HOMOTOPICAL DYNAMICS IN SYMPLECTIC TOPOLOGY.

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ABSTRACT. This is mainly a survey of recent work on algebraic ways to "measure" moduli spaces of connecting trajectories in Morse and Floer theories as well as related applications to symplectic topology. The paper also contains some new results. In particular, we show that the methods of [1] continue to work in general symplectic manifolds (without any connectivity conditions) but under the bubbling threshold.

CONTENTS

1. INTRODUCTION

The main purpose of this paper is to survey a number of Morse-theoretic results which show how to estimate algebraically the high-dimensional moduli spaces of Morse flow lines and to describe some of their recent applications to symplectic topology. We also deduce some new applications.

The paper starts with a brief discussion of the various proofs showing that the differential in the Morse complex is indeed a differential. With this occasion we introduce the main concepts in Morse theory and, in particular, the notion of connecting manifold (or, equivalently, the moduli space of flow lines connecting two critical points) which is the main object of interest in our further constructions. Moreover, an extension of one of these proofs leads naturally to an important result of John Franks [13] which describes the framed cobordism class of connecting manifolds between consecutive critical points as a certain relative attaching map. After describing Franks' result, we proceed to a stronger result initially proved in [6] which computes a framed bordism class naturally associated to the same connecting manifolds in terms of certain Hopf invariants. While these results only

apply to consecutive critical points we then describe a recent method to estimate general connecting manifolds by means of the Serre spectral sequence of the path-loop fibration having as base the ambient manifold [1]. Some interesting topological consequences of these results are briefly mentioned as well as some other methods used in the study of these problems.

The third section discusses a number of symplectic applications. We start with some results which first appeared in [7]. These use the non-vanishing of certain Hopf invariants to deduce the existence of bounded orbits of hamiltonian flows (obviously, inside non-compact manifolds). This is a very "soft" type of result even if difficult to prove. We then continue in §3.2 by describing how to use the Serre spectral sequence result to detect pseudo-holomorphic strips as well as some consequences of the existence of the strips. Most of the results of this part have first appeared in [1] but there are some that are new: we discuss explicitly the detection of pseudoholomorphic strips passing through some submanifold and we present a way to construct in a coherent fashion our theory for lagrangians in general symplectic manifolds as long as we remain *under a bubbling threshold*. Notice that even the analogue of the classical Floer theory (which is a very particular case of our construction) has not been explicited in the literature in the Lagrangian case even if all the necessary ideas are present in some form - see [25] for the hamiltonian case.

The paper contains a number of open problems and ends with a conjecture which is supported by the results in §3.2 as well as by recent joint results of the second author with François Lalonde.

2. Elements of Morse theory

Assume that M is a compact, smooth manifold without boundary of dimension n. Let $f : M \to \mathbb{R}$ be a smooth Morse function and let $\gamma : M \times \mathbb{R} \to M$ be a negative gradient Morse-Smale flow associated of f. In particular, f is strictly decreasing along any non-constant flow line of γ and the stable manifolds

$$W^s(P) = \{x \in M : \lim_{t \to \infty} \gamma_t(x) = P\}$$

and the unstable manifolds

$$W^u(Q) = \{x \in M : \lim_{t \to -\infty} \gamma_t(x) = Q\}$$

of any pair of critical points P and Q of f intersect transversally. One of the most useful and simple tools that can be defined in this context is the Morse complex

$$C(\gamma) = (\mathbb{Z}/2 < Crit(f) >, d) .$$

Here $\mathbb{Z}/2 < S >$ is the $\mathbb{Z}/2$-vector space generated by the set S, the vector space $\mathbb{Z}/2 < Crit(f) >$ has a natural grading given by $|P| = ind_f(P), \forall P \in Crit(f)$ and d is the differential of the complex which is defined by

$$dx = \sum_{|y|=|x|-1} a_y^x y$$

so that the coefficients $a_y^x = \#((W^u(x) \cap W^s(y))/\mathbb{R})$. This definition makes sense because the set $W^u(P) \cap W^s(Q)$ which consists of all the points situated on some flow line joining P to Q is invariant by the $\mathbb{R}$-action given by the flow. Moreover, $W^u(P)$ and $W^s(P)$ are homeomorphic to open disks which implies that the set $M_Q^P = (W^u(P) \cap W^s(Q))/\mathbb{R}$ has the structure of a smooth (in general, non-compact)

manifold of dimension $|P| - |Q| - 1$. We call this space the moduli space of flow lines joining P to Q. It is not difficult to understand the reasons for the non-compactness of M_Q^P when M is compact as in our setting: this is simply due to the fact that a family of flow lines joining P to Q may approach a third, intermediate, critical point R. For this to happen it is necessary (and sufficient - see Smale [27] or Franks [13]) to have some flow line which joins P to R and some other joining R to Q. This implies that when $|P| = |Q| + 1$ the set M_Q^P is compact and thus the sum above is finite. For further use let's define also the unstable sphere of a critical point P as $S_a^u(P) = W^u(P) \cap f^{-1}(a)$ as well as the stable sphere $S_a^s(P) = W^s(P) \cap f^{-1}(a)$ where a is a regular value of f. It should be noted that this names are slightly abusive as these two sets are spheres, in general, only if a is sufficiently close to $f(P)$. In that case $S_a^u(P)$ is homeomorphic to a sphere of dimension $|P| - 1$ and $S_a^s(P)$ is homeomorphic to a sphere of dimension $n - |P| - 1$. With this notations ths moduli space M_Q^P is homeomorphic to $S_a^u(P) \cap S_a^s(P)$ for any $a \in (f(Q), f(P))$ which is a regular value of f.

The main properties of the object defined above are that:

$$d^2 = 0 \text{ and } H_*(C(\gamma)) \approx H_*(M; \mathbb{Z}/2) \ .$$

We will sometimes denote this complex by $C(f)$ and will call it the (classical) Morse complex of f. The flow γ may be in fact even a pseudo- (negative) gradient flow of f. There also exists a version of this complex over $\mathbb{Z}$ in which the counting of the elements in M_y^x takes into account orientations.

There are essentially four methods to prove these properties:

(i) Deducing the equation $\sum_y a_y^x a_z^y = 0$ (which is equivalent to $d^2 = 0$) from the properties of the moduli spaces M_z^x with $|x| - |z| = 2$.
(ii) Comparing a_y^x with a certain relative attaching map.
(iii) Expressing a_y^x in terms of a connection map in Conley's index theory.
(iv) A method based on a deformation of the de Rham complex (clearly, in this case the coefficients are required to be in $\mathbb{R}$).

For the rest of this paper the two approaches that are of the most interest are (i) and (ii). Therefore we shall first say a few words on the other two methods and will then describe in more detail the first two. Method (iii) consists in regarding two critical points x, y so that $|x| = |y| + 1$ as an attractor-repellor pair and to apply the general Conley theory of Morse decompositions to this situation [22]. Method (iv) has been introduced by Witten in [30] and is based on a deformation of the differential of the de Rham complex which provides a new differential with respect to which the harmonic forms are in bijection with the critical points of f.

Method (i) has been probably folklore for a long time but it first appeared explicitly in Witten's paper. It is based on noticing that the moduli space M_Q^P admits a compactification $\overline{M}_Q^P$ which is a compact, topological manifold with boundary so that the boundary verifies the formula:

(1) $$\partial \overline{M}_Q^P = \bigcup_R \overline{M}_R^P \times \overline{M}_Q^R \ .$$

There are two main ways to prove this formula. One is analytic and regards a flow line from P to Q as a solution of a differential equation $\dot{x} = -\nabla f(x)$ and studies the properties of such soultions (this method is presented in the book of Schwarz [24]).

A second approach is more topological/dynamical in nature as is described in detail by Weber [29]. Clearly, from formula (1) we immediately deduce $\sum_y a_y^x a_z^y = 0$ and hence $d^2 = 0$. Just a little more work is needed to deduce from here the second property.

Method (ii) was the one best known classically and it is essentially implicit in Milnor's h-cobordism book [17]. It is based on the observation that a_y^x can be viewed as follows. First, to simplify slightly the argument assume that the only critical points in $f^{-1}([f(y), f(x)])$ are x and y. It is well known that for $a \in (f(y), f(x))$, there exists a deformation retract

$$r : M(a) = f^{-1}(-\infty, a] \to M(f(y) - \epsilon) \cup_{\phi_y} D^{|y|} = M'$$

where the attaching map

$$(2) \qquad\qquad \phi_y : S^u_{f(y)-\epsilon}(y) \to M(f(y) - \epsilon)$$

is just the inclusion and ϵ is small. This deformation retract follows the flow till reaching of $U(W^u(y)) \cup M(f(y) - \epsilon)$ where $U(W^u(y)$ is a tubular neighbourhood of $W^u(y)$ so that the flow is transverse to its boundary and then collapses this neighbourhood to $W^u(y)$ by the canonical projection. Clearly, applying this remark to each critical point of f provides a CW-complex of the same homotopy type as that of M and with one cell $\bar{x}$ for each element of $x \in Crit(f)$. To this cellular decomposition we may associate a cellular complex $(C'(f), d')$ with the property that $d\bar{x} = \sum k_{\bar{y}}^{\bar{x}} \bar{y}$ where $k_{\bar{y}}^{\bar{x}}$ is, by definition, the degree of the composition:

$$(3) \qquad\qquad \psi_y^x : S_a^u(x) \xrightarrow{\phi_x} M(a) \xrightarrow{r} M' \xrightarrow{u} M'/M(f(y) - \epsilon)$$

with $\phi_x : S_a^u(x) \to M(a)$ again the inclusion and where the last map, u, is the projection onto the respective topological quotient space (which is homeomorphic to the sphere $S^{|y|}$).

Notice now that $M_y^x \subset S_a^u(x)$ is a finite union of points say $P_1, \ldots, P_k$. Imagine a small disk $D_i \subset S_a^u(x)$ around P_i. The key (but geometrically clear) remark is that the composition of the flow γ together with the retraction r transports D_i (if it is chosen sufficiently small) homeomorphically onto a neighbourhood of y inside $W^u(y)$. Therefore, the degree of $deg(\psi_y^x) = a_y^x$ and thus $d = d'$ which shows that d is a differential and that the homology it computes agrees with the homology of M.

As we shall see further, the points of view reflected in the approaches at (i) and (ii) lead to interesting applications which go much beyond "classical" Morse theory. Method (iv), while striking and inspiring appears for now not to have been exploited efficiently.

2.1. **Connecting Manifolds.** One way to look to the Morse complex is by viewing the coefficients a_y^x of the differential as a measure of the 0-dimensional manifold M_y^x. The question we discuss here is in what way we can measure algebraically the similar higher dimensional moduli spaces. This is clearly a significant issue because, obviously, only a very superficial part of the dynamics of the negative gradient flow of f is encoded in the 0-dimensional moduli spaces of connecting flow lines.

As a matter of terminology, the space M_Q^P when viewed inside the unstable sphere $S_a^u(P)$ (with $f(P) - a$ positive and very small) is also called [13] the connecting manifold of P and Q.

It was mentioned above that, in general, a connecting manifold M_Q^P is not closed. However, if the critical points P and Q are *consecutive* in the sense that there does not exist a critical point R so that $M_R^P \times M_Q^R \neq \emptyset$, then M_Q^P is closed.

2.1.1. *Framed Cobordism Classes.* An important remark of John Franks [13] is that connecting manifolds are canonically framed. First recall that a framed manifold V is a submanifold $V \hookrightarrow S^n$ which has a trivial normal bundle together with a trivialization of this bundle. Two such trivializations are equivalent (and will generally be identified) if they are restrictions of a trivialization of the normal bundle of $V \times [0,1]$ inside $S^n \times [0,1]$. We also recall the Thom-Pontryagin construction in this context [18]. Assuming $V \hookrightarrow S^n$ is framed we define a map

$$\phi_V : S^n \to S^{codim(V)}$$

as follows: consider a tubular neighbourhood $U(V)$ of V, use the framing to define a homeomorphism $\psi : U(V) \to D^k \times V$ where D^k is the closed disk of dimension $k = codim(V)$, consider the composition $\psi' : U(V) \xrightarrow{\psi} D^k \times V \xrightarrow{p_1} D^k \to D^k/S^{k-1} = S^k$ and define ϕ_V by extending ψ' outside $U(V)$ by sending each $x \in S^n \backslash U(V)$ to the base point in $D^k/S^{k-1} = S^k$. The homotopy class of this map is the same if two framings are equivalent. It is easy to see that two framed manifolds (of the same dimension) are cobordant iff their associated Thom maps are homotopic.

We return now to Franks' remark and notice that the manifolds M_Q^P are framed. First, we make the convention to view M_Q^P as a submanifold of the unstable sphere of P, $S_a^u(P)$ (the other choice would have been to use $S^s(Q)$ as ambient manifold). Notice that we have

$$M_Q^P = S_a^u(P) \cap W^s(Q)$$

and this intersection is transversal. Clearly, as $W^s(Q)$ is homeomorphic to a disk, its normal bundle in M is trivial and any two trivializations of this bundle are equivalent. This implies that the normal bundle of $M_Q^P \hookrightarrow S_a^u(P)$ is also trivial and a trivialization of the normal bundle of $W^s(Q)$ provides a trivialization of this bundle which is unique up to equivalence.

