New trends in Air Traffic Complexity

D. Delahaye and S. Puechmorel

Applied Math Laboratory, ENAC

March 8, 2010
Agenda

- Why complexity metrics are needed?
- Dynamical system modeling of aircraft trajectories
- How a complexity map is built?
- Results
Agenda

- Why complexity metrics are needed?
- Dynamical system modeling of aircraft trajectories
- How a complexity map is built?
- Results
Why complexity metrics are needed?

Airspace design

- Airspace comparison (US-Europe)
- Evaluation of Airspace Organisation Schemes
- Implementation of flexible use of airspace policies
Why complexity metrics are needed?

**Airspace design**
- Airspace comparison (US-Europe)
- Evaluation of Airspace Organisation Schemes
- Implementation of flexible use of airspace policies

**4D contract framework**
- 4D Trajectory design.
- Forecasting of potentially hazardous traffic situations.
- Automated Conflict Solver enhancement (robustness of the solution).
Workload

- Related to cognitive processes for human controllers.
- Easy/Hard forecasting of conflict occurrence.
- Monitoring is a non negligible part of the workload.

Complexity vs Workload

Complexity

- Related to traffic structure.
- Measure of intrinsic disorder of a set of trajectories.
- Increases with:
  - Sensitivity to uncertainties.
  - Interdependance of conflicts.
Complexity vs Workload

Workload
- Related to cognitive processes for human controllers.
- Easy/Hard forecasting of conflict occurrence.
- Monitoring is a non negligible part of the workload.

Complexity
- Related to traffic structure.
- Measure of intrinsic disorder of a set of trajectories.
- Increases with:
  - Sensitivity to uncertainties.
  - Interdependence of conflicts.
Sensitivity

- Uncertainties on positions, wind speed, intents induce inaccuracy on trajectory prediction.
- Depending on the situation, prediction error can grow exponentially fast!
- In such a case, the situation is complex because nearly impossible to forecast.
Intrinsic part of complexity

Sensitivity

- Uncertainties on positions, wind speed, intents induce inaccuracy on trajectory prediction.
- Depending on the situation, prediction error can grow exponentially fast!
- In such a case, the situation is complex because nearly impossible to forecast.

Interdependance

- Solving a conflict may induce other ones.
- Even in conflict-less situations, interactions between trajectories can rise the perceived level of complexity.
- Complexity is related to mixing behaviour.
Sensitivity-interdependance

- **No Sensitivity**
  - No conflict
  - Easy situation

- **High sensitivity**
  - Potential conflicts without interaction between solutions
  - Potential conflicts with interactions between solutions
  - Hard situation
Agenda

- Why complexity metrics are needed?
- Dynamical system modeling of aircraft trajectories
- How a complexity map is built?
- Results
The key idea is to model the set of aircraft trajectories by a linear dynamical system which is defined by the following equation:

$$\dot{\vec{X}} = A \cdot \vec{X} + \vec{B}$$

where $\vec{X}$ is the state vector of the system:

$$\vec{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Matrix $A$ and vector $B$ are the parameters of the model.
Regression of a Linear Dynamical System
Regression of a Linear Dynamical System
Regression of a Linear Dynamical System

Based on a set of observations (positions and speeds), one has to find a dynamical system which fits those observations. Suppose that $N$ observations are given:

**Positions:**

$$\vec{X}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

**and speeds:**

$$\vec{V}_i = \begin{bmatrix} vx_i \\ vy_i \\ vz_i \end{bmatrix}$$
Regression of a Linear Dynamical System

- Based on a set of observations (positions and speeds), one has to find a dynamical system which fits those observations. Suppose that $N$ observations are given:

  Positions:
  
  $$\vec{X}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

  and speeds:

  $$\vec{V}_i = \begin{bmatrix} vx_i \\ vy_i \\ vz_i \end{bmatrix}$$

- A LMS procedure is applied in order to extract the matrix $A$ and the vector $\vec{B}$. 

D. Delahaye and S. Puechmorel (Applied Math Laboratory, ENAC)

New trends in Air Traffic Complexity

March 8, 2010 13 / 36
Properties of the matrix $A$

- When real part of the eigenvalues of matrix $A$ is positive, the system is in expansion mode and when they are negative, the system is in contraction mode.
- Furthermore, the imaginary part of such eigenvalues are related with curl intensity of the field.
Properties of the matrix $\mathbf{A}$

- Convergence
- Full curl
- Imaginary axis
- Divergence
- Real Axis

Organized Traffic
Linear Dynamical System Modeling: An example

<table>
<thead>
<tr>
<th>Situation 1</th>
<th>Situation 2</th>
<th>Situation 3</th>
<th>Situation 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallel trajectories (relative speed=0)</td>
<td>Convergent trajectories</td>
<td>Divergent trajectories</td>
<td>Round about trajectories (relative speed=0)</td>
</tr>
</tbody>
</table>

Position of the eigenvalues of matrix A in the complex coordinate system

- Imaginary
  - 1
  - -1
  - Unit v/d
- Real
  - 1
  - -1
Linear Model limitations

- Give a global tendency of the traffic situation.
- Do not fit exactly with all traffic situations.

