On the Distribution of the Maximum of a Gaussian Field with d Parameters.*

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Abstract

Let I be a compact d-dimensional manifold, $X : I \to \mathcal{R}$ a Gaussian process with regular paths and $F_I(u)$, $u \in \mathcal{R}$ the probability distribution function of $\sup_{t\in I} X(t)$.

We prove that under certain regularity and non-degeneracy conditions, F_I is a C^1-function and F'_I is absolutely continuous, and that F'_I F''_I satisfy certain implicit equations that permit to give bounds for their values and to compute their asymptotic behaviour as $u \to +\infty$. This is a partial extension of previous results by the authors in the case $d = 1$.

Our methods use strongly the so-called Rice formulae for the moments of the number of roots of an equation of the form $Z(t) = x$, where $Z : I \to \mathcal{R}^d$ is a random field and x a fixed point in $\mathcal{R}^d$. We also give proofs for this kind of formulae, which have their own interest beyond the present application.

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1 Introduction and notations.

Let I be a d-dimensional compact manifold and $X : I \to \mathcal{R}$ a Gaussian process with regular paths defined on some probability space $(\Omega, \mathcal{A}, \mathrm{P})$. Define $M_I = \sup_{t \in I} X(t)$ and $F_I(u) = \mathrm{P}\{M_I \leq u\}$, $u \in \mathcal{R}$ the probability distribution function of the random variable M_I. Our aim is to study the regularity of the function F_I when $d > 1$.

There exist a certain number of general results on this subject, starting from the papers by Ylvisaker (1968) and Tsirelson (1975) (see also Weber (1985), Lifshits (1995), Diebolt and Posse (1996) and references therein). The main purpose of this paper is to extend to $d > 1$ some of the results about the regularity of the function $u \rightsquigarrow F_I(u)$ in Azaïs & Wschebor (2001), which concern the case $d = 1$.

Our main tool here is Rice Formula for the moments of the number of roots $N_u^Z(I)$ of the equation $Z(t) = u$ on the set I, where $\{Z(t) : t \in I\}$ is an $\mathcal{R}^d$-valued Gaussian field, I is a subset of $\mathcal{R}^d$ and u a given point in $\mathcal{R}^d$. For $d > 1$, even though it has been used in various contexts, as far as the authors know, a full proof of Rice Formula for the moments of $N_u^Z(I)$ seems to have only been published by R. Adler (1981) for the first moment of the number of critical points of a real-valued stationary Gaussian process with a d-dimensional parameter, and extended by Azaïs and Delmas (2002) to the case of processes with constant variance. Cabaña (1985) contains related formulae for random fields; see also the PHD thesis of Konakov cited by Piterbarg (1996b). In the next section we give a more general result which has an interest that goes beyond the application of the present paper. At the same time the proof appears to be simpler than previous ones. We have also included the proof of the formula for higher moments, which in fact follows easily from the first moment. Both extend with no difficulties to certain classes of non-Gaussian processes.

It should be pointed out that the validity of Rice Formula for Lebesgue-almost every $u \in \mathcal{R}^d$ is easy to prove (Brillinger, 1972) but this is insufficient for a certain number of standard applications. For example, assume $X : I \rightsquigarrow \mathcal{R}$ is a real-valued random process and one is willing to compute the moments of the number of critical points of X. Then, we must take for Z the random field $Z(t) = X'(t)$ and the formula one needs is for the precise value $u = 0$ so that a formula for almost every u does not solve the problem.

We have added Rice Formula for processes defined on smooth manifolds. Even though Rice Formula is local, this is convenient for various applications. We will need a formula of this sort to state and prove the implicit formulae for the derivatives of the distribution of the maximum (see Section 3).

The results on the differentiation of F_I are partial extensions of Azaïs & Wschebor (2001). They concern only the first two derivatives and remain quite far away from what is known for $d = 1$. The main result in that paper states that if X is a real-valued Gaussian process defined on a certain compact interval I of the real line, has $\mathcal{C}^{2k}$ paths (k integer, $k \geq 1$) and satisfies a non-degeneracy condition, then the distribution of M_I is of class $\mathcal{C}^k$.

For Gaussian fields defined on a d-dimensional regular manifold ($d > 1$) and possessing regular paths we obtain some improvements with respect to classical and general results due to Tsirelson (1975) for Gaussian sequences. An example is Corollary 6.1, that provides an asymptotic formula for $F_I'(u)$ as $u \to +\infty$ which is explicit in terms of the covariance of the process and can be compared with Theorem 4 in Tsirelson (1975) where an implicit expression depending on the function F itself is given.

We use the following notations:

If Z is a smooth function $U \rightsquigarrow \mathcal{R}^{d'}$, U a subset of $\mathcal{R}^d$, its successive derivatives are denoted Z', Z'',...$Z^{(k)}$ and considered respectively as linear, bilinear, ..., $k-$linear forms on $\mathcal{R}^d$. For example, $X^{(3)}(t)\{v_1, v_2, v_3\}$ is the value of the third derivative at point t applied to the triplet (v_1, v_2, v_3). The same notation is used for a derivative on a $\mathcal{C}^\infty$ manifold.

$\dot{I}, \partial I$ and $\bar{I}$ are respectively the interior, the boundary and the closure of the set I. If ξ is a random vector with values in $\mathcal{R}^d$, whenever they exist, we denote by $p_\xi(x)$ the value of the density of ξ at the point x, by $\mathrm{E}(\xi)$ its expectation and by $\mathrm{Var}(\xi)$ its variance-covariance matrix. λ is Lebesgue measure.

If u, v are points in $\mathcal{R}^d$, $\langle u, v\rangle$ denotes their usual scalar product and $\|u\|$ the Euclidean norm of u.

For M a $d \times d$ real matrix, we denote

$$\|M\| = \sup_{\|x\|=1} \|Mx\|$$

We put $\lambda_1^2 = \lambda_{min}, ...\lambda_d^2$ for the eigenvalues of MM^T, $0 \leq \lambda_1 \leq ... \leq \lambda_d$. Then, $\|M\| = \lambda_d$ and if M is non-singular, $\|M^{-1}\| = \frac{1}{\lambda_1} = \frac{1}{\lambda_{min}(M)}$.

Also for symmetric M, $M \succ 0$ (respectively $M \prec 0$) denotes that M is positive definite (resp. negative definite).

$\binom{m}{n}$ is the usual combinatorial number, i.e. $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ if m, n are non-negative integers, $m \geq n$ and $\binom{m}{n} = 0$ otherwise.

A^c denotes the complement of the set A. For real x, $x^+ = \sup(x, 0)$, $x^- = \sup(-x, 0)$

2 Rice formulae

Our main results in this section are the following:

Theorem 2.1 *Let $Z : I \rightsquigarrow \mathcal{R}^d$, I a compact subset of $\mathcal{R}^d$, be a random field and $u \in \mathcal{R}^d$.*

Assume that:

A0: Z is Gaussian,

A1: $t \rightsquigarrow Z(t)$ is a.s. of class $\mathcal{C}^1$,

A2: for each $t \in I$, $Z(t)$ has a non degenerate distribution (i.e. $\mathrm{Var}\big(Z(t)\big) \succ 0$),

A3: $\mathrm{P}\{\exists t \in \dot{I}, Z(t) = u, \det\big(Z'(t)\big) = 0\} = 0$

A4: $\lambda(\partial I) = 0$.

Then

$$\mathrm{E}\left(N_u^Z(I)\right) = \int_I \mathrm{E}\left(|\det(Z'(t))|/Z(t) = u\right)\ p_{Z(t)}(u)dt, \tag{1}$$

and both members are finite.

Theorem 2.2 *Let k, $k \geq 2$ be an integer. Assume the same hypotheses as in Theorem (2.1) excepting for A2 that is replaced by*

A'2 : for $t_1, ..., t_k \in I$ pairwise different values of the parameter, the distribution of

$$\big(Z(t_1), ..., Z(t_k)\big)$$

does not degenerate in $(\mathcal{R}^d)^k$. Then

$$\mathrm{E}\left[\big(N_u^Z(I)\big)\big(N_u^Z(I)-1\big)...\big(N_u^Z(I)-k+1\big)\right] = \int_{I^k} \mathrm{E}\left(\prod_{j=1}^{k} |\det\big(Z'(t_j)\big)|/Z(t_1) = ... = Z(t_k) = u\right) p_{Z(t_1),...,Z(t_k)}(u, ..., u)dt_1...dt_k, \tag{2}$$

where both members may be infinite.

Remark.

Note that Theorem 2.1 (resp 2.2) remains valid, excepting for the finiteness of the expectation in Theorem (2.1), if I is open and hypotheses A0,A1,A2 (resp A'2) and

A3 are verified. This follows immediately from the above statements. A standard extension argument shows that (1) holds true if one replaces I by any Borel subset of $\dot{I}$

Sufficient conditions for hypotheses A3 to hold are given by the next proposition.

Proposition 2.1 *Let $Z : I \rightsquigarrow \mathcal{R}^d$, I a compact subset of $\mathcal{R}^d$ be a random field with paths of class $\mathcal{C}^1$ and $u \in \mathcal{R}^d$. Assume that*

- *$p_{Z(t)}(x) \leq C$ for all $t \in I$ and x in some neighbourhood of u.*
- *at least one of the two following hypotheses is satisfied:*
  a) a.s. $t \rightsquigarrow Z(t)$ is of class $\mathcal{C}^2$
  b)
  $$\alpha(\delta) = \sup_{t \in I, x \in V(u)} \mathrm{P}\big\{|\det(Z'(t))| < \delta / Z(t) = x\big\} \to 0$$
  as $\delta \to 0$, where $V(u)$ is some neighbourhood of u.

Then A3 holds true.

Proof. If condition a) holds true, the result is Lemma 5 in Cucker and Wschebor (2003).

To prove it under condition b), assume with no loss of generality that $I = [0, 1]^d$ and that $u = 0$. Put $G_I = \big\{\exists t \in I, Z(t) = 0, \det\big(Z'(t)\big) = 0\big\}$ Choose $\varepsilon > 0$, $\eta > 0$; there exists a positive number M such that

$$\mathrm{P}(E_M) = \mathrm{P}\big\{\sup_{t \in I} \|Z'(t)\| > M\big\} \leq \varepsilon.$$

Denote by $\omega_{\det}$ the modulus of continuity of $|\det(X'(.))|$ and choose m large enough so that

$$\mathrm{P}(F_{m,\eta}) = \mathrm{P}\big\{\omega_{\det}(\frac{\sqrt{d}}{m}) \geq \eta\big\} \leq \varepsilon.$$

Consider the partition of I into m^d small cubes with sides of length $1/m$. Let $C_{i_1 \ldots i_d}$ such a cube and $t_{i_1 \ldots i_d}$ its centre $(1 \leq i_1, ..., i_d \leq m)$. Then

$$\mathrm{P}(G_I) \leq \mathrm{P}(E_M) + \mathrm{P}(F_{m,\eta}) + \sum_{1 \leq i_1 \ldots i_d \leq m} \mathrm{P}\Big(G_{C_{i_1 \ldots i_d}} \cap E_M^c \cap F_{m,\eta}^c\Big) \tag{3}$$

When the event in the term corresponding to $i_1 ... i_d$ of the last sum occurs, we have:

$$|Z_j(t_{i_1 ... i_d})| \leq \frac{M}{m}\sqrt{d} \quad j = 1, ..., d$$

where Z_j denotes the j-th coordinate of Z, and:

$$\left|\det\left(Z'(t_{i_1 ... i_d})\right)\right| < \eta.$$

So, if m is chosen sufficiently large so that $V(0)$ contains the ball centred at 0 with radius $\frac{M\sqrt{d}}{m}$, one has:

$$\mathrm{P}(G_I) \leq 2\epsilon + m^d(\frac{2M}{m}\sqrt{d})^d C\alpha(\eta)$$

Since ε and η are arbitrarily small, the result follows. □

Lemma 2.1 *With the notations of Theorem (2.1), suppose that A1 and A4 hold true and that*

$$p_{Z(t)}(x) \leq C \text{ for all } t \in I \text{ and } x \text{ in some neighbourhood of } u$$

Then $\mathrm{P}\left\{N_u^Z(\partial I) \neq 0\right\} = 0$

Proof: We use the notation of Proposition 2.1, with the same definition of E_M excepting that we do not suppose that $I = [0,1]^d$.

Since ∂I has zero measure, for each positive integer m, it can be covered by $h(m)$ cubes $C_1, ..., C_{h(m)}$ with centres $t_1, ... t_{h(m)}$ and side lengths $s_1, ... s_{h(m)}$ smaller than $1/m$, such that

$$\sum_{i=1}^{h(m)} (s_i)^d \to 0 \quad \text{as} \quad m \to +\infty.$$

So,

$$\begin{aligned} \mathrm{P}\left\{N_u^Z(\partial I) \right. &\neq \left. 0\right\} \leq \mathrm{P}(E_M) + \sum_{i=1}^{h(m)} \mathrm{P}\left(\left(N_u^Z(C_i) \neq 0\right) \cap E_M^c\right) \\ &\leq \varepsilon + \sum_{i=1}^{h(m)} \mathrm{P}\left\{ |Z_j(t_i) - u_j| \leq M s_i \frac{\sqrt{d}}{2} \ \forall \ j = 1, ..., d\right\} \leq \varepsilon + C\sum_{i=1}^{h(m)} (\sqrt{d} M s_i)^d \end{aligned}$$

This gives the result. □

Lemma 2.2 *Let $Z : I \to \mathcal{R}^d$, I a compact subset of $\mathcal{R}^d$, be a $\mathcal{C}^1$ function and u a point in $\mathcal{R}^d$. Assume that*

a) $\inf_{t \in Z^{-1}(\{u\})} \Big(\lambda_{min}\big(Z'(t)\big)\Big) \geq \Delta > 0$

b) $\omega_{Z'}(\eta) < \Delta/d$

where $\omega_{Z'}$ is the continuity modulus of Z', defined as the maximum of the continuity moduli of its entries and η a positive number.