Recall that if P and Q are consecutive critical points, then M_Q^P is closed. As we have seen that it is also framed we may associate to it a framed cobordism class

$$\widetilde{M_Q^P} \in \pi_{|P|-1}(S^{|Q|}) \ .$$

Moreover, it is easy to see that the function f may be perturbed without modifying the dynamics of the the negative gradient flow so that the cell attachments corresponding to the critical points Q and P are in succession. Therefore the map

$$\psi_Q^P : S^{|P|-1} \to S^{|Q|}$$

defined as in formula (3) is still defined. The main result of Franks in [13] is:

Theorem 2.1. [13] *Assume P and Q are consecutive critical points of f. Up to sign $\widetilde{M_Q^P}$ coincides with the homotopy class of ψ_Q^P.*

The idea of proof of this result is quite simple. All that is required is to make even more precise the constructions used in the approach (ii) used to show $d^2 = 0$ for the Morse complex. For this we fix for $W^s(Q)$ a normal framing o which is invariant by translation along the flow γ and which at $Q \in W^s(Q)$ is given by a basis e of $T_Q W^u(Q)$ (this is possible because $W^u(Q)$ and $W^s(Q)$ intersect transversally at Q). We also fix the tubular neighbourhood $U(W^u(Q))$ so that the projection $r'' : U(W^u(Q)) \to W^u(Q)$ has the property that $(r'')^{-1}(Q) = W^s(Q) \cap U(W^u(Q))$ and, for any point $y \in (r'')^{-1}(Q)$, we have $(r'')_*(o_y) = e$. Moreover, we may assume that the normal bundle of M_Q^P in $S_a^u(P)$ is just the restriction of the normal bundle of $W^s(Q)$ (in fact, the two are, in general, only isomorphic and not equal but this is just a minor issue).

Now, follow what happens with the framing of M_Q^P along the composition $u \circ r$. For this we write $r = r'' \circ r'$ where r' follows the flow till reaching $U(W^u(Q))$. Now pick a point in $x \in M_Q^P$ together with its normal frame o_x at x. After applying r', the pair (x, o_x) is taken to a pair $(x', o_{x'})$ with $x' \in \partial((r'')^{-1}(Q))$. Applying now r'', the image of $(x', o_{x'})$ is (Q, e). Take now V a tubular neighbourhood of $M_Q^P \hookrightarrow S_a^u(P)$ together with an identification $V \approx D^{|Q|} \times M_Q^P$ which is provided by the framing o. The argument above implies that if the constant ϵ used to construct the map $u : M(f(Q) - \epsilon) \cup W^u(Q) = M' \to M'/M(f(Q) - \epsilon)$ is very small, then the composition $u \circ r'' \circ r'$ coincides with the relevant Thom-Pontryagin map.

2.1.2. Framed Bordism Classes and Hopf invariants.

It it natural to wonder whether, besides their framing, there are some other properties of the connecting manifolds which can be detected algebraically.

A useful point of view in this respect turns out to be the following: imagine the elements of M_Q^P as path or loops on M. The fact that they are paths is obvious (we parametrize them by the value of $-f$; the negative sign gives the flow lines the orientation coherent with the negative gradient) but they can be transformed into loops very easily. Indeed, fix a simple path in M which joins all the critical points of f and contract this to a point thus obtaining a quotient space $\widehat{M}$ which has the same homotopy type as M. Let $q : M \to \widehat{M}$ be the quotient map. We denote by ΩM the space of based loops on M and keep the notation q for the induced map $\Omega M \to \Omega \widehat{M}$. This discussion shows that there are continuous maps

$$j_Q^P : M_Q^P \to \Omega \widehat{M} \ .$$

These maps have first been defined and used in [6] and they have some interesting properties.

For example, given such a map j_Q^P and assuming that P and Q are consecutive it is natural to ask whether the homology class $[M_Q^P] \in H_{|P|-|Q|-1}(\Omega M; \mathbb{Z})$ is computable (here $[M_Q^P]$ is the image by j_Q^P of the fundamental class of M_Q^P). We shall see that quite a bit more is indeed possible: the full framed bordism class associated to j_Q^P and to the canonical framing on M_Q^P can be expressed as a relative Hopf invariant.

To explain this result first recall that if $V \hookrightarrow S^n$ is framed and $l : V \to X$ is a continuous map with V a closed manifold we may construct a richer Thom-Pontryagin map as follows. We again consider a tubular neighbourhood $U(V)$ of V in S^n together with an identification $U(V) \approx D^k \times V$ where $k = codim(V)$ which

is provided by the framing. We now define a map

$$\bar{\phi}_V : U(V) \xrightarrow{j} D^k \times V \xrightarrow{id \times l} D^k \times X \to (D^k \times X)/(S^{k-1} \times X)$$

where $j : U(V) = D^k \times V \to V$ is the projection and the last map is just the quotient map (which identifies $S^{k-1} \times X$ to the base point). Notice that $\bar{\phi}_V(\partial U(V)) \subset S^{k-1} \times X$. Therefore, we may extend the definition above to a map

$$\bar{\phi}_V : S^n \to (D^k \times X)/(S^{k-1} \times X)$$

by sending all the points in the complement of $U(V)$ to the base point. It is well-known (and a simple exercise of elementary homotopy theory) that there exists a (canonical) homotopy equivalence $(D^k \times X)/(S^{k-1} \times X) \simeq \Sigma^k(X^+)$ where ΣX is the (reduced) suspension of X, Σ^i is the suspension iterated i-times and X^+ is the space X with an added disjoint point (notice also that $\Sigma^k(X^+) = \Sigma^k X \vee S^k$ where $\vee$ denotes the *wedge* or the one point union of spaces). This allows us to view the map $\bar{\phi}_V$ as a map with values in $\Sigma^k(X^+)$. The framed bordism class of V is simply the homotopy class $[\bar{\phi}_V] \in \pi_n(\Sigma^k(X^+))$. This is independent of the various choices made in the construction. Two pairs of data (framings included) (V, l) and (V', l') admit an extension to a manifold $W \subset S^n \times [0, 1]$ with $\partial W = V \times \{0\} \cup V' \times \{1\}$ iff $\bar{\phi}_V \simeq \bar{\phi}_{V'}$. Notice also that to an element $\alpha \in \pi_n(\Sigma^k X^+)$ we may associate a homology class $[\alpha] \in H_{n-k}(X)$ obtained by applying the Hurewicz homomorphism, desuspending k-times and projecting on the $H_*(X)$ term in $H_*(X^+)$.

Returning now to our connecting manifolds M_Q^P we again focus on the case when P and Q are consecutive. The map j_Q^P together with the canonical framing provide a homotopy class $\{M_Q^P\} \in \pi_{|P|-1}(\Sigma^{|Q|}(\Omega M^+))$ (to simplify notation we have replaced $\widehat{M}$ with M here - the two are homotopy equivalent).

As indicated above, it turns out that this class can be computed in terms of a relative Hopf invariant. We shall now discuss how this invariant is defined.

Assume that $S^{q-1} \xrightarrow{\alpha} X_0 \to X'$ and $S^{p-1} \xrightarrow{\beta} X' \to X''$ are two successive cell attachments and that X'' is a subspace of some larger space X. In particular, $X' = X_0 \cup_\alpha D^q$, $X'' = X' \cup_\beta D^p$. Let $S \subset D^q$ be the $q-1$-sphere of radius $1/2$. There is an important map called the coaction associated to α which is defined by the composition

$$\nabla : X' \to X'/S \approx S^q \vee X'$$

where the first map identifies all the points of S to a single one and the second is a homeomorphism (in practice it is convenient to also assume that all the maps and spaces involved are pointed and in that case we view D^q as the reduced cone over S^{q-1}).

We consider the composition

$$\psi(\beta, \alpha) : S^{p-1} \xrightarrow{\beta} X' \xrightarrow{\nabla} S^q \vee X' \xrightarrow{id \vee j} S^q \vee X'' \hookrightarrow S^q \vee X$$

and notice that if $p_2 : S^q \vee X \to X$ is the projection on the second factor, then the composition $p_2 \circ \psi(\beta, \alpha)$ is null-homotopic. This is due to the fact that this composition is homotopic to $S^{p-1} \xrightarrow{\beta} X' \to X'' \hookrightarrow X$ which is clearly null-homotopic. We now consider the map $p_2 : S^q \vee X \to X$. It is well-known in

homotopy theory that any map may be transformed into a fibration. In our case
this comes down to considering the free path fibration

$$t : \widetilde{P}X \to X$$

where $\widetilde{P}X$ is the set of all continuous path in X parametrized by $[0,1]$, $t(\gamma) = \gamma(0)$.
We take the pull back of this fibration over p_2. The total space $\widetilde{E}$ of the resulting
fibration has the same homotopy type as $S^q \vee X$ and it is endowed with a canonical
projection

$$\widetilde{p} : \widetilde{E} \to X$$

which replaces p_2, $\widetilde{p}(z, \gamma) = \gamma(1)$. It is an exercise in homotopy theory to see that
the fibre of the fibration $\widetilde{p}$ is homotopic to $\Sigma^q((\Omega X)^+)$ and that, moreover, the
inclusion of this fibre in the total space is injective in homotopy. As the compo-
sition $p_2 \circ \psi(\beta, \alpha)$ is homotopically trivial, the homotopy exact sequence of the
fibration $\widetilde{E} \to X$ implies that $\psi(\beta, \alpha)$ admits a lift to $\bar{\psi}(\alpha, \beta) : S^{p-1} \to \Sigma^q((\Omega X)^+)$
whose homotopy class does not depend on the choice of lift. We let $H(\alpha, \beta) \in$
$\pi_{p-1}(\Sigma^q((\Omega X)^+))$ be equal to this homotopy class and we call it the relative Hopf
invariant associated to the attaching maps α and β (for a discussion of the relations
between this Hopf invariants and other variants see Chapters 6 and 7 in [9]).

To return to Morse theory, recall from (2) that passing through the two consec-
utive critical points Q and P leads to two successive attaching maps $\phi_Q : S^{|Q|-1} \to$
$M(f(Q) - \epsilon)$ and $\phi_P : S^{|P|-1} \to M(f(P) - \epsilon)$ (we assume again - as we may - that
the set $f^{-1}([f(Q), f(P)])$ does not contain any other critical points besides P and
Q). Moreover, as we know the inclusion $M' = M(f(Q)-\epsilon) \cup W^u(Q) \hookrightarrow M(f(P)-\epsilon)$
is a homotopy equivalence. Therefore, the construction above can be applied to ϕ_Q
and ϕ_P and it leads to a relative Hopf invariant

$$H(P, Q) \in \pi_{|P|-1}(\Sigma^{|Q|}(\Omega M^+)) \ .$$

With these constructions our statement is:

Theorem 2.2. [6][7] *The homotopy class $H(P, Q)$ coincides (up to sign) with the
bordism class $\{M_Q^P\}$. In particular, the homology class $[M_Q^P]$ equals (up to sign and
desuspension) the Hurewicz image of $H(P, Q)$.*

The proof of this result can be found in [7] (a variant proved by a slightly different
method appears in [6]). The proof is considerably more complicated than the proof
of Theorem 2.1 so we will only present a rough justification here. To simplify
notation we let $M_0 = M(f(Q) - \epsilon)$. Let $M_1 = M(f(Q) - \epsilon) \cup U(W^u(Q))$. Recall,
that the inclusions $M' \hookrightarrow M_1 \hookrightarrow M(f(P) - \epsilon)$ are homotopy equivalences.

Let $\mathcal{P} : \Omega M \to PM \to M$ be the path-loop fibration (of total space the *based*
paths on M and of fibre the space of based loops on M). We denote by E_0 the total
space of the pull-back of the fibration $\mathcal{P}$ over the inclusion $M_0 \subset M$. Similarly,
we let E_1 be the total space of the pull-back of $\mathcal{P}$ over the inclusion $M_1 \to M$.
The key remark is that the attaching map $\phi_P : S^u(P) \to M_1$ admits a natural lift
to a map $\widetilde{\phi}_P : S^u(P) \to E_1$. Indeed, we assume here that all the critical points
are identified to the base point. The space E_1 consists of the based paths in M
that end at points in M_1. But each element of the image of ϕ_P corresponds to
precisely such a path which is explicitly given by the corresponding flow line (we
need to use here Moore paths and loops which are paths parametrized by arbitrary

intervals $[0, a]$ and not only the interval $[0, 1]$). Consider the inclusion $E_0 \hookrightarrow E_1$. It is not difficult to see that the quotient topological space E_1/E_0 admits a canonical homotopy equivalence $\eta : E_1/E_0 \to \Sigma^{|Q|}(\Omega M^+)$. Therefore, we may consider the composition $\eta' = \eta \circ \widetilde{\phi_P}$. It is possible to show that this map η' is homotopic to $H(P, Q)$. At the same time, we see that the restriction of $\widetilde{\phi_P}$ to M_Q^P coincides with j_Q^P. Moreover, by making explicit η it is also possible to see that $\widetilde{\phi_P}$ is homotopic to the Thom-Pontryagin map associated to M_Q^P.

2.1.3. *Some topological applications.* We now describe a couple of topological applications of Theorem 2.2. The idea behind both of them is quite simple: the function $-f$ is also a Morse function and the critical points Q, P are consecutive critical points for $-f$. Therefore, the connecting manifold M_P^Q is well defined as well as its associated bordism homotopy class $\{M_P^Q\}$. Clearly, the underlying space for both M_Q^P and M_P^Q is the same. The map j_P^Q is different from j_Q^P just by reversing the direction of the loops. The two relevant framings may also be different. The relation between them is somewhat less straightforward but it still may be understood by considering M embedded inside a high dimensional euclidean space and taking into account the twisting induced by the stable normal bundle. In all cases, this establishes a relationship between the two Hopf invariants $H(P, Q)$ and $H(Q, P)$.

