LIE BRACKETS AND SINGULAR TYPE OF REAL HYPERSURFACES

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Abstract. This note is the sequel of [1], where the regular type of a real hyper surface H in $\mathbb{C}^n$ (eventually endowed with an almost complex structure) was characterized in terms of Lie brackets of complex tangent vector fields on H. This note extends these results to the singular type.

Let J be a smooth almost complex structure on $\mathbb{C}^n$, H a smooth real hypersurface, $p \in H$, and let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a function such that 0 is a regular value of ρ and $H = \{\rho = 0\}$.

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$, where the variable will be denoted by $t = x + iy$. A complex curve will be a J-holomorphic map $u : \mathbb{D} \to \mathbb{C}^n$. The valuation of u at 0 is the integer $\nu(u)$ such that $u^{(k)}(0) = 0$ for $k < \nu(u)$ and $u^{(\nu(u))}(0) \neq 0$. In the same way, suppose $u(0) = p$ and define $\nu(\rho(u))$ as the vanishing order of $\rho(u)$ at 0.

Following [3], let

$$\Delta(u, H) = \frac{\nu(\rho(u))}{\nu(u)},$$

and define the 1-type of H at 0 as

$$\Delta_1(H, p) = \max_{\{u/u(0)=p\}} \Delta(u, H). \tag{1}$$

This notion 1-type plays is well known in complex analysis and plays a crucial role for instance when trying to estimate the regularity of the Cauchy-Riemann equation. D'Angelo proved in [3] that points of finite type form an open set in H. More precisely, he proved that

$$\Delta_1(H, p) \leq 2\,\Delta_1^{n-1}(H, p_0)$$

for p close to p_0. It would be nice to know if this inequality still holds in the almost complex case, or if the finite type condition remains open. In particular, in D'Angelo's proof's, curves are seen as common zeros of holomorphic functions, and the result rests on algebraic study of the multiplicity of ideals of such functions. Of course, no such function exist in general in the almost complex case, and the proof can not be directly extended to this case. A proof in this setting would hence also provide a different point of view on the integrable case.

Restricting to regular curves only in the definition (1) leads to the so called regular type Δ_1^{reg}. The object of [1] was to characterize this type in term of Lie brackets or jets of J-tangent vectors on H, as this was a question of Bloom and Graham [2].

In this note, we extend these results to the singular case. Notice that the results are stated in the almost complex setting for the sake of generality, but are new even in the integrable case.

The main argument here is to turn singular curves into regulars by considering their graph: this allows the results of [1] to be applied almost directly to this new situation.

1. Main definitions and statement

We first recall some notations from [1]. In particular, given a non vanishing vector field $X \in \mathbb{C}^n$, it's (p,q)-th derivative is the vector field $(JX)^q X^{p+1}$ obtained by differentiating X p times in its own direction and then q times in the JX direction (we use the Euclidean connection on $\mathbb{C}^n$ to define the derivation along X, but if a different connection ∇ would define a different jet, the valuation proposed in definition 1.1 would be the same).

Notice that if X is a vector field on $T^J H$ and $\hat{X}$ is an extension of X to $\mathbb{C}^n$, then, on H, $(J\hat{X})^q \hat{X}^{p+1}$ depends on X only.

Let $\tilde{H} = \mathbb{C} \times H \subset \mathbb{C}^{n+1}$. Let $\tilde{J} = i \oplus J$. The coordinates on $\mathbb{C}^{n+1}$ will be denoted by (τ, z).

Definition 1.1. *Given a regular vector field $\tilde{X} \in \Gamma(T^{\tilde{J}}\tilde{H})$, use the decomposition $T^{\tilde{J}}\tilde{H} = \mathbb{C}\partial_\tau \oplus T^J H$ to write $\tilde{X}$ as a sum $\tilde{X} = \lambda\partial_\tau + X$. The vector field $\tilde{X}$ is said to have a contact order at least k with the line $\mathbb{C}\partial_\tau$ at a point $\tilde{p}$ if*

$$(\tilde{J}\tilde{X})^q \tilde{X}^{p+1} = (i\lambda\partial_\tau)^q (\lambda\partial_\tau)^{p+1} \text{ for } p+q<k \tag{2}$$

Define then the τ-valuation of $\tilde{X}$ as $\nu_{\tilde{p}}(\tilde{X}) = \max\{k \in \mathbb{N}, (2) \text{ holds}\}$.

