Nodal symplectic spheres in $\mathbb{CP}^2$ with positive self intersection

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Abstract [1]: *Let ω be the canonical Kähler structure on $\mathbb{CP}^2$ We prove that any ω-symplectic sphere with positive ordinary double points is isotopic, among such curves, to an algebraic curve. In other words, if J is a ω-tamed almost complex structure on $\mathbb{CP}^2$, the rational J-curves with ordinary double points are isotopic to algebraic curves. Moreover, calling N_d the number of degree d rational J-curves going through $3d-1$ points "in general position", it follows from the proof that N_d does not depend on the almost complex structure J.*

Since they were introduced by M. Gromov in 1985 [1], pseudo-holomorphic curves have proven to be powerful tools in global symplectic geometry. By comparing their spaces of curves, one can compare symplectic manifolds. The aim of this paper is to compare some pseudo-holomorphic and algebraic curves in $\mathbb{CP}^2$.

The result we propose here can nevertheless be formulated without the formalism of pseudo-holomorphic curves. Let ω denote the canonical Kähler form on $\mathbb{CP}^2$. We have the following theorem:

Theorem 1. *Any symplectic sphere in $(\mathbb{CP}^2, \omega)$ having only positive ordinary double point singularities is symplectically isotopic (among symplectic spheres with such singularities) to an algebraic curve.*

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Let $\mathcal{J}_\omega$ denote the space of ω-tamed almost complex structures on $\mathbb{CP}^2$. For $J \in \mathcal{J}_\omega$, a J-holomorphic sphere with ordinary double points is clearly a symplectic sphere. It follows from ideas of [1] that the reverse is true : take a symplectic sphere C with positive ordinary double points. A ω-tamed structure $J : T\mathbb{CP}^2{}_{|C} \longrightarrow T\mathbb{CP}^2{}_{|C}$ can be built along C such that $J(TC) = TC$. Because $\mathcal{J}_\omega$ is the space of the sections of some fiber bundle with contractible fibers, this J can now be extended to all $\mathbb{CP}^2$. We can then reformulate theorem 1:

Theorem 2. *Let J be an ω-tamed almost complex structure on $\mathbb{CP}^2$ and C be a rational J-curve having only ordinary double point singularities. There exists a path (J_t, C_t), where C_t is a J_t-curve with ordinary double point singularities, joining (J, C) to an algebraic curve.*

In [1], M. Gromov proved that any pseudo-holomorphic curve of degree 1 is isotopic to an algebraic line, and sketched the proof that any pseudo-holomorphic curve of degree 2 is isotopic to a conic. Our proof is modeled on his work.

Sketch of the proof. The proof of M. Gromov for the degree 1 curves rests on the following facts. Let $\mathcal{U}_1$ be the space of degree 1 maps from $\mathbb{CP}^1$ to $\mathbb{CP}^2$ (with regularity $C^{k+1,\alpha}$ for example). In the space $\mathcal{U}_1 \times \mathcal{J}_\omega$, define

$$\begin{aligned} \mathcal{P} &= \{(u, J) \in \mathcal{U}_1 \times \mathcal{J}_\omega \ / \ \ u \text{ is } J\text{-holomorphic}\}, \\ \mathcal{M} &= \mathcal{P} / \operatorname{Aut}(\mathbb{CP}^1). \end{aligned}$$

We then have:

- $\mathcal{P}$ is a Banach manifold,
- the projection $\pi : \mathcal{P} \longrightarrow \mathcal{J}_\omega$ is a local submersion,
- the projection $\mathcal{M} \longrightarrow \mathcal{J}_\omega$ is proper.

The last fact is a consequence of the compactness theorem in [1]. If (J_n) is a sequence in $\mathcal{J}_\omega$ converging to a structure J_∞, then any sequence (C_n), where C_n is a J_n-line, has a subsequence "converging" to a finite collection of J_∞-curves. But the degree 1 homology class is indecomposable. The subsequence must therefore converge to a J_∞-line.

According to the three properties above, any J-line can be deformed to a standard line. The positivity of intersection for pseudo-holomorphic curves implies now that J-lines are smooth. The deformation is therefore an isotopy.

In the situation we are interested in, we have to prove more: the corresponding space $\mathcal{P}$ is still a manifold, but there is no reason for the projection π to be everywhere a local submersion, nor for the projection $\mathcal{M} \longrightarrow \mathcal{J}_\omega$ to be proper. Moreover, in the case where those "accidents" can be avoided, a deformation of C to an algebraic curve exists, but this deformation may fail to be an isotopy if some new singularity appears along the path, namely, if there is a curve in the deformation which has a non-imbedded point, a triple point, or a double point which is not ordinary.

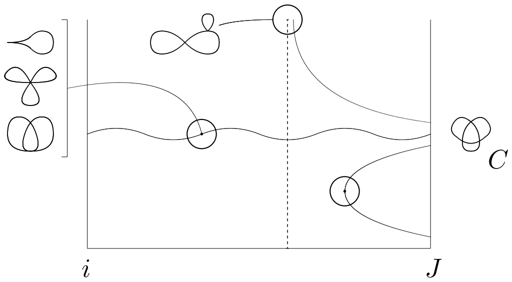

Figure 1: The accidents to be avoided in order to build an isotopy from C to an algebraic curve

In order to study the deformations of our curve, we fix $3d-1$ points p_1, ... , p_{3d-1} in $\mathbb{CP}^2$, and we restrict our attention to the curves going through those points. The idea is that the expected dimension for the corresponding space of curves is zero and that if we choose the points in a "general enough" position, accidents will be forbidden.