A. The first application [6] concerns the construction of examples of non-smoothable, simply-connected, Poincaré duality spaces. The idea is as follows: we construct Poincaré duality spaces which have a simple CW-decomposition and with the property that for certain two successively attached cells e, f the resulting Hopf invariant H and the Hopf invariant H' associated to the dual cells f', e' are not related in the way described above. If the respective Poincaré duality space is smoothable, then the given cell decomposition can be viewed as associated to an appropriate Morse function and this leads to a contradiction. The obstructions to smoothability constructed in this way are obstructions to the lifting of the Spivak normal bundle to BO. This is an obstruction theory problem but one which can be very difficult to solve directly in the presence of many cells. Thus, this approach is quite efficient to construct examples.

B. The second application [8] concerns the detection of obstructions to the embedding of CW-complexes in euclidean spaces in low codimension. The argument in this case goes roughly as follows. If the CW-complex X embedds in S^n, then we may consider a neighbourhood $U(X)$ of X which is a smooth manifold with boundary. We consider a smooth Morse function $f : U(X) \to \mathbb{R}$ which is constant, maximal and regular on the boundary of $U(X)$. If P and Q are two consecutive critical points for this funtion we obtain that $\Sigma^k H(P, Q) = \Sigma^{k'} H(Q, P)$ for certain values of k and k' which can be estimated explicitly - the main reason for this equality is that the Morse function in question is defined on the sphere so all the questions involving the stable normal bundle become irrelevant. If X admits some reasonably explicit cell-decomposition it is possible to express $H(P, Q)$ as the Hopf invariant H of some successive attachment of two cells e, f and $H(Q, P)$ as $\Sigma^{k''} H'$ where H' is another similar Hopf invariant. The obstructions to embedding appear because the low codimension condition translates to the fact that $k' + k'' > k$.

This can be viewed as an obstruction because it means that after k suspensions the homotopy class of H has to de-suspend more than k-times.

Remark 2.3. The applications at A and B are purely of homotopical type. It is natural to expect that the Morse theoretical arguments that were used to establish these statements can be replaced by purely homotopical ones but this has not been done till now.

2.1.4. *The Serre spectral sequence.* Theorem 2.2 provides considerable information on connecting manifolds for pairs of *consecutive* critical points. However, it does not shed any light on the case of non-consecutive ones. Clearly, if the critical points are not consecutive the respective connecting manifold is not closed and thus no bordism or cobordism class can be directly associated to it. However, after compactification, the boundary of this connecting manifold has a special structure reflected by equation (1). As we shall see following [1], this structure is sufficient to construct an algebraic invariant which provides an efficient "measure" of all connecting manifolds.

This construction is based on the fact that the maps

$$j_Q^P : M_Q^P \to \Omega M$$

are compatible with compactification and with the formula (1) in the following sense. Recall that here ΩM are the based Moore loops on M (these are loops parametrized by intervals $[0, a]$), the critical points of f have been identified to a single point and, moreover, in the definition of j_Q^P we use the parametrization of the flow lines by the values of $-f$. Recall that we have a product given by the concatenation of loops

$$\mu : \Omega M \times \Omega M \to \Omega M \ .$$

With these notations it is easy to see that we have the following formula:

$$(4) \qquad\qquad j_Q^P(u, v) = \mu(j_R^P(u), j_Q^R(v))$$

where $(u, v) \in \overline{M}_R^P \times \overline{M}_Q^R \subset \partial \overline{M}_Q^P$.

We proceed with our consruction. Let $C_*(X)$ be the (reduced) cubical complex of X with coefficients in $\mathbb{Z}/2$. Notice that there is a natural map

$$C_k(X) \otimes C_{k'}(Y) \to C_{k+k'}(X \times Y) \ .$$

A family of cubical chain $s_y^x \in C_{|x|-|y|-1}(\overline{M}_y^x)$, $x, y \in Crit(f)$ is called a *representing chain system* for the moduli spaces M_y^x if for each pair of critical points x, z we have:

    i.

$$ds_z^x = \sum_y s_y^x \otimes s_z^y$$

    ii. s_z^x represents the fundamental class in $H_{|x|-|z|-1}(M_z^x, \partial M_z^x)$.

It is easy to show by induction on the index difference $|x| - |z|$ that such representing chain systems exist. We now fix such a representing chain system $\{s_y^x\}$ and

we define $a_y^x \in C_{|x|-|y|-1}(\Omega M)$ by

$$a_y^x = (j_y^x)_*(s_y^x) \ .$$

Notice that this definition extends the definition of these coefficients in the usual Morse case when $|x| - |y| - 1 = 0$. We have a product map

$$\cdot : C_k(\Omega M) \otimes C_{k'}(\Omega M) \longrightarrow C_{k+k'}(\Omega M \times \Omega M) \xrightarrow{C_*(\mu)} C_{k+k'}(\Omega M)$$

which makes $C_*(\Omega M)$ into a differential ring. The discussion above shows that inside this ring we have the formula

$$da_z^x = \sum_y a_z^x \cdot a_y^z \ .$$

An elegant way to rephrase this formula is to group these coefficients in a matrix $A = (a_y^x)$ and then we have

$$(5) \qquad\qquad\qquad\qquad dA = A^2 \ .$$

We now define a new chain complex $\mathcal{C}(f)$ associated to f by

$$(6) \qquad \mathcal{C}(f) = (C_*(\Omega M) \otimes \mathbb{Z}/2 < Crit(f) > , d) \ \ , \ \ dx = \sum_y a_y^x \otimes y \ .$$

We shall call this complex *the extended Morse complex of f*. Here, $C_*(\Omega M) \otimes \mathbb{Z}/2 < Crit(f) >$ is viewed as a graded $C_*(\Omega M)$-module and d respects this structure in the sense that it verifies $d(a \otimes x) = (da) \otimes x + a(dx)$ (the grading on $Crit(f)$ is given, as before, by the Morse index). Choosing orientations on all the stable manifolds of all the critical points induces a co-orientation on all the unstable manifolds, and hence an orientations on the intersections $W^u(P) \cap W^s(Q)$ and finally on all the moduli spaces M_Q^P : we may then use $\mathbb{Z}$-coefficients for this complex as well as, of course, for the classical Morse complex. In this case appropriate signs appear in the formulae above. Clearly, $d^2 = 0$ due to (5).

By definition, the coefficients a_y^x represent the moduli spaces M_y^x. However, these coefficients are not invariant with respect to the choices made in their construction. Therefore, it is remarkable that there is a natural construction which extracts from this complex a useful algebraic invariant which is not just the homology of the complex - as it happens, this homology is not too interesting as it coincides with that of a point.

Consider the obvious differential filtration which is defined on this complex by

$$F^k\mathcal{C}(f) = C_*(\Omega M) \otimes \mathbb{Z}/2 < x \in Crit(f) : \ ind_f(x) \leq k > \ .$$

Denote the associated spectral sequence by

$$E(f) = (E_{p,q}^r(f), d^r) \ .$$

Theorem 2.4. [1] *When M is simply connected and if $r \geq 2$ the spectral sequence $E(f)$ coincides with the Serre spectral sequence of the path-loop fibration*

$$\mathcal{P} : \Omega M \to PM \to M \ .$$

Remark 2.5. a. A similar result can be established even in the absence of the simple-connectivity condition which has been assumed here to avoid some technical complications.

b. The Serre spectral sequence of the path-loop fibration of a space X contains considerable information on the homotopy type of the space. In particular, there are spaces with the same cohomology and cup-product but which may be distinguished by their respective Serre spectral sequences.

To outline the proof of the theorem we start by recalling the construction of the Serre spectral sequence in the form which will be of use here. We shall assume here that the Morse function f is self-indexed (in the sense that for each critical point x we have $ind_f(x) = k \Rightarrow f(x) = k$) and that it has a single minimum denoted by m. Let $M_k = f^{-1}((-\infty, k + \epsilon])$. We have

$$M_k = M_{k-1} \bigcup_{\phi_y} D_y^k$$

where the union is taken over all the critical points $y \in Crit_k(f)$ and $\phi_y : S^u(y) \to M_{k-1}$ are the respective attaching maps. Denote by E_k the total space of the fibration induced by pull-back over the inclusion $M_k \hookrightarrow M$ from the fibration $\mathcal{P}$. Consider the filtration of $C_*(PM)$ given by $F^k P = Im(C_*(E_k) \longrightarrow C_*(PM))$. The spectral sequence induced by this filtration is invariant after the second page and is precisely the Serre spectral sequence (this spectral sequence may be constructed as above but by using an arbitrary skeletal filtration $\{X_k\}$ of a space X which has the same homotopy type as that of M; in our case the particular filtration given by the sets M_k is a natural choice).

For further use, we also notice that there is an obvious action of ΩM on PM and this action induces one on each E_k. Therefore, we may view $C_*(E_k)$ as a $C_*(\Omega M)$-module.

The first step in proving the theorem is to consider a certain compactification of the unstable manifolds of the critical points of f. Recall that f is self-indexed and that m is the unique minimum critical point of f. Fix $x \in Crit(f)$ and define the following equivalence relation on the set $\overline{M}_m^x \times [0, f(x)]$:

$$(a, t) \sim (a', t') \text{ iff } t = t' \text{ and } a(-\tau) = a'(-\tau) \ \forall \tau \geq t \ .$$

Here the elements of $\overline{M}_m^x$ are viewed as paths in M parametrized by the value of $-f$ (so that $f(a(-\tau)) = \tau$).

Denote by $\widehat{W}(x)$ the resulting quotient topological space. Notice that, if $y \in \overline{W^u(x)}$, then there exists some $a \in \overline{M}_m^x$ so that y is on the (possibly broken) flow line represented by a. Or, in other words, so that $a(-f(y)) = y$. This path a might not be unique. Indeed, inside $\widehat{W}(x)$ there is precisely one equivalence class $[a, f(y)]$ (with $a(-f(y)) = y$) for each (possibly) broken flow line joining x to y. Clearly, if $y \in W^u(x)$, then there is just one such flow line and so the natural surjection

$$\pi : \widehat{W}(x) \longrightarrow \overline{W^u(x)}, \ \pi([a, t]) = a(-t)$$

is a homeomorphism when restricted to $\pi^{-1}(W^u(x))$. Thus we may view $\widehat{W}(x)$ as a special compactification of $W^u(x)$ or as a desingularization of $\overline{W^u(x)}$. It is not difficult to believe (but harder to show and will not be proven here see [1]) that

$\widehat{W}(x)$ is a topological manifold with boundary and moreover

$$(7) \qquad \partial\widehat{W}(x) = \bigcup_y M_y^x \times \widehat{W}(y) \ .$$

We continue the proof of the Theorem 2.4 with the remark that there are obvious maps

$$h_x : \widehat{W}(x) \to PM$$

which associate to $[a, t]$ the path in M which follows a from x to $a(-t)$. These maps and the maps j_y^x are compatible with formula (7) in the sense that

$$(8) \qquad h_x(a', [a'', t]) = j_y^x(a') \cdot h_y([a'', t])$$

where $(a', [a'', t]) \in M_y^x \times \widehat{W}(y)$ and $\cdot$ represents the action of ΩM on PM.

Now, we may obviously rewrite (7) as:

$$\partial\widehat{W}(x) = \bigcup_y \overline{M}_y^x \times \widehat{W}(y) \ .$$

Given the representing chain system $\{s_y^x\}$ it is easy to construct an associated representing chain system for $\widehat{W}(x)$. This is a system of chains $v(x) \in C_{|x|}(\widehat{W}(x))$ so that $v(x)$ represents the fundamental class of $C_{|x|}(\widehat{W}(x), \partial\widehat{W}(x))$ and we have the formula

$$dv(x) = \sum_y s_y^x \otimes v(y) \ .$$

Finally, we define a $C_*(\Omega M)$-module chain map

$$\alpha : \mathcal{C}(f) \to C_*(PM)$$

by

$$\alpha(x) = (h_x)_*(v(x)) \ .$$

The formulas above show that we have

$$d[(h_x)_*(v(x))] = \sum_y a_y^x \cdot (h_y)_*(v(y))$$

and so α is a chain map. It is clear that the map α is filtration preserving and it is not difficult to see that it induces an isomorphism at the E^2 level of the induced spectral sequences and this concludes the proof of Theorem 2.4.

Remark 2.6. a. Another important but immediate property of $\widehat{W}(x)$ is that it is a contractible space. Indeed, all the points in $\overline{M}_m^x \times \{f(x)\}$ are in the same equivalence class. Moreover, each point $[a, t] \in \widehat{W}(x)$ has the property that it is related by the path $[a, \tau]$, $\tau \in [t, f(x)]$ to $* = [a, f(x)]$. The contraction of $\widehat{W}(x)$ to $*$ is obtained by deforming $\widehat{W}(x)$ along these paths. Given that $\widehat{W}(x)$ is a contractible topological manifold with boundary, it is natural to suspect that $\widehat{W}(x)$ is homeomorphic to a disk. This is indeed the case as is shown in [1] and is an interesting fact in itself because it implies that the union of the unstable manifolds of a self-indexed Morse-Smale function gives a CW-decomposition of M. The attaching map of the cell $\widehat{W}(x)$ is simply the restriction of π to $\partial\widehat{W}(x)$.

b. The Serre spectral sequence result above and the bordism result in Theorem 2.2 are obviously related via the central role of the maps j_Q^P. There is also a more explicit relation. Indeed, (a stable version of) the Hopf invariants appearing

in Theorem 2.2 can be interpreted as differentials in the Atyiah-Hirzebruch-Serre spectral sequence of the path-loop fibration with coefficients in the stable homotopy of ΩM. Moreover, the relation (1) can be understood as also keeping track of the framings. This leads to a type of extended Morse complex in which the coefficients of the differential are stable Hopf invariants [6]. All of this strongly suggests that the construction of the complex $\mathcal{C}(f)$ can be enriched so as to include the framings of the connecting manifolds and, by the same method as above, that the whole Atyiah-Hirzebruch-Serre spectral sequence should be recovered from this construction.

c. Another interesting question, open even for consecutive critical points P, Q, is whether there are some additional constraints on the topology of the connecting manifolds M_Q^P besides those imposed by Theorem 2.2.

d. Yet another open question is how this machinery can be adapted to the Morse-Bott situation and how it can be extended to general Morse-Smale flows (not only gradient-like ones).

e. It is natural to wonder what is the richest level of information that one can extract out of the moduli spaces of Morse flow lines. At a naive level, the union of all the points situated on the flow lines of f is precisely the whole underlying manifold M so we expect that there should exist some assembly process producing the manifold M out of these moduli spaces. Such a machine has been constructed by Cohen, Jones and Segal [4][5]. They show that one can form a category out of the moduli spaces of connecting trajectories and that the classifying space of this category is of the homeomorphism type of the underlying manifold. In their construction an essential point is that the glueing of flow lines is associative. This approach is quite different from the techniques above and does not imply the results concerning the extended Morse complex or the Hopf invariants that we have presented. The two points of view are, essentially, complementary.

