New trends in Air Traffic Complexity

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Agenda

- Why complexity metrics are needed ?
- Dynamical system modeling of aircraft trajectories
- How a complexity map is built ?
- Results

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Why complexity metrics are needed ?

Airspace design

- Airspace comparison (US-Europe)
- Evaluation of Airspace Organisation Schemes
- Implementation of flexible use of airspace policies

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4D contract framework

- 4D Trajectory design.
- Forecasting of potentially hazardous traffic situations.
- Automated Conflict Solver enhancement (robustness of the solution).

Complexity vs Workload

Workload

- Related to cognitive processes for human controllers.
- Easy/Hard forecasting of conflict occurence.
- Monitoring is a non negligible part of the workload.

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Complexity

- Related to traffic structure.
- Measure of intrinsic disorder of a set of trajectories.
- Increases with :
  - Sensitivity to uncertainties.
  - Interdependance of conflicts.

Intrinsic part of complexity

Sensitivity

- Uncertainties on positions, wind speed, intents induce inaccuracy on trajectory prediction.
- Depending on the situation, prediction error can grow exponentially fast !
- In such a case, the situation is complex because nearly impossible to forecast.

Intrinsic part of complexity

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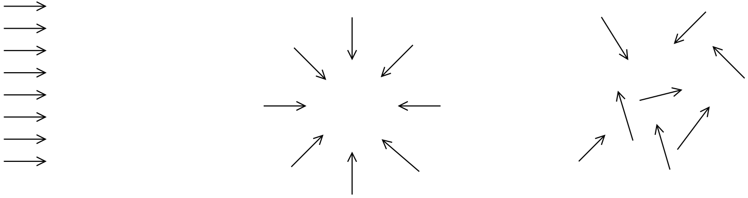
Interdependance

- Solving a conflict may induce other ones.
- Even in conflict-less situations, interactions between trajectories can rise the perceived level of complexity.
- Complexity is related to mixing behaviour.

Sensitivity-interdependance



No Sensitivity

No conflict

Easy situation

Hight sensitivity

Potential conflicts without interaction between solutions

Hight sensitivity

Potential conflicts with interactions between solutions

Hard situation

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Linear Dynamical System Modeling

The key idea is to model the set of aircraft trajectories by a linear dynamical system which is defined by the following equation :

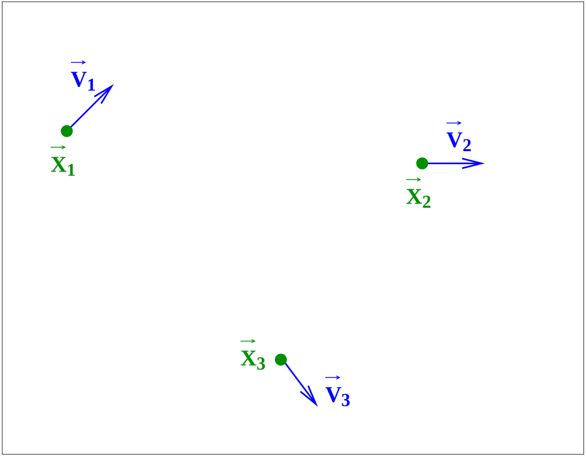
$$\dot{\vec{X}} = \mathbf{A} \cdot \vec{X} + \vec{B}$$

where $\vec{X}$ is the state vector of the system :

$$\vec{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Matrix $\mathbf{A}$ and vector $\vec{B}$ are the parameters of the model.