More precisely, we consider the following spaces. Let $\mathcal{U}_d$ denote the space of degree d maps from $\mathbb{CP}^1$ to $\mathbb{CP}^2$ (with regularity $C^{k+1,\alpha}$ for example), and

$$\mathcal{P}_d = \{(u, J) \in \mathcal{U}_d \times \mathcal{J}_\omega \ / \ \ u \text{ is } J\text{-holomorphic}\} \tag{1}$$

$$\mathcal{P}_{\frown} = \Big\{(u, J, z_1, \dots, z_{3d-1}) \in \mathcal{P}_d \times (\mathbb{CP}^1)^{3d-1} \ / \ \ u(z_i) = p_i\Big\} \tag{2}$$

$$\mathcal{P}_{\prec} = \Big\{(u, J, z) \in \mathcal{P}_{\frown} \times \mathbb{CP}^1 \ / \ \ du(z) = 0\Big\} \tag{3}$$

$$\mathcal{P}_{\divideontimes} = \Big\{(u, J, z_1, z_2, z_3) \in \mathcal{P}_{\frown} \times (\mathbb{CP}^1)^3 \setminus \Delta \ / \ \ u(z_1) = u(z_2) = u(z_3)\Big\} \tag{4}$$

$$\mathcal{P}_{\asymp} = \Big\{(u, J, z_1, z_2) \in \mathcal{P}_{\frown} \times (\mathbb{CP}^1)^2 \setminus \Delta \ / \ \ u(z_1) = u(z_2) \text{ and } \dots \\ \operatorname{im} du(z_1) = \operatorname{im} du(z_2)\Big\} \tag{5}$$

and for each $d' \leq d$ and each subset $\{p_{i_1}, \dots, p_{i_{3d'}}\} \subset \{p_1, \dots, p_{3d-1}\}$:

$$\mathcal{P}_{\perp} = \Big\{(u, J, z) \in \mathcal{P}_{d'} \times (\mathbb{CP}^1)^{3d'} \ / \ \ \forall j \ \ u(z_j) = p_{i_j}\Big\} \tag{6}$$

Let finally $\mathcal{M}_{\frown}$, $\mathcal{M}_{\prec}$, $\mathcal{M}_{\divideontimes}$, $\mathcal{M}_{\asymp}$ and $\mathcal{M}_{\perp}$ denote the quotient of those spaces by the natural action of $\mathrm{Aut}(\mathbb{CP}^1)$.

The space $\mathcal{P}_{\frown}$ is the (extended) space of parametrized curves going through our points p_i. Any non irreducible curve which is a limit of curves of $\mathcal{P}_{\frown}$ has necessarily a component in one of the $\mathcal{P}_{\perp}$ spaces, so we will use these spaces to study the properness of $\mathcal{M}_{\frown} \to \mathcal{J}_\omega$. Finally, the spaces $\mathcal{P}_{\prec}$, $\mathcal{P}_{\divideontimes}$ and $\mathcal{P}_{\asymp}$, are the spaces of the curves that should be avoided for a deformation to be an isotopy.

We will prove (see 2.1) that

Lemma 1. *The spaces $\mathcal{P}_{\frown}$, $\mathcal{P}_{\prec}$, $\mathcal{P}_{\divideontimes}$, $\mathcal{P}_{\asymp}$ and $\mathcal{P}_{\perp}$ are Banach manifolds.*

Definition 1. Let us call a structure $J \in \mathcal{J}_\omega$ *generic* for the degree d and the points (p_i) if J is a regular value of the projections of the spaces $\mathcal{P}_{\frown}$, $\mathcal{P}_{\prec}$, $\mathcal{P}_{\divideontimes}$, $\mathcal{P}_{\asymp}$ and $\mathcal{P}_{\perp}$ to $\mathcal{J}_\omega$.

We will check (see 2.2) that, as one can expect, the space of generic structures $J \in \mathcal{J}_\omega$ is a dense G_δ set in $\mathcal{J}_\omega$. Theorem 2 is implied by the following lemma :

Lemma 2. *The space of generic structures for the degree d and the points $p_1, \dots, p_{3d-1}$ is path connected.*

The main tools we will use in the proofs are presented in section 1. Section 2 is dedicated to the proof of lemmas 1 and 2. Theorem 2 is then proven in section 3.

1 Main tools

1.1 Operators D and D^N

We will use the local description of the space of J-curves of given genus and homology proposed by S. Ivashkovich and V. Shevchishin [3].

Let C be a rational J-curve in $\mathbb{CP}^2$, and let $\mathcal{M}(J)$ denote the space of all rational J-curves of the same degree as C.

Recall that a map $u : \mathbb{CP}^1 \to \mathbb{CP}^2$ is J-holomorphic if and only if it satisfies

$$du + J(u)\, du\, i = 0. \tag{7}$$

Let ∇ be a complex connection on $\mathbb{CP}^2$. The linearisation of Φ with respect to u at $(u, J) \in \Phi^{-1}(0)$, is the classical Gromov's operator D on the sections of $E = u^* T\mathbb{CP}^2$. It does not depend on ∇, and can be written in the form $D = \bar\partial + a$, where a is an $\mathbb{R}$-linear operator of order 0. The map du commutes with the $\bar\partial$ operator on $T\mathbb{CP}^1$ and D on E. The diagram