To end this section it is useful to make explicit a relation between Theorems 2.4 and 2.2 (we assume as above that M is simply-connected).

Proposition 2.7. *Assume that there are $q, p \in \mathbb{N}$ so that in the Serre spectral sequence of the path loop fibration of M we have $E_{k,s}^2 = 0$ for $q < k < p$ and there is an element $a \in E_{p,0}^2$ so that $d^{p-q}a \neq 0$, then any Morse-Smale function on M has a pair of consecutive critical points P, Q of indexes at least q and at most p so that the homology class $[M_Q^P] \in H_{|P|-|Q|-1}(\Omega M) \neq 0$.*

Clearly, Theorem 2.4 directly implies that, even without any restriction on E^2, if we have $d^r a \neq 0$ with $a \in E_{p,0}^r$, then for any Morse-Smale function f there are critical points P and Q with $|P| = p$ and $|Q| = p - r$ so that $M_Q^P \neq \emptyset$. Indeed, if this would not be the case, then all the coefficients a_y^x in the extended Morse complex of f are null whenever $|x| = p$, $|y| = p - r$. By the construction of the associated spectral sequence, this leads to a contradiction. However, the pair P, Q resulting from this argument might have a connecting manifold which is not closed so that its homology class is not even defined and, thus, Proposition 2.7 provides a stronger conclusion. The proof of the Proposition is as follows. Recall that $E_{s,r}^2 \approx H_s(M) \otimes H_r(\Omega M)$ and so $H_*(M) = 0$ for $q \leq 0 \leq p$. If there are some points $P, Q \in Crit(f)$ with $q \leq |Q|, |P| \leq p$ so that the differential of P in the classical Morse complex contains Q with a non-trivial coefficient then this pair P, Q may be taken as the one we are looking for. If all such differentials in the classical Morse complex are trivial it follows that the critical points of index p and

q are consecutive. In this case, the geometric arguments used in the proofs of either Theorem 2.2 or 2.4 imply that if for all pairs P, Q with $|P| = p$, $|Q| = q$ we would have $[M_Q^P] = 0$, then the differential d^{p-q} would vanish on $E_{p,0}^{p-q}$.

Remark 2.8. Notice that the pair of critical points P and Q constructed in the proposition verify the property that $|P|$ and $|Q|$ are consecutive inside the set $\{ind_f(x) : x \in Crit(f)\}$.

2.2. **Operations.** We discuss here a different and, probably, more familiar approach to understanding connecting manifolds as well as other related Morse theoretic moduli spaces. This point of view has been used extensively by many authors - Fukaya [14], Betz and Cohen [2] being just a few of them. For this reason we shall review this technique very briefly.

Given two consecutive critical points x, y notice that the set $T_y^x = W^u(x) \cap W^s(y)$ is homeomorphic to the un-reduced suspension of M_y^x. Therefore, we may see this as an obvious inclusion

$$i_y^x : \Sigma M_y^x \to M$$

and we may consider the homology class $[T_y^x] = (i_y^x)_*(s[M_y^x])$ where s is suspension and $[M_y^x]$ is the fundamental class. There exists an obvious evaluation map

$$e : \Sigma \Omega M \to M$$

which is induced by $\Omega M \times [0,1] \to M$, $(\beta, t) \to \beta(t)$ (the loops here are parametrized by the interval $[0, 1]$ but this is a minor technical difficulty). It is easy to see, by the definition of this evaluation map, that $[T_y^x] = e_*((j_y^x)_*([M_y^x]))$. In general the map e_* is not injective in homology. Clearly, the full bordism class $\{M_y^x\}$ carries much more information than the homology class $[T_y^x]$. Still, there is a direct way to determine $[T_y^x]$ without passing through a calculation of $\{M_y^x\}$ and we will now describe it.

Consider a second Morse-Smale function $g : M \to \mathbb{R}$ so that its associated unstable and stable manifolds $W_g^u(-)$, $W_g^s(-)$ intersect transversally the stable and unstable manifolds of f and, except if they are of top dimension, they avoid the critical points of f.

Fix $x, y \in Crit(f)$ and $s \in Crit(g)$ so that $|x| - |y| - ind_g(s) = 0$. We may define $k(x, y; s) = \#(T_y^x \cap W_g^s(s)$ (where the counting takes into account the relevant orientations if we work over $\mathbb{Z}$). We now put

$$\bar{k}_y^x = \sum_s k(x, y; s)s \in C(g) \ .$$

The essentially obvious claim is that:

Proposition 2.9. *The chain $\bar{k}_y^x$ is a cycle whose homology class is $[T_y^x]$.*

Indeed we have $\sum_s k(x, y; s)h_z^s = 0$ where $ind_g(z) = ind_g(s) - 1$ and h_z^s are the coefficients in the classical Morse complex of g. This equality is valid because we may consider the 1-dimensional space $T_y^x \cap W_g^s(z)$. This is an open 1-dimensional manifold whose compactification is a 1-manifold whose boundary points are counted precisely by the formula $\sum_s k(x, y; s)h_z^s = 0$. By basic intersection theory it is immediate to see that the homology class represented by this cycle is $[T_y^x]$.

While this construction does not shed a lot of light on the properties of M_y^x its role is important once we use it to recover the various homological operations of M. To see how this is done from our perspective notice that the intersection

$$T_y^x \cap W_g^s(s) = W_f^u(x) \cap W_f^s(y) \cap W_g^s(s)$$

can be viewed as a particular case of the following situation: assume that f_1, f_2, f_3 are three Morse-Smale functions in general position and define

$$T_z^{x,y} = \left(W_{f_1}^u(x) \cap W_{f_2}^u(y) \cap W_{f_3}^s(z)\right).$$

If we assume that $|x| + |y| - |z| - n = 0$ we may again count the points in $T_z^{x,y}$ with appropriate signs and we may define coefficients $t_z^{x,y} = \#T_z^{x,y}$.

This leads to an operation [2][14]

$$C(f_1) \otimes C(f_2) \to C(f_3)$$

given as a linear extension of

$$x \otimes y \to \sum_z t_z^{x,y} z.$$

It is easy to see that this operation descends in homology and that it is in fact the dual of the $\cup$-product. Moreover, the space $T_z^{x,y}$ may be viewed as obtained by considering a graph formed by three oriented edges meeting into a point with the first two entering the point and the other one exiting it and considering all the configurations obtained by mapping this graph into M so that to the first edge we associate a flow line of f_1 which exits x, to the second edge a flow line of f_2 which exits y and to the third a flow line of f_3 which enters z. Clearly, this idea may be pushed further by considering other, more complicated graphs and understanding what are the operations they correspond to as was done by Betz and Cohen [2].

3. Applications to Symplectic topology

We start with some applications that are rather "soft" even if difficult to prove and we shall continue in the main part of the section,§3.2, with some others that go deeper.

3.1. **Bounded orbits.** We fix a symplectic manifold (M, ω) which is not compact. Assume that $H : M \to \mathbb{R}$ is a smooth hamiltonian whose associated hamiltonian vector field is denoted by X_H. One of the main questions in hamiltonian dynamics is whether a given regular hypersurface $A = H^{-1}(a)$ of H has any closed caracteristics, or equivalently, whether the hamiltonian flow of H has any periodic orbits in A. As M is not compact, from the point of view of dynamical systems, the first natural question is whether X_H has any *bounded* orbits in A. Moreover, there is a remarkable result of Pugh and Robinson [20], the C^1-closing lemma, which shows that, for a generic choice of H, the presence of bounded orbits insures the existence of some periodic orbits. Therefore, we shall focus in this subsection on the detection of bounded orbits. It should be noted however that the detection of periodic orbits in this way is not very effective because the periods of the orbits found can not be estimated. Moreover, there is no reasonable test to decide whether a given hamiltonian belongs to the generic family to which the C^1-closing lemma applies. Finally, it will be clear from the methods of proof described below that these results are also soft in the sense that they are not truly specific to Hamiltonian flows but rather they apply to many other flows.

An example of a bounded orbit result is the following statement [7].

Theorem 3.1. *Assume that H is a Morse-Smale function with respect to a riemannian metric g on M so that M is metrically complete and there exists an ϵ and a compact set $K \subset M$ so that $||\nabla_g H(x)|| \geq \epsilon$ for $x \notin K$. Suppose that P and Q are two critical points of H so that $|P|$ and $|Q|$ are successive in the set $\{ind_H(x) : x \in Crit(H)\}$. If the stabilization $[H(P,Q)] \in \pi^S_{|P|-|Q|-1}(\Omega M)$ of the Hopf invariant $H(P,Q)$ is not trivial then there are regular values $v \in (H(Q), H(P))$ so that $H^{-1}(v)$ contains bounded orbits of X_H.*

Before describing the proof of this result let's notice that the theorem is not difficult to apply. Indeed, one simple way to verify that there are pairs P, Q as required is to use Proposition 2.7 together with Remark 2.8 with a minor adaptation required in a non-compact setting. This adaptation consists of replacing the Serre spectral sequence of the path loop fibration with the Serre spectral sequence of a relative fibration $\Omega M \to (E_1, E_0) \to (N_1, N_0)$ where N_1 is an isolating neighbourhood for the gradient flow of H which contains K and N_0 is a (regular) exit set for this neighbourhood (to see the precise definition of these Conley index theoretic notions see [22]). The fibration is induced by pull-back from the path-loop fibration $\Omega M \to PM \to M$ over the inclusion $(N_1, N_0) \hookrightarrow (M, M)$. In short, because the gradient of H is away from 0 outside of a compact set, pairs (N_1, N_0) as above are easy to produce and if the pair (N_1, N_0) has some interesting topology it is easy to deduce the existence of non-constant bounded orbits. Here is a concrete example.

Corollary 3.2. *Assume that M is the contangent bundle of some closed, simply-connected manifold N of dimension $k \geq 2$ and ω is an arbitrary symplectic form. Assume that $H : M \to \mathbb{R}$ is Morse and that outside of some compact set containing the 0 section, H restricts to each fibre of the bundle to a non-degenerate quadratic form. Then, X_H has bounded, non constant orbits.*

Of course, this result is only interesting when there are no compact level hypersurfaces of H. This does happen if the quadratic form in question has an index which is neither 0 nor k. The proof of this result comes down to the fact that as N is closed and not a point there exists a lowest dimensional homology class $u \in H_t(N)$ which is transgressive in the Serre spectral sequence (this means $d^t u \neq 0$). Using the structure of function quadratic at infinity of H it is easy to construct a pair (N_1, N_0) where N_1 is is homotopic to a disk bundle of base N and N_0 is the associated sphere bundle. The spectral sequence associated to this pair can be related by the Thom isomorphism to the Serre spectral sequence of the path-loop fibration over N and the element $\bar{u} \in H_*(N_1, N_0)$ which corresponds to u by the Thom isomorphism will have a non-vanishing differential. This means that Proposition 2.7 may be used to show the non-triviality of a homology class $[M^P_Q]$ for P and Q as in Theorem 3.1. By Theorem 2.2, $[M^P_Q]$ is the same up to sign as the homology class of the Hopf invariant $H(P,Q)$ so Theorem 3.1 is applicable to detect bounded orbits.

We now describe the proof of the theorem. The basic idea of the proof is simpler to present in the particular case when $H^{-1}(H(Q), H(P))$ does not contain any critical value. In this case let $A = H^{-1}(a)$ where $a \in (H(Q), H(P))$. We intend to show that A contains some bounded orbits of X_H. To do this notice that the two

sets $S_1 = W^u(P) \cap A$, $S_2 = W^s(Q) \cap A$ are both diffeomorphic to spheres. We now assume that no bounded orbits exist and we consider a compact neighbourhood U of $S_1 \cup S_2$. Assume that we let S_2 move along the flow X_H. As this flow has no bounded orbits, each point of S_2 will leave U at some moment. Suppose that we are able to perturb the flow induced by X_H to a new deformation $\eta : M \times \mathbb{R} \to M$ so that for some finite time T *all* the points in S_2 are taken simultaneously outside U (in other words $\eta_T(S_2) \cap U = \emptyset$) and so that η leaves Q fixed. It is easy to see that this implies that $S_1 \cap S_2$ is bordant to the empty set which, by Theorem 2.2, is impossible because $H(P, Q) \neq 0$. This perturbation η is in fact not hard to construct by using some elements of Conley's index theory and the fact that the maximal invariant set of X_T inside U is the empty set (the main step here is to possibly modify also U so that it admits a regular exit region $U_0 \subset U$ and we then construct η so that it follows the flow lines of X_T but stops when reaching U_0, this eliminates the problem of "bouncing" points which first exit U but later re-enter it).