Then, if t_1, t_2 are two distinct roots of the equation $Z(t) = u$ such that the segment $[t_1, t_2]$ is contained in I, the Euclidean distance between t_1 and t_2 is greater than η.

Recall that $\lambda_{min}\big(Z'(t)\big)$ is the inverse of $\|\big(Z'(t)\big)^{-1}\|$.

Proof: Set $\tilde{\eta} = \|t_1 - t_2\|$, $v = \frac{t_1 - t_2}{\|t_1 - t_2\|}$. Using the mean value theorem, for $i = 1, ..., d$, there exists $\xi_i \in [t_1, t_2]$ such that

$$\big(Z'(\xi_i)v\big)_i = 0$$

Thus

$$\begin{aligned} |\big(Z'(t_1)v\big)_i| &= |\big(Z'(t_1)v\big)_i - \big(Z'(\xi_i)v\big)_i| \\ &\leq \sum_{k=1}^{d} |Z'(t_1)_{ik} - Z'(\xi_i)_{ik}||v_k| \leq \omega_{Z'}(\tilde{\eta}) \sum_{k=1}^{d} |v_k| \leq \omega_{Z'}(\tilde{\eta})\sqrt{d} \end{aligned}$$

In conclusion

$$\Delta \leq \lambda_{min}\big(Z'(t_1)\big) \leq \|Z'(t_1)v\| \leq \omega_{Z'}(\tilde{\eta})d,$$

that implies $\tilde{\eta} > \eta$. □

Proof of Theorem 2.1: Consider a continuous non-decreasing function F such that

$$\begin{aligned} F(x) = 0 &\qquad \text{for } x \leq 1/2 \\ F(x) = 1 &\qquad \text{for } x \geq 1. \end{aligned}$$

Let Δ and η be positive real numbers. Define the random function

$$\alpha_{\Delta,\eta}(u) = F\Big(\frac{1}{2\Delta} \inf_{s \in I} \big[\lambda_{min}\big(Z'(s)\big) + \|Z(s) - u\|\big]\Big) \times \Big(1 - F\big(\frac{d}{\Delta}\omega_{Z'}(\eta)\big)\Big), \qquad (4)$$

and the set $I_{-\eta} = \{t \in I : \|t - s\| \geq \eta, \forall s \notin I\}$. If $\alpha_{\Delta,\eta}(u) > 0$ and $N_u^Z(I_{-\eta})$ does not vanish, conditions a) and b) in Lemma 2.2 are satisfied. Hence, in each ball with diameter $\frac{\eta}{2}$ centred at a point in $I_{-\eta}$ there is at most one root of the equation $Z(t) = u$, and a compactness argument shows that $N_u^Z(I_{-\eta})$ is bounded by a constant $C(\eta, I)$, depending only on η and on the set I.

Take now any real-valued non-random continuous function $f : \mathcal{R}^d \to \mathcal{R}$ with compact support. Because of the coarea formula (Federer, 1969, Th 3.2.3), since a.s. Z is Lipschitz and $\alpha_{\Delta,\eta}(u).f(u)$ is integrable:

$$\int_{\mathcal{R}^d} f(u) N_u^Z(I_{-\eta}) \alpha_{\Delta,\eta}(u) du = \int_{I_{-\eta}} |\det(Z'(t))| f\big(Z(t)\big) \alpha_{\Delta,\eta}\big(Z(t)\big) dt.$$

Taking expectations in both sides,

$$\int_{\mathcal{R}^d} f(u) \mathrm{E}\left(N_u^Z(I_{-\eta}) \alpha_{\Delta,\eta}(u)\right) du =$$
$$\int_{\mathcal{R}^d} f(u) du \int_{I_{-\eta}} \mathrm{E}\left(|\det(Z'(t))| \alpha_{\Delta,\eta}(u) / Z(t) = u\right) \ p_{Z(t)}(u) dt.$$

It follows that the two functions

$$(i) : \mathrm{E}\left(N_u^Z(I_{-\eta}) \alpha_{\Delta,\eta}(u)\right)$$

$$(ii) : \int_{I_{-\eta}} \mathrm{E}\left(|\det(Z'(t))| \alpha_{\Delta,\eta}(u) / Z(t) = u\right) \ p_{Z(t)}(u) dt,$$

coincide Lebesgue-almost everywhere as functions of u.

Let us prove that both functions are continuous, hence they are equal for every $u \in \mathcal{R}^d$.

Fix $u = u_0$ and let us show that the function in (i) is continuous at $u = u_0$. Consider the random variable inside the expectation sign in (i). Almost surely, there is no point t in $Z^{-1}(\{u_0\})$ such that $\det(Z'(t))$ =0. By the local inversion theorem, $Z(.)$ is invertible in some neighbourhood of each point belonging to $Z^{-1}(\{u_0\})$ and the distance from $Z(t)$ to u_0 is bounded below by a positive number for $t \in I_{-\eta}$ outside of the union of these neighbourhoods. This implies that, a.s., as a function of u, $N_u^Z(I_{-\eta})$ is constant in some (random) neighbourhood of u_0. On the other hand, it is clear from its definition that the function $u \rightsquigarrow \alpha_{\Delta,\eta}(u)$ is continuous and bounded. We may now apply dominated convergence as $u \to u_0$, since $N_u^Z(I_{-\eta}) \alpha_{\Delta,\eta}(u)$ is bounded by a constant that does not depend on u.

For the continuity of (ii), it is enough to prove that, for each $t \in I$ the conditional expectation in the integrand is a continuous function of u. Note that the random variable $|\det(Z'(t))|\alpha_{\Delta,\eta}(u)$ is a functional defined on $\{(Z(s),Z'(s)) : s \in I\}$. Perform a Gaussian regression of $(Z(s),Z'(s)) : s \in I$ with respect to the random variable $Z(t)$, that is, write

$$\begin{aligned} Z(s) &= Y^t(s) + \alpha^t(s)Z(t) \\ Z'_j(s) &= Y^t_j(s) + \beta^t_j(s)Z(t),\ j = 1,...,d \end{aligned}$$

where $Z'_j(s)$ $(j = 1,...,d)$ denote the columns of $Z'(s)$, $Y^t(s)$ and $Y^t_j(s)$ are Gaussian vectors, independent of $Z(t)$ for each $s \in I$, and the regression matrices $\alpha^t(s)$, $\beta^t_j(s)$ $(j = 1,...,d)$ are continuous functions of s,t (take into account A2). Replacing in the conditional expectation we are now able to get rid of the conditioning, and using the fact that the moments of the supremum of an a.s. bounded Gaussian process are finite, the continuity in u follows by dominated convergence.

So, now we fix $u \in \mathcal{R}^d$ and make $\eta \downarrow 0$, $\Delta \downarrow 0$ in that order, both in (i) and (ii). For (i) one can use Beppo Levi's Theorem. Note that almost surely

$$N^Z_u(I_{-\eta}) \uparrow N^Z_u(\dot{I}) = N^Z_u(I),$$

where the last equality follows from Lemma 2.1. On the other hand, the same Lemma 2.1 plus A3 imply together that,almost surely:

$$\inf_{s\in I}\Big[\lambda_{min}\big(Z'(s)\big) + \|Z(s) - u\|\Big] > 0$$

so that the first factor in the right-hand member of (4) increases to 1 as Δ decreases to zero. Hence by Beppo Levi's Theorem:

$$\lim_{\Delta\downarrow 0}\lim_{\eta\downarrow 0}\mathrm{E}\Big(N^Z_u(I_{-\eta})\alpha_{\Delta,\eta}(u)\Big) = \mathrm{E}\Big(N^Z_u(I)\Big).$$

For (ii), one can proceed in a similar way after de-conditioning obtaining (1). To finish the proof, remark that standard Gaussian calculations show the finiteness of the right-hand member of (1). □

Proof of Theorem 2.2: For each $\delta > 0$, define the domain

$$D_{k,\delta}(I) = \{(t_1,...,t_k) \in I^k,\ \|t_i - t_j\| \geq \delta\ if\ \ i \neq j, i,j = 1,...,k\}$$

and the process $\widetilde{Z}$

$$(t_1,...,t_k) \in D_{k,\delta}(I) \rightsquigarrow \widetilde{Z}(t_1,...,t_k) = \big(Z(t_1),...,Z(t_k)\big).$$

It is clear that $\widetilde{Z}$ satisfies the hypotheses of Theorem 2.1 for every value $(u, ..., u) \in (\mathcal{R}^d)^k$. So,

$$\mathrm{E}\left[N^{\widetilde{Z}}_{(u,\dots,u)}\big(D_{k,\delta}(I)\big)\right] = \int_{D_{k,\delta}(I)}$$

$$\mathrm{E}\Big(\prod_{j=1}^{k} |\det\big(Z'(t_j)\big)|/Z(t_1) = ... = Z(t_k) = u\Big) p_{Z(t_1),\dots,Z(t_k)}(u, ..., u) dt_1 ... dt_k \quad (5)$$

To finish, let $\delta \downarrow 0$, note that $\big(N_u^Z(I)\big)\big(N_u^Z(I)-1\big)...\big(N_u^Z(I)-k+1\big)$ is the monotone limit of

$$N^{\widetilde{Z}}_{(u,\dots,u)}\big(D_{k,\delta}(I)\big),$$

and that the diagonal $D_k(I) = \big\{(t_1, ..., t_k) \in I^k, t_i = t_j \text{ for some pair } i, j, i \neq j\big\}$ has zero Lebesgue measure in $(\mathcal{R}^d)^k$. □

Remark Even thought we will not use this in the present paper, we point out that it is easy to adapt the proofs of Theorems 2.1 and 2.2 to certain classes of non-Gaussian processes.

For example, the statement of Theorem 2.1 remains valid if one replaces hypotheses A0 and A2 respectively by the following B0 and B2:

B0 : $Z(t) = H(Y(t))$ for $t \in I$ where
$Y : I \to \mathcal{R}^n$ is a Gaussian process with $\mathcal{C}^1$ paths such that for each $t \in I$, $Y(t)$ has a non-degenerate distribution and $H : \mathcal{R}^n \to \mathcal{R}^d$ is a $\mathcal{C}^1$ function.

B2 : for each $t \in I$, $Z(t)$ has a density $p_{Z(t)}$ which is continuous as a function of (t, u).

Note that B0 and B2 together imply that $n \geq d$. The only change to be introduced in the proof of the theorem is in the continuity of (ii) where the regression is performed on $Y(t)$ instead of $Z(t)$

Similarly, the statement of Theorem 2.2 remains valid if we replace A0 by B0 and add the requirement the joint density of $Z(t_1), ..., Z(t_k)$ to be a continuous function of $t_1, ..., t_k, u$ for pairwise different $t_1, ..., t_k$

Now consider a process X from I to $\mathcal{R}$ and define

$$M^X_{u,1}(I) = \sharp\{t \in I, X(.) \text{ has a local maximum at the point } t,\ X(t) > u\}$$

$$M^X_{u,2}(I) = \sharp\{t \in I, X'(t) = 0,\ X(t) > u\}$$

The problem of writing Rice Formulae for the factorial moments of these random variables can be considered as a particular case of the previous one and the proofs are the same, mutatis mutandis. For further use, we state as a theorem, Rice Formula for the expectation. For short we do not state the equivalent of Theorem (2.2) that holds true similarly.

Theorem 2.3 *Let $X : I \rightsquigarrow \mathcal{R}$, I a compact subset of $\mathcal{R}^d$, be a random field. Let $u \in \mathcal{R}$, define $M^X_{u,i}(I)$, $i = 1, 2$ as above. For each $d \times d$ real symmetric matrix M, we put $\delta^1(M) := |\det(M)| 1\!\!1_{M \prec 0}$, $\delta^2(M) := |\det(M)|$.*

Assume:

A0: X is Gaussian,

A"1: a.s. $t \rightsquigarrow X(t)$ is of class $\mathcal{C}^2$,

A"2: for each $t \in I$, $X(t), X'(t)$ has a non degenerate distribution in $\mathcal{R}^1 \times \mathcal{R}^d$,

A"3: either

$$\text{a.s. } t \rightsquigarrow X(t) \text{ is of class } \mathcal{C}^3$$

or

$$\alpha(\delta) = \sup_{t \in I, x' \in V(0)} \mathrm{P}\big(|\det\big(X''(t)\big)| < \delta / X'(t) = x'\big) \to 0$$

as $\delta \to 0$, where $V(0)$ denotes some neighbourhood of 0,

A4: ∂I has zero Lebesgue measure.

Then, for $i = 1, 2$:

$$\mathrm{E}\left(M^X_{u,i}(I)\right) = \int_u^{\infty} dx \int_I \mathrm{E}\left(\delta^i\big(X''(t)\big) / X(t) = x, X'(t) = 0\right) p_{X(t),X'(t)}(x, 0) dt$$

and both members are finite.