$$\begin{array}{ccccccc}
0 \to & \Gamma^{k+1,\alpha}(T\mathbb{P}^1) & \xrightarrow{du} & \Gamma^{k+1,\alpha}(E) & \to & \Gamma^{k+1,\alpha}(E)/\operatorname{im} du & \to 0 \\
& \downarrow{\scriptstyle \bar\partial} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle \overline{D}} & \\
0 \to & \Gamma^{k,\alpha}(\Lambda^{01} T\mathbb{P}^1) & \xrightarrow{du} & \Gamma^{k,\alpha}(\Lambda^{01}\mathbb{P}^1 \otimes E) & \to & \Gamma^{k,\alpha}(\Lambda^{01}\mathbb{P}^1 \otimes E)/\operatorname{im} du & \to 0
\end{array} \tag{8}$$

defines a linear map $\overline{D}$, which is the key of the local description of $\mathcal{M}(J)$ near C. Indeed, when $\mathcal{M}(J)$ is a manifold, we have

$$T\mathcal{M}(J) = \ker \overline{D}.$$

Let $\mathcal{O}_D(E)$ denote the sheaf defined by the kernel of D. Let also $H^0_D(E) = \ker D$ and $H^1_D(E) = \operatorname{coker} D$. Because $du \circ \bar\partial = D \circ du$, we get the following exact sequence:

$$0 \to \mathcal{O}_{\bar\partial}(T\mathbb{CP}^1) \xrightarrow{du} \mathcal{O}_D(E) \to \mathcal{N} \to 0. \tag{9}$$

The sections in the sheaf $\mathcal{O}_D(E)$ are not holomorphic, but their expansion near a point begins with an holomorphic term. An order of annulation for each zero of du can therefore be defined. Let $A = \sum_{du=0} \mu_i z_i$ be the divisor associated to the zeroes of du. The map du extends to the sections of $T\mathbb{CP}^1 \otimes A$.

Let N denote the normal bundle of C ($N = E/TC = E/\overline{\operatorname{im} du}$). Let D^N be the projection of D on N. We have the following exact sequence:

$$0 \to \mathcal{O}(T\mathbb{CP}^1 \otimes A) \xrightarrow{du} \mathcal{O}_D(E) \to \mathcal{O}_{D^N}(N) \to 0. \tag{10}$$

From (9) and (10) we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(T\mathbb{CP}^1) & \longrightarrow & \mathcal{O}_D(E) & \longrightarrow & \mathcal{N} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & \mathcal{O}(T\mathbb{CP}^1 \otimes A) & \longrightarrow & \mathcal{O}_D(E) & \longrightarrow & \mathcal{O}_{D^N}(N) & \longrightarrow & 0 \end{array} \tag{11}$$

Define

$$\mathcal{N}_1 := \ker \pi = \bigoplus_{du=0} \mathbb{C}^{\mu_i}_{z_i}.$$

The sheaf $\mathcal{N}_1$ is the skyscraper sheaf having stalk $\mathbb{C}^{\mu_i}$ at each zero z_i of du, μ_i being its multiplicity.

We have $\mathcal{N} \simeq \mathcal{O}_{D^N}(N) \oplus \mathcal{N}_1$, and the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(T\mathbb{CP}^1) & \xrightarrow{du} & H^0_D(E) & \longrightarrow & H^0_{D^N}(N) \oplus H^0(\mathcal{N}_1) & \longrightarrow & \\ & \longrightarrow & H^1(T\mathbb{CP}^1) & \longrightarrow & H^1_D(E) & \longrightarrow & H^1_{D^N}(N) & \longrightarrow & 0\ , \end{array} \tag{12}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(T\mathbb{CP}^1 \otimes A) & \xrightarrow{du} & H^0_D(E) & \longrightarrow & H^0_{D^N}(N) & \longrightarrow & \\ & \longrightarrow & H^1(T\mathbb{CP}^1 \otimes A) & \longrightarrow & H^1_D(E) & \longrightarrow & H^1_{D^N}(N) & \longrightarrow & 0\ . \end{array} \tag{13}$$

Let $\mathcal{K}\mathrm{er}\,\overline{D}$ denote the kernel of $\overline{D}$ as a sheaf.

Lemma 3. *$\mathcal{N}$ and $\mathcal{K}\mathrm{er}\,\overline{D}$ are isomorphic.*

Proof. We have

$$\mathcal{N} = \mathcal{O}_D(E)/du(\mathcal{O}(T\mathbb{CP}^1)),$$

and

$$\mathcal{K}\mathrm{er}\,\overline{D} = D^{-1}(du.\Gamma^{k,\alpha}(\Lambda^{01}\mathbb{CP}^1))/du.\Gamma^{k+1,\alpha}(T\mathbb{CP}^1)$$

($\Gamma^{k,\alpha}(\Lambda^{01}\mathbb{CP}^1)$ and $\Gamma^{k+1,\alpha}(T\mathbb{CP}^1)$ are seen as sheafs). Hence, there is a natural map $\theta : \mathcal{N} \longrightarrow \mathcal{K}\mathrm{er}\,\overline{D}$.

The map θ is injective: let $v \in \mathcal{O}_D(E) \cap du.\Gamma^{k+1,\alpha}(T\mathbb{CP}^1)$. Then there is ξ in $\Gamma^{k+1,\alpha}(T\mathbb{CP}^1)$ such that $v = du.\xi$. Because $Dv = 0$, we get $du.\bar{\partial}\xi = 0$, and $\bar{\partial}\xi = 0$ except maybe on a finite set. We have $\xi \in \Gamma^{k+1,\alpha}(T\mathbb{CP}^1)$, so ξ is continuous and hence holomorphic and $v \in du\big(\mathcal{O}(T\mathbb{CP}^1)\big)$.