For each regular vector field $\tilde{X} \in \Gamma(T^{\tilde{J}}\tilde{H})$, we define $\mathrm{Stab}_{\tilde{p}}(\tilde{X})$ as the maximal integer k such that all iterated Lie brackets of length at most k, involving $\tilde{X}$ and $\tilde{J}\tilde{X}$, belong to the complex line spanned by $\tilde{X}$ at $\tilde{p}$.

Definition 1.2. *For a regular vector field $\tilde{X} \in \mathbb{C}^{n+1}$ such that $\tilde{X}_{|\tilde{H}} \in \Gamma(T^{\tilde{J}}\tilde{H})$ and a point $\tilde{p} \in \tilde{H}$, we put*

$$\Delta(\tilde{X}, \tilde{p}) = \frac{\mathrm{Stab}_{\tilde{p}}(\tilde{X}) + 1}{\nu_{\tilde{p}}(\tilde{X}) + 1}$$

(with the convention that if $\nu_{\tilde{p}}(\tilde{x}) = +\infty$, then $\Delta(\tilde{X}, \tilde{p}) = 0$). To a point $p \in H$, we associate then the number

$$\Delta_1^{\mathrm{Lie}}(H,p) = \max(\Delta(\tilde{X}, (0,p))), \tilde{X} \in \Gamma(T^{\tilde{J}}\tilde{H}))$$

The main statement is the following:

Theorem 1.3. *With the previous notations, we have:*

$$\Delta_1(H,p) = \Delta_1^{\mathrm{Lie}}(H,p)$$

Remark 1.4. *As explained in* [1] *(theorem 1), the regular type of H can be characterized in term of stability order of complex line sub-bundles of $T^J H$. It turns out that this interpretation still holds in the singular case. In fact, the definition 1.1 only depends on the complex line sub-bundle spanned by $\tilde{X}$ rather than on $\tilde{X}$ itself.*

Proof. Let $\tilde{X}$ be some vector field in $\Gamma(T^{\tilde{J}}\tilde{H})$, and (p,q) some integers such that $p+q < \nu_{\tilde{p}}(\tilde{X})$.

Let α be some non vanishing complex valued function, and $\tilde{X}' = \alpha\tilde{X}$. Then at $\tilde{p}$, $(\tilde{J}\tilde{X}')^q \tilde{X}'^{p+1} - \alpha^{p+q+1}(\tilde{J}\tilde{X})^q\tilde{X}^{p+1}$ is a linear combination of the vectors $(\tilde{J}\tilde{X})^s\tilde{X}^{r+1}$

with $r+s<p+q$ and the coefficients are polynomials in the derivatives of α in the directions of such vectors. Since $(\tilde{J}\tilde{X})^s\tilde{X}^{r+1}=(i\lambda\partial_\tau)^s(\lambda\partial_\tau)^{r+1}$, we deduce that $(\tilde{J}\tilde{X}')^q\tilde{X}'^{p+1}=(i\lambda'\partial_\tau)^q(\lambda'\partial_\tau)^{p+1}$, with $\lambda'=\alpha\lambda$. □

As a consequence, all the (non vanishing) sections of a given complex line sub-bundle have the same τ-valuation and we can consider the following definition:

Definition 1.5. *Let $\tilde{L}$ be a complex line sub-bundle of $T^{\tilde{J}}\tilde{H}$ and $\tilde{p}\in\tilde{H}$. We then define*

$$\nu_{\tilde{p}}(\tilde{L})=\nu_{\tilde{p}}(\tilde{X})$$

where $\tilde{X}$ is some section of $\tilde{L}$.