The case when there are critical points in $H^{-1}(H(Q), H(P))$ follows the same idea but is considerably more difficult. The main difference comes from the fact that the sets S_1 and S_2 might not be closed manifolds. Even their closures $\bar{S}_1$ and $\bar{S}_2$ are not closed manifolds in general but might be singular sets. To be able to proceed in this case we first replace P and Q with a pair of critical points of the same index so that for any critical point $Q' \in H^{-1}(H(Q), H(P))$ with $ind_H(Q') = ind_H(Q)$ we have $[H(P, Q')] = 0$. We then take a very close to $H(Q)$ so that S_2 at least is diffeomorphic to a sphere. We then first study the stratification of $\bar{S}_1$: there is a top stratum of dimension $|P| - 1$ which is S_1 and a singular stratum S' of dimension $|Q| - 1$ which is the union of the sets $W^u(Q') \cap A$ for all Q' so that $M_{Q'}^P \neq \emptyset$ and $|Q'| = |Q|$. Notice that the way to construct the null-bordism of $S_1 \cup S_2$ is to consider in $A \times [0, T]$ the submanifold $W = (\eta_t(\bar{S}_2), t)$ and intersect it with $W' = \bar{S}_1 \times [0, T]$ - we assume here $\eta_T(S_2) \cap S_1 = \emptyset$. Clearly, we need this intersection to be transverse and this can be easily achieved by a perturbation of η. The main technical difficulty is that L might intersect the singular part, $S' \times [0, T]$. Indeed, $dim(W) = n - q$, $dim(S') = q - 1$, $dim(A) = n - 1$ and so generically the intersection I between W and $S' \times [0, T]$ is 0-dimensional and not necessarily void. By studying the geometry around each of the points of I it can be seen that $S_1 \cap S_2$ is bordant to the union of the $M_{Q'}^P$'s where $Q' \in H^{-1}(H(Q), H(P))$ (roughly, this follows by eliminating from the singular bordism $W \cap W'$ a small closed, cone-like neighbourhood around each singular point and showing that the boundary of this cone-like neighbourhood is homeomorphic to a $M_{Q'}^P$). We now use the fact that all the stable bordism classes of the $M_{Q'}^P$'s vanish (because $[H(P, Q')] = 0$) and this leads us to a contradiction. Notice also that, at this point, we need to use stable Hopf invariants (or bordism classes) $\in \pi^S(\Omega M)$ because, by contrast to the stable case, the unstable Thom-Pontryagin map associated to a disjoint union is not necessarily equal to the sum of the Thom-Pontryagin maps of the terms in the union and hence, unstably, even if we know $H(P, Q') = 0, \forall Q'$ we still can not deduce $H(P, Q) = 0$.

Remark 3.3. It would be interesting to see whether, under some additional assumptions, a condition of the type $[H(P, Q)] \neq 0$ implies the existence of periodic orbits and not only bounded ones.

3.2. **Detection of pseudoholomorphic strips and Hofer's norm.** In this subsection we shall again use the Morse theoretic techniques described in §2 and, in particular, Theorem 2.4 to study some symplectic phenomena by showing that Floer's complex can be enriched in a way similar to the passage from the classical Morse complex to the extended one.

3.2.1. *Elements of Floer's theory.* We start by recalling very briefly some elements from Floer's construction (for a more complete exposition see, for example, [23]).

We shall assume from now on that (M, ω) is a symplectic manifold - possibly non-compact but in that case convex at infinity - of dimension $m = 2n$. We also assume that L, L' are closed (no boundary, compact) Lagrangian submanifolds of M which intersect transversally.

To start the description of our applications it is simplest to assume for now that L, L' are simply-connected and that $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$. Cotangent bundles of simply-connected manifolds offer immediate examples of manifolds verifying these conditions.

We fix a path $\eta \in \mathcal{P}(L, L') = \{\gamma \in C^\infty([0, 1], M) : \gamma(0) \in L, \gamma(1) \in L'\}$ and let $\mathcal{P}_\eta(L, L')$ be the path-component of $\mathcal{P}(L, L')$ containing η. This path will be trivial homotopically in most cases, in particular, if L is hamiltonian isotopic to L'. We also fix an almost complex structure J on M compatible with ω in the sense that the bilinear form $X, Y \to \omega(X, JY) = \alpha(X, Y)$ is a Riemannian metric. The set of all the almost complex structures on M compatible with ω will be denoted by $\mathcal{J}_\omega$. Moreover, we also consider a smooth 1-periodic Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$ which is constant outside a compact set and its associated 1-periodic family of hamiltonian vector fields X_H determined by the equation

$$\omega(X_H^t, Y) = -dH_t(Y) \ , \ \forall Y \ .$$

we denote by ϕ_t^H the associated Hamiltonian isotopy. We also assume that $\phi_1^H(L)$ intersects transversally L'.

In our setting, the action functional below is well-defined:

(9) $$\mathcal{A}_{L,L',H} : \mathcal{P}_\eta(L, L') \to \mathbb{R} \ , \ x \to -\int \overline{x}^*\omega + \int_0^1 H(t, x(t))dt$$

where $\overline{x}(s, t) : [0, 1] \times [0, 1] \to M$ is such that $\overline{x}(0, t) = \eta(t), \overline{x}(1, t) = x(t), \forall t \in [0, 1]$, $x([0, 1], 0) \subset L, x([0, 1], 1) \subset L'$. The critical points of $\mathcal{A}$ are the orbits of X_H that start on L, end on L' and which belong to $\mathcal{P}_\eta(L, L')$. These orbits are in bijection with a subset of $\phi_1^H(L) \cap L'$ so they are finite in number. If H is constant these orbits coincide with the intersection points of L and L' which are in the class of η. We denote the set of these orbits by $I(L, L'; \eta, H)$ or shorted $I(L, L')$ if η and H are not in doubt.

We now consider the solutions u of Floer's equation:

(10) $$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t, u) = 0$$

with

$$u(s, t) : \mathbb{R} \times [0, 1] \to M \ , u(\mathbb{R}, 0) \subset L \ , \ u(\mathbb{R}, 1) \subset L' \ .$$

When H is constant, these solutions are called *pseudo-holomorphic strips*.

For any strip $u \in \mathcal{S}(L, L') = \{u \in C^\infty(\mathbb{R} \times [0,1], M) : u(\mathbb{R}, 0) \subset L , \ u(\mathbb{R}, 1) \subset L'\}$ consider the energy

$$(11) \qquad E_{L,L',H}(u) = \frac{1}{2} \int_{\mathbb{R} \times [0,1]} ||\frac{\partial u}{\partial s}||^2 + ||\frac{\partial u}{\partial t} - X_H^t(u)||^2 \ ds \ dt .$$

For a generic choice of J, the solutions u of (10) which *are of finite energy*, $E_{L,L',H}(u) < \infty$, behave like negative gradient flow lines of $\mathcal{A}$. In particular, $\mathcal{A}$ decreases along such solutions. We consider the moduli space

$$(12) \qquad \mathcal{M}' = \{u \in \mathcal{S}(L, L') : u \text{ verifies}(10) , \ E_{L,L',H}(u) < \infty\} .$$

The translation $u(s,t) \to u(s+k, t)$ obviously induces an $\mathbb{R}$ action on $\mathcal{M}'$ and we let $\mathcal{M}$ be the quotient space. For each $u \in \mathcal{M}'$ there exist $x, y \in I(L, L'; \eta, H)$ such that the (uniform) limits verify

$$(13) \qquad \lim_{s \to -\infty} u(s,t) = x(t) , \ \lim_{s \to +\infty} u(s,t) = y(t) .$$

We let $\mathcal{M}'(x,y) = \{u \in \mathcal{M}' : u \text{ verifies } (13)\}$ and $\mathcal{M}(x,y) = \mathcal{M}'(x,y)/\mathbb{R}$ so that $\mathcal{M} = \bigcup_{x,y} \mathcal{M}(x,y)$. If needed, to indicate to which pair of Lagrangians, to what Hamiltonian and to what almost complex structure are associated these moduli spaces we shall add L and L', H, J as subscripts (for example, we may write $\mathcal{M}_{L,L',H,J}(x,y)$).

For $x, y \in I(L, L'; \eta, H)$ we let

$$\mathcal{S}(x,y) = \{u \in C^\infty([0,1] \times [0,1], M) : \quad u([0,1], 0) \subset L, u([0,1], 1) \subset L', \\ u(0,t) = x(t), u(1,t) = y(t)\} .$$

To each $u \in S(x,y)$ we may associate its Maslov index $\mu(u) \in \mathbb{Z}$ [28] and it can be seen that, in our setting, this number only depends on the points x, y. Thus, we let $\mu(x,y) = \mu(u)$. Moreover, we have the formula

$$(14) \qquad \mu(x, z) = \mu(x,y) + \mu(y,z) .$$

According to these relations, the choice of an arbitrary intersection point x_0 and the normalization $|x_0| = 0$, defines a grading $|.|$ such that :

$$\mu(x, y) = |x| - |y|.$$

There is a notion of regularity for the pairs of (H, J) so that, when regularity is assumed, the spaces $\mathcal{M}'(x,y)$ are smooth manifolds (generally non-compact) of dimension $\mu(x,y)$ and in this case $\mathcal{M}(x,y)$ is also a smooth manifold of dimension $\mu(x,y) - 1$. Regular pairs (H, J) are generic and, in fact, they are so even if L and L' are not transversal (but in that case H can not be assumed to be constant), for example, when $L = L'$.

Floer's construction is natural in the following sense. Let $L'' = (\phi_1^H)^{-1}(L')$. Consider the map $b_H : \mathcal{P}(L, L'') \to \mathcal{P}(L, L')$ defined by $(b_H(x))(t) = \phi_t^H(x(t))$. Let $\eta' \in \mathcal{P}(L, L'')$ be such that $\eta = b_H(\eta')$. Clearly, b_H restricts to a map between $\mathcal{P}_{\eta'}(L, L'')$ and $\mathcal{P}_\eta(L, L')$ and it restricts to a bijection $I(L, L''; \eta', 0) \to I(L, L'; \eta, H)$.

It is easy to also check

$$\mathcal{A}_{L,L',H}(b_H(x)) = \mathcal{A}_{L,L'',0}(x)$$

and that the map b_H identifies the geometry of the two action functionals. Indeed for $u : \mathbb{R} \times [0,1] \to M$ with $u(\mathbb{R}, 0) \subset L$, $u(\mathbb{R}, 1) \subset L''$, $\tilde{u}(s, t) = \phi_t(u(s, t))$, $\tilde{J} = \phi^* J$ we have

$$\phi_* \left(\frac{\partial u}{\partial s} + \tilde{J} \frac{\partial u}{\partial t} \right) = \frac{\partial \tilde{u}}{\partial s} + J \left(\frac{\partial \tilde{u}}{\partial t} - X_H \right) .$$

Therefore, the map b_H induces diffeomorphisms:

$$b_H : \mathcal{M}_{L,L'',\tilde{J},0}(x, y) \to \mathcal{M}_{L,L',J,H}(x, y)$$

where we have identified $x, y \in L \cap L''$ with their orbits $\phi_t^H(x)$ and $\phi_t^H(y)$.

Finally, the non-compactness of $\mathcal{M}(x, y)$ for $x, y \in I(L, L'; \eta, H)$ is due to the fact that, as in the Morse-Smale case, a sequence of strips $u_n \in \mathcal{M}(x, y)$ might "converge" (in the sense of Gromov) to a broken strip. There are natural compactifications of the moduli spaces $\mathcal{M}(x, y)$ called Gromov compactifications and denoted by $\overline{\mathcal{M}}(x, y)$ so that each of the spaces $\overline{\mathcal{M}}(x, y)$ is a topological manifold with corners whose boundary verifies:

$$(15) \qquad \partial \overline{\mathcal{M}}(x, y) = \bigcup_{z \in I(L,L';\eta,H)} \overline{\mathcal{M}}(x, z) \times \overline{\mathcal{M}}(z, y) .$$

A complete proof of this fact can be found in [1] (when $dim(\mathcal{M}_y^x) = 1$ the proof is due to Floer and is now classical).

3.2.2. *Pseudoholomorphic strips and the Serre spectral sequence.* We will now construct a complex $\mathcal{C}(L, L'; H, J)$ by a method that mirrors the construction of $\mathcal{C}(f)$ in §2.1.4.

This complex, called the *extended Floer complex* associated to L, L', H, J has the form:

$$\mathcal{C}(L, L'; H, J) = (C_*(\Omega L) \otimes \mathbb{Z}/2 < I(L, L'; \eta, H) >, D)$$

where the cubical chains $C_*(\Omega L)$ have, as before, $\mathbb{Z}/2$-coefficients. If needed, the moduli spaces $\mathcal{M}(x, y)$ can be endowed with orientations which are compatible with formula (15), and so we could as well use $\mathbb{Z}$-coefficients.

To define the differential we first fix a simple path w in L which joins all the points $\gamma(0)$, $\gamma \in I(L, L'; H)$ and we identify all these points to a single one by collapsing this path to a single point. We shall continue to denote the resulting space by L to simplify notation.

For each moduli space $\mathcal{M}(x, y)$ there is a continuous map

$$l_y^x : \mathcal{M}(x, y) \to \Omega L$$

which is defined by associating to $u \in \mathcal{M}(x, y)$ the path $u(\mathbb{R}, 0)$ parametrized by the (negative) values of the action functional $\mathcal{A}$. This is a continuous map and it is seen to be compatible with formula (15) in the same sense as in (8).