The map θ is surjective: let $w \in D^{-1}(du.\Gamma^{k,\alpha}(\Lambda^{01}\mathbb{CP}^1))$. Then, there is α in $\Gamma^{k,\alpha}(\Lambda^{01}\mathbb{CP}^1)$ such that $Dw = du.\alpha$. We can solve $\bar{\partial}\xi = \alpha$ on a small enough open set. Let $v = w - du.\xi$, we have $Dv = 0$, and $w = \theta(v)$. □

From $\mathcal{N} \simeq \mathcal{O}_{D^N}(N) \oplus \mathcal{N}_1$, we get the following theorem:

Theorem 3 (S. Ivashkovich, V. Shevchishin). *The space* $\ker\overline{D}$ *is isomorphic to $H^0_{D^N}(N) \oplus H^0(\mathcal{N}_1)$ and* $\mathrm{coker}\,\overline{D}$ *to $H^1_{D^N}(N)$.*

This result can be used to describe the space $\mathcal{M}(J)$ near C:

Theorem 4 (S. Ivashkovich, V. Shevchishin). *If $H^1(N) = 0$, then, in the neighborhood of C, the space $\mathcal{M}(J)$ is a manifold of finite dimension and*

$$T_C\mathcal{M}(J) \cong H^0_{D^N}(N) \oplus H^0(\mathcal{N}_1).$$

Remark 1. This isomorphism can be made somewhat more precise.

$$\begin{aligned}\mathcal{N}_1 &= \mathcal{O}\big(du(T\mathbb{CP}^1 \otimes A)\big)/du\big(\mathcal{O}(T\mathbb{CP}^1)\big)\\ &= du\big(\mathcal{O}(T\mathbb{CP}^1 \otimes A)/\mathcal{O}(T\mathbb{CP}^1)\big)\\ &= du(\widetilde{\mathcal{N}}_1)\end{aligned}$$

The sheaf $\widetilde{\mathcal{N}}_1$ is of course isomorphic to $\mathcal{N}_1$ but gives rise to the exact sequence

$$0 \longrightarrow H^0(\widetilde{\mathcal{N}}_1) \xrightarrow{du} T_C\mathcal{M}(J) \longrightarrow H^0_{D^N}(N) \longrightarrow 0,$$

where the arrows are canonical.

1.2 Automatic genericity

We will only recall a theorem of [2]. The facts described in this theorem will be refered to as the phenomenon of "automatic genericity". The situation is the following. Let Σ be a Riemann surface of genus g, and L a complex line bundle over Σ. Let D be a differential operator on L of the form $\bar\partial + a$. The operator D is elliptic, and, using the Riemann-Roch theorem to compute its index, we get $\mathrm{ind}_{\mathbb{R}} D = 2\big(c_1(L) + 1 - g\big)$.

If this index is positive (resp. negative), a perturbation of D can make it surjective (resp. injective). The automatic genericity ensures that if the index is positive (resp. negative) enough, D is automatically surjective (resp. injective).

Theorem 5 ([2]). *With the previous notations, we have*

- *If* $c_1(L) \geq 2g-1$ *then* $\operatorname{coker} D = 0$*.*
- *If* $c_1(L) \leq -1$ *then* $\ker D = 0$*.*

We stress that L has to be a rank 1 fiber bundle over Σ. The two cases are equivalent under some duality argument, and in the case $c_1(L) \leq -1$, the proof rests on a positivity of intersection argument applied to a section and the null section.

1.3 Operator $\tilde{D}$

To define the spaces of curves we are interested in, we used evaluation maps which we study now.

Let F be a complex vector bundle over a Riemann surface Σ and D be a differential operator on F of the form $\bar\partial + a$. Let $z_1, \dots, z_m$ be m distinct points on Σ. Consider the evaluation map at each z_i:

$$\mathcal{O}_D(F) \xrightarrow{\tau} \oplus_{i=1}^m F_{z_i} \longrightarrow 0.$$

Let P be the divisor $-\sum_{i=1}^m z_i$, and $\tilde{F} = F \otimes P$.

Lemma 4. *There exists a unique operator* $\tilde{D} = \bar{\partial} + \tilde{a}$ *on* $\tilde{F}$ *making the following diagram commutative:*

$$\begin{array}{ccc} L^{1,p}(\tilde{F}) & \xrightarrow{\iota} & L^{1,p}(F) \\ {\scriptstyle \tilde{D}}\big\downarrow & & {\scriptstyle D}\big\downarrow \\ L^p(\Lambda^{0,1}\Sigma \otimes \tilde{F}) & \longrightarrow & L^p(\Lambda^{0,1}\Sigma \otimes F). \end{array} \tag{14}$$

Proof. There is no difficulty (nor any choice) in defining D away from the points z_i. In a neighborhood of a point z_i, if such an operator exists, it must satisfy:

$$\begin{aligned} \tilde{D}s &= z^{-1} Dz\, s \\ &= z^{-1}\bar{\partial} z\, s + z^{-1} az\, s \\ &= (\bar{\partial} + z^{-1}az)\, s. \end{aligned}$$

Piecing together the operators $z^{-1}az$, we define an operator $\tilde{a}$ of order 0. The operator $\tilde{D} = \bar{\partial} + \tilde{a}$ is then well defined. Moreover, $\tilde{a}$ is L^∞ in z, which is sufficient to apply classical results about elliptic operators to $\tilde{D}$ (in particular $\tilde{D}$ is Fredholm). □

The sequence

$$0 \to \mathcal{O}_{\tilde{D}}(\tilde{F}) \to \mathcal{O}_D(F) \xrightarrow{\tau} \oplus_{i=1}^m F_{z_i} \to 0$$

is exact and gives rise to the long exact sequence of cohomology

$$0 \to \ker \tilde{D} \to \ker D \xrightarrow{\tau} \oplus_{i=1}^m F_{z_i} \to \operatorname{coker} \tilde{D} \to \operatorname{coker} D \to 0. \tag{15}$$

This sequence will be useful to study the surjectivity of evaluation maps.