⇒ Non Linear Extension
Non Linear Extension in Space

\[ \dot{\vec{X}} = \vec{f}(\vec{X}) \]

**Optimization problem**

- \( \vec{f} \) ? such that:

\[
minE = \sum_{i=1}^{i=N} \| \vec{V}_i - \vec{f}(\vec{X}_i) \|^2
\]

- and

\[
min \int_{\mathbb{R}^3} \| \Delta \vec{f}(\vec{x}) \|^2 d\vec{x} \quad \text{with} \quad \Delta \vec{f} = \begin{bmatrix}
\frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_x}{\partial y^2} + \frac{\partial^2 f_x}{\partial z^2} \\
\frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_y}{\partial z^2} \\
\frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2}
\end{bmatrix}
\]
Non Linear Extension in Space

**Exact Solution (Amodei)**

\[
\tilde{f}(\tilde{X}) = \sum_{i=1}^{N} \Phi(\|\tilde{X} - \tilde{X}_i\|).\tilde{a}_i + A.\tilde{X} + B
\]

with

\[
\Phi(\|\tilde{X} - \tilde{X}_i\|) = Q(\|\tilde{X} - \tilde{X}_i\|^3)
\]

and

\[
Q = \begin{bmatrix}
\partial_{xx}^2 + \partial_{yy}^2 + \partial_{zz}^2 & 0 & 0 \\
0 & \partial_{xx}^2 + \partial_{yy}^2 + \partial_{zz}^2 & 0 \\
0 & 0 & \partial_{xx}^2 + \partial_{yy}^2 + \partial_{zz}^2
\end{bmatrix}
\]
Non Linear Extension in Space and time

\[ \dot{\vec{X}} = \vec{f}(\vec{X}, t) \]

Optimization problem

- We are looking for \( \vec{f} \) such that:

\[
\min E = \sum_{i=1}^{i=N} \sum_{k=1}^{k=K} \| \vec{V}_i(t_k) - \vec{f}(\vec{X}_i, t_k) \|^2
\]

- and

\[
\min \int_{\mathbb{R}^3} \int_t \| \Delta \vec{f}(\vec{x}) \|^2 + \| \frac{\partial \vec{f}}{\partial t} \|^2 d\vec{x} dt
\]
Non Linear Extension in Space and time

Exact Solution (Puechmorel and Delahaye)

\[ \vec{f}(\vec{X}, t) = \sum_{i=1}^{N} \sum_{k=1}^{K} \Phi(\| \vec{X}(t) - \vec{X}_i(t_k) \|, |t - t_k|).\vec{a}_{i,k} + A.\vec{X} + \vec{B} \]

with

\[ \Phi(r, t) = \text{diag} \left( \frac{\sigma}{r.\sqrt{\pi}}.\text{erf} \left( \frac{r}{\sigma}.\frac{1}{\sqrt{2 + \theta.|t|}} \right) \right) \]
Non Linear Extension in Space and time

**Gradient close form**

\[
\frac{\partial \Phi(r, t)}{\partial x} = (\alpha - \beta).x
\]

with

\[
\alpha = \frac{2.\sigma}{r^2.\pi} \cdot \frac{1}{\sqrt{2 + \theta.|t|}} \cdot e^{-\frac{r^2}{\sigma^2.(2+\theta.|t|)}}
\]

\[
\beta = \frac{\Phi(r, t)}{r^2}
\]

\[
r = \sqrt{x^2 + y^2 + z^2}
\]
Agenda

- Why complexity metrics are needed?
- Dynamical system modeling of aircraft trajectories
- How a complexity map is built?
- Results
How fast two neighboring dynamical system trajectories diverge?

- Let $\gamma(t, s), t: [a, b], s \in V$ be a family of trajectories of the dynamical system in the neighborhood $V$ of a given point $s_0$.
- We assume that the nominal trajectory is $t \mapsto \gamma(t, s_0)$.
- A perturbed trajectory is $t \mapsto \gamma(t, s)$ with $s \in V$.
- Divergence to nominal trajectory with respect to time is thus $\|\gamma(t, s_0) - \gamma(t, s)\| = D(t, s)$. 
Characterization of sensitivity

How fast two neighboring dynamical system trajectories diverge?

