# MATHEMATICAL NOTES (14): "EVERY POLYNOMIAL HAS A ROOT"

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1. Any proof of this theorem† must appeal by the nature of the case to limit-processes lying outside algebra proper. The true algebraist, indeed, profiting by the crime of others, is not interested in the sordid details, and any appeal my remarks may have will probably be to the analyst.

If limit processes are to be reduced to a minimum we must exclude the circular functions‡, and what I have to say presupposes that they are absolutely barred. This disallows the proof by

$$\text{“}\Delta \arg f(z) = 2n\pi \text{ for a large circle”}.$$

It also disallows the summary proof that $z^n = a+ib$ has a root (and this is important for the sequel).

There is a proof, one of Gauss's§, which appeals only to the principles:

(A 1) *the square root of a positive number exists*;

(A 2) *an equation of odd degree with real coefficients has a real root*;

and is otherwise algebraical. This would seem to be the last word from a purely logical standpoint, but for that very reason it is rather sophisticated, and any proof that is really easy to follow and to remember is worth at least passing attention.

There is a proof|| which begins very congenially to the analyst. Besides (A 1) and (A 2) it appeals to:

(A 3) *a continuous function attains its lower bound* (*in a closed set*).

If $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ has no root, then, since $|f(z)|$ is large when $|z|$ is large, it attains its lower bound, different from 0, for some value

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† In its ordinary sense: the variable and the coefficients are complex numbers and some complex number is to be a root.

‡ Whose foundation requires either integration or infinite power series.

§ It is given (considerably disguised for the amateur by its modern dress) in van der Waerden's *Moderne Algebra*, I, § 67.

|| Also due to Gauss. See Weber's *Algebra*, I, §§ 35 and 41.

$z_0$ of $z$. We aim now at the contradiction that $|f(z)|$ is decreased if $z$ moves a small distance from $z_0$ in an appropriate "direction". Let

$$f(z_0+\zeta) = A_0+A_1\zeta+\ldots+\zeta^n,$$

where $A_0=f(z_0) \neq 0$. If $A_1 \neq 0$ there is no difficulty in specifying the "direction"; we take $\zeta = -(A_0/A_1)\delta$, when

$$|f(z_0+\zeta)| = |A_0|\left(1-\delta+O(\delta^2)\right),$$

and is less than $|A_0| = |f(z_0)|$ if $\delta$ is small and positive. Since, however, $A_1$ may be so unfair as to vanish, we must consider the general case when $A_m$ is the first of $A_1, A_2, \ldots, A_n = 1$ that does not vanish. Now we can proceed practically as before*, *provided* we can solve the general equation of the type $z^n = a+ib$, to which problem our task is accordingly reduced. The fact that $z^2 = a+ib$ is soluble is well known†, and we may consequently suppose that $n$ is odd.

2. So far all is unforgettable, once known. But when, at long intervals, I have wished to reconstruct the complete proof, I have stuck at the next step, long enough at least to make me look it up; other analysts have probably had the same experience.

The usual argument is: the solution of $z^n = a+ib$ where $b$ is not zero) is equivalent to that of the simultaneous equations (in real $x$, $y$)

$$(1) \qquad (x^2+y^2)^n = a^2+b^2,$$

$$(2) \qquad \frac{(x+iy)^n\,(a-ib)-(x-iy)^n\,(a+ib)}{2i} = 0.$$

Now (2) is an equation in $y/x$ of odd degree with real coefficients, and has a real root; $x$ and $y$ can then be obtained from (1)‡.

The observation I have now to make is that we can escape this argument by repeating the original idea, the details being forced by the necessities of the case: it is only necessary to apply the "lower bound" method to

* Taking $\zeta = \delta u$, where $u^m = -A_0/A_m$, so that

$$|f(z_0+\zeta)| = |A_0|\left(1-\delta^m+O(\delta^{m+1})\right).$$

† $\pm\zeta$ are solutions, where

$$\zeta = \sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2+b^2)}\}}+i\,\operatorname{sgn} b\,\sqrt{\{-\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2+b^2)}\}}.$$

‡ $r^n-c=0$ bein soluble for odd $n$ and positive $c$ as a special case of (A 2).

the polynomial $f(z) = z^n - (a+ib)$ itself, $a$ and $b$ not being both zero. If there is no root, $|f|$ attains a non-zero lower bound when $z = z_0$, say. If $z_0$ is not 0, the number $A_1$ is not 0, and all is well. If finally $z_0 = 0$ we have to show that $|f(z)| < |f(0)|$ for some small $z$.* If $a$ is not zero $f(\pm\delta)$ has the same imaginary part $b$ as $f(0)$, and has the real part $R = (\pm 1)^n \delta^n - a$: since $n$ is odd, one of the two signs must make $|R|$ less than $|a|$ for small $\delta$, and so $|f(z)| < |a+ib|$. If $a$ is and $b$ is not zero we argue similarly with $f(\pm i\delta)$.