Remark 2. Multiplicities can be assigned to the points z_i, from which an operator $\tilde{D}$ can be derived exactly in the same way (there are only a few technical words to be said about the space of "D-holomorphic" jets, which differs from the space of holomorphic ones, but have same dimension).

2 Proof of lemma 2

2.1 The spaces $\mathcal{P}$ are Banach manifolds.

Recall that $\mathcal{U}_d$ denotes the space of degree d maps from $\mathbb{CP}^1$ to $\mathbb{CP}^2$, and $\mathcal{P}_d$ that of couples $(u, J) \in \mathcal{U}_d \times \mathcal{J}_\omega$ for which u is J-holomorphic. It is well known (see for example [4]) that $\mathcal{P}_d$ is a Banach manifold. Let us extend now the classical proof to our spaces $\mathcal{P}_{\frown}$, $\mathcal{P}_{\prec}$, $\mathcal{P}_{\times}$, $\mathcal{P}_{\asymp}$ and $\mathcal{P}_{\underline{\ell}}$.

Consider the vector bundle $\mathcal{F}_d$ over $\mathcal{B}_d = \mathcal{U}_d \times \mathcal{J}_\omega$ having fiber $\Gamma^{k,\alpha}(\Lambda^{01}\mathbb{CP}^1 \otimes u^* T\mathbb{CP}^2)$ at (u, J). The map

$$\Phi(u, J) = du \, + \, J(u) \, du \, i \tag{16}$$

defines a section of $\mathcal{F}_d$, and $\mathcal{P}_d = \Phi^{-1}(0)$.

Consider now the following vector bundles:

$$\begin{tikzcd} \mathcal{F}_{\frown} = \mathcal{F}_d \oplus (\mathbb{CP}^2)^{3d-1} \arrow[d] \\ \mathcal{B}_{\frown} = \mathcal{B}_d \times (\mathbb{CP}^1)^{3d-1} \arrow[u, bend right, "\Phi_{\frown}"'] \end{tikzcd} \quad \begin{tikzcd} \mathcal{F}_{\prec} = \mathcal{F}_{\frown} \oplus \mathcal{T}^1_J \arrow[d] \\ \mathcal{B}_{\prec} = \mathcal{B}_{\frown} \times (\mathbb{CP}^1) \arrow[u, bend right, "\Phi_{\prec}"'] \end{tikzcd} \quad \begin{tikzcd} \mathcal{F}_{\times} = \mathcal{F}_{\frown} \oplus (\mathbb{CP}^2)^3 \arrow[d] \\ \mathcal{B}_{\times} = \mathcal{B}_{\frown} \times (\mathbb{CP}^1)^3 \arrow[u, bend right, "\Phi_{\times}"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F}_{\asymp} = \mathcal{F}_{\frown} \oplus (\mathcal{T}^1_J)^2 \arrow[d] \\ \mathcal{B}_{\asymp} = \mathcal{B}_{\frown} \times (\mathbb{CP}^1)^2 \arrow[u, bend right, "\Phi_{\asymp}"'] \end{tikzcd} \quad \begin{tikzcd} \mathcal{F}_{\underline{\ell}} = \mathcal{F}_d \oplus (\mathbb{CP}^2)^{3d'} \arrow[d] \\ \mathcal{B}_{\underline{\ell}} = \mathcal{B}_{d'} \times (\mathbb{CP}^1)^{3d'} \arrow[u, bend right, "\Phi_{\underline{\ell}}"'] \end{tikzcd}$$

where $\mathcal{T}^1_J$ has fiber $\Lambda^{10}\mathbb{CP}^1 \otimes u^* T\mathbb{CP}^2$ at (u, J, z). Let $\Phi_{\frown}, \ldots, \Phi_{\underline{\ell}}$ be the direct sum of Φ and the natural evaluation maps $(u, z) \mapsto \tau^0_z u = u(z)$ or $\tau^1_z u = (u(z), [du(z)]^{10})$.

We then have $\mathcal{P}_{\frown} = \Phi^{-1}_{\frown}(0 \times \mathcal{S}_{\frown})$, with $\mathcal{S}_{\frown} = \{(p_1, \ldots, p_{3d-1})\}$, and similar relations for the other spaces with:

$$\mathcal{S}_{\prec} = \mathcal{S}_{\frown} \times 0_{\mathcal{T}^1} \tag{17}$$

$$\mathcal{S}_{\times} = \mathcal{S}_{\frown} \times \Delta \tag{18}$$

$$\mathcal{S}_{\asymp} = \mathcal{S}_{\frown} \times \big\{ \tau_1, \tau_2 \in \mathcal{T}^1_J \ / \ \ \pi(\tau_1) = \pi(\tau_2), \ \mathrm{rg}_{\mathbb{C}}(\tau_1, \tau_2) \leq 1 \big\} \tag{19}$$

$$\mathcal{S}_{\underline{\ell}} = \big\{ (p_{i_1}, \ldots, p_{i_{3d'}}) \big\} \tag{20}$$

Our problem is now to study, in each case, the transversality of Φ and the associated manifold $\mathcal{S}$.