2.1 Processes defined on a smooth manifold.

Let U be a differentiable manifold (by differentiable we mean infinitely differentiable) of dimension d. We suppose that U is orientable in the sense that there exists a non-vanishing differentiable d-form Ω on U. This is equivalent to assuming that there exists an atlas $\big((U_i, \phi_i); i \in I\big)$ such that for any pair of intersecting charts (U_i, ϕ_i), (U_j, ϕ_j), the Jacobian of the map $\phi_i \circ \phi_j^{-1}$ is positive.

We consider a Gaussian stochastic process with real values and $\mathcal{C}^2$ paths $X = \{X(t) : t \in U\}$ defined on the manifold U. In this subsection, our aim is to write Rice Formulae for this kind of processes under various geometric settings for U. More precisely we will consider three cases: first, when U is a manifold without any additional structure on it; second, when U has a Riemannian metric; third, when it is embedded in an Euclidean space. We will make use of these formulae in Section 3 but they have an interest in themselves. (See Taylor and Adler (2002) for other details or similar results).

We will assume that in every chart $X(t)$ and $DX(t)$ have a non-degenerate joint distribution and that hypothesis A"3 is verified. For S a Borel subset of $\dot{U}$, the following quantities are well defined and measurable : $M_{u,1}^X(S)$, the number of local maxima and $M_{u,2}^X(S)$, the number of critical points.

2.1.1 Abstract manifold

Proposition 2.2 *For $k = 1, 2$ the quantity which is expressed in every chart ϕ with coordinates $s_1, ..., s_d$ as*

$$\int_u^{+\infty} dx \mathrm{E}\left(\delta^k(Y''(s))/Y(s) = x, Y'(s) = 0\right) p_{Y(s),Y'(s)}(x, o) \wedge_{i=1}^d ds_i, \qquad (6)$$

where $Y(s)$ is the process X written in the chart : $Y = X \circ \phi^{-1}$, defines a d-form Ω^k on $\dot{U}$ and for every Borel set $S \subset \dot{U}$

$$\int_S \Omega^k = \mathrm{E}\left(M_{u,k}^X(S)\right).$$

Proof: Note that a d-form is a measure on $\dot{U}$ whose image in each chart is absolutely continuous with respect to Lebesgue measure $\wedge_{i=1}^d ds_i$,. To prove that (6) defines an d-form, it is sufficient to prove that its density with respect to $\wedge_{i=1}^d ds_i$, satisfies locally the change-of-variable formula. Let (U_1, ϕ_1), (U_2, ϕ_2) two intersecting charts and set

$$U_3 := U_1 \cap U_2 \ ; \ Y_1 := X \circ \phi_1^{-1} \ ; \ Y_2 := X \circ \phi_2^{-1} \ ; \ H := \phi_2 \circ \phi_1^{-1}.$$

Denote by s_i^1 and s_i^2, $i = i, ..., d$ the coordinates in each chart. We have

$$\frac{\partial Y_1}{\partial s_i^1} = \sum_{i'} \frac{\partial Y_2}{\partial s_{i'}^2} \frac{\partial H_{i'}}{\partial s_i^1}$$

$$\frac{\partial^2 Y_1}{\partial s_i^1 \partial s_j^1} = \sum_{i',j'} \frac{\partial^2 Y_2}{\partial s_{i'}^2 \partial s_{j'}^2} \frac{\partial H_{i'}}{\partial s_i^1} \frac{\partial H_{j'}}{\partial s_j^1} + \sum_{i'} \frac{\partial Y_2}{\partial s_{i'}^2} \frac{\partial^2 H_{i'}}{\partial s_i^1 \partial s_j^1}.$$

Thus at every point

$$Y_1'(s^1) = \left(H'(s^1)\right)^T Y_2'(s^2),$$

$$p_{Y_1(s^1),Y_1'(s^1)}(x,0) = p_{Y_2(s^2),Y_2'(s^2)}(x,0)|\det(H'(s^1)|^{-1}$$

and at a singular point

$$Y_1''(s^1) = \big(H'(s^1)\big)^T Y_2''(s^2)H'(s^1).$$

On the other hand, by the change of variable formula

$$\wedge_{i=1}^d ds_i^1 = |\det(H'(s^1)|^{-1} \wedge_{i=1}^d ds_i^2.$$

Replacing in the integrand in (6), one checks the desired result.

For the second part again it suffices to prove it locally for an open subset S included in a unique chart. Let (S,ϕ) a chart and let again $Y(s)$ be the process written in this chart, it suffices to check that

$$\mathrm{E}\left(M_{u,k}^X(S)\right) = \int_{\phi(S)} d\lambda(s) \int_u^{+\infty} dx\ \mathrm{E}\left(\delta^k(Y''(s))/Y(s)=x, Y'(s)=0\right) p_{Y(s),Y'(s)}(x,0). \quad (7)$$

Since $M_{u,k}^X(S)$ is equal to $M_{u,k}^Y\{\phi(S)\}$ we see that the result is a direct consequence of Theorem (2.3)

2.1.2 Riemannian manifold

The form in (6) is intrinsic (in the sense that it does not depend on the parametrization) but the terms inside the integrand are not. It is possible to give a complete intrinsic expression in the case when U is a equipped with a Riemannian metric. When such a Riemannian metric is not given, it is always possible to use the metric g induced by the process itself (see Taylor and Adler, 2002) by setting

$$g_s(Y,Z) = \mathrm{E}\Big(\big(Y(X)\big)\big(Z(X)\big)\Big),$$

for Y,Z belonging the tangent space $T(s)$ at $s \in U$. $Y(X)$, (resp. $Z(X)$) denotes the action of the tangent vector Y (resp. Z) on the function X. This metric leads to very simple expressions for centred variance-1 Gaussian processes.

The main point is that at a singular point of X the second order derivative D^2X is intrinsic since it defines locally the Taylor expansion. Given the Riemannian metric g_s the second differential can be represented by an endomorphism that will be denoted $\nabla^2 X(s)$.

$$D^2X(s)\{Y,Z\} = Y(Z(X) = Z(Y(X) = g_s(\nabla^2 X(s)Y,Z). \quad (8)$$

In fact, at a singular point the definition given by formula (8) coincide with the definition of the Hessian read in and orthonormal basis. This endomorphism is intrinsic and of course its determinant. So in a chart

$$\det\left(\nabla^2 X(s)\right) = \det(D^2 X(s)) \det(g_s)^{-1}, \tag{9}$$

and $\nabla^2 X(s)$ is negative definite if and only if $D^2 X(s)$ is. Hence

$$\delta^k\left(\nabla^2 X(s)\right) = \delta^k\left(D^2 X(s)\right) \det(g_s)^{-1} \ ; \ (k = 1, 2)$$

We turn now to the density in (6). The gradient at some location s is defined as the unique vector $\nabla X(s) \in T(s)$ such that $g_s(\nabla X(s), Y) = DX(s)\{Y\}$. In a chart the vector of coordinates of the gradient in the basis $\partial x_i, i = 1, d$ is given by $\left(g_s\right)^{-1} DX(s)$ where $DX(s)$ is now the vector of coordinates of the derivative in the basis $dx^i, i = 1, d$. The joint density at $(x, 0)$ of $\left(X(s), \nabla X(s)\right)$is intrinsic only if read in an orthonormal basis of the tangent space. In that case the vector of coordinates is given by

$$\widetilde{\nabla X}(s) = \left(g_s\right)^{1/2} \nabla X(s) = \left(g_s\right)^{-1/2} DX$$

By the change of variable formula :

$$p_{X(s),\widetilde{\nabla X}(s)}(x, 0) = p_{X(s),DX(s)}(x, 0)\sqrt{\det(g_s)}$$

Remembering that the Riemannian volume Vol satisfies

$$Vol = \sqrt{\det(g_s)} \wedge_{i=1}^d ds_i^2$$

we can rewrite expression (6) as

$$\int_u^{+\infty} dx \ \mathrm{E}\big(\delta^k(\nabla^2 X(s)/X(s) = x, \nabla X(s) = 0\big) p_{X(s),\nabla X(s)}(x, 0) \ Vol \tag{10}$$

where we have omitted the tilde above $\nabla X(s)$ for simplicity. This is the Riemannian intrinsic expression.

2.1.3 Embedded manifold

In most practical applications, U is naturally embedded in an Euclidean space $\mathcal{R}^m$. Examples of such situations are given by U being a sphere or the boundary of a domain in $\mathcal{R}^m$. In such a case we look for an expression for (10) as a function of the natural derivative on $\mathcal{R}^m$. The manifold is equipped with the metric induced by the Euclidean metric in $\mathcal{R}^m$. Considering the form (10), clearly the Riemannian volume is just the geometric measure σ on U.

Following Milnor (1965), we assume that the process X_t is defined on an open neighbourhood of U so that the ordinary derivatives $X'(s)$ and $X''(s)$ are well defined for $s \in U$. Denoting the projector onto the tangent and normal spaces by $P_{T(s)}$ and $P_{N(s)}$, we have.

$$\nabla X(s) = P_{T(s)}(X'(s)).$$

Wee now define the second fundamental form II of U embedded in $\mathcal{R}^m$ than can be defined in our simple case as the bilinear application (see Kobayashi Nomizu 199? T 2, chap. 7 for details).

$$Y, Z \in T(s) \rightsquigarrow P_{N(s)}(\nabla'_X Y).$$

where $\nabla'_X Y$ is the Levi-Civita connection on $\mathcal{R}^n$. The next formula is well known, or easy to check at a singular point, and gives the expression of the Hessian on U.

$$Y, Z \in T(s) \rightsquigarrow X''(s)\{Y, Z\} + < \mathrm{II}\{Y, Z\}, X'(s) >, \tag{11}$$

The determinant of the bilinear form given by (11), expressed in an orthonormal basis, gives the value of det $\big(\nabla^2 X(s)\big)$. As a conclusion we get the expression of every terms involved in (10).

Examples:

Codimension 1: with a given orientation we get

$$\nabla^2 X = X''_T + \mathrm{II}.X'_N$$

where X''_T is the tangent projection of the second derivative and X'_N the normal component of the gradient.

Sphere: When U is a sphere of radius $r > 0$ in $\mathcal{R}^{d+1}$ oriented towards the inside

$$\nabla^2 X = X''_T + r(Id)_d X'_N \tag{12}$$

Curve: When the manifold is a curve parametrized by arc length

$$\mathrm{E}\big(M_u^k(U)\big) = \int_u^{+\infty} dx \int_0^L dt$$
$$\mathrm{E}\left(\delta^k\big(X''_T(t) + C(t)X'_N(t)/X(t) = x, X'_T(t) = 0\big)\, p_{X(t),X'_T(t)}(x, 0),\right. \tag{13}$$

Where $C(t)$ is the curvature at location t and $X'_N(t)$ is the derivative taken is the direction of the normal to the curve at point t.

Remark: One can consider a number of variants of Rice formulae, in which we may be interested in computing the moments of the number of roots of the equation $Z(t) = u$ under some additional conditions. This has been the case in the statement of Theorem 2.3 in which we have given formulae for the first moment of the number of zeroes of X' in which X is bigger than u (i=2) and also the real-valued process X has a local maximum (i=1).

We just consider below two additional examples of variants that we state here for further reference. We limit the statements to random fields defined on subsets of $\mathcal{R}^d$. Similar statements hold true when the parameter set is a general smooth manifold. Proofs are essentially the same as the previous ones.

Variant 1: Assume that Z_1, Z_2 are $\mathcal{R}^d$-valued random fields defined on compact subsets I_1, I_2 of $\mathcal{R}^d$ and suppose that (Z_i, I_i) $(i = 1, 2)$ satisfy the hypotheses of Theorem 2.1 and that for every $s \in I_1$ and $t \in I_2$, the distribution of $\big(Z_1(s), Z_2(t)\big)$ does not degenerate. Then, for each pair $u_1, u_2 \in \mathcal{R}^d$:

$$\mathrm{E}\left(N_{u_1}^{Z_1}(I_1) N_{u_2}^{Z_2}(I_2)\right)$$
$$= \int_{I_1 \times I_2} dt_1 dt_2 \mathrm{E}\left(|\det(Z_1'(t_1))||\det(Z_2'(t_2))|/Z_1(t_1) = u_1, Z_2(t_2) = u_2\right)\ p_{Z_1(t_1),Z_2(t_2)}(u_1, u_2), \tag{14}$$

Variant 2: Let Z, I be as in Theorem 2.1 and ξ a real-valued bounded random variable which is measurable with respect to the σ-algebra generated by the process Z. Assume that for each $t \in I$, there exists a continuous Gaussian process $\{Y^t(s) : s \in I\}$, for each $s, t \in I$ a non-random function $\alpha^t(s) : \mathcal{R}^d \to \mathcal{R}^d$ and a Borel-measurable function $g : \mathcal{C} \to \mathcal{R}$ where $\mathcal{C}$ is space of real-valued continuous functions on I equipped with the supremum norm, such that:

1. $\xi = g\big(Y^t(.) + \alpha^t(.)Z(t)\big)$
2. $Y^t(.)$ and $Z(t)$ are independent
3. for each $u_0 \in \mathcal{R}$, almost surely the function
$$u \rightsquigarrow g\big(Y^t(.) + \alpha^t(.)u\big)$$
is continuous at $u = u_0$

Then the formula :

$$\mathrm{E}\left(N_u^Z(I)\xi\right) = \int_I \mathrm{E}\left(|\det(Z'(t))|\xi/Z(t) = u\right)\ p_{Z(t)}(u)dt,$$

holds true.

We will be particularly interested in the function $\xi = \mathbb{1}_{M_I<v}$ for some $v \in \mathcal{R}$. We will see that later on that it satisfies the above conditions under certain hypotheses on the process Z.