It will be observed that the complete proof now uses one only of (A 2) and (A 3) (the latter).

3. The arguments can be knit more closely, and combined with induction†. We thus obtain a proof, somewhat artificial in appearance, but very concise.

Suppose the theorem false, let $n$ be the least index of failure, and let $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ be a polynomial without root. Since $|f|$ is large when $|z|$ is large, $|f|$ attains its non-zero lower bound when $z = z_0$, say. Let

$$\phi(\zeta) = f(z_0 + \zeta) = A_0 + A_1 \zeta + \ldots + \zeta^n;$$

$|\phi|$ has a minimum $|A_0| \neq 0$ at $\zeta = 0$. Let $n = 2^k m$, where $m$ is odd, and let $A_r$ be the first of $A_1, A_2, \ldots$ different from 0. If $r < n$, $\zeta^r = -A_0/A_r$ has a root, $u$ say, and

$$|\phi(u\delta)| = |A_0|\left(1 - \delta^r + O(\delta^{r+1})\right) < |A_0|$$

for small positive $\delta$, a contradiction. If $r = n$, $\phi(\zeta)$ is $A_0 + \zeta^n$. If $\mathfrak{R} A_0 \neq 0$, let $v$ be a root‡ of

$$v^{2^k} = \pm 1$$

according as $\mathfrak{R} A_0 \lesseqgtr 0$; if $\mathfrak{I} A_0 \neq 0$, let $w$ be a root§ of

$$w^{2^k} = \pm i(-1)^{\frac{1}{2}(m-1)}$$

* Prof. G. H. Hardy points out to me that the proof of this corresponds to the fact, geometrically intuitive, that if $P$ and $Q$ are distinct points and $P_1, P_2, P_3, P_4$ are near $P$ and respectively N., S., E., and W. of $P$, then one of the distances $P_i Q$ is strictly less than $PQ$.

† At a lecture in which I was giving the proof of § 2, Mr. J. T. Wiltshire raised the question, "Can the theorem be proved by induction?". The present proof is a result.

‡ Existent since every equation $z^2 = a + ib$ has a root.

§ The cases overlap; we take the second only in the special case $\mathfrak{R} A_0 = 0$.

according as $\Im A_0 \lessgtr 0$. In the first case we take $\zeta = v\delta$, in the second $\zeta = w\delta$, where $\delta$ is small and positive. In the change from $z = 0$ to $z = \zeta$, $\phi$ has in the first case its real part decreased in absolute value and its imaginary part left unaltered; in the second it has its real part left unaltered and its imaginary part decreased in absolute value. In either case $|\phi|$ is decreased, and we obtain the desired contradiction.

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# NOTE ON THE PRODUCT OF THREE HOMOGENEOUS LINEAR FORMS

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## *Introduction.*

Let $L_1$, $L_2$, $L_3$ be three homogeneous linear forms in $u$, $v$, $w$ with real coefficients and determinant 1. Let $M$ denote the lower bound of

$$|L_1 L_2 L_3|$$

for integral values of $u$, $v$, $w$ other than 0, 0, 0. I proved‡ some time ago that

$$M \leqslant \tfrac{1}{7}, \tag{1}$$

and that the constant on the right is best possible. In the present note (which is self-contained) I give a new and simple proof§ of (1). In a later paper the method will be developed to give a much deeper result concerning the "second minimum" of $|L_1 L_2 L_3|$.

## *The lemma.*

*Suppose that* $0 \leqslant \epsilon < \frac{1}{10}$. *Let* $a_1$, $a_2$, $a_3$ *be real numbers such that*

$$|(n-a_1)(n-a_2)(n-a_3)| \geqslant 1-\epsilon \tag{2}$$

*for all integers* $n$. *Then*

$$S = (a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2 \geqslant 14-10\epsilon. \tag{3}$$

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‡ *Proc. London Math. Soc.* (2), 44 (1938), 412–431.

§ This proof is not related to the one given recently by Mordell, *Proc. London Math. Soc.* (not yet published).