Regression of a Linear Dynamical System



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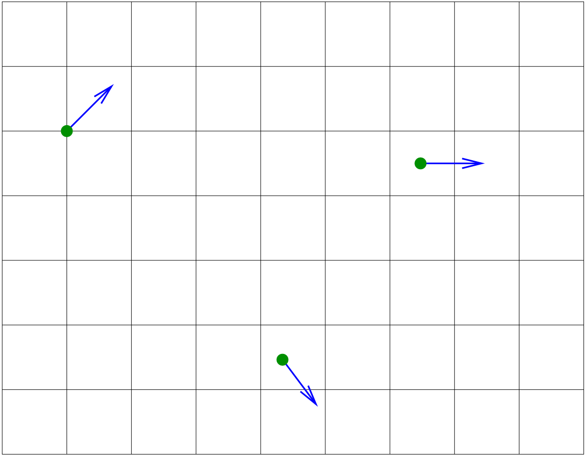

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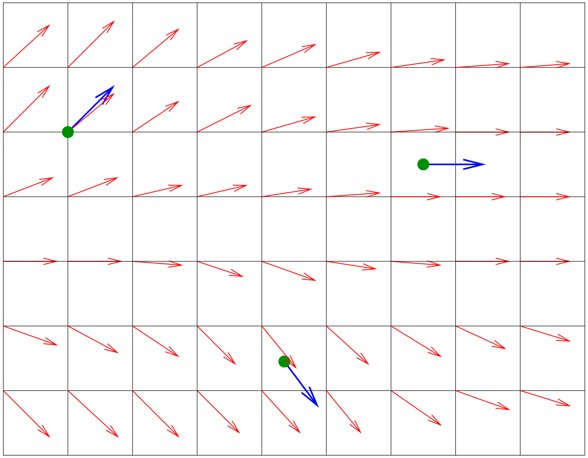

Regression of a Linear Dynamical System

- Based on a set of observations (positions and speeds), one has to find a dynamical system which fits those observations.Suppose that N obervations are given :
  Positions :

$$\vec{X}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

  and speeds :

$$\vec{V}_i = \begin{bmatrix} vx_i \\ vy_i \\ vz_i \end{bmatrix}$$

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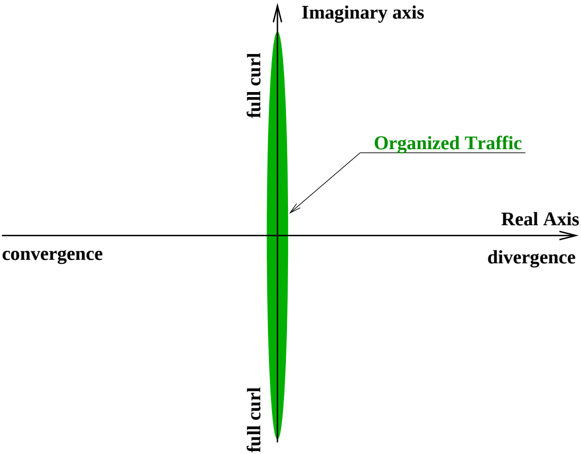
  and speeds :
  $$\vec{V}_i = \begin{bmatrix} vx_i \\ vy_i \\ vz_i \end{bmatrix}$$
- A LMS precedure is applied in order to extract the matrix $\mathbf{A}$ and the vector $\vec{B}$.

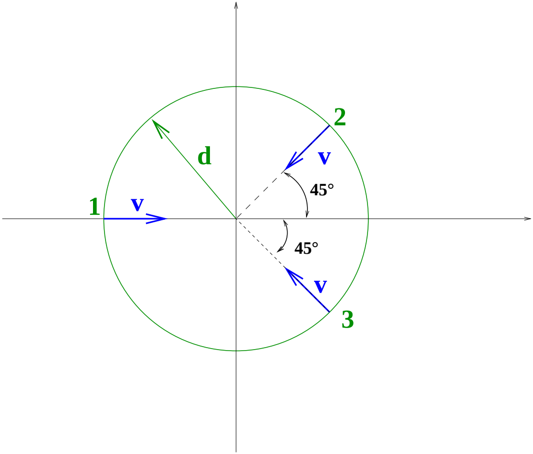
Properties of the matrix **A**

- When real part of the eigenvalues of matrix **A** is positive, the system is in expansion mode and when they are negative, the system is in contraction mode.
- Furthermore, the imaginary part of such eigenvalues are related with curl intensity of the field.