3 First Derivative, First Form.

Our main goals in this and the next section are to prove existence and regularity of the derivatives of the function $u \rightsquigarrow F_I(u)$ and, at the same time, that they satisfy some implicit formulae that can be used to provide bounds on them. In the following we assume that I is a d-dimensional $\mathcal{C}^\infty$ manifold embedded in $\mathcal{R}^N$, $N \geq d$. σ and $\tilde{\sigma}$ are respectively the geometric measures on I and ∂I. Unless explicit statement of the contrary, the topology on I will be he relative topology.

In this section we prove formula (17) for $F_I'(u)$. -that we call "first form"- which is valid for λ-almost every u, under strong regularity conditions on the paths of the process X. In fact, the hypothesis that X is Gaussian is only used in Rice formula itself and in Lemma 3.1 which gives a bound for the joint density

$$p_{X(s),X(t),X'(s),X'(t)}.$$

In both places, one can substitute Gaussianity by appropriate conditions that permit to obtain similar results.

More generally, it is easy to see that inequality (15) below is valid under quite general non Gaussian conditions and implies that we have the following upper bound for the density of the distribution of the random variable M_I.

$$F_I'(u) \leq \int_I \mathrm{E}\left(\delta^1(X''(t))/X(t) = u, X'(t) = 0\right) p_{X(t),X'(t)}(u,0)\sigma(dt) + \int_{\partial I} \mathrm{E}\left(\delta^1(\widetilde{X}''(t))/X(t) = u, \widetilde{X}'(t) = 0\right) p_{X(t),\widetilde{X}'(t)}(u,0)\tilde{\sigma}(dt), \quad (15)$$

where the function δ^1 has been defined in the statement of Theorem 2.3 and $\widetilde{X}$ denotes the restriction of X to the boundary ∂I.

Even for $d = 1$ (one parameter processes) and X Gaussian and stationary, inequality (15) provides reasonably good upper bounds for $F_I'(u)$ (see Diebolt and Posse (1996), Azaïs and Wschebor (2001). We will see an example for $d = 2$ at the end of this section.

In the next section, we are able to prove that $F_I(u)$ is a $\mathcal{C}^1$ function and that formula (17) can be essentially simplified by getting rid of the conditional expectation, thus obtaining the "second form" for the derivative. This is done under weaker regularity conditions but the assumption that X is Gaussian becomes essential.

In case the dimension d of the parameter is equal to 1, this is the starting point to continue the differentiation procedure and under hypotheses H_{2k} one is able to prove that F_I is a $\mathcal{C}^k$ function and to obtain implicit formulae for $F_I^{(k)}$ (see Azaïs & Wschebor, 2001)

When $d > 1$, a certain number of difficulties arise and it is not clear that the process can continue beyond $k = 2$. With the purpose of establishing such formula for F_I'', we introduce in Section 4 the "helix-processes" which appear in a natural way in these formulae and have paths possessing singularities of a certain form that will be described precisely in that section.

Definition 3.1 *Let $X : I \to \mathcal{R}$ be real-valued stochastic process defined on a subset of $\mathcal{R}^d$. We will say that X satisfies condition (H_k), k a positive integer, if the following three conditions hold true:*

- *X is Gaussian;*
- *a.s. the paths of X are of class C^k;*
- *for any choice of pairwise different values of the parameter $t_1, ..., t_n$ the joint distribution of the random variables*

$$X(t_1), ..., X(t_n), X'(t_1), ..., X'(t_n), ..., X^{(k)}(t_1), ..., X^{(k)}(t_n) \tag{16}$$

has maximum rank. Note that the number of distinct real-valued Gaussian variables belonging to this set (16), on account of exchangeability of the order of differentiation, is equal to

$$n \left[1 + \binom{d}{d-1} + \binom{d+1}{d-1} + ... + \binom{k+d-1}{d-1} \right]$$

The next proposition shows that there exist processes that satisfy (H_k).

Proposition 3.1 *Let $X = \{X(t) : t \in \mathcal{R}^d\}$ be a centred stationary Gaussian process having continuous spectral density f^X. Assume that $f^X(x) > 0$ for every $x \in \mathcal{R}^d$ and that for any $\alpha > 0$ $f^X(x) \le C_\alpha \|x\|^{-\alpha}$ holds true for some constant C_α and all $x \in \mathcal{R}^d$.*

Then, X satisfies (H_k) for every $k = 1, 2, ...$

Proof: The proof is an adaptation of the proof of a related result for $d = 1$ (Cramer & Leadbetter (1967), p. 203).

It is well-known that the hypothesis implies that the paths are C^k for every $k = 1, 2, ...$ As for the non-degeneracy condition, let $t_1, ..., t_n$ be pairwise different values of the parameter. Denote by $\partial_{k_1,k_2...,k_d}X$ the partial derivative of X k_1 times with respect to the first coordinate, k_2 times with respect to the second, ..., k_d times with respect to the d-th coordinate. We want to prove that, for any $k = 1, 2, ...$ the centred Gaussian joint distribution of the random variables

$$\partial_{k_1,k_2...,k_d}X(t_h)$$

where the d-tuple $(k_1, ..., k_d)$ varies on the set of non-negative integers such that $k_1 + ...k_d \leq d$ and t_h varies in the set $\{t_1, ...t_n\}$, is non-degenerate. For this purpose, it suffices to show that if we put

$$Z = \sum_{h=1}^{n} \sum\nolimits_k \lambda_{k_1,k_2...,k_d,h} \; \partial_{k_1,k_2...,k_d}X(t_h)$$

where $\sum_k$ denotes summation over all the d-tuples of non-negative integers $k_1, k_2..., k_d$ such that $k_1+k_2+..+k_d \leq k$ and $\lambda_{k_1,k_2...,k_d,h}$ are complex numbers, then $\mathrm{E}\left(|Z|^2\right) = 0$ implies $\lambda_{k_1,k_2...,k_d,h} = 0$ for any choice of the indices $k_1, k_2..., k_d, h$ in the sum. Using the spectral representation, and denoting $x = (x_1, ..., x_d)$,

$$\begin{aligned} \mathrm{E}\left(|Z|^2\right) &= \sum_{h,h'=1}^{n} \sum \lambda_{k_1,k_2...,k_d,h} \overline{\lambda_{k'_1,k'_2...,k'_d,h'}} \int_{\mathcal{R}^d} (ix_1)^{k_1}...(ix_d)^{k_d}(ix_1)^{k'_1}...(ix_d)^{k'_d}. \\ &\quad . \exp\left[i \langle x, t_h - t_{h'}\rangle\right] \; f^X(x)dx \end{aligned}$$

where the inner sum is over all $2d-$tuples of non-negative integers $k_1, k_2..., k_d, k'_1, k'_2..., k'_d$ such that $k_1 + k_2 + .. + k_d \leq k$, $k'_1 + k'_2 + .. + k'_d \leq k$. Hence,

$$\mathrm{E}\left(|Z|^2\right) = \int_{\mathcal{R}^d} \left| \sum_{h=1}^{n} \sum\nolimits_k \lambda_{k_1,k_2...,k_d,h} \;\; (ix_1)^{k_1}...(ix_d)^{k_d} \exp\left[i \langle x, t_h\rangle\right] \right|^2 \; f^X(x)dx$$

The hypothesis on f^X implies that if $\mathrm{E}\left(|Z|^2\right) = 0$, then

$$\sum_{h=1}^{n} \sum\nolimits_k \lambda_{k_1,k_2...,k_d,h} \;\; (ix_1)^{k_1}...(ix_d)^{k_d} \exp\left[i \langle x, t_h\rangle\right] = 0 \text{ for all } x \in \mathcal{R}^d.$$

The result follows from the fact that the set of functions $x_1^{k_1}...x_d^{k_d} \exp\left[i \langle x, t_h\rangle\right]$ where $k_1, k_2..., k_d, h$ vary as above, is linearly independent. $\square$

Theorem 3.1 (First derivative, first form) *Let $X : I \to \mathcal{R}$ be a Gaussian process, I a $\mathcal{C}^\infty$ compact d-dimensional manifold .*

Assume that X verifies H_k for every $k = 1, 2, ...$

Then, the function $u \rightsquigarrow F_I(u)$ is absolutely continuous and its Radon-Nikodym derivative is given for almost every u by:

$$F_I'(u) = (-1)^d \int_I \mathrm{E}\left(\det(X''(t))\ \mathbb{1}_{M_I \le u} / X(t) = u, X'(t) = 0\right) p_{X(t),X'(t)}(u, 0)\sigma(dt) +$$
$$(-1)^{d-1} \int_{\partial I} \mathrm{E}\left(\det(\widetilde{X}''(t))\ \mathbb{1}_{M_I \le u} / X(t) = u, \widetilde{X}'(t) = 0\right) p_{X(t),\widetilde{X}'(t)}(u, 0)\tilde{\sigma}(dt). \quad (17)$$

Proof : For $u < v$ and S (respectively $\widetilde{S}$) a subset of I (resp. ∂I), let us denote

$$\begin{aligned} M_{u,v}(S) &= \sharp\{t \in S : u < X(t) \le v,\ X'(t) = 0,\ X''(t) \prec 0\} \\ \widetilde{M}_{u,v}(\widetilde{S}) &= \sharp\left\{t \in \widetilde{S} : u < X(t) \le v,\ \widetilde{X}'(t) = 0,\ \widetilde{X}''(t) \prec 0\right\} \end{aligned}$$

Step 1. Let $h > 0$ and consider the increment

$$F_I(u) - F_I(u-h) = \mathrm{P}\left(\{M_I \le u\} \cap \left[\left\{M_{u-h,u}(\dot{I}) \ge 1\right\} \cup \left\{\widetilde{M}_{u-h,u}(\partial I) \ge 1\right\}\right]\right).$$

Let us prove that

$$\mathrm{P}\left(M_{u-h,u}(\dot{I}) \ge 1, \widetilde{M}_{u-h,u}(\partial I) \ge 1\right) = o(h) \text{ as } h \downarrow 0. \quad (18)$$

In fact, for $\delta > 0$:

$$\mathrm{P}\left(M_{u-h,u}(\dot{I}) \ge 1, \widetilde{M}_{u-h,u}(\partial I) \ge 1\right)$$
$$\le \mathrm{E}\left(M_{u-h,u}(I_{-\delta})\widetilde{M}_{u-h,u}(\partial I)\right) + E\left(M_{u-h,u}(I \setminus I_{-\delta})\right) \quad (19)$$

The first term in the right-hand member of (19) can be computed by means of a Rice-type Formula, and it can be expressed as:

$$\int_{I_{-\delta} \times \partial I} \sigma(dt)\widetilde{\sigma}(d\widetilde{t}) \iint_{u-h}^{u} dx d\widetilde{x}$$
$$\mathrm{E}\left(\delta^1(X''(t))\delta^1(\widetilde{X}''(\widetilde{t}))/X(t) = x, \widetilde{X}(\widetilde{t}) = \widetilde{x}, X'(t) = 0, \widetilde{X}'(\widetilde{t}) = 0\right)$$
$$p_{X(t),\widetilde{X}(\widetilde{t}),X'(t),\widetilde{X}'(\widetilde{t})}(x, \widetilde{x}, 0, 0),$$

where the function δ^1 has been defined in Theorem 2.3.

Since in this integral $\left\|t - \widetilde{t}\right\| \geq \delta$, the integrand is bounded and the integral is $O(h^2)$.

For the second term in (19) we apply Rice formula again. Taking into account that the boundary of I is smooth and compact, we get:

$$\begin{aligned}
&\mathrm{E}\left(M_{u-h,u}(I \setminus I_{-\delta})\right\} \\
&\quad = \int_{I\setminus I_{-\delta}} \sigma(dt) \int_{u-h}^{u} \mathrm{E}\left(\delta^1(X''(t))/X(t) = x, X'(t) = 0\right) p_{X(t),X'(t)}(x,0)\ dx \\
&\qquad\qquad \leq (const)\ h\ \sigma(I \setminus I_{-\delta}) \leq (const)\ h\delta.,
\end{aligned}$$

where the constant does not depend on h and δ. Since $\delta > 0$ can be chosen arbitrarily small, (18) follows and we may write:

$$\begin{aligned}
&F_I(u) - F_I(u-h) \\
&\quad = \mathrm{P}\left(M_I \leq u, M_{u-h,u}(\dot{I}) \geq 1\right) + \mathrm{P}\left(M_I \leq u, \widetilde{M}_{u-h,u}(\partial I) \geq 1\right) + o(h)
\end{aligned}$$

as $h \to 0$.

Note that the foregoing argument also implies that F_I is absolutely continuous with respect to Lebesgue measure and that the density is bounded above by the right-hand member of (17). In fact:

$$\begin{aligned}
F_I(u) - F_I(u-h) &\leq \mathrm{P}\left(M_{u-h,u}(\dot{I}) \geq 1\right) + \mathrm{P}\left(\widetilde{M}_{u-h,u}(\partial I) \geq 1\right) \\
&\leq \mathrm{E}\left(M_{u-h,u}(\dot{I})\right) + E\left(\widetilde{M}_{u-h,u}(\partial I)\right)
\end{aligned}$$

and it is enough to apply Rice Formula to each one of the expectations on the right-hand side.

The delicate part of the proof consists in showing that we have equality in (17).