Let us start with the space $\mathcal{P}_{\frown}$. Let $(u, J, z) \in \mathcal{P}_{\frown}$. We want to prove that the linearization of $\Phi_{\frown}$ at (u, J, z) is surjective, namely that for all $\alpha \in \Gamma^{k,\alpha}(\Lambda^{01}\mathbb{CP}^1 \otimes u^*T\mathbb{CP}^2)$ and $v_i \in T_{p_i}\mathbb{CP}^2$ the following equations (in $(v, \delta J, \xi)$) can be solved

$$\begin{cases} D.v + \delta J dui = \alpha & \textit{(i)} \\ v(z_i) + d_{z_i}u.\xi_i = \delta p_i & \textit{(ii).} \end{cases} \tag{21}$$

For the spaces $\mathcal{P}_{\lessdot}, \mathcal{P}_{\ast}, \mathcal{P}_{\asymp}$, some more equations of the same form as *(ii)* or maybe of the form

$$L\tau^1(v, \delta J, \xi) = \beta_i \quad \textit{(ii')} \tag{22}$$

have to be added to the system (21).

In a neighborhood U of every singular point of $C = u(\mathbb{CP}^1)$ and of every point z_i, we can "solve the $\bar\partial$ operator" and hence solve (21) (and (22)) with $\delta J = 0$. Let v denote this local solution. We extend v to all $\mathbb{CP}^1$. Only the following equation has to be solved left:

$$\delta J\, du\, i = \tilde\alpha, \tag{23}$$

where $\tilde\alpha = \alpha - D.v$. The section $\tilde\alpha$ vanishes on U. Outside U, we have $du \neq 0$. The equation (23) determines the restriction of δJ as a map from TC to $T\mathbb{CP}^2{}_{|C}$. Any anti-complex map from TC to $T\mathbb{CP}^2{}_{|C}$ can be extended to an anti-complex map from $T\mathbb{CP}^2{}_{|C}$ to itself, and then extended to all $\mathbb{CP}^2$. Finally, the equations (21) (and (22)) always have solutions.

This means that $\Phi_{\frown}$ (resp. $\Phi_{\lessdot}, \ldots, \Phi_{\bot}$) is transverse to $\mathcal{S}_{\frown}$ (resp. $\mathcal{S}_{\lessdot}, \ldots, \mathcal{S}_{\bot}$), and the spaces $\mathcal{P}_{\frown}, \ldots, \mathcal{P}_{\bot}$ are Banach manifolds.

2.2 The projections $\mathcal{P} \to \mathcal{J}_\omega$ are Fredholm maps

Consider a point $(u, J, z) \in \mathcal{P}_{\frown}$ (resp. $\mathcal{P}_{\lessdot}, \ldots, \mathcal{P}_{\bot}$), and the linearization (still denoted by π) of the projection $\pi : \mathcal{P}_{\frown} \to \mathcal{J}_\omega$ at this point. We have

$$T\mathcal{P}_{\frown} = \{(v, \delta J, \delta z) \ / \ Dv + \delta J\, du\, i = 0 \text{ and } \tau.(v, \delta z) \in T\mathcal{S}_{\frown}\},$$

(resp. $\tau.(v, \delta z) \in T\mathcal{S}_{\prec}, \dots, T\mathcal{S}_{\wp}$) and the following commutative diagram:

$$\begin{array}{ccccc} T\mathcal{P}_{\frown} & \xrightarrow{\pi_1} & \Gamma(u^*T\mathbb{CP}^2) \oplus (T\mathbb{CP}^1)^{3d-1} & \xrightarrow{\tau} & (T\mathbb{CP}^2)^{3d-1} \\ \pi\downarrow & & D\oplus 0 \downarrow & & \bigcup \\ & & & & T\mathcal{S}_{\frown} \\ T\mathcal{J}_\omega & \xrightarrow{\alpha} & \Gamma(\Lambda^{01}\mathbb{CP}^1 \otimes u^*T\mathbb{CP}^2) & & \end{array} \tag{24}$$

with $\alpha.\delta J = \delta J \, du \, i$ (and here, $T\mathcal{S}_{\frown} = 0$).

We have

$$\ker \pi = \big(\ker D \oplus (T\mathbb{CP}^1)^{3d-1} \big) \cap \tau^{-1}(T\mathcal{S}_{\frown}). \tag{25}$$

Consequently $\ker \pi$ is finite dimensional. We also have $\ker \alpha \subset \operatorname{im} \pi$, so $\operatorname{coker} \pi \simeq \operatorname{im} \alpha / \operatorname{im}(\alpha \circ \pi)$.

$$\operatorname{im}(\alpha \circ \pi) = \operatorname{im} \alpha \cap \operatorname{im} D \cap D(\tau^{-1}(T\mathcal{S}_{\frown}))$$

and, because $\alpha \oplus D \oplus \tau$ is surjective,

$$\operatorname{im} \alpha + \big(D(\tau^{-1}(T\mathcal{S}_{\frown})\big) = \Gamma(\Lambda^{01} \otimes u^*T\mathbb{CP}^2).$$

Finally:

$$\operatorname{coker} \pi_J \simeq \Gamma(\Lambda^{01} \otimes u^*T\mathbb{CP}^2) / D(\tau^{-1}(T\mathcal{S}_{\frown})). \tag{26}$$

Because $\tau^{-1}(T\mathcal{S}_{\frown})$ has finite codimension in $\Gamma(u^*T\mathbb{CP}^2)$, $\operatorname{coker} \pi$ has finite dimension.