- Let $\gamma(t, s), t: [a, b], s \in V$ be a family of trajectories of the dynamical system in the neighborhood $V$ of a given point $s_0$.
- We assume that the nominal trajectory is $t \mapsto \gamma(t, s_0)$.
- A perturbed trajectory is $t \mapsto \gamma(t, s)$ with $s \in V$.
- Divergence to nominal trajectory with respect to time is thus $\|\gamma(t, s_0) - \gamma(t, s)\| = D(t, s)$.

Computing $D(t, s)$

Main idea: when $t \mapsto \gamma(t, s)$ is the solution of a differential equation with initial condition $\gamma(0, s) = s$, it is possible to show that $D$ itself satisfies a differential equation.
local behaviour of trajectories

\[ \gamma(t,s) \quad \gamma(t,s_0) \]

\[ D(t,s) \]

\[ V \]

\[ s_0 \quad s \]

D. Delahaye and S. Puechmorel (Applied Math Laboratory, ENAC)

New trends in Air Traffic Complexity

March 8, 2010 26 / 36
Local behaviour of trajectories

The variational equation I

- The value of $D(t,s)$ is the result of a cumulative process.
- Assume that $\gamma(t,s)$ is defined to be a flow:
  \[
  \frac{\partial \gamma(t,s)}{\partial t} = F(t, \gamma(t,s)) \quad \gamma(0,s) = s
  \]
  with $F$ a smooth vector field.
- Given a nonimal trajectory $\gamma(t,s_0)$, then divergence of nearby trajectories can be found up to order one in $\|s - s_0\|$ by solving:
  \[
  \frac{\partial D(t,s)}{\partial t} = DF(t, \gamma(t,s_0)).D(t,s) \quad D(0,s) = \|s - s_0\|
  \]
  with $DF$ the jacobian matrix of $F$ (with respect to $s$).
The variational equation II

- Since the previous equation is linear, it can be described by a matrix $M(t)$ that obeys:

$$\frac{dM(t)}{dt} = DF(t, \gamma(t, s_0)).M(t) \quad M(0) = Id$$

This equation is called the variational equation of the flow.

- The variational equation describes in some sense a linear dynamical system “tangent” to the original one.
Lyapunov exponents

Let $U^t(t)\Sigma(t)V(t) = M(t)$ be the SVD decomposition of $M(t)$.

The Lyapunov exponents are mean values of the logarithms of the diagonal elements of $\Sigma(t)$:

$$\kappa(s) = -\frac{1}{T} \int_0^T \log(\Sigma_{ii}(t)) dt \quad \forall \Sigma_{ii}(t) \leq 1$$
Based on trajectories observations, a 4D non linear dynamical system model is built.

The associated 4D vector field is then computed on a cube of airspace.

Lyapunov exponents are computed on each point of the cube in order to built a complexity map.
Given an initial point, the Lyapunov exponents and the associated SVD decomposition provide us with a decomposition of space in principal directions and corresponding convergence/divergence rate.

It is a localized version of the complexity based on linear systems.
Agenda

- Why complexity metrics are needed ?
- Dynamical system modeling of aircraft trajectories
- How a complexity map is built ?
- Results
Results
Results
An intrinsic trajectories complexity metric has been developed.
This metric can be computed as a map.
Give the areas of airspace where the traffic is organized and the ones where there is disorder.
Parallel computation by the mean of Local linear Model

\[
\bar{V}_i(\bar{X}_i, t_i) = \bar{f}(\bar{X}, t) + \frac{\partial \bar{f}(\bar{X}, t)}{\partial t} (t_i - t) + \bar{f}(\bar{X}, t) + \frac{\partial \bar{f}(\bar{X}, t)}{\partial \bar{X}} (\bar{X}_i - \bar{X}) + \\
O(|t_i - t| + \|\bar{X}_i - \bar{X}\|)
\]

\[
= \bar{a} + \bar{b}(t_i - t) + C.(\bar{X}_i - \bar{X}) + O(|t_i - t| + \|\bar{X}_i - \bar{X}\|)
\]

\[
\min_{\bar{a}, \bar{b}, C} \sum_{i=1}^{N} \| \bar{V}_i(\bar{X}_i, t_i) - \bar{a} + \bar{b}(t_i - t) + C.(\bar{X}_i - \bar{X}) \|^2 \psi(t_i - t, \bar{X}_i - \bar{X})
\]

Robustness improvement by taking into account uncertainties on time position of aircraft on their trajectories