Step 2. For $g : I \to \mathcal{R}$ we put

$$\|g\|_\infty = \sup_{t \in I} |g(t)|$$

and if k is a non-negative integer,

$$\|g\|_{\infty,k} = \sup_{k_1+k_2+..+k_d \leq k} \|\partial_{k_1,k_2...,k_d} g\|_\infty .$$

For fixed $\gamma > 0$ (to be chosen later on) and $h > 0$,we denote by E_h the event:

$$E_h = \left\{ \|X\|_{\infty,4} \leq h^{-\gamma} \right\}$$

Because of the Landau-Shepp-Fernique inequality (see Landau-Shepp, 1970 or Fernique, 1975) there exist positive constants C_1, C_2 such that

$$\mathrm{P}(E_h^C) \leq C_1 \exp\left[-C_2 h^{-2\gamma}\right] = o(h) \text{ as } h \to 0$$

so that to have (17) it suffices to show that, as $h \to 0$:

$$\mathrm{E}\left(\left[M_{u-h,u}(\dot{I}) - 1\!\!1_{M_{u-h,u}(\dot{I}) \geq 1} \right] 1\!\!1_{M_I \leq u} 1\!\!1_{E_h} \right) = o(h) \tag{20}$$

$$\mathrm{E}\left(\left[\widetilde{M}_{u-h,u}(\partial I) - 1\!\!1_{\widetilde{M}_{u-h,u}(\partial I) \geq 1} \right] 1\!\!1_{M_I \leq u} 1\!\!1_{E_h} \right) = o(h) \tag{21}$$

We prove (20). (21) can be proved in a similar way.

Put $M_{u-h,u} = M_{u-h,u}(\dot{I})$. We have:

$$\mathrm{E}\left(\left[M_{u-h,u} - \ 1\!\!1_{M_{u-h,u} \geq 1} \right] \ 1\!\!1_{M_I \leq u} \ 1\!\!1_{E_h} \right) \leq \mathrm{E}\left(M_{u-h,u}(M_{u-h,u} - 1) \ 1\!\!1_{E_h} \right)$$
$$= \iint_{I \times I} \sigma(s) \ \sigma(t) \iint_{u-h}^{u} \ dx_1 dx_2$$
$$\mathrm{E}\left(\delta^1(X''(s)) \delta^1(X''(t)) \ 1\!\!1_{E_h} / X(s) = x_1, X(t) = x_2, X'(s) = 0, X'(t) = 0 \right)$$
$$\cdot p_{X(s),X(t),X'(s),X'(t)}(x_1, x_2, 0, 0), \tag{22}$$

on applying Rice formula for the second factorial moment.

Our goal is to prove that the integrand in the right member of (22), that is:

$$A_{s,t} = \iint_{u-h}^{u} \ dx_1 dx_2$$
$$\mathrm{E}\left(|\det(X''(s) \det(X''(t)| \ 1\!\!1_{X''(s) \prec 0, X''(t) \prec 0} \ 1\!\!1_{E_h} / X(s) = x_1, X(t) = x_2, X'(s) = 0, X'(t) = 0 \right)$$
$$\cdot p_{X(s),X(t),X'(s),X'(t)}(x_1, x_2, 0, 0), \tag{23}$$

is $o(h)$ as $h \downarrow 0$ uniformly on s, t. Note that when s, t vary in a domain of the form $D_\delta := \{t, s \in I : \|t - s\| > \delta\}$ for some $\delta > 0$, then the Gaussian distribution in (23) is non-degenerate and $A_{s,t}$ is bounded by $(const)h^2$, the constant depending on the minimum of the determinant:

$$\det \mathrm{Var}\big((X(s), X(t), X'(s), X'(t)\big),$$

for $s,t \in D_\delta$.

So it is enough to prove that $A_{s,t} = o(h)$ for $\|t-s\|$ small, and we may assume that s and t are in the same chart (U,ϕ). Writing the process in this chart we may assume that I is a ball or a half ball in $\mathcal{R}^d$. Let s,t two such points, define the process $Y = Y^{s,t}$ by

$$Y(\tau) = X\big(s + \tau(t-s)\big) \quad ; \quad \tau \in [0,1].$$

Under the conditioning one has:

$$Y(0) = x_1, \quad Y(1) = x_2, \quad Y'(0) = Y'(1) = 0$$

$$Y''(0) = X''(s)[(t-s),(t-s)] \;\; ; \;\; Y''(1) = X''(t)[(t-s),(t-s)].$$

Consider the interpolation polynomial Q of degree 3 such that

$$Q(0) = x_1, \quad Q(1) = x_2, \quad Q'(0) = Q'(1) = 0$$

Check that

$$Q(y) = x_1 + (x_2 - x_1)\; y^2(3-2y),\; Q''(0) = -Q''(1) = 6(x_2 - x_1)$$

Denote

$$Z(\tau) = Y(\tau) - Q(\tau) \quad 0 \le \tau \le 1.$$

Under the conditioning, one has:

$$Z(0) = Z(1) = Z'(0) = Z'(1) = 0$$

and if also the event E_h occurs, an elementary calculation shows that for $0 \le \tau \le 1$:

$$|Z''(\tau)| \le \sup_{\tau\in[0,1]} \frac{|Z^{(4)}(\tau)|}{2!} = \sup_{\tau\in[0,1]} \frac{|Y^{(4)}(\tau)|}{2!} \le (const)\|t-s\|^4 h^{-\gamma}. \qquad (24)$$

On the other hand, check that if A is a positive semi-definite symmetric $d \times d$ real matrix and v_1 is a vector of Euclidean norm equal to 1, then the inequality

$$\det(A) \le \langle Av_1, v_1\rangle \;\; \det(B) \qquad (25)$$

holds true, where B is the $(d-1)\times(d-1)$ matrix

$$B = ((\langle Av_j, v_k\rangle))_{j,k=2,\dots,d}$$

and $\{v_1, v_2, ..., v_d\}$ an orthonormal basis of R^d containing v_1.

Assume $X"(s)$ is negative definite, and that the event E_h occurs. We can apply (25) to the matrix $A = -X"(s)$ and the unit vector

$$v_1 = \frac{t-s}{\|t-s\|}.$$

Note that in that case, the elements of matrix B are of the form $\langle -X"(s)v_j, v_k\rangle$ hence bounded by $(const)h^{-\gamma}$. So,

$$\det\left[-X''(s)\right] \le \langle -X''(s)v_1, v_1\rangle \ C_d \ h^{-(d-1)\gamma} = C_d \ \left[Y''(0)\right]^- \|t-s\|^{-2} h^{-(d-1)\gamma}$$

the constant C_d depending only on the dimension d.

Similarly, if $X"(t)$ is negative definite, and the event E_h occurs, then:

$$\det\left[-X''(t)\right] \le C_d \ \left[Y''(1)\right]^- \|t-s\|^{-2} h^{-(d-1)\gamma}$$

Hence, if $\mathcal{C}$ is the condition $\{X(s) = x_1, X(t) = x_2, X'(s) = 0, X'(t) = 0\}$:

$$\begin{aligned}
& \mathrm{E}\left(|\det(X''(s))\det(X''(t))| \ \mathbb{I}_{X''(s)\prec 0, X''(t)\prec 0} \ \mathbb{I}_{E_h}/\mathcal{C}\right) \\
\le \ & C_d^2 \ h^{-2(d-1)\gamma}\|t-s\|^{-4}\mathrm{E}\left(\left[Y''(0)\right]^- \left[Y''(1)\right]^- \ \mathbb{I}_{E_h}/\mathcal{C}\right) \\
\le \ & C_d^2 \ h^{-2(d-1)\gamma}\|t-s\|^{-4}\mathrm{E}\left(\left[\frac{Y''(0)+Y''(1)}{2}\right]^2 \ \mathbb{I}_{E_h}/\mathcal{C}\right) \\
= \ & C_d^2 \ h^{-2(d-1)\gamma}\|t-s\|^{-4}\mathrm{E}\left(\left[\frac{Z''(0)+Z''(1)}{2}\right]^2 \ \mathbb{I}_{E_h}/\mathcal{C}\right) \\
\le \ & (const) \ C_d^2 \ h^{-2d\gamma} \ \|t-s\|^4
\end{aligned}$$

We now turn to the density in (22) using the following Lemma which is similar to Lemma 4.3., p. 76, in Piterbarg (1996).

Lemma 3.1 *For all $s, t \in I$:*

$$\|t-s\|^{d+3} \ p_{X(s),X(t),X'(s),X'(t)}(0,0,0,0) \le D \tag{26}$$

where D is a constant.

Proof. Assume that (26) does not hold, i.e., that there exist two convergent sequences $\{s_n\}$, $\{t_n\}$ in I , $s_n \to s^*$, $t_n \to t^*$ such that

$$\|t_n - s_n\|^{d+3} \ p_{X(s_n),X(t_n),X'(s_n),X'(t_n)}(0,0,0,0) \to +\infty \tag{27}$$

If $s^* \neq t^*$, (27) can not hold, since the non degeneracy condition assures that this sequence has the finite limit $\|t^* - s^*\|^{d+3} \, p_{X(s^*),X(t^*),X'(s^*),X'(t^*)}(0,0,0,0)$. So, $s^* = t^*$.

Since one can assume with no loss of generality that I is a ball or a half ball, the segment $[s_n, t_n]$ is contained in I. Denote the unit vector $e_{1,n} = \frac{t_n - s_n}{\|t_n - s_n\|}$,complete it to an orthonormal basis $\{e_{1,n}, e_{2,n}, ..., e_{d,n}\}$ of $\mathcal{R}^d$ and take a subsequence of the integers $\{n_k\}$ so that $e_{j,n_k} \to e_j^*$ as $k \to +\infty$ for $j = 1, ..., d$. In what follows, without loss of generality, we assume that $\{n_k\}$ is the sequence of all positive integers. For each $\tau \in \mathcal{R}^d$ we denote $\tau_{1,n}, ..., \tau_{d,n}$ the coordinates of τ in the basis $\{e_{1,n}, ..., e_{d,n}\}$. Note that $t_n - s_n$ has coordinates $(t_{1,n} - s_{1,n}, 0, ..., 0) = (\|t_n - s_n\|, 0, ..., 0)$.

Also, we denote $\tau_1^*, ..., \tau_d^*$ the coordinates of τ in the basis $\{e_1^*, ..., e_d^*\}$

The following computation is similar to the proof of Lemma 3.2. in Azaïs & Wschebor (2001). We have:

$$\begin{aligned}
\Delta_n &= \det \operatorname{Var}\left(X(s_n), X(t_n), X'(s_n), X'(t_n)\right) \\
&= \det \operatorname{Var}\left(X(s_n), X(t_n), \frac{\partial X}{\partial \tau_{1,n}}(s_n), \frac{\partial X}{\partial \tau_{1,n}}(t_n), ..., \frac{\partial X}{\partial \tau_{d,n}}(s_n), \frac{\partial X}{\partial \tau_{d,n}}(t_n)\right) \\
&= \det \operatorname{Var}\left(X(s_n), \frac{\partial X}{\partial \tau_{1,n}}(s_n), Y_{1,n}, Z_{1,n}, \frac{\partial X}{\partial \tau_{2,n}}(s_n), Z_{2,n}, ..., \frac{\partial X}{\partial \tau_{d,n}}(s_n), Z_{d,n}\right)
\end{aligned}$$

where

$$Y_{1,n} = X(t_n) - X(s_n) - \frac{\partial X}{\partial \tau_{1,n}}(s_n)(t_{1,n} - s_{1,n})$$

$$Z_{1,n} = \frac{\partial X}{\partial \tau_{1,n}}(t_n) - \frac{\partial X}{\partial \tau_{1,n}}(s_n) - \frac{2}{t_{1,n} - s_{1,n}} Y_{1,n}$$

$$Z_{2,n} = \frac{\partial X}{\partial \tau_{2,n}}(t_n) - \frac{\partial X}{\partial \tau_{2,n}}(s_n), ..., Z_{d,n} = \frac{\partial X}{\partial \tau_{d,n}}(t_n) - \frac{\partial X}{\partial \tau_{d,n}}(s_n)$$

Using now Taylor expansions and taking into account the integrability of the supremum of bounded Gaussian process, we have:

$$Y_{1,n} = \frac{(t_{1,n} - s_{1,n})^2}{2} \frac{\partial^2 X}{\partial \tau_{1,n}^2}(s_n) + \alpha_{1,n}(t_{1,n} - s_{1,n})^3$$

$$Z_{1,n} = \frac{(t_{1,n} - s_{1,n})^2}{6} \frac{\partial^3 X}{\partial \tau_{1,n}^3}(s_n) + \beta_n (t_{1,n} - s_{1,n})^3$$

$$\begin{aligned}
Z_{2,n} &= (t_{1,n} - s_{1,n}) \frac{\partial^2 X}{\partial \tau_{2,n} \partial \tau_{1,n}}(s_n) + \alpha_{2,n}(t_{1,n} - s_{1,n})^2, ..., \\
Z_{d,n} &= (t_{1,n} - s_{1,n}) \frac{\partial^2 X}{\partial \tau_{d,n} \partial \tau_{1,n}}(s_n) + \alpha_{d,n}(t_{1,n} - s_{1,n})^2
\end{aligned}$$

where the random variables $\alpha_{1,n}, \alpha_{2,n}, ..., \alpha_{d,n}, \beta_n$ are uniformly bounded in L^2 of the underlying probability space.