We pick a representing chain system $\{k_y^x\}$ for the moduli spaces $\mathcal{M}(x, y)$ and we let

$$m_y^x = (l_y^x)_*(k_y^x) \in C_*(\Omega L)$$

and

$$(16) \qquad Dx = \sum_y m_y^x \otimes y .$$

As in the case of the extended Morse complex the fact that $D^2 = 0$ is an immediate consequence of formula (15).

Remark 3.4. a. There is an apparent assymetry between the roles of L and L' in the definition of this extended Floer complex. In fact, the coefficients of this complex belong naturally to an even bigger and more symmetric ring than $C_*(\Omega L)$. Indeed, consider the space $T(L, L')$ which is the homotopy-pullback of the two inclusions $L \hookrightarrow M$, $L' \hookrightarrow M$. This space is homotopy equivalent to the space of all the continuous paths $\gamma : [0, 1] \rightarrow M$ so that $\gamma(0) \in L$, $\gamma(1) \in L'$. By replacing both L and M by the respective spaces obtained by contracting the path w to a point, we see that there are continous maps $\mathcal{M}(x, y) \rightarrow \Omega(T(L, L'))$. We may then use these maps to construct a complex with coefficients in $C_*(\Omega(T(L, L')))$. Clearly, there is an obvious map $T(L, L') \rightarrow L$ and it is precisely this map which, after looping, changes the coefficients of this complex into those of the extended Floer complex.

b. At this point it is worth mentioning why using representing chain systems is useful in our constructions. Indeed, for the extended Morse complex representing chain systems are not really essential: the moduli spaces M_y^x are triangulable in a way compatible with the boundary formula and so, to represent this moduli space inside the loop space ΩM, we could use instead of the chain a_y^x a chain given by the sum of the top dimensional simplexes in such a triangulation. This is obviously, a simpler and more natural approach but it has the disadvantage that it does not extend directly to the Floer case. The reason is that it is not known whether the Floer moduli spaces $\mathcal{M}(x, y)$ admit coherent triangulations (even if this is likely to be the case).

The chain complex $\mathcal{C}(L, L'; H)$ admits a natural degree filtration which is given by

(17) $F^k \mathcal{C}(L, L'; H, J) = C_*(L) \otimes \mathbb{Z}/2 < x \in I(L, L'; \eta, H) : |x| \leq k > .$

It is clear that this filtration is differential. Therefore, there is an induced spectral sequence which will be denoted by $\mathcal{E}(L, L'; H, J) = (\mathcal{E}_{p,q}^r, D^r)$. We write $\mathcal{E}(L, L'; J) = \mathcal{E}(L, L'; 0, J)$. For convenience we have omitted η from this notation (the relevant components of the paths spaces $\mathcal{P}(L, L')$ will be clear below).

Here is the main result concerning this spectral sequence.

Theorem 3.5. *For any two regular pairs* $(H, J), (H', J')$, *the spectral sequences* $\mathcal{E}(L, L'; H, J)$ *and* $\mathcal{E}(L, L'; H', J')$ *are isomorphic up to translation for* $r \geq 2$. *Moreover, if* ϕ *is a hamiltonian diffeomorphism , then* $\mathcal{E}(L, L'; J)$ *and* $\mathcal{E}(L, \phi(L'); J')$ *are also isomorphic up to translation for* $r \geq 2$ *(whenever defined). The second term of this spectral sequence is* $\mathcal{E}^2(L, L'; H, J) \approx H_*(\Omega L) \otimes HF_*(L, L')$ *where* $HF_*(-, -)$ *is the Floer homology. Finally, if* L *and* L' *are hamiltonian isotopic, then* $\mathcal{E}(L, L'; J)$ *is isomorphic up to translation to the Serre spectral sequence of the path-loop fibration* $\Omega L \rightarrow PL \rightarrow L$.

Isomorphism up to translation of two spectral sequences $E_{p,q}^r$, $F_{p,q}^r$ means that there exists a $k \in \mathbb{Z}$ and chain isomorphisms $\phi^r : E_{p,q}^r \rightarrow F_{p+k,q}^r$. This notion appears naturally here because the choice of the element $x_0 \in I(L, L'; H)$ with $|x_0| = 0$ is arbitrary. A different choice will simply lead to a translated spectral sequence. As follows from the discussion in §3.2.3 B, it is possible to replace this choice of grading with one that only depends on the path η. However, this might

make the absolute degrees fractionary and, as the choice of η is not canonical, the resulting spectral sequence will still be invariant only up to translation.

The outline of the proof of this theorem is as follows (see [1] for details). First, in view of the naturality properties of Floer's construction, it is easy to see that the second invariance claim in the statement is implied by the first one. Now, we consider a homotopy G between H and H' as well as a one-parameter family of almost complex structures $\bar{J}$ relating J to J'. For $x \in I(L, L'; H, J)$ and $y \in I(L, L'; H', J')$ we define moduli spaces $\mathcal{N}(x, y)$ which are solutions of an equation similar to (10) but replaces H with G, J with $\bar{J}$ (and takes into account the additional parameter - this is a standard construction in Floer theory). These moduli spaces have properties similar to the $\mathcal{M}(x, y)$'s. In particular they admit compactifications which are manifolds with boundary so that the following formula is valid

$$\partial\overline{\mathcal{N}}(x, y) = \bigcup_{z \in I(L, L'; H)} \overline{\mathcal{M}}(x, z) \times \overline{\mathcal{N}}(z, y) \cup \bigcup_{z' \in I(L, L'; H')} \overline{\mathcal{N}}(x, z') \times \overline{\mathcal{M}}(z', y) \ .$$

The representing chain idea can again be used in this context and it leads to coefficients $n_y^x \in C_*(\Omega L)$. If we group these coefficients in a matrix $\mathcal{B}$ and we group the coefficients of the differential of $\mathcal{C}(L, L'; H, J)$ in a matrix $\mathcal{A}$ and the coefficients of $\mathcal{C}(L, L'; H', J')$ in a matrix $\mathcal{A}'$, then the relation above implies that we have:

$$(18) \qquad\qquad \partial\mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B} \cdot \mathcal{A}' \ .$$

It follows that the module morphism

$$\phi_{G, \bar{J}} : \mathcal{C}(L, L'; H, J) \rightarrow \mathcal{C}(L, L'; H, J)$$

which is the unique extension of

$$\phi_{G, \bar{J}}(x) = \sum_y r_y^x \otimes y, \ \forall x \in I(L, L'; H)$$

is a chain morphism. Moreover, the chain morphism constructed above preserves filtrations (of course, to for this it is required that the choices for the point x_0 with $|x_0| = 0$ for our two sets of data be coherent - this is why the isomorphisms are "up to translation"). After verifying that $\mathcal{E}^2 \approx H_*(\Omega L) \otimes FH_*(L)$ for both spectral sequences it is not difficult to see that ϕ_G induces an isomorphism at the $\mathcal{E}^2$-level of these spectral sequences [1]. Hence it also induces an isomorphism for $r > 2$.

For the last point of the theorem we use Floer's reduction of the moduli spaces $\mathcal{M}_{J,L,L'}(x, y)$ of pseudoholomorphic strips to moduli spaces of Morse flow lines $M_y^x(f)$. In short, this shows [11] [12] that for certain choices of J, f and L' which is hamiltonian isotopic to L we have homeomorphisms $\psi_{x,y} : \mathcal{M}(x, y) \rightarrow M_y^x$ which are compatible with the compactification and with the boundary formulae. This means that with these choices we have an isomorphism $\mathcal{C}(L, L') \rightarrow \mathcal{C}(f)$ and it is easy to see that this preserves the filtrations of these two chain complexes. By Theorem 2.4 this completes the outline of proof.

Remark 3.6. It is shown in [1] that the $\mathcal{E}^1$-term of this spectral sequence has also some interesting invariance properties.

3.2.3. *Applications.* We will discuss here a number of direct corollaries of Theorem 3.5 most (but not all)of which appear in [1].

A. Localization and Hofer's metric. An immediate adaptation of Theorem 2.4 provides a statement which is much more flexible. This is a "change of coefficients" or "localization" phenomenon that we now describe.

Assume that $f : L \to X$ is a smooth map. Then we can consider the induced map $\Omega f : \Omega L \to \Omega X$ and we may use this map to change the coefficients of $\mathcal{C}(L, L'; H, J)$ thus getting a new complex

$$\mathcal{C}_X(L, L'; H, J) = (C_*(\Omega X) \otimes \mathbb{Z}/2 < I(L, L'; H) >, D_X)$$

so that $D_X(x) = \sum_y (\Omega f)_*(m_y^x) \otimes y$ where m_y^x are the coefficients in the differential D of $\mathcal{C}(L, L'; H, J)$ (compare with (16)). This complex behaves very much like the one studied in Theorem 3.5. In particular, this complex admits a similar filtration and the resulting spectral sequence, $\mathcal{E}_X(L, L')$, has the same invariance properties as those in the theorem and, moreover, for L, L' hamiltonian isotopic this spectral sequence coincides with the Serre spectral sequence of the fibration $\Omega X \to E \to L$ which is obtained from the path-loop fibration $\Omega X \to PX \to X$ by pull-back over the map f.

In particular, the homology of this complex coincides with the singular homology of E. If X is just one point, $\odot$, it is easy to see that the complex $\mathcal{C}_\odot(L, L'; H, J)$ coincides with the Floer complex.

The complex $\mathcal{C}_X(L, L'; H, J)$ may also be viewed as a sort of localization in the following sense. Assume that we are interested to see what pseudo-holomorphic strips pass through a region $A \subset L$. Then we may consider the closure C of the complement of this region and the space L/C obtained by contracting C to a point. There is the obvious projection map $L \to L/C$ which can be used in place of f above. Now, if some non-vanishing differentials appear in $\mathcal{E}_{L/C}(L, L')$ for $r \geq 2$, then it means that there are some coefficients m_y^x so that $|(m_y^x)| > 0$ and $(\Omega f)_*(m_y^x) \neq 0$. This means that the map

$$\mathcal{M}(x, y) \xrightarrow{l_y^x} \Omega L \xrightarrow{\Omega f} \Omega(C/L)$$

carries the representing chain of $\mathcal{M}(x, y)$ to a nonvanishing chain in $C_*(\Omega(L/C))$. But this means that the intersection $\mathcal{M}'(x, y) \cap A$ is of dimension equal to $\mu(x, y)$.

The typical choice of region A is a tubular neighbourhood of some submanifold $V \hookrightarrow L$. In that case L/C is the associated Thom space.

Let

$$\nabla(L, L') = \inf_{H, \phi_1^H(L)=L'} \left(\max_{x,t} H(x, t) - \min_{x,t} H(x, t) \right) .$$

be the Hofer distance between Lagrangians. It has been shown to be non-degenerate by Chekanov [3] for symplectic manifolds with geometry bounded at infinity.

Corollary 3.7. *Let $a \in H_k(L)$ be a non-trivial homology class. If a closed submanifold $V \hookrightarrow L$ represents the class a, then for any generic J and any L' hamiltonian isotopic to L, there exists a pseudoholomorphic strip u of Maslov index at most $n - k$ which passes through V and which verifies:*

$$\int u^*\omega \leq \nabla(L, L') .$$

In view of the discussion above, the proof is simple (we are using here a variant of that used in [1]).

We start with a simple topological remark. Take A to be a tubular neighbourhood of V. Then $L/C = TV$ is the associated Thom space. In the Serre spectral sequence of $\Omega(TV) \to P(TV) \to TV$ we have that $d^{n-k}(\tau) \neq 0$ where $\tau \in H_{n-k}(TV)$ is the Thom class of V. By Poincaré duality, there is a class $b \in H_{n-k}(TV)$ which is taken to τ by the projection map $L \to TV$ (b corresponds to the Poincaré dual a^* of a via the isomorphism $H^{n-k}(L) \approx H_k(L)$). This means that $D^{n-k}(b)$ is not zero in $\mathcal{E}_{TV}(L, L'; H)$ (for any Hamiltonian H).

To proceed with the proof notice that, by the naturality properties of the Floer moduli spaces, it is sufficient to show that for any Hamiltonian H (and any generic J) so that $\phi_1^H(L) = L'$ there exists an element $u \in \mathcal{M}'_{L,L,J,H}$ which is of Maslov index $(n-k)$ and so that $u(\mathbb{R}, 0) \cap A \neq \emptyset$ and $E_{L,L,H}(u) \leq ||H||$ where $||H|| = \max_{x,t} H(x,t) - \min_{x,t} H(x,t)$. We may assume that $\min_{x,t} H(x,t) = 0$ and we let $K = \max_{x,t} H(x,t)$. We consider a Morse function $f : L \to \mathbb{R}$ which is very small in C^2 norm and we extend it to a function (also denoted by f) which is defined on M and remains C^2 small. In particular, we suppose $\min_x f(x) = 0$ and $\max_x f(x) < \epsilon$. We denote $\underline{f} = f - \epsilon$ and $\overline{f} = f + K$. It follows that we may construct monotone homotopies $G : \overline{f} \simeq H$ and $G' : H \simeq \underline{f}$. Consider the following action filtration of $\mathcal{C}_{TV}(L, L; H)$

$$F_v\mathcal{C}_{TV}(L, L; J, H) = C_*(\Omega TV) \otimes \mathbb{Z}/2 < x \in I(L, L; H) : \mathcal{A}_{L,L,H}(x) \leq v >$$

and similarly on the complexes $\mathcal{C}_{TV}(L, L; \overline{f})$ and $\mathcal{C}_{TV}(L, L; \underline{f})$. It is obvious that this is a differential filtration and, if the choice of path η (used to define the action functional, see (9)) is the same for all the three hamiltonians involved, these monotone homotopies preserve these filtrations.