Properties of the matrix **A**



Linear Dynamical System Modeling : An example



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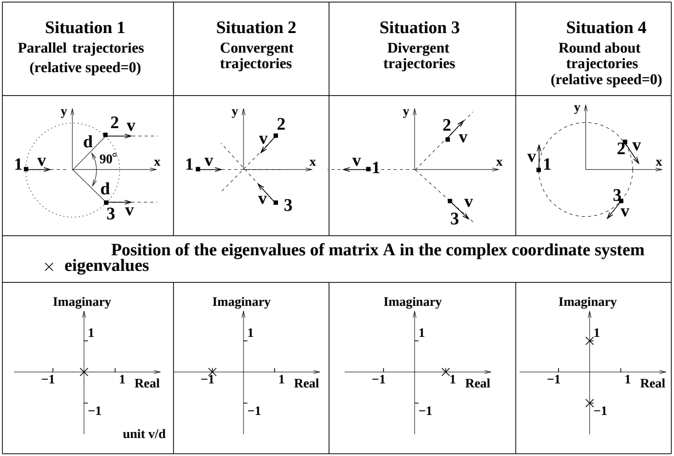

Linear Model limitations

- Give a global tendency of the traffic situation.
- Do not fit exactly with all traffic situations.
- $\Rightarrow$ **Non Linear Extension**

Non Linear Extension in Space

$$\dot{\vec{X}} = \vec{f}(\vec{X})$$

Optimization problem

- $\vec{f}$? such that :

$$minE = \sum_{i=1}^{i=N} \|\vec{V}_i - \vec{f}(\vec{X}_i)\|^2$$

- and

$$min \int_{\mathbb{R}^3} \|\Delta \vec{f}(\vec{x})\|^2 d\vec{x} \quad \text{with} \quad \Delta \vec{f} = \begin{bmatrix} \frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_x}{\partial y^2} + \frac{\partial^2 f_x}{\partial z^2} \\ \frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_y}{\partial z^2} \\ \frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2} \end{bmatrix}$$

Non Linear Extension in Space

Exact Solution (Amodei)

$$\vec{f}(\vec{X}) = \sum_{i=1}^{N} \mathbf{\Phi}(\|\vec{X} - \vec{X}_i\|).\vec{a}_i + \mathbf{A}.\vec{X} + \vec{B}$$

with

$$\mathbf{\Phi}(\|\vec{X} - \vec{X}_i\|) = \mathbf{Q}(\|\vec{X} - \vec{X}_i\|^3)$$

and

$$Q = \begin{bmatrix} \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz} & 0 & 0 \\ 0 & \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz} & 0 \\ 0 & 0 & \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz} \end{bmatrix}$$

Non Linear Extension in Space and time

$$\dot{\vec{X}} = \vec{f}(\vec{X}, t)$$

Optimization problem

- We are looking for $\vec{f}$ such that :
- $$minE = \sum_{i=1}^{i=N} \sum_{k=1}^{k=K} \| \vec{V}_i(t_k) - \vec{f}(\vec{X}_i, t_k) \|^2$$
- and
$$min \int_{\mathbb{R}^3} \int_t \| \Delta \vec{f}(\vec{x}) \|^2 + \| \frac{\partial \vec{f}}{\partial t} \|^2 d\vec{x} dt$$

Non Linear Extension in Space and time

Exact Solution (Puechmorel and Delahaye)

$$\vec{f}(\vec{X},t) = \sum_{i=1}^{N}\sum_{k=1}^{K} \mathbf{\Phi}(\|\vec{X}(t) - \vec{X}_i(t_k)\|, |t - t_k|).\vec{a}_{i,k} + \mathbf{A}.\vec{X} + \vec{B}$$

with

$$\mathbf{\Phi}(r,t) = \mathbf{diag}\left(\frac{\sigma}{r.\sqrt{\pi}}.erf\left[\frac{r}{\sigma}.\frac{1}{\sqrt{2+\theta.|t|}}\right]\right)$$