Substituting into Δ_n it follows that:

$$144\ (t_{1,n} - s_{1,n})^{-[8+2(d-1)]}\Delta_n \rightarrow$$
$$\det \operatorname{Var}\left(X(s^*), \frac{\partial X}{\partial \tau_1^*}(s^*), ..., \frac{\partial^3 X}{\partial\left(\tau_1^*\right)^3}(s^*), \frac{\partial X}{\partial \tau_2^*}(s^*), \frac{\partial^2 X}{\partial \tau_2^* \partial \tau_1^*}(s^*), ..., \frac{\partial X}{\partial \tau_d^*}(s^*), \frac{\partial^2 X}{\partial \tau_d^* \partial \tau_1^*}(s^*) \right)$$

and this limit is bounded below by a positive constant, independent of s^*, because of the non-degeneracy assumption. Since $t_{1,n} - s_{1,n} = \|t_n - s_n\|$, this contradicts (27) and finishes the proof of the Lemma. □

Returning to the proof of Theorem 3.1.

To bound the expression in (22) we use Lemma 3.1 and the bound on the conditional expectation, thus obtaining

$$\begin{aligned} & \mathrm{E}\left(M_{u-h,u}(M_{u-h,u} - 1)\mathbb{1}_{E_h}\right) \\ \leq\ & (const) C_d^2\ h^{-2d\gamma} D \iint_{I\times I} \|t-s\|^{-d+1}\, ds\ dt \iint_{u-h}^{u} dx_1 dx_2 \\ \leq\ & (const)\ h^{2-2d\gamma} \end{aligned}$$

since the function $(s,t) \rightsquigarrow \|t-s\|^{-d+1}$ is Lebesgue-integrable in $I \times I$. The last constant depends only on the dimension d and the set I, Taking γ small enough (20) follows. □

An example: Let $\{X(s,t)\}$ be a real-valued two-parameter Gaussian, centred stationary isotropic process with covariance Γ. Assume that its spectral measure μ is absolutely continuous with density

$$\mu(ds, dt) = f(\rho) ds dt, \quad \rho = (s^2 + t^2)^{\frac{1}{2}}.$$

So that

$$2\pi \int_0^{+\infty} \rho f(\rho) d\rho = 1.$$

Assume further that $J_k = \int_0^{+\infty} \rho^k f(\rho) d\rho < \infty$, for $1 \leq k \leq 5$. Our aim is to give an explicit upper bound for the density of the probability distribution of M_I where I is the unit disc i. e.

$$I = \{(s,t) : s^2 + t^2 \leq 1\}$$

Using (15) which is a consequence of Theorem 3.1 and the invariance of the law of the process, we have

$$F_I'(u) \leq \pi \mathrm{E}\Big(\delta^1(X''(0,0))/X(0,0) = u, X'(0,0) = (0,0)\Big) p_{X(0,0),X'(0,0)}(u,(0,0))$$
$$+ 2\pi \mathrm{E}\Big(\delta^1(\tilde{X}''(1,0))/X(1,0) = u, \tilde{X}'(1,0) = 0\Big) p_{X(1,0),\tilde{X}'(1,0)}(u,0) = I_1 + I_2. \quad (28)$$

We denote by X, X', X'' the value of the different processes at some point (s,t); by $X''_{ss}, X''_{st}, X''_{tt}$ the entries of the matrix X'' and by φ and Φ the standard normal density and distribution.

One can easily check that:

- X' is independent of X and X'', and has variance $\pi J_3 I_d$
- X''_{st} is independent of X, X' X''_{ss} and X''_{tt}, and has variance $\frac{\pi}{4} J_5$
- Conditionally on $X = u$, the random variables X''_{ss} and X''_{tt} have
  - expectation: $-\pi J_3$
  - variance: $\frac{3\pi}{4} J_5 - (\pi J_3)^2$
  - covariance: $\frac{\pi}{4} J_5 - (\pi J_3)^2$.

Using an elementary computation we get that the expectation of the negative part of a Gaussian variable with expectation μ and variance σ^2 is equal to

$$\sigma\varphi(\frac{\mu}{\sigma}) - \mu\Phi(\frac{-\mu}{\sigma}).$$

We obtain

$$I_2 = \sqrt{\frac{2}{J_3}}\varphi(u) \ \Big[\big(\frac{3\pi}{4}J_5 - (\pi J_3)^2\big)^{\frac{1}{2}}\varphi(bu) + \pi J_3 u \Phi(bu)\Big],$$

with

$$b = \frac{\pi J_3}{\big(\frac{3\pi}{4}J_5 - (\pi J_3)^2\big)^{\frac{1}{2}}}.$$

As for I_1 we remark that, conditionally on $X = u$, $X''_{ss} + X''_{tt}$ and $X''_{ss} - X''_{tt}$ are independent, so that a direct computation gives:

$$I_1 = \frac{1}{8\pi J_3}\varphi(u)\mathrm{E}\Big[\big(\alpha\eta_1 - 2\pi J_3 u\big)^2 - \frac{\pi J_5}{4}(\eta_2^2 + \eta_3^2)$$
$$\mathbb{1}_{\{\alpha\eta_1 < 2\pi J_3 u\}} \ \mathbb{1}_{\{\big(\alpha\eta_1 - 2\pi J_3 u\big)^2 - \frac{\pi J_5}{4}(\eta_2^2+\eta_3^2) > 0\}}\Big], \quad (29)$$

Where η_1, η_2, η_3 are standard independent normal random variables and $\alpha^2 = 2\pi J_5 - 4\pi^2 J_3^2$. Finally we get

$$I_1 = \frac{\sqrt{2\pi}}{8\pi J_3}\varphi(u)\int_0^\infty \Big[(\alpha^2+a^2-c^2x^2)\Phi(a-cx)+[2a\alpha-\alpha^2(a-cx)]\varphi(a-cx)\Big]x\varphi(x)dx,$$

with $a = 2\pi J_3 u, \;\; c = \sqrt{\frac{\pi J_5}{4}}$.

4 First derivative, second form

We choose, once for all along this section a finite atlas $\mathcal{A}$ for I. Then, to every $t \in I$ it is possible to associate a fixed chart that will be denoted (U_t, ϕ_t). When $t \in \partial I$, $\phi_t(U_t)$ can be chosen to be a half ball with $\phi_t(t)$ belonging to the hyperplane limiting this half ball. For $t \in I$, let V_t an open neighbourhood of t whose closure is included in U_t and ψ_t a $\mathcal{C}^\infty$ function such that

$$\psi_t \equiv 1 \quad \text{on} \quad V_t \tag{30}$$
$$\psi_t \equiv 0 \quad \text{on} \quad U_t^c \tag{31}$$

- For every $t \in \dot{I}$ and $s \in I$ we define the normalization $n(t,s)$ in the following way:
  - for $s \in V_t$, we set "in the chart" (U_t, ϕ_t)

    $$n_1(t,s) = \frac{1}{2}\|s-t\|^2. \tag{32}$$

    By "in the chart" we mean that $\|s-t\|$, is in fact $\|\phi_t(t) - \phi_t(s)\|$.
  - for general s we set

    $$n(t,s) = \psi_t(s)n_1(t,s) + \big(1 - \psi_t(s)\big)$$

  Note that in the flat case (d=N) the simpler definition $n(t,s) = \frac{1}{2}\|s-t\|^2$ works.

- For every $t \in \partial I$ and $s \in I$, we set instead of formula (32)

  $$n_1(t,s) = |(s-t)_N| + \frac{1}{2}\|s-t\|^2.$$

  where $(s-t)_N$ is the normal component of $(s-t)$ with respect to the hyperplane delimiting the half ball $\phi_t(U_t)$. The rest of the definition is the same.

Definition 4.1 *We will say that f is an helix-function - or an h-function - on I with pole $t \in I$ satisfying hypothesis $H_{t,k}$, k integer $k > 1$ if*

- *f is a bounded $\mathcal{C}^k$ function on $I \backslash \{t\}$.*
- *$\underline{f}(s) := n(t,s)f(s)$ can be prolonged as function of class $\mathcal{C}^k$ on I.*

Definition 4.2 *In the same way X is called an h-process with pole $t \in I$ satisfying hypothesis $H_{t,k}$, k integer $k > 1$ if*

- *Z is a Gaussian process with $\mathcal{C}^k$ paths on $I \backslash \{t\}$.*
- *for $t \in \dot{I}$; $\underline{Z}(s) := n(t,s)Z(s)$ can be prolonged as a process of class $\mathcal{C}^k$ on I, with $\underline{Z}(t) = 0$ $\underline{Z}'(t) = 0$. If $s_1, ..., s_m$ are pairwise different points of $I \backslash \{t\}$ then the distribution of*

$$\underline{Z}^{(2)}(t), ..., \underline{Z}^{(k)}(t), \underline{Z}(s_1), ..., \underline{Z}^{(k)}(s_1), ..., \underline{Z}^{(k)}(s_m)$$

  does not degenerate.

- *for $t \in \partial I$; $\underline{Z}(s) := n(t,s)Z(s)$ can be prolonged as a process of class $\mathcal{C}^k$ on I with $\underline{Z}(t) = 0$ $\tilde{\underline{Z}}'(t) = 0$ and if $s_1, ..., s_m$ are pairwise different points of $I \backslash \{t\}$ then the distribution of*

$$\underline{Z}'_N(t), \underline{Z}^{(2)}(t), ..., \underline{Z}^{(k)}(t), \underline{Z}(s_1), ..., \underline{Z}^{(k)}(s_1), ..., \underline{Z}^{(k)}(s_m)$$

  does not degenerate. $\underline{Z}'_N(t)$ is the derivative normal to the boundary of I at t.

We use the terms "h-function" and "h-process" since the function and the paths of the process need not to extend to a continuous function at the point t. However, the definition implies the existence of radial limits at t. So the process may take the form of a helix around t.

Lemma 4.1 *Let X be a process satisfying H_k, $k \geq 2$, and f be a $\mathcal{C}^k$ function $I \to \mathcal{R}$*
(A) For $t \in \dot{I}$, set for $s \in I, s \neq t$

$$X(s) = a_s^t X(t) + < b_s^t, X'(t) > + n(t,s) X^t(s),$$

where a_s^t and b_s^t are the regression coefficients.
In the same way, set

$$f(s) = a_s^t f(t) + < b_s^t, f'(t) > + n(t,s) f^t(s),$$

using the regression coefficients associated to X.

(B) For $t \in \partial I$, $s \in T, s \neq t$ set

$$X(s) = \tilde{a}_s^t X(t) + < \tilde{b}_s^t, \tilde{X}'(t) > + n(t,s) X^t(s)$$

and

$$f(s) = \tilde{a}_s^t f(t) + < \tilde{b}_s^t, \tilde{f}'(t) > + n(t,s) f^t(s),$$

Then $s \rightsquigarrow X^t(s)$ and $s \rightsquigarrow f^t(s)$ are respectively a h-process and a h-function with pole t satisfying $H_{t,k}$.

Proof: We give the proof in the case $t \in \dot{I}$, the other one being similar. In fact, the quantity denoted by $\underline{X}^t(s)$ is just $X(s) - a_s^t X(t) - < b_s^t, X'(t) >$. On $L^2(\Omega, P)$, let Π be the projector on the orthogonal complement to the subspace generated by $X(t), X'(t)$. Using a Taylor expansion

$$X(s) = X(t) + < (s-t), X'(t) > + \|t-s\|^2 \int_0^1 X''\big((1-\alpha)t + \alpha s\big)\big[v,v\big](1-\alpha)d\alpha,$$

With $v = \frac{s-t}{\|s-t\|}$. This implies that

$$\underline{X}^t(s) = \Pi\Big[\|t-s\|^2 \int_0^1 X''\big((1-\alpha)t + \alpha s\big)\big[v,v\big](1-\alpha)d\alpha\Big], \tag{33}$$

which gives the result due to the non degeneracy condition. □

We state now an extension of Ylvisaker's Theorem (1968) on the regularity of the distribution of the maximum of a Gaussian process which we will use in the proof of Theorem 4.2 and might have some interests in itself.

Theorem 4.1 *Let $Z : T \longrightarrow \mathcal{R}$ a Gaussian separable process on some parameter set T and denote by $M^Z = \sup_{t \in T} Z(t)$ which is a random variable taking values in $\mathcal{R} \cup \{+\infty\}$. Assume that there exists $\sigma_0 > 0$, $m_- > -\infty$ such that*

$$m(t) = \mathrm{E}(Z_t) \geq m_-$$

$$\sigma^2(t) = \mathrm{Var}(Z_t) \geq \sigma_0^2$$

for every $t \in T$. Then the distribution of the random variable M^Z is the sum of an atom at $+\infty$ and a-possibly defective-probability measure on $\mathcal{R}$ which has a locally bounded density.

Proof: Suppose first that $X : T \longrightarrow \mathcal{R}$ a Gaussian separable process satisfying

$$\mathrm{Var}(X_t) = 1 \ ; \ \mathrm{E}(X_t) \geq 0,$$

for every $t \in T$. A close look at Ylvisaker's proof (1968) shows that the distribution of the supremum M^X has a density p_{M^X} that satisfies

$$p_{M^X}(u) \leq \psi(u) = \frac{\exp(-u^2/2)}{\int_u^\infty \exp(-v^2/2)dv} \text{ for every u} \in \mathcal{R} \tag{34}$$

Let now Z satisfy the hypotheses of the theorem. For given $a, b \in \mathcal{R}, a < b$, choose $A \in \mathcal{R}^+$ so that $|a| < A$ and consider the process:

$$X(t) = \frac{Z(t) - a}{\sigma(t)} + \frac{|m_-| + A}{\sigma_0}.$$

Clearly for every $t \in T$:

$$\mathrm{E}\big(X(t)\big) = \frac{m(t) - a}{\sigma(t)} + \frac{|m_-| + A}{\sigma_0} \geq -\frac{|m_-| + |a|}{\sigma_0} + \frac{|m_-| + A}{\sigma_0} \geq 0,$$

and

$$\mathrm{Var}\big(X(t)\big) = 1.$$

So that (34) holds for the process X.