In the same way, all the (non vanishing) sections of a given complex line sub-bundle have the same stability order at a given point, so we can set $\mathrm{Stab}_{\tilde{p}}(\tilde{L})=\mathrm{Stab}_{\tilde{p}}(\tilde{X})$ where $\tilde{X}$ is any section of $\tilde{L}$.

Definition 1.6. *Let $\tilde{L}$ be a complex line sub-bundle of $T^{\tilde{J}}\tilde{H}$ and $\tilde{p}\in$ tildeH. We then define*

$$\Delta(\tilde{L},\tilde{p})=\frac{\mathrm{Stab}_{\tilde{p}}(\tilde{L})+1}{\nu_{\tilde{p}}(\tilde{L})+1}$$

These definitions lead to the following rephrasing of theorem 1.3:

Theorem 1.7. *With the previous notations, we have:*

$$\Delta_1(H,p)=\max_{\tilde{L}}\{\Delta(\tilde{L},(0,p))\}$$

where $\tilde{L}$ runs through all the complex line sub-bundles of $T^{\tilde{J}}\tilde{H}$.

2. Proof of theorem 1.3

The main idea in the proof is to turn singular curves into regulars by considering their graph. In the sequel, we fix p at 0 in H. Moreover, we define $\tilde{\rho}(\tau,z)=\rho(z)$ so that $\tilde{H}=\tilde{\rho}^{-1}(0)$.
Proof of $\Delta_1(H,p)\leq\Delta_1^{\mathrm{Lie}}(H,p)$. :

Proof. Let us start with a possibly singular curve $u:\mathbb{D}\to\mathbb{C}^n$ such that $u(0)=0$ and $\rho(u(t))=O(|t|^k)$ and $\nu(u)=\nu$ so that $\Delta(H,u)\geq\frac{k}{\nu}$ (note that we can suppose $\frac{k}{\nu}\geq 2$, if it is not, there is nothing to do) .

Let $\tilde{u}:D\to\mathbb{C}^{n+1}$ the map $\tilde{u}(t)=(t,u(t))$. It is a regular curve in $\mathbb{C}^{n+1}$ and $\tilde{\rho}(\tilde{u}(t))=\rho(u(t))=O(|t|^k)$, so that $\tilde{u}$ has contact order at least k with $\tilde{H}$.

From [1] (corollary 11) we deduce that there is a (regular) germ of vector field $\tilde{X}\in T^{\tilde{J}}\tilde{H}$ at 0, such that

$$\mathrm{Stab}(\tilde{X},0)\geq k-1$$

More precisely, $\tilde{X}$ can be chosen in such a way that for $p+q+1\leq k-1$:

$$(\tilde{J}\tilde{X})^q\tilde{X}^{p+1}(0)=\frac{\partial^{p+q}}{\partial x^p\partial y^q}\frac{\partial\tilde{u}}{\partial x}(0)\text{ (with }t=x+iy)$$

If u is regular, the last equality (with $p+q=0$) says $\nu_0(\tilde{X})=0$ and $\Delta_1^{\mathrm{Lie}}(H,0)\geq k$. If u is singular, the same equality gives : in particular, $\tilde{X}(0)=\partial_\tau(0)$, and for $1<p+q+1\leq\nu-1$, $(\tilde{J}\tilde{X})^q\tilde{X}^{p+1}(0)=(0,0)$, and $\tilde{X}^\nu(0)=(0,u^{(\nu)}(0))\neq(0,0)$. We deduce $\nu_0(\tilde{X})=\nu-1$, and hence $\Delta_1^{\mathrm{Lie}}(H,0)\geq\frac{k}{\nu}$. □

Proof of $\Delta_1^{\rm Lie}(H,p) \leq \Delta_1(H,p).$:

Proof. Consider a (regular) vector field $\tilde{X} \in \Gamma(T^{\tilde{J}}\tilde{H})$ and let $k = \mathrm{Stab}(\tilde{X},0)$ and $\nu = \nu_0(\tilde{X})$ (we can suppose $\frac{k+1}{\nu+1} \geq 2$, because $\Delta_1(H,0) \geq 2$.