Thus the projection $\pi : \mathcal{P}_{\frown} \to \mathcal{J}_\omega$ is Fredholm.

Let us compute the (real) index of π using relations (25) and (26). Consider a decomposition

$$T\mathcal{U} \simeq \overbrace{K \oplus \underbrace{K_\cap}_{} }^{\ker D} \!\!\!\!\!\!\underbrace{}_{\tau^{-1}(T\mathcal{S}_{\frown})}\!\!\!\!\!\! \oplus \overbrace{F \oplus G}^{\operatorname{im} D} \quad \text{and} \quad \Gamma(\Lambda^{01}\mathbb{CP}^1 \otimes u^*T\mathbb{CP}^2) \simeq F \oplus G \oplus \operatorname{coker} D.$$

We then have $\ker D \simeq K_\cap \oplus (T\mathbb{CP}^2)^{3d-1}$ and $\operatorname{coker} D \simeq G \oplus \operatorname{coker} D$: $\operatorname{ind} \pi = \dim K_\cap - \dim G - \dim \operatorname{coker} D$. Moreover,

$$\begin{aligned} \dim K_\cap \quad &+ \quad \dim K \ = \dim \ker D, \\ \dim G \quad &+ \quad \dim K \ = \operatorname{codim} \tau^{-1}(T\mathcal{S}). \end{aligned}$$

Because τ is surjective, we obtain

$$\operatorname{ind} \pi = \operatorname{ind} D + \dim (T\mathbb{C}\mathbb{P}^1)^{3d-1} - \operatorname{codim} T\mathcal{S}_{\frown}. \tag{27}$$

Exactly in the same way, it can be proven that the same results hold for the spaces $\mathcal{P}_{\prec}, \dots, \mathcal{P}_{\underline{\circ}}$ and the manifolds $\mathcal{S}_{\prec}, \dots, \mathcal{S}_{\underline{\circ}}$.

Namely, we have

$$\begin{aligned}
\operatorname{ind}_{\mathbb{R}} \pi_{\frown} &= 2\big(3d+2 + 3d-1 - 2(3d-1)\big) = 6,\\
\operatorname{ind}_{\mathbb{R}} \pi_{\prec} &= 2\big(3d+2 + 3d-1+3 - 2(3d-1)+4\big) = 4,\\
\operatorname{ind}_{\mathbb{R}} \pi_{\bowtie} &= 2\big(3d+2 + 3d-1+2 - 2(3d-1)+3\big) = 4,\\
\operatorname{ind}_{\mathbb{R}} \pi_{\rtimes} &= 2\big(3d+2 + 3d-1+1 - 2(3d-1)+2\big) = 4,\\
\operatorname{ind}_{\mathbb{R}} \pi_{\underline{\circ}} &= 2\big(3d'+2 + 3d' - 2(3d')\big) = 4.
\end{aligned}$$

Using sufficiently regular almost complex structures, those projections can be made as smooth as desired, and the Sard-Smale theorem can hence be applied: the set of generic structures is a G_δ dense set.

Remark 3. A structure J is a regular value of a projection $\pi_{\prec}$, $\pi_{\bowtie}$, $\pi_{\rtimes}$ or $\pi_{\underline{\circ}}$ if and only if it is not reached by this projection. In fact, suppose that there is a rational J-curve of degree d going through the $3d-1$ points p_i and having a singularity which is not an ordinary double point, or a rational J-curve of degree $d' < d$ going through $3d'$ of the points p_i. Let $(u, J, z) \in \mathcal{P}$ be a parametrization associated to it. Letting $\operatorname{Aut}(\mathbb{C}\mathbb{P}^1)$ act on (u, z), we get a family of curves of real dimension 6 in the fiber of the projection. This projection being of real index 4, its cokernel is of real dimension at least 2, thus J is not a regular value of the projection.

2.3 The space of generic structures is path connected

The Sard-Smale theorem implies that along a generic path $(J_t)_{t\in[0,1]}$, either the structure J_t is a regular value of projections $\mathcal{P}_{\frown}, \dots, \mathcal{P}_{\underline{\circ}} \to \mathcal{J}_\omega$, or $\dim_{\mathbb{R}} \operatorname{coker} \pi = 1$ for one of these projections.

We already proved that $\dim_{\mathbb{R}} \operatorname{coker} \pi = 1$ is impossible for the spaces $\mathcal{P}_{\prec}$, $\dots$,$\mathcal{P}_{\underline{\circ}}$. Let us deal with the space $\mathcal{P}_{\frown}$.

Let $(u, J_t, z) \in \mathcal{P}_{\frown}$. We know that J_t is a regular value of the projections $\pi : \mathcal{P}_{<}, \ldots, \mathcal{P}_{\perp} \to \mathcal{J}_\omega$. So the curve $C = u(\mathbb{CP}^1)$ has only ordinary double point singularities.