Non Linear Extension in Space and time

Gradient close form

$$\frac{\partial \Phi(r,t)}{\partial x} = (\alpha - \beta).x$$

with

$$\alpha = \frac{2.\sigma}{r^2.\pi}.\frac{1}{\sqrt{2+\theta.|t|}}.e^{-\frac{r^2}{\sigma^2.(2+\theta.|t|)}}$$

$$\beta = \frac{\Phi(r,t)}{r^2}$$

$$r = \sqrt{x^2+y^2+z^2}$$

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Characterization of sensitivity

How fast two neighboring dynnamical system trajectories diverge ?

- Let $\gamma(t,s), t\colon [a,b], s \in V$ be a family of trajectories of the dynamical system in the neighborood V of a given point s_0 .
- We assume that the nominal trajectory is $t \mapsto \gamma(t, s_0)$.
- A perturbed trajectory is $t \mapsto \gamma(t,s)$ with $s \in V$.
- Divergence to nominal trajectory with respect to time is thus $\|\gamma(t,s_0) - \gamma(t,s)\| = D(t,s)$.

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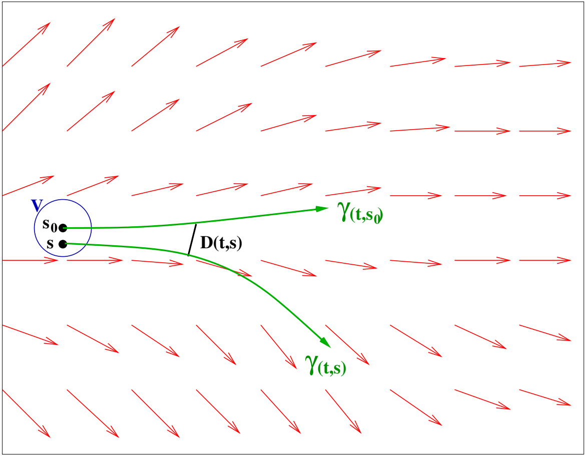
Computing $D(t, s)$

Main idea : when $t \mapsto \gamma(t, s)$ is the solution of a differential equation with initial condition $\gamma(0, s) = s$, it is possible to show that D itself satisfies a differential equation.

local behaviour of trajectories



Local behaviour of trajectories

The variational equation I

- The value of $D(t, s)$ is the result of a cumulative process.
- Assume that $\gamma(t, s)$ is defined to be a flow :

$$\frac{\partial \gamma(t, s)}{\partial t} = F(t, \gamma(t, s)) \quad \gamma(0, s) = s$$

  with F a smooth vector field.
- Given a nonimal trajectory $\gamma(t, s_0)$, then divergence of nearby trajectories can be found up to order one in $\|s - s_0\|$ by solving :

$$\frac{\partial D(t, s)}{\partial t} = DF(t, \gamma(t, s_0)).D(t, s) \quad D(0, s) = \|s - s_0\|$$

  with DF the jacobian matrix of F (with respect to s).

The variational equation II

- Since the previous equation is linear, it can be described by a matrix $M(t)$ that obeys :

$$\frac{dM(t)}{dt} = DF(t, \gamma(t, s_0)).M(t) \quad M(0) = Id$$

  This equation is called the variational equation of the flow.
- The variational equation describes in some sense a linear dynamical system "tangent" to the original one.

Lyapunov exponents

- Let $U^t(t)\Sigma(t)V(t) = M(t)$ be the SVD decomposition of $M(t)$.
- The Lyapunov exponents are mean values of the logarithms of the diagonal elements of $\Sigma(t)$:

$$\kappa(s) = -\frac{1}{T}\int_0^T \log(\Sigma_{ii}(t))dt \quad \forall \Sigma_{ii}(t) \leq 1$$

Summary

- Based on trajectories observations, a 4D non linear dynamical system model is buit.
- The associated 4D vector field is then computed on a cube of airspace.
- Lyapunov exponents are computed on each point of the cube in order to built a complexity map.