Write $\tilde{X} = \lambda\partial_\tau + X_H$.

Suppose $\lambda(0) \neq 0$: Notice that $\mathrm{Stab}(\frac{1}{\lambda}\tilde{X}) = \mathrm{Stab}(\tilde{X})$, and $\nu_0(\frac{1}{\lambda}\tilde{X}) = \nu_0(\tilde{X})$, so that $\tilde{X}$ can be replaced by $\frac{1}{\lambda}\tilde{X}$ without loss of generality, and hence be written in the form:

$$\tilde{X} = \partial_\tau + X_H$$

From [1] (theorem 1) we infer the existence of a regular curve $\tilde{u}(t) = (\tau(t), z(t))$, such that:

$$\begin{cases} \rho(\tilde{u}(t)) = O(|t|^{k+1}) \\ \frac{\partial^{p+1}\tilde{u}}{\partial x^p}(0) = \tilde{X}^{(p+1)}(0) \text{ (with } p+1 \leq k) \end{cases}$$

Since $\nu_0(\tilde{X}) = \nu$, we have $\tilde{u}(t) = (t,0) + O(|t|^{\nu+1})$.

Let $u(t) = \pi_2(\tilde{u}(t)) = z(t)$. Due to the particular form of the almost complex structure$\tilde{J}$, u is a J-holomorphic curve in C^n, and $\nu_0(u) = \nu + 1$.

Finally $\rho(u(t)) = \tilde{\rho}(\tau(t), u(t)) = O(|t|^{k+1})$. As a consequence:

$$\Delta_1(H,0) \geq \frac{k+1}{\nu+1}$$

This ends the proof in the this case.

Suppose $\lambda(0) = 0$: (in particular $\nu_0(\tilde{X}) = 0$). Since $\tilde{X}(0) \neq 0$, we have $X_H(0) \neq 0$. Using the same argument as before, we have a regular curve $\tilde{u}(t) = (\tau(t), z(t))$, such that:

$$\begin{cases} \rho(\tilde{u}(t)) = O(|t|^{k+1}) \\ \frac{\partial^{p+1}\tilde{u}}{\partial x^p}(0) = \tilde{X}^{(p+1)}(0) \text{(with } p+1 \leq k) \end{cases}$$

and projecting it on H, we get J-holomorphic curve $u(t)$ that is *regular* at 0. The same computations as before show that

$$\Delta_1(H,0) \geq k+1 = \Delta(\tilde{X},0)$$

This ends the proof in the this case. □

3. Type realtif to a complex foliation of dimension 1:

The definition 1.1 does not depend on the particular form of the vector field ∂_τ, and defines in fact a contact order of two (non vanishing) vector fields, or more geometrically, of the two complex line sub-bundles they span. More precisely, given a vector field X and a complex tangent line sub-bundle L, define $\nu_p(X,L)$ as the maximal k such that there is a section Y of L such that X and Y have the same jet up to order $\nu_p(X,L)$.

This observation allows the construction to be brought to the situation where $\tilde{H}$ is in fact a real hyper-surface M in $\mathbb{C}^{n+1}$ endowed with a foliation $\mathcal{F}$ by smooth complex curves.

Given a smooth vector field X, define

$$\Delta(X,\mathcal{F},p) = \frac{\mathrm{Stab}_p(X)+1}{\nu_p(X,T\mathcal{F})+1} \qquad \text{and} \qquad \Delta_1^{\rm Lie}(M,\mathcal{F},p) = \max_X(\Delta(X,\mathcal{F},p))$$

This number $\Delta_1^{\rm Lie}(M,\mathcal{F},p)$ can be viewed as a notion of one dimensional type modulo the foliation $\mathcal{F}$. In particular, in the case where M is bi-holomorphic to a product $H \times \mathbb{C}$ and $\mathcal{F}$ to the foliation given by the $\mathbb{C}$ factor, one recovers the previous definitions, and hence the singular type of H.

References

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