konstantopoulos (2003)). *Let $p \in (0,1)$, if $(\xi_n, n \geq 1)$ is distributed as a sequence of i.i.d. Geometric random variables with parameter p, then*

$$\lim_{n\to\infty} \frac{F_n}{n} = C(p) \quad a.s.$$

Using this coupling, as well as an analogue to Lemma 5.1 for infinite-bin models, Foss and Konstantopoulos obtained upper and lower bounds for the function C. In [M19] and [M15], Mallein and Ramassamy extended the analysis of infinite-bin models to refine these estimates. In particular, they proved that C is analytic on $(0,1]$, with an explicit formula for the asymptotic expansion around $p = 1$.

Theorem 5.C. *The function $p \mapsto C(p)$ is analytic on $(0,1]$. Moreover, for $p > 1/2$, we have*

$$C(p) = \sum_{n=0}^{\infty} \sum_{|u|=n} \varepsilon(u)(1-p)^n p^{R(u)},$$

with $R(u) = \sum_{j=1}^{|u|} u(j)$ and ε an explicit function $\mathcal{U} \to \{-1,0,1\}$.

In particular, the function C admits the following expansion as $p \to 1$

$$\begin{aligned} C(p) = 1 &- (1-p) + (1-p)^2 - 3(1-p)^3 + 7(1-p)^4 - 15(1-p)^5 + 29(1-p)^6 \\ &- 54(1-p)^7 + 102(1-p)^8 - 192(1-p)^9 + 375(1-p)^{10} + O\left((1-p)^{11}\right). \end{aligned}$$

The first 18 terms of this asymptotic expansion are known, and can be found as sequence A321309 of the OEIS [174]. We believe that the sequence is alternate and increasing in absolute value, but this is not yet known. The radius of convergence of C around $p = 1$ is larger than $1/4$, but smaller than 1. This result is mostly used by making a perturbation analysis of patterns appearing in the sequence $(\xi_n, n \in \mathbb{N})$ as $p \to 1$.

The asymptotic behaviour of C in the sparse graph limit, i.e. as $p \to 0$, is obtained in a very different fashion. Here, it is noted in [M19] that the infinite-bin model is liked to a branching random walk with selection. We first observe that if in an infinite-bin model balls are added at rate 1, the process can be described alternatively as follows: each ball currently present in the system gives birth independently to offspring one unit of space to its right, in such a way that the kth rightmost ball give birth to offspring at rate $p(1-p)^k$. Thus, it can be thought-of as a rank-based branching process.

In the limit $p \to 0$, this branching process becomes comparable with an $\left\lfloor \frac{1}{p} \right\rfloor$-branching random walk. Using the Brunet-Derrida behaviour of this branching random walk, it then becomes possible to prove the following result.

Theorem 5.D (Mallein and Ramassamy (2019)). *We have $C(p) = pe\left(1 - \frac{\pi^2}{2(\log p)^2}\right) + o(p(\log p)^{-2})$ as $p \to 0$.*

The computation of an explicit value for $C(p)$ can be attained through different methods. In particular, as $(L_{n+1}, n \geq 0)$ is a subadditive sequence of random variables, we have

$$C(p) = \inf_{n \geq 1} \frac{\mathbb{E}(L_{n+1})}{n} = \lim_{n \to \infty} \frac{L_n}{n} \quad \text{a.s.}$$

Hence, C can be approached by simulating an infinite-bin model for a long time, or using Monte-Carlo simulations. However, both these methods give a biased estimate for C, that they typically overestimate. In [M39], Foss, Konstantopoulos, Mallein and Ramassamy describe a method for constructing a Bernoulli variable $X(p)$ with parameter $C(p)$. In words, this method consists in constructing a stationary version of the infinite-bin model, and identifying renovation events in its past, so that one can compute explicitly in terms of the sequence $(\xi_{-T}, \ldots, \xi_0, \xi_1)$ the variable $X(p)$ which is equal to 1 if and only if the ball added at time 1 is added in a previously empty bin.

Theorem 5.E (Foss et al. (2021))**.** *Let*

$$T := \inf\{k \in \mathbb{N} : \xi_{-k} = 1 \text{ and } \xi_{-k+j} \leq j \text{ for } 1 \leq j \leq k\},$$

there exists an explicit measurable function F such that $X(p) = F(\xi_{-T}, \ldots, \xi_0, \xi_1)$. Additionally, we have $\mathbb{E}(X(p)) = C(p)$.

Using this construction, it becomes possible to estimate $C(p)$ by Monte-Carlo methods. We complete this short tour of Barak-Erdős graph by a mention of an estimate on the length of the *shortest* path between vertices 1 and n in these graphs, obtained in [M41].

Theorem 5.F (Mallein and Tesemnikov (2022))**.** *Let $\gamma \in (0, 1)$ and $\theta > 0$. We denote by R_n the length of the longest path between vertices 1 and n in a Barak-Erdős graph with parameters n and $\frac{\theta}{n^\gamma}$. We have*

$$\lim_{n \to \infty} \mathbb{P}(R_n \leq k) = \begin{cases} 0 & \text{if } k < \frac{1}{1-\gamma}, \\ e^{-\frac{\theta^k}{k!}} & \text{if } k = \frac{1}{1-\gamma}, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, we mention that for $\gamma = 1$, the Barak-Erdős graph becomes disconnected, so that $R_n = \infty$ with high probability for n large enough. These estimates are obtained by Chen-Stein's method, showing that the number of paths of length k between 1 and n converge to a Poisson random variable.

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