From (10), we derive

$$0 \to \mathcal{O}(T\mathbb{CP}^1) \to \mathcal{O}_D(u^* T\mathbb{CP}^2) \to \mathcal{O}_{D^N}(N) \to 0.$$

Consider the associated long exact sequence of cohomology and the evaluation map at the points z_i. We get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(T\mathbb{CP}^1) & \xrightarrow{du} & \ker D & \xrightarrow{p} & \ker D^N & \longrightarrow & 0 \\ & & \tau_1 \big\downarrow & & \tau \big\downarrow & & \tau_2 \big\downarrow & & \\ 0 & \longrightarrow & \mathbb{C}^{3d-1} & \longrightarrow & (\mathbb{C}^2)^{3d-1} & \longrightarrow & \mathbb{C}^{3d-1} & \longrightarrow & 0\ , \end{array} \tag{28}$$

Let us write the sequence (14) for N and the points z_i. We have

$$0 \to \mathcal{O}_{\widetilde{D^N}}(\tilde{N}) \to \mathcal{O}_{D^N}(N) \to \oplus \mathbb{C}_{z_i} \to 0. \tag{29}$$

We obtain the long exact sequence of cohomology:

$$0 \to \ker \widetilde{D^N} \to \ker D^N \xrightarrow{\tau_2} \mathbb{C}^{3d-1} \to \operatorname{coker} \widetilde{D^N} \to \operatorname{coker} D^N \to 0. \tag{30}$$

We have $\operatorname{ind} \widetilde{D^N} = 0$. The automatic genericity (theorem 5) then ensures that $\ker \widetilde{D^N} = 0$ and $\operatorname{coker} \widetilde{D^N} = 0$. Consequently, in (28), the map τ_2 is an isomorphism.

Finally

$$\ker \pi \simeq \ker \tau_2 \circ p \simeq H^0(T\mathbb{CP}^1). \tag{31}$$

In particular, $\dim \ker \pi = \operatorname{ind} \pi$, so that π is a local submersion. This completes the proof of lemma 2.

Remark 4. The group $\operatorname{Aut}(\mathbb{CP}^1)$ acts on $\mathcal{P}_{\frown}$ and the quotient, which is a manifold, is the space $\mathcal{M}_{\frown}$ of non parametrized curves going through the points p_i. The equation (31) shows that the tangent space to the action at one point is exactly the kernel of π: we deduce that the projection $\mathcal{M}_{\frown} \to \mathcal{J}_\omega$ is a local diffeomorphism.

3 Proof of theorem 2

Let C be a rational J-curve of degree d, having only ordinary double point singularities. We want to build an isotopy from C to an algebraic curve.

The construction rests directly on lemma 2. Technically, the structures at the ends of the isotopy are fixed, so the proof is divided into two parts: the typical case, and the technical adaptations at the ends.

Isotopy associated to two generic structures. Let $p_1, \dots, p_{3d-1}$ be $3d-1$ distinct points in $\mathbb{CP}^2$. Let J_0 and J_1 be two generic structures for these points and degree d. Let finally C be a J_0-curve going through the points p_i.

Consider now a path $(J_t)_{t\in[0,1]}$ of generic structures joining J_0 to J_1. Let $\mathcal{P}_{\frown}$ still denote the restriction of $\mathcal{P}_{\frown}$ to this path, and $\mathcal{M}_{\frown} = \mathcal{P}_{\frown}/\operatorname{Aut}(\mathbb{CP}^1)$ the quotient space.

Let us prove that $\mathcal{M}_{\frown}$ is a compact set. Let (J_{t_n}, C_n) be a sequence in $\mathcal{M}_{\frown}$. It has a subsequence for which $J_{t_n} \to J_\infty$ and C_n converges to a cycle $C_\infty = \sum_{i=1}^N Z_i$ consisting of rational J_∞-curves. Let d_i be the degree of Z_i and k_i the number of points of $\{p_1, \dots, p_{3d-1}\}$ Z_i is going through. Because J_∞ is generic, J_∞ is not reached by the projection $\mathcal{P}_{\wp} \to \mathcal{J}_\omega$, thus

$$\forall i,\ k_i \leq 3d_i - 1.$$

We also have $\sum d_i = d$ and $\sum k_i \geq 3d-1$, so $1 \leq N \leq 1$. The cycle C_∞ is a rational J-curve of degree d going through the points $p_1, \dots, p_{3d-1}$. Thus $C_\infty \in \mathcal{M}_{\frown}$, and $\mathcal{M}_{\frown}$ is a compact set.

Let K be the connected component of C in $\mathcal{M}_{\frown}$. It is a compact set so $\pi(K)$ is compact. It is an open set and the map π is a local diffeomorphism, so $\pi(K)$ is an open set. We then have $\pi(K) = [0,1]$.

This means that there is a homotopy joining the J_0-curve C to a J_1-curve. Moreover, all the structures J_t are generic, so the homotopy is an isotopy.

Ends of the isotopy. Choose $3d-1$ distinct points on C. The structure J itself might not be "globally" generic for these points, but is so at least locally, for the linearization of the projection $\pi_J : \mathcal{P}_{\frown} \longrightarrow \mathcal{J}_\omega$ at (C, J) is surjective (see (31)). We can build a first deformation (C_t) of C to a J'-curve C', where J' is now generic for the chosen points.

The standard structure i of $\mathbb{CP}^2$ might not be generic for the chosen points. Let then $(p'_1, \dots, p'_{3d-1})$ be points for which it is generic. Let (ϕ_t) be a path of diffeomorphisms of $\mathbb{CP}^2$ such that $\phi_0 = \mathrm{Id}$ and $\phi_1(p'_i) = p_i$.

The construction in the typical case can now be applied to the structures $\phi_{1*}i$ and J', and then pulled back by (ϕ_t) to give the isotopy we were looking for.

This brings the proof of theorem 2 to its end.

Remark 5. Since $\pi : \mathcal{M} \longrightarrow \mathcal{J}_{[0,1]}$ is a local diffeomorphism, this proof leads to a more precise result: the number N_d of degree d rational J-curves going through $3d - 1$ generic points does not depend on J.

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