We now denote

$$\mathcal{C} = F_{K+\epsilon}\mathcal{C}_{TV}(L, L'; \overline{f})/F_{-\epsilon}\mathcal{C}_{TV}(L, L'; \overline{f}),$$

$$\mathcal{C}' = F_{K+\epsilon}\mathcal{C}_{TV}(L, L'; H)/F_{-\epsilon}\mathcal{C}_{TV}(L, L'; H),$$

$$\mathcal{C}'' = F_{K+\epsilon}\mathcal{C}_{TV}(L, L'; \underline{f})/F_{-\epsilon}\mathcal{C}_{TV}(L, L'; \underline{f})$$

These three complexes inherit degree filtrations and there are associated spectral sequences $\mathcal{E}(\mathcal{C})$, $\mathcal{E}(\mathcal{C}')$, $\mathcal{E}(\mathcal{C}'')$. We have induced morphisms $\phi_G : \mathcal{C} \to \mathcal{C}'$ and $\phi_{G'} : \mathcal{C}' \to \mathcal{C}''$ which also induce morphisms among these spectral sequences. Moreover, as $\mathcal{C}$ and $\mathcal{C}''$ both coincide with $\mathcal{C}_{TV}(f)$ (because f is very C^2-small and $0 \leq f(x) < \epsilon$), the composition $\phi_{G'} \circ \phi_G$ induces an isomorphism of spectral sequences for $r \geq 2$ (here $\mathcal{C}_{TV}(f)$ is the extended Morse complex obtained from $\mathcal{C}(f)$ by changing the coefficients by the map $L \to TV$). But, as the class b has the property that its D^{n-k} differential is not trivial in $\mathcal{E}(\mathcal{C})$, this implies that $D^{n-k}(\phi_G(b)) \neq 0$ which is seen to immediately imply that there is some moduli space $\mathcal{M}'(x, y)$ of dimension $n - k$ with $\mathcal{A}_{L,L';H}(x), \mathcal{A}_{L,L';H}(y) \in [-\epsilon, K + \epsilon]$ and $\mathcal{M}'(x, y) \cap A \neq \emptyset$. Therefore there are J-strips passing through A which have Maslov index $n - k$ and area less than $||H|| + 2\epsilon$. By letting A tend to V and $\epsilon \to 0$, $||H|| \to \nabla(L, L')$, these strips converge to strips with the properties desired.

We may apply this even to $1 \in H_0(L)$ and Corollary 3.7 shows in this case that through each point of L passes a strip of Maslov index at most n (again, for J generic) and of area at most $\nabla(L, L')$. The case $V = pt$ was discussed explicitly in [1].

Remark 3.8. a. It is clear that the strips detected in this corollary actually have a symplectic area which is no larger than $c(b; H) - c(1; H)$ where $c(x; H)$ is the spectral value of the homology class x relative to H,

$$c(x; H) = \inf\{v \in \mathbb{R} : x \in Im(H_*(FC_{\leq v})(L, L; H) \to FH_*(L, L; H))\}$$

where $FC_{\leq v}(L, L; H)$ is the Floer complex of L, L, H generated by all the elements of $I(L, L; \eta, H)$ of action smaller or equal than v; $FH_*(L, L; H)$ is the Floer homology. Under our assumptions we have a canonical isomorphism (up to translation) between $HF_*(L, L, H)$ and $H_*(L)$ so we may view $b \in HF_*(L, L; H)$.

b. Clearly, in view of Gromov compactness our result also implies that for any J (even non regular) and for any L' hamiltonian isotopic to L and for any $x \in L \backslash L'$ there exists a J-holomorphic strip passing through x which has area less than $\nabla(L, L')$. This result (without the area estimate) also follows from independent work of Floer [10] and Hofer [15]. Another method has been mentioned to us by Dietmar Salamon. It is based on starting with disks with their boundary on L and which are very close to be constant maps. Therefore, an appropriate evaluation defined on the moduli space of these disks is of degree 1. Each of these disks is made out of two semi-disks which are pseudo-holomorphic and which are joined by a short semi-tube verifying the non-homogenous Floer equation for some given Hamiltonian H. This middle region is then allowed to expand till, at some point, it will necessarily produce a semi-tube belonging to some $\mathcal{M}'_H(x, y)$. It is also possible to use the pair of pants product to produce Floer orbits joining the "top and bottom classes" [26]. Still, having simultaneous area and Maslov index estimates appears to be more difficult by methods different from ours. Of course, detecting strips of lower Maslov index so that they meet some fixed submanifold is harder yet.

Corollary 3.7 has a nice geometric application.

Corollary 3.9. *Assume that, as before, L and L' are hamiltonian isotopic. For any symplectic embedding $e : (B_r, \omega_0) \to M$ so that $e^{-1}(L) = \mathbb{R}^n \cap B(r)$ and $e(B_r) \cap L' = \emptyset$ we have $\pi r^2 / 2 \leq \nabla(L, L')$.*

This is proven (see [1]) by using a variant of the standard isoperimetric inequality: a J_0-pseudoholomorphic surface in the standard ball (B_r, ω_0) of radius r whose boundary is on $\partial B_r \cup \mathbb{R}^n$ has area at least $\pi r^2 / 2$.

Clearly, this implies the non-degeneracy result of Chekanov that was mentioned before under the connectivity conditions that we have always assumed till this point.

B. Relaxing the connectivity conditions. We have worked till now under the assumption that

(19) L, L' are simply $-$ connected and $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$

These requirements were used in a few important places: in the definition of the action functional, the definition of the Maslov index, the boundary product formula (because they forbid bubbling). Of these, only the bubbling isssue is in fact

essential: the boundary formula is precisely the reason why $d^2 = 0$ as well as the cause of the invariance of the resulting homology.

We proceed below to extend the corollaries and techniques discussed above to the case when all the connectivity conditions are dropped but we assume that L and L' are hamiltonian isotopic and only work below the minimal energy that could produce some bubbling (this is similar to the last section of [1] but goes beyond the cases treated there).

First, for a time dependent almost complex structure J_t, $t \in [0,1]$, we define $\delta_{L,L'}(J)$ as the infimmum of the symplectic areas of the following three types of objects:

- the J_t-pseudoholomorphic spheres in M (for $t \in [0,1]$).
- the J_0-pseudoholomorphic disks with their boundary on L.
- the J_1-pseudoholomorphic disks with their boundary on L'.

By Gromov compactness this number is strictly positive.

We will proceed with the construction in the case when $L = L'$ and in the presence of a hamiltonian H. We shall assume that the pair (H, J) is regular in the sense that the moduli spaces of strips defined below, $\mathcal{M}(x, y)$, are regular.

We take the fixed reference path η to be a constant point in L (see (9)). Denote $\mathcal{P}_0(L, L) = \mathcal{P}_\eta(L, L)$ and consider in this space the base point given by η. Notice that there is a morphism $\omega : \pi_1 \mathcal{P}_0(L, L) \to \mathbb{R}$ obtained by integrating ω over the disk represented by the element $z \in \pi_1 \mathcal{P}_0(L, L)$ (such an element can be viewed as a disk with boundary in L). Similarly, let $\mu : \pi_1 \mathcal{P}_0(L, L) \to \mathbb{Z}$ be the Maslov morphism. Let $\mathcal{K}$ be the kernel of the morphism

$$\omega \times \mu : \pi_1 \mathcal{P}_0(L, L) \to \mathbb{R} \times \mathbb{Z} .$$

The group

$$\pi = \pi_1(\mathcal{P}_0(L, L))/\mathcal{K}$$

is an abelian group (as it is a subgroup of $\mathbb{R} \times \mathbb{Z}$) and is of finite rank. Let's also notice that this group is the quotient of $\pi_2(M, L)$ by the equivalence relation $a \sim b$ iff $\omega(a) = \omega(b), \mu(a) = \mu(b)$ (with this definition this group is also known as the Novikov group). This is a simple homotopical result. First $\mathcal{P}(L, L)$ is the homotopy pull-back of the map $L \to M$ over the map $L \to M$. But this means that we have a fibre sequence $F \to \mathcal{P}(L, L) \to L$ with F the homotopy fibre of $L \to M$ and that this fibre sequence admits a canonical section. This implies that

$$\pi_1 \mathcal{P}_0(L, L) \approx \pi_1(F) \times \pi_1(L) .$$

But $\pi_1(F) = \pi_2(M, L)$. As both ω and μ are trivial on $\pi_1(L)$ the claim follows. It might not be clear at first sight why μ is null on $\pi_1(L)$ here. The reason is that the term $\pi_1(L)$ in the product above is the image of the map induced in homotopy by

$$j_L : L \to \mathcal{P}_0(L, L) .$$

This map associates to a point in L the constant path. Consider a loop $\gamma(s)$ in L. Then $j_L \circ \gamma$ is a loop in $\mathcal{P}_0(L, L)$ which at each moment s is a constant path. We now need to view this loop as the image of a disk and $\mu([j_L(\gamma)])$ is the Maslov index of this disk. But this disk is null homotopic so $\mu([j_L(\gamma)]) = 0$.

Consider the regular covering $p : \mathcal{P}'_0(L, L) \to \mathcal{P}_0(L, L)$ which is associated to the group $\mathcal{K}$. We fix an element $\eta_0 \in \mathcal{P}'_0(L, L)$ so that $p(\eta_0) = \eta$. Clearly, the action functional

$$\mathcal{A}'_{L,L,H} : \mathcal{P}'_0(L, L) \to \mathbb{R}$$

may be defined by essentially the same formula as in (9):

$$\mathcal{A}'_{L,L,H}(x) = -\int (p \circ u)^* \omega + \int_0^1 H(t, (p \circ x)(t))dt$$

where $u : [0, 1] \to \mathcal{P}'_0(L, L)$ is such that $u(0) = \eta_0$, $u(1) = x$ and is now well-defined.

Let $I'(L, L, H) = p^{-1}(I(L, L, H))$. For $x, y \in I'(L, L, H)$ we may define $\mu(x, y) = \mu(p \circ u)$ where $u : [0, 1] \to \mathcal{P}'_0(L, L)$ is a path that joins x to y. This is again well defined. For each $x \in I'(L, L, H)$ we consider a path $v_x : [0, 1] \to \mathcal{P}'_0(L, L)$ so that $v_x(0) = \eta_0$ and $v_x(1) = x$. The composition $p \circ v_x$ can be viewed as a "semi-disk" whose boundary is resting on the orbit $p(x)$ and on L. Therefore, we may associate to it a Maslov index $\mu(v_x)$ [21] and it is easy to see that this only depends on x. Thus we define $\mu(x) = \mu(v_x)$ and we have $\mu(x, y) = \mu(x) - \mu(y)$ for all $x, y \in I'(L, L, H)$.

To summarize what has been done till now: once the choices of η and η_0 are made, both the action functional $\mathcal{A}' : \mathcal{P}'_0(L, L, H) \to \mathbb{R}$ and the "absolute" Maslov index $\mu(-) : I'(L, L, H) \to \mathbb{Z}$ are well-defined.

Fix an almost complex structure J. Consider two elements $x, y \in I'(L, L, H)$. We may consider the moduli space which consists of all paths $u : \mathbb{R} \to \mathcal{P}'_0(L, L)$ which join x to y and are so that $p \circ u$ satisfies Floer's equation (10) modulo the $\mathbb{R}$-action. If regularity is achieved, the dimension of this moduli space is precisely $\mu(x, y) - 1$. The action functional $\mathcal{A}'$ decreases along such a solution u and the energy of u (which is defined as the energy of $p \circ u$) verifies, as in the standard case, $E(u) = \mathcal{A}'(x) - \mathcal{A}'(y)$. Bubbling might of course be present in the compactification of these moduli spaces. As we only intend to work below the minimal bubbling energy $\delta_{L,L}(J)$ we *artificially* put:

$$\mathcal{M}(x, y) = \emptyset \text{ if } \mathcal{A}'(x) - \mathcal{A}'(y) \geq \delta_{L,L}(J)$$

and, of course, for $\mathcal{A}'(x) - \mathcal{A}'(y) < \delta_{L,L}(J)$, $\mathcal{M}(x, y)$ consists of the elements u mentioned above. We only require these moduli spaces to be regular.

With this convention, for all $x, y \in I'(L, L, H)$ so that $\mathcal{M}(x, y)$ is not void we have the usual boundary formula (15). Notice at the same time that this formula is false for general pairs x, y (and so there is no way to define a Floer type complex at this stage).

Now consider a map $f : L \to X$ so that X is simply-connected (the only reason to require this is to insure that the Serre spectral sequence does not require local coefficients).

We consider the group:

$$\mathcal{C}(L, L, H; X) = C_*(\Omega X) \otimes \mathbb{Z}/2 < I'(L, L, H) > .$$

For $w \geq v \in \mathbb{R}$, we denote $I'_{v,w} = \{x \in I'(L, L, H) \; : \; w \geq \mathcal{A}'(x) \geq v\}$ and we define the subgroup

$$\mathcal{C}_{w,v}(L, L, H; X) = C_*(\Omega X) \otimes \mathbb{Z}/2 < I'_{v,w} > .$$

Suppose that $w - v \leq \delta_{L,L}(J) - \epsilon$. We claim that in this case we may define a differential on $\mathcal{C}_{v,w}(L, L, H; X)$ by the usual procedure. Consider representing chain systems for all the moduli spaces $\mathcal{M}(x, y)$ and let the image of these chains inside $C_*(\Omega X)$ be respectively $\bar{m}_y^x$. Let D be the linear extension of the map given by

$$Dx = \sum_{y \in I'_{v,w}} \bar{m}_y^x \otimes y .$$

Proposition 3.10. *The linear map D is a differential. A generic monotone homotopy G between two hamiltonians H and H'*

$$\phi_G^X : \mathcal{C}_{v,w}(L, L, H, J; X) \to \mathcal{C}_{v,w}(L, L, H', J; X) .$$

A monotone homotopy between monotone homotopies G and G' induces a chain homotopy between ϕ_G^X and $\phi_{G'}^X$ so that $H_*(\phi_G^X) = H_*(\phi_{G'}^X)$.