Interpretation of Lyapunov exponents

- Given an initial point, the Lyapunov exponents and the associated SVD decomposition provide us with a decomposition of space in principal directions and corresponding convergence/divergence rate.
- It is a localized version of the complexity based on linear systems.

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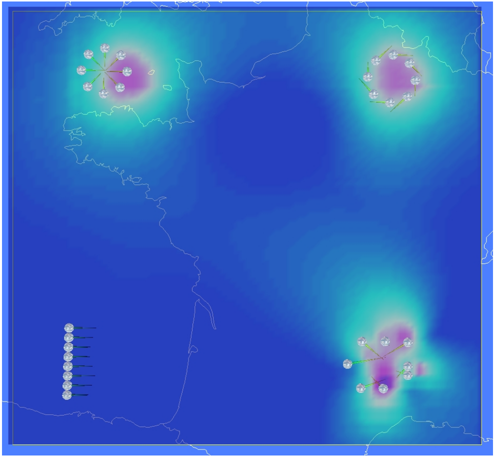

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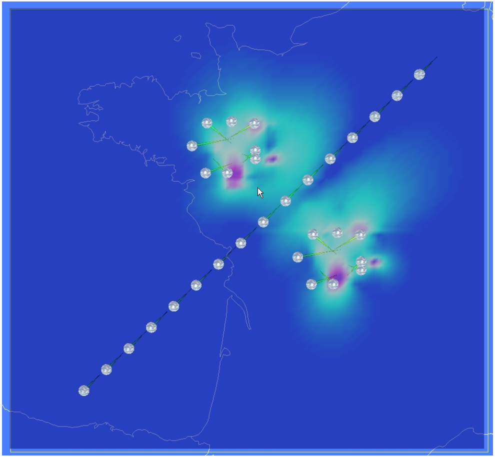

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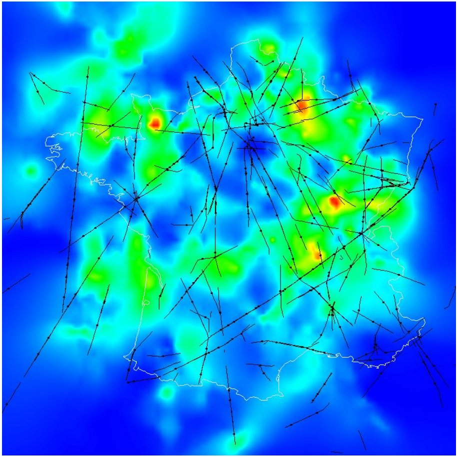

Conclusion

- An intrinsic trajectories complexity metric has been developped.
- This metric can be computed as a map.
- Give the areas of airspace where the traffic is organized and the ones where there is desorder

Current work

- Parallel computation by the mean of Local linear Model

$$\vec{V}_i(\vec{X}_i, t_i) = \vec{f}(\vec{X}, t) + \frac{\partial \vec{f}(\vec{X}, t)}{\partial t}(t_i - t) + \vec{f}(\vec{X}, t) + \frac{\partial \vec{f}(\vec{X}, t)}{\partial \vec{X}}(\vec{X}_i - \vec{X}) +$$

$$O(|t_i - t| + \|\vec{X}_i - \vec{X}\|)$$

$$= \vec{a} + \vec{b}(t_i - t) + C.(\vec{X}_i - \vec{X}) + O(|t_i - t| + \|\vec{X}_i - \vec{X}\|)$$

$$\min_{\vec{a},\vec{b},C} \sum_{i=1}^{N} \|\vec{V}_i(\vec{X}_i, t_i) - \vec{a} + \vec{b}(t_i - t) + C.(\vec{X}_i - \vec{X})\|^2 . \psi(t_i - t, \vec{X}_i - \vec{X})$$

- Robustness improvement by taking into account uncertainties on time position of aircraft on their trajectories