On the other hand:

$$\{a < M^Z \leq b\} \subset \{\frac{|m_-| + A}{\sigma_0} < M^X \leq \frac{|m_-| + A}{\sigma_0} + \frac{b-a}{\sigma_0}\}.$$

And it follows that

$$\mathrm{P}\big\{a < M^Z \leq b\big\} \leq \int_{\frac{|m_-|+A}{\sigma_0}}^{\frac{|m_-|+A}{\sigma_0}+\frac{b-a}{\sigma_0}} \psi(u)du = \int_a^b \frac{1}{\sigma_0}\psi\Big(\frac{v - a + |m_-| + A}{\sigma_0}\Big)dv.$$

which shows the statement. □

Set now $\beta(t) \equiv 1$. The key point is that, due to regression formulae, under the condition $\big\{X(t) = u, X'(t) = 0\big\}$ the event

$$A_u(X, \beta) := \big\{X(s) \leq u, \forall s \in I\big\}$$

coincides with the event

$$A_u(X^t, \beta^t) := \big\{X^t(s) \leq \beta^t(s)u, \forall s \in I \backslash \{t\}\big\},$$

where X^t and β^t are the h-process and the h-function defined in Lemma 4.1.

Theorem 4.2 (First derivative, second form) *Let $X : I \to \mathcal{R}$ be a Gaussian process, I a C^∞compact manifold contained in $\mathcal{R}^d$.*

Assume that X has paths of class C^2 and for $s \neq t$ the triplet $(X(s), X(t), X'(t))$ in $\mathcal{R} \times \mathcal{R} \times \mathcal{R}^d$ has a non-degenerate distribution.

Then, the result of Theorem 3.1 is valid, the derivative $F'_I(u)$ given by relation (17) can be written as

$$F'_I(u) = \big(-1\big)^d \int_I \mathrm{E}\Big[\det\big(\underline{X}^{t\prime\prime}(t) - \underline{\beta}^{t\prime\prime}(t)u\big)\ \mathbb{I}_{A_u(X^t,\beta^t)}\Big] p_{X(t),X'(t)}(u,0)\sigma(dt)$$
$$+ \big(-1\big)^{d-1} \int_{\partial I} \mathrm{E}\Big[\det\big(\tilde{\underline{X}}^{t\prime\prime}(t)\big) - \tilde{\underline{\beta}}^{t\prime\prime}(t)u\ \mathbb{I}_{A_u(X^t,\beta^t)}\Big] p_{X(t),\tilde{X}'(t)}(u,0)\tilde{\sigma}(dt), \quad (35)$$

and this expression is continuous as a function of u.

The notation $\tilde{\underline{X}}^{t\prime\prime}(t)$ should be understood in the sense that we first define $\underline{X}^t$ and then calculate its second derivative along ∂I.

Proof: As a first step, assume that the process X satisfies the hypotheses of theorem 3.1, which are stronger that those in the present theorem.

We prove that the first term in (17) can be rewritten as the first term in (35). One can proceed in a similar way with the second term, mutatis mutandis. For that purpose, use the remark just before the statement of Theorem 4.2 and the fact that under the condition

$$\big\{X(t) = u, X'(t) = 0\big\}$$

, $X''(t)$ is equal to

$$\underline{X}^{t\prime\prime}(t) - \underline{\beta}^{t\prime\prime}(t)u.$$

Replacing in the conditional expectation in (17) and on account of the Gaussianity of the process, we get rid of the conditioning and obtain the first term in (35).

We now study the continuity of $u \rightsquigarrow F'_I(u)$. The variable u appears at three locations

- in the density $p_{X(t),X'(t)}(u,0)$ which is clearly continuous
- in
$$\mathrm{E}\Big[\det\big(\underline{X}^{t\prime\prime}(t) - \underline{\beta}^{t\prime\prime}(t)u\big)\ \mathbb{I}_{A_u(X^t,\beta^t)}\Big]$$
  where it occurs twice: in the first factor and in the indicator function.

Due to the integrability of the supremum of bounded Gaussian processes, it is easy to prove that this expression is continuous as a function of the first u.

As for the u in the indicator function, set

$$\xi_v := \det\left(\underline{X}^{t\prime\prime}(t) - \underline{\beta}^{t\prime\prime}(t)v\right) \tag{36}$$

and, for $h > 0$, consider the quantity

$$\mathrm{E}\Big[\xi_v \ \mathbb{1}_{A_u(X^t,\beta^t)}\Big] - \mathrm{E}\Big[\xi_v \ \mathbb{1}_{A_{u-h}(X^t,\beta^t)}\Big]$$

which is equal to

$$\mathrm{E}\Big[\xi_v \ \mathbb{1}_{A_u(X^t,\beta^t)\backslash A_{u-h}(X^t,\beta^t)}\Big] - \mathrm{E}\Big[\xi_v \ \mathbb{1}_{A_{u-h}(X^t,\beta^t)\backslash A_u(X^t,\beta^t)}\Big] \tag{37}$$

Apply Schwarz's inequality to the first term in (37).

$$\mathrm{E}\Big[\xi_v \ \mathbb{1}_{A_u(X^t,\beta^t)\backslash A_{u-h}(X^t,\beta^t)}\Big] \leq \big[\mathrm{E}(\xi_v^2)\mathrm{P}\{A_u(X^t,\beta^t)\backslash A_{u-h}(X^t,\beta^t)\}\big]^{1/2}$$

The event $A_u(X^t,\beta^t)\backslash A_{u-h}(X^t,\beta^t)$ can be described as

$$\forall s \in I\backslash\{t\} : \ X^t(s) - \beta^t(s)u \leq 0 \ ; \ \exists s_0 \in I\backslash\{t\} : \ X^t(s_0) - \beta^t(s_0)(u-h) > 0$$

This implies that $\beta^t(s_0) > 0$ and that

$$-\|\beta^t\|_\infty h \leq \sup_{s\in I\backslash\{t\}} X^t(s) - \beta^t(s)u \leq 0.$$

Now, observe that our improved version of Ylvisaker's theorem (Theorem 4.1),applies to the process $s \rightsquigarrow X^t(s) - \beta^t(s)u$ defined on $I\backslash\{t\}$. This implies that the first term in (37) tends to zero as $h \downarrow 0$. An analogous argument applies to the second term. Finally, the continuity of $F_I'(u)$ follows from the fact that one can pass to the limit under the integral sign in (35).

To finish the proof we still have to show that the added hypotheses are in fact unnecessary for the validity of the conclusion. Suppose now that the process X satisfies only the hypotheses of the theorem and define

$$X^\epsilon(t) = Z_\epsilon(t) + \epsilon\, Y(t) \tag{38}$$

where for each $\epsilon > 0$, Z_ϵ is a real-valued Gaussian process defined on I, measurable with respect to the σ-algebra generated by $\{X(t) : t \in I\}$, possessing $\mathcal{C}^\infty$

paths and such that almost surely $Z_\epsilon(t),\ Z'_\epsilon(t),\ Z''_\epsilon(t)$ converge uniformly on I to $X(t), X'(t), X''(t)$ respectively $as\epsilon \downarrow 0$. One standard form to construct such an approximation process Z_ϵ is to use a $\mathcal{C}^\infty$ partition of the unity on I and to approximate locally the composition of a chart with the function X by means of a convolution with a $\mathcal{C}^\infty$ kernel.

In (38), Y denotes the restriction to I of a Gaussian centred stationary process satisfying the hypotheses of proposition 3.1, defined on $\mathcal{R}^N$, and independent of X. Clearly X^ϵ satisfies condition (H_k) for every k, since it has $\mathcal{C}^\infty$ paths and the independence of both terms in (38) ensures that X^ϵ inherits from Y the non-degeneracy condition in Definition 3.1. So, if

$$M_I^\epsilon = \max_{t\in I} X^\epsilon(t) \text{ and } F_I^\epsilon(u) = \mathrm{P}\{M_I^\epsilon \le u\}$$

one has

$$\begin{aligned} F_I^{\epsilon\prime}(u) = \big(-1\big)^d \int_I \mathrm{E}\Big[\det\big(\underline{X}^{\epsilon t\prime\prime}(t) - \underline{\beta}^{\epsilon t\prime\prime}(t)u\big)\ \mathbb{I}_{A_u(X^{\epsilon t},\beta^{\epsilon,t})}\Big] p_{X^\epsilon(t),X^{\epsilon\prime}(t)}(u,0)\sigma(dt) \\ + \big(-1\big)^{d-1} \int_{\partial I} \mathrm{E}\Big[\det\big(\tilde{\underline{X}}^{\epsilon t\prime\prime}(t)\big) - \tilde{\underline{\beta}}^{\epsilon t\prime\prime}(t)u\ \mathbb{I}_{A_u(X^{\epsilon t},\beta^{\epsilon t})}\Big] p_{X^\epsilon(t),\tilde{X}^{\epsilon\prime}(t)}(u,0)\tilde{\sigma}(dt), \quad (39) \end{aligned}$$

We want to pass to the limit as $\epsilon \downarrow 0$ in (39). We prove that the right-hand member is bounded if ϵ is small enough and converges to a continuous function of u as $\epsilon \downarrow 0$. Since $M_I^\epsilon \to M_I$, this implies that the limit is continuous and coincides with $F_I'(u)$ by a standard argument on convergence of densities. We consider only the first term in (39), the second is similar.

The convergence of X_ϵ and its first and second derivative, together with the non-degeneracy hypothesis imply that uniformly on $t \in I$, as $\epsilon \downarrow 0$:

$$p_{X^\epsilon(t),X^{\epsilon\prime}(t)}(u,0) \to p_{X(t),X'(t)}(u,0).$$

The same kind of argument can be used for

$$\det\big(\underline{X}^{\epsilon t\prime\prime}(t) - \underline{\beta}^{\epsilon t\prime\prime}(t)u\big),$$

on account of the form of the regression coefficients and the definitions of $\underline{X}^t$ and $\underline{\beta}^t$. The only difficulty is to prove that, for fixed u:

$$\mathrm{P}\{C_\epsilon \Delta C\} \to 0 \text{ as } \epsilon \downarrow 0, \quad (40)$$

where

$$C_\epsilon = A_u(X^{\epsilon t}, \beta^{\epsilon t})$$

$$C = A_u(X^t, \beta^t)$$

We prove that

$$\text{a. s.} \quad \mathbb{1}_{C_\epsilon} \to \mathbb{1}_C \text{ as } \epsilon \downarrow 0, \tag{41}$$

which implies (40).

First of all, note that the event

$$L = \Big\{ \sup_{s \in I \setminus \{t\}} \big(X^t(s) - \beta^t(s) u \big) = 0 \Big\}$$

has zero probability, as already mentioned.

Second, from the definition of $X^t(s)$ and the hypothesis, it follows that , as $\epsilon \downarrow 0$, $X^{\epsilon,t}(s), \beta^{\epsilon,t}(s)$ converge to $X^t(s), \beta^t(s)$ uniformly on $I \setminus \{t\}$. Now, if $\omega \notin C$, there exists $\bar{s} = \bar{s}(\omega) \in I \setminus \{t\}$ such that

$$X^t(\bar{s}) - \beta^t(\bar{s}) u > 0$$

and for $\epsilon > 0$ small enough, one has

$$X^{\epsilon t}(\bar{s}) - \beta^{\epsilon t}(\bar{s}) u > 0,$$

which implies that $\omega \notin C_\epsilon$.

On the other hand, let $\omega \in C \setminus L$. This implies that

$$\sup_{s \in I \setminus \{t\}} \big(X^t(s) - \beta^t(s) u \big) < 0.$$

From the above mentioned uniform convergence, it follows that if $\epsilon > 0$ is small enough, then

$$\sup_{s \in I \setminus \{t\}} \big(X^{\epsilon t}(s) - \beta^{\epsilon t}(s) u \big) < 0,$$

hence $\omega \in C_\epsilon$. (41) follows.

So, we have proved that the limit as $\epsilon \downarrow 0$ of the first term in (39) is equal to the first term in (35). Since,if $\epsilon > 0$ is small enough the integrand is bounded for $t \in I$ and u in a compact interval of the real line.

It remains only to prove that the first term in (35) is a continuous function of u. For this purpose, it suffices to show that the function

$$u \rightsquigarrow \mathrm{P}\{A_u(X^t, \beta^t)\}.$$

is continuous. This is a consequence of the inequality

$$\Big|\mathrm{P}\{A_{u+h}(X^t,\beta^t)\}-\mathrm{P}\{A_u(X^t,\beta^t)\}\Big| \le \mathrm{P}\Big\{\big|\sup_{s\in I\backslash\{t\}}\big(X^t(s)-\beta^t(s)u\big)\big| \le |h| \sup_{s\in I\backslash\{t\}}|\beta^t(s)|\Big\}$$

and of Theorem 4.1, applied once again to the process $s \rightsquigarrow X^t(s) - \beta^t(s)u$ defined on $I\backslash\{t\}$.