Now, $D^2 x = \sum_z (\sum_y \bar{m}_y^x \cdot \bar{m}_z^y + \partial \bar{m}_z^x) \otimes z$. In this formula we have $\mathcal{A}'(x) - \mathcal{A}'(z) \leq \delta_{L,L}(J)$ and, because the usual boundary formula (15) is valid in this range, all the terms vanish so that $D^2(x) = 0$. The same idea may be applied to a monotone homotopy as well as to a monotone homotopy between monotone homotopies and it implies the claim.

Remark 3.11. a. If we take for the space X a single point $*$ we get a chain complex whose differential only takes into account the 0-dimensional moduli spaces and which is a truncated version of Floer homology.

b. The complex $\mathcal{C}_{v,w}(L, L, H, J; X)$ admits a degree filtration which is perfectly similar to the one given by (17). Let $\mathcal{E}\mathcal{C}_{v,w}(L, L, H, J; X)$ be the resulting spectral sequence. Then, under the restrictions in the Proposition 3.10, a monotone homotopy G induces a morphism of spectral sequences $\mathcal{E}_X(\phi_G)$ and two such homotopies G, G' which are monotonously homotopic have the property that they induce the same morphism $\mathcal{E}_X^r(\phi_G) = \mathcal{E}_X^r(\phi_{G'})$ for $r \geq 2$. This last fact follows from Proposition 3.10 by computing $\mathcal{E}_X^2(\phi_G) = id_{H_*(\Omega X)} \otimes H_*(\phi_G^*) = \mathcal{E}_X^2(\phi_{G'})$ where $\phi_G^* : \mathcal{C}_{v,w}(L, L, H; *) \to \mathcal{C}_{v,w}(L, L, H'; *)$.

Naturally, the next step is to compare our construction with its Morse theoretical analogue. Consider the map $j_L : L \to \mathcal{P}_0(L, L)$ and consider $p : \tilde{L} \to L$ the regular covering obtained by pull-back from $\mathcal{P}'_0(L, L) \to \mathcal{P}_0(L, L)$. Notice that, because both compositions $\omega \circ \pi_1(j_L)$ and $\mu \circ \pi_1(j_L)$ are trivial, it follows that the covering $\tilde{L} \to L$ is trivial. Let $\bar{f} : \tilde{L} \to \mathbb{R}$ be defined by $\bar{f} = f \circ p$ and consider $\mathcal{C}(\bar{f}; X)$ the extended Morse complex of $\bar{f}$ with coefficients changed by the map $\Omega \tilde{L} \to \Omega X$. Notice that, in general, the group π acts on $I'(L, L, H)$ and we have the formula:

$$\mathcal{A}'(gy) = \omega(g) + \mathcal{A}'(y), \ \mu(gy) = \mu(g) + \mu(y), \forall y \in I'(L, L, H), \forall g \in \Pi .$$

In our particular case, when $H = f$, we have $I'(L, L, H) = Crit(\bar{f})$. For each point $x \in Crit(f)$ let $\bar{x} \in Crit(\bar{f})$ be the element of $p^{-1}(x)$ which belongs to the component of $\tilde{L}$ which also contains η_0. We then have $\mathcal{A}'(\bar{x}) = f(x)$ and $\mu(\bar{x}) = ind_{\bar{f}}(\bar{x})$. The extended Morse complex $\mathcal{C}(\bar{f}; X)$ is therefore isomorphic to $\mathcal{C}(f; X) \otimes \mathbb{Z}[\pi]$ and the action filtration is determined by writting $\mathcal{A}'(x \otimes g) = f(x) + \omega(g)$. The degree filtration induces, as usual, a spectral sequence which will be denoted by $\mathcal{E}\mathcal{C}(f; X)$.

The remarks above together with Theorem 3.5 show that this spectral sequence consists of copies the Serre spectral sequence of $\Omega X \to E \to L$: one copy for each connected component of $\tilde{L}$. We denote by $\mathcal{C}_0(f; X)$ and $\mathcal{EC}_0(f; X)$ the copies of the extended complex and of the spectral sequence that correspond to the connected component L_0 of $\tilde{L}$ which contains η_0.

Proposition 3.12. *Suppose* $||H||_0 < \delta_J(L, L)$. *There exists a chain morphisms* $\phi : \mathcal{C}_0(f; X) \to \mathcal{C}_{0,||H||}(L, L, H, J; X)$ *and* $\psi : \mathcal{C}_{0,||H||}(L, L, H, J; X) \to \mathcal{C}_0(f; X)$ *which preserve the respective degree filtrations and so that* $\psi \circ \phi$ *induces an isomorphism at the* E^2 *level of the respective spectral sequences.*

To prove this proposition we shall use a different method than the one used in Corollary 3.7. The comparison maps ϕ, ψ will be constructed by the method introduced in [19] and later used in [25] and [26]. Compared to Proposition 3.10 this is particularly efficient because it avoids the need to control the bubbling thershold along deformations of J.

We fix as before the Morse function f as well as the pair H, J. To simplify the notation we shall assume that $\inf H(x, t) = 0$. The construction of ϕ is based on defining certain moduli spaces $\mathcal{W}(x, y)$ with $x \in Crit(\bar{f})$ and $y \in I'(L, L, H)$. They consist of pairs (u, γ) where $u : \mathbb{R} \to \mathcal{P}'_0(L, L)$, $\gamma : (-\infty, 0] \to \tilde{L}$ and if we put $u' = p(u)$, $\gamma' = p(\gamma)$ ($p : \mathcal{P}'_0(L, L) \to \mathcal{P}_0(L, L)$ is the covering projection) then we have:

$$u'(R \times \{0, 1\}) \subset L \ , \ \partial_s(u') + J(u')\partial_t(u') + \beta(s)\nabla H(u', t) = 0 \ , u(+\infty) = y$$

and

$$\frac{d\gamma'}{dt} = -\nabla_g f(\gamma') \ , \ \gamma(-\infty) = x \ , \ \gamma(0) = u(-\infty) \ .$$

Here g is a Riemannian metric so that (f, g) is Morse-Smale and β is a smooth cutt-off function which is increasing and vanishes for $s \leq 1/2$ and equals 1 for $s \geq 1$. It is useful to view an element (u, γ) as before as a semi-tube connecting x to y. Under usual regularity assumptions these moduli spaces are manifolds of dimension $\mu(x) - \mu(y)$.

The energy of such an element (u, γ) is defined in the obvious way by $E(u, \gamma) = \int ||\partial_s u'||^2 ds dt$. A simple computation shows that:

$$E(u, \gamma) = I(u) + \int_{\mathbb{R} \times [0,1]} (u')^* \omega - \int_0^1 H(y(t)) dt$$

where $I(u) = \int_{\mathbb{R} \times [0,1]} \beta'(s) H(u'(s), t) ds dt$. If $x \in L_0$, then the energy verifies

$$E(u, \gamma) = I(u) - \mathcal{A}'(y) \leq \sup(H) - \mathcal{A}'(y) \ .$$

As before, we only want to work here under the bubbling threshold and we are only interested in the critical points $x \in L_0$ so we put $\mathcal{W}(x, y) = \emptyset$ for all those pairs (x, y) with either $y \in I'(L, L, H)$ so that $\mathcal{A}'(y) \notin [0, ||H||]$ or with $x \notin L_0$. This means that there is no bubbling in our moduli spaces. Thus we may apply the usual procedure: compactification, representing chain systems, representation in the loop space (for this step we need to choose a convenient way to parametrize the paths represented by the elements (u, γ)). Notice that the boundary of $\overline{\mathcal{W}}(x, y)$ is the union of two types of pieces $\overline{M}_z^x \times \overline{\mathcal{W}}(z, y)$ and $\overline{\mathcal{W}}(x, z) \times \overline{\mathcal{M}}(z, y)$. We then

define $\phi(x) = \sum w_y^x \otimes y$ where w_y^x is a cubical chain representing the moduli space $\mathcal{W}(x,y)$. This is a chain map as desired.

We now proceed to construct the map ψ. The construction is prefectly similar: we define moduli spaces $\mathcal{W}'(y,x)$, $y \in I'(L,L,J)$, $x \in Crit(\bar{f})$ except that the pairs (u,γ) considered here, start as semi-tubes and end as flow lines of f. The equation verified by u is similar to the one before but istead of the cut-off function β we use the cut-off function $-\beta$. For $x \in L_0$ the energy estimate in this case gives $E(u,\gamma) \leq \mathcal{A}'(y)$. By the same method as above we define $\mathcal{W}'(y,x)$ to be void whenever $x \notin L_0$ or $\mathcal{A}'(y) > ||H||$ and we define $\psi(y) = \sum \bar{w}_x^y \otimes x$ where $\bar{w}_x^y$ is a cubical chain representing the moduli space $W'(y,x)$. Notice that because $E(u,\gamma) \leq \mathcal{A}'(y)$ this map ψ does in fact vanish on $\mathcal{C}_0(L,L,H,J;X)$ and so it induces a chain map (also denoted by ψ) as desired.

The next step is to notice that the composition $\psi \circ \phi$ induces an isomorphism at E^2. This is equivalent to showing that $H_*(\psi \circ \phi)$ is an isomorphism for $X = *$. In turn, this fact follows by now standard deformation arguments as in [19].

Corollary 3.13. *Assume that L and L' are hamiltonian isotopic and suppose that J is generic. If $\nabla(L,L') < \delta_{L,L'}(J)$, then the statement of Corollary 3.7 remains true (for J) without the connectivity assumption (19).*

Notice that if H is a hamiltonian so that $\phi_1^H(L) = L'$ and $J_H = (\phi^H)_*(J)$ then, by the naturality described in §3.2, we have:

$$\delta_{L,L}(J_H) = \delta_{L,L'}(J) \ .$$

This implies, again by this same naturality argument, that the problem reduces to finding appropriate semi-tubes whose detection comes down to showing the non-vanishing of certain differentials in $\mathcal{EC}_{0,w}(L,L,H;TV)$ for some well chosen $w < \delta_{L,L}(J_H)$. But this immediatly follows from Proposition 3.12 by the same topological argument as the one used in the proof of Corollary 3.7

We formulate the geometric consequence which corresponds to Corollary 3.9. For two lagrangians L and L' the following number has been introduced in [1]: $B(L,L')$ is the supremum of the numbers $r \geq 0$ so that there exists a symplectic embedding

$$e : (B(r),\omega_0) \to (M,\omega)$$

so that $e^{-1}(L) = \mathbb{R}^n \cap B(r)$ and $Im(e) \cap L' = \emptyset$.

Corollary 3.14. *There exists an almost complex structure J so that we have the inequality:*

$$\nabla(L,L') \geq \min\{\delta_{L,L'}(J) \ , \ \frac{\pi}{2}B(L,L')^2\} \ .$$

Clearly, this implies that $\nabla(-,-)$ is non-degenerate in full generality (and recovers, in particular, the fact that the usual Hofer norm for Hamiltonians is non-dgenerate). It is useful to also notice as in [1] that this result is a Lagrangian version of the usual capacity - displacement energy inequality [16]. Indeed, this inequality (with the factor $\frac{1}{2}$) is implied by the following statement which has been conjectured to hold for any two compact lagrangians in a symplectic manifold [1]:

$$(20) \qquad\qquad \nabla(L,L') \geq \frac{\pi}{2}B(L,L')^2 \ .$$

This remains open. An even stronger conjecture is the following:

Conjecture 3.15. *For any two hamiltonian isotopic closed lagrangians $L, L' \subset (M, \omega)$ and for any almost complex structure J which compatible with ω and any point $x \in L \backslash L'$ there exists a pseudoholomorphic curve u which is either a strip resting on L and on L' or a pseudoholomorphic disk with boundary in L so that $x \in Im(u)$ and $\int u^* \omega \leq \nabla(L, L')$.*

By the isoperimetric inequality used earlier in this paper, it follows that this statement implies (20). There is a substantial amount of evidence in favor of this conjecture:

- as explained in this paper, in the absence of any pseudoholomorphic disks (that is, when $\omega|_{\pi_2(M,L)} = 0$) it was proven in [1].
- the statement in Corollary 3.14 shows that the area estimate is not unreasonable.
- one striking consequence of Conjecture 3.15 is that if the disjunction energy of the lagrangian L is equal to $E_0 < \infty$, then, for any J as in the statement and any $x \in L$ there is a pseudoholomorpic disk of area at most E_0 which passes through x. Under certain assumptions, this is indeed true and follows from recent work of the second author joint with François Lalonde. By the same geometric argument as above we deduce a nice consequence. Define the relative (or *real*) Gromov radius of L, $Gr(L)$, to be the supremum of the positive numbers r so that there exists a symplectic embedding $e : (B(r), \omega_0) \rightarrow (M, \omega)$ with the property that $e^{-1}(L) = \mathbb{R}^n \cap B(r)$, then $\pi(Gr(L))^2/2 \leq E_0$ (where E_0, as before, is the disjunction energy of L). It is also useful to note that if L is the zero section of a cotangent bundle, then $Gr(L) = \infty$.

There are numerous other interesting consequences of Conjecture 3.15 besides (20). To conclude, 3.15 appears to be a statement worth investigating.

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