5 Second derivative

Theorem 5.1 *Suppose now that I is a d-dimensional smooth manifold without boundary and that the process X satisfies hypothesis H_4 then the distribution of M_I admits a density which is absolutely continuous and its derivative satisfies :*

$$F_I''(u) = \big(-1\big)^d \int_I \mathrm{E}\Big[\det\big(\underline{X}^{t\prime\prime}(t) - \underline{\beta}^{t\prime\prime}(t)u\big)\ \mathbb{1}_{A_u}\Big] p^{(1,0)}_{X(t),X'(t)}(u,0)dt$$
$$-\big(-1\big)^d \int_I \mathrm{E}\Big(\sum_{i,j=1}^{n}\big(\underline{\beta}^{t\prime\prime}(t)\big)_{i,j} C_{i,j}(u)\ \mathbb{1}_{A_u}\Big) p_{X(t),X'(t)}(u,0)dt$$
$$+\int_I dt \int_I ds\ \beta^t(s)$$
$$\mathrm{E}\Big(\det\big(X^{t\prime\prime}(s)-\beta^{t\prime\prime}(s)u\big)\det\big(\underline{X}^{t\prime\prime}(t)-\underline{\beta}^{t\prime\prime}(t)u\big)\ \mathbb{1}_{A_u}/X^t(s)=\beta^t(s)u, X^{t\prime}(s)=\beta^{t\prime}(s)u\Big)$$
$$p_{X^t(s),X^{t\prime}(s)}\big(\beta^t(s)u,\beta^{t\prime}(s)u\big)p_{X(t),X'(t)}\big(u,0\big)+$$
$$\int_{S^{d-1}}\sigma(dw)\mathrm{E}\big[\det\big(\underline{X}^{t\prime\prime}_T(w)-\underline{\beta}^{t\prime\prime}_T(w)x\big)\xi_v\ \mathbb{1}_{A_u}/\underline{X}^{t\prime\prime}_N(w)=\underline{\beta}^{t\prime\prime}_N(w)x, \underline{X}^{t\prime\prime}_{NT}(w)=\underline{\beta}^{t\prime\prime}_{NT}(w)x\big]$$
$$p_{\underline{X}^{t\prime\prime}_N(w),\underline{X}^{t\prime\prime}_{NT}(w)}\big(\underline{\beta}^{t\prime\prime}_N(w)x,\underline{\beta}^{t\prime\prime}_{NT}(w)x\big) \quad (42)$$

Where A_u stands for $A_u(X^t,\beta^t)$ and $p^{(1,0)}_{X(t),X'(t)}$, $C_{i,j}(u)$, $\underline{X}^{t\prime\prime}_T$, $\underline{X}^{t\prime\prime}_N$, $\underline{X}^{t\prime\prime}_{NT}$, $\underline{\beta}^{t\prime\prime}_T$, $\underline{\beta}^{t\prime\prime}_N$ and $\underline{\beta}^{t\prime\prime}_{NT}$, are defined in the proof.

Proof: We have to check that the expression given in Theorem 4.2 that now takes the form

$$F_I'(u) = \big(-1\big)^d \int_I \mathrm{E}\Big[\det\big(\underline{X}^{t\prime\prime}(t) - \underline{\beta}^{t\prime\prime}(t)u\big)\ \mathbb{1}_{A_u(X^t,\beta^t)}\Big] p_{X(t),X'(t)}(u,0)\sigma(dt) \quad (43)$$

is differentiable with respect to u. A sufficient condition is the integrand itself to be differentiable with a derivative integrable in t, u , $t \in I$, u in a compact interval.

The derivative of the integrand in (43) is the sum of the three derivatives corresponding to the three locations where the variable u appears, namely :

- in the density $p_{X(t),X'(t)}(u,0)$ which is clearly differentiable with bounded derivative :
$$p^{(1,0)}_{X(t),X'(t)}(u,0).$$
This gives the first term in (42).

- In the derivative with respect to the first occurrence of u in
$$\mathrm{E}\Big[det\big(\underline{X^{t}}''(t)-\underline{\beta^{t}}''(t)u\big)\ \mathbb{1}_{A_u(X^t,\beta^t)}\Big].$$
The derivative of which is
$$-\mathrm{E}\Big(\sum_{i,j=1}^{d}\big(\underline{\beta^{t}}''(t)\big)_{i,j}C_{i,j}(u)\ \mathbb{1}_{A_u(X^t,\beta^t)}\Big),$$
where $C_{i,j}(u)$ is the cofactor of location (i,j) in the matrix $\underline{X^{t}}''(t)-\underline{\beta^{t\prime\prime}}(t)u$. This quantity is uniformly bounded when u varies in a compact interval, which follows easily from an expression of the type (33). This gives the second term in (42).

- in the derivative with respect to the second occurrence of u in
$$\mathrm{E}\Big[det\big(\underline{X^{t}}''(t)-\underline{\beta^{t}}''(t)u\big)\ \mathbb{1}_{A_u(X^t,\beta^t)}\Big].$$

To evaluate this derivative define ξ_v as in (36) and set for δ sufficiently small:
$$I^{-\delta}:=I\backslash B(t,\delta)\ ;\ A_u^{\delta}=A_u^{\delta}(X^t,\beta^t):=\big\{X^t(s)\le\beta^t(s)u,\forall s\in I^{-\delta}\big\},$$
$B(t,\delta)$ being the ball with center t and radius δ in the chart (ϕ_t,U_t). By dominated convergence
$$\mathrm{E}\big[\xi_v\big(\ \mathbb{1}_{A_{u+h}(X^t,\beta^t)}-\ \mathbb{1}_{A_u(X^t,\beta^t)}\big)\big]=\lim_{\delta\to 0}\mathrm{E}\big[\xi_v\big(\ \mathbb{1}_{A^{\delta}_{u+h}(X^t,\beta^t)}-\ \mathbb{1}_{A^{\delta}_u(X^t,\beta^t)}\big)\big]$$
On $I^{-\delta}$, X^t is a process satisfying $H4$. In the same manner as in Lemma 3.3 of Azaïs and Wschebor (2001), we can generalize the proof of Theorem 3.1 to the case

of a non constant function β^t and a non constant random variable ξ_v to obtain

$$\mathrm{E}\big[\xi_v(\,\mathbb{1}_{A^\delta_{u+h}} - \,\mathbb{1}_{A^\delta_u})\big]$$

$$\int_u^{u+h} dx \int_{I^{-\delta}} \sigma(ds)(-1)^d \beta^t(s)\mathrm{E}\big[\det\big(Y^{t\prime\prime}(s)\big)\xi_v\,\mathbb{1}_{A^\delta_x}/X^t(s) = \beta^t(s)x, X^{t\prime}(s) = \beta^{t\prime}(s)x\big]$$

$$p_{X^t(s),X^{t\prime}(s)}\big(\beta^t(s)x, \beta^{t\prime}(s)x\big)$$

$$+\int_u^{u+h} dx \int_{S(t,\delta)} \tilde{\sigma}(ds)(-1)^{d-1}\beta^t(s)\mathrm{E}\big[det\big(\tilde{Y}^{t\prime\prime}(s)\big)\xi_v\,\mathbb{1}_{A^\delta_x}/X^t(s) = \beta^t(s)x, \tilde{X}^{t\prime}(s) = \tilde{\beta}^{t\prime}(s)x\big]$$

$$p_{X^t(s),\tilde{X}^{t\prime}(s)}\big(\beta^t(s)x, \tilde{\beta}^{t\prime}(s)x\big) = I_1^\delta + I_2^\delta, \quad (44)$$

where $S(t,\delta)$ is the sphere with centre t and radius δ , $Y^{t\prime\prime}(s) = X^{t\prime\prime}(s) - \beta^{t\prime\prime}(s)x$, $\tilde{Y}^{t\prime\prime}(s) = \tilde{X}^{t\prime\prime}(s) - \tilde{\beta}^{t\prime\prime}(s)x$.
Let us prove that the first integral converges as $\delta \to 0$. The only problem is the behaviour around t. So it is sufficient to prove the convergence locally around t in the chart (ϕ_t, U_t) with s in V_t which implies that $n(s,t) = \frac{1}{2}\|t-s\|^2$. Without loss of generality we may assume that the representation of t in this chart is the point 0 in $\mathcal{R}^d$. To study the behaviour of the integrand as $s \to 0$, we choose an orthonormal basis with $\frac{s}{\|s\|}$ as first vector and set $s = (\rho, 0, ...0)^T$. At $s = 0$ the process $\underline{X}^t$ and its derivative have the following expansions (for short, derivatives are indicated by sub-indices).

$$\underline{X}^t(s) = \frac{1}{2}\rho^2\underline{X}_{11} + \frac{\rho^3}{6}\underline{X}_{111} + \frac{\rho^4}{24}\underline{X}_{1111} + o(\rho^4) \quad (45)$$

$$\frac{\partial \underline{X}^t}{\partial s_1}(s) = \underline{X}_1^t(s) = \rho\underline{X}_{11} + \frac{\rho^2}{2}\underline{X}_{111} + \frac{\rho^3}{6}\underline{X}_{1111} + o(\rho^3) \quad (46)$$

$$\frac{\partial^2 \underline{X}^t}{\partial^2 s_1}(s) = \underline{X}_{11}^t(s) = \underline{X}_{11} + \rho\underline{X}_{111} + \frac{\rho^2}{2}\underline{X}_{1111} + o(\rho^2) \quad (47)$$

$$\frac{\partial \underline{X}^t}{\partial s_j}(s) = \underline{X}_j^t(s) = \rho\underline{X}_{1j} + O(\rho^2) \quad (j \neq 1), \quad (48)$$

where $\underline{X}_{ij} = \underline{X}^t_{ij}(0)$, $\underline{X}_{111} = \frac{\partial^3 \underline{X}^t}{\partial^3 s_1}(0)$ and $\underline{X}_{1111} = \frac{\partial^4 \underline{X}^t}{\partial^4 s_1}(0)$ Since

$$X^t(s) = m(s).\underline{X}^t$$

with

$$m(s) := 1/n(s,0) = \frac{2}{s_1^2 + ... + s_d^2} \;;\; \frac{\partial m(s)}{\partial s_1}(\rho, 0...0) = \frac{-4}{\rho^3}$$

$$\frac{\partial m(s)}{\partial s_i}(\rho,0...0) = 0 \ (i \neq 1) \ ; \ \frac{\partial^2 m(s)}{\partial s_1^2}(\rho,0...0) = \frac{12}{\rho^4}$$
$$\frac{\partial^2 m(s)}{\partial s_i^2}(\rho,0...0) = \frac{-4}{\rho^4} \ (i \neq 1) \ ; \ \frac{\partial^2 m(s)}{\partial s_i \partial s_j}(\rho,0...0) = 0 \ (i \neq j).$$

Using derivation rules, we get

$$X^t(s) = \underline{X}_{11} + \frac{\rho}{3}\underline{X}_{111} + O(\rho^2) \quad (49)$$
$$\frac{\partial X^t}{\partial s_1}(s) = X^t_1(s) = \frac{1}{3}\underline{X}_{111} + O(\rho) \quad (50)$$
$$\frac{\partial^2 X^t}{\partial^2 s_1}(s) = X^t_{11}(s) = \frac{1}{6}\underline{X}_{1111} + O(\rho) \quad (51)$$
$$\frac{\partial X^t}{\partial s_j}(s) = X^t_j(s) = \frac{2}{\rho}\underline{X}_{1j} + O(1) \quad (j \neq 1) \quad (52)$$
$$\frac{\partial^2 X^t}{\partial s_i \partial s_j}(s) = \frac{2}{\rho^2}\underline{X}_{ij} + O(\rho^{-1}) \quad (j \neq 1) \quad (i \neq 1) \quad (i \neq j) \quad (53)$$

from that we deduce that

$$p_{X^t(s),X^{t\prime}(s)}\big(\beta^t(s)x, \beta^{t\prime}(s)x\big) \leq (const)\rho^{(d-1)}$$

Since if A^δ_x occurs and $X^t(s) = \beta^t(s)x$, then a.s. the matrix $X^{t\prime\prime}(s) - \beta^{t\prime\prime}(s)x$ is definite negative, and using relation (25), we have

$$\big|\det\big(X^{t\prime\prime}(s) - \beta^{t\prime\prime}(s)x\big)\big| \leq \prod_{i=1}^{d} |X^t_{ii}(s) - \beta^t_{ii}(s)x|$$

where the notation $\beta^{t\prime\prime}_{ii}(s)$ has an obvious meaning. The condition

$$C(s) = \big\{X^t(s) = \beta^t(s)x \ ; \ X^{t\prime}(s) = \beta^{t\prime}(s)x\big\}$$

converges as $s \to 0$ to the condition

$$\big\{\underline{X}_{11} = \underline{\beta}_{11}x \ ; \ \underline{X}_{111} = \underline{\beta}_{111}x \ ; \ \underline{X}^t_i(0) = \underline{\beta}^t_i(0)x \ (i = 2, d)\big\}$$

which is non singular (again the notations $\underline{\beta}_{11}$, $\underline{\beta}_{111}$, $\underline{\beta}^t_i$ are obvious). Consider a Gaussian variable which is measurable with respect to the process and which is bounded in probability. then its distribution conditional to $C(s)$ remains bounded in probability. Since for $i \neq 1$,

$$X_{ii}^t(s) - \beta_{ii}^t(s)x = \frac{2}{\rho^2}\underline{X}_{ii} + \frac{1}{\rho^2}\underline{X}_{11} - \frac{2}{\rho^2}\underline{\beta}_{ii}x - \frac{1}{\rho^2}\underline{\beta}_{11}x = O_p(\rho^{-2}),$$

this variable has the same order of magnitude under $C(s)$.

On the other hand, under $C(s)$

$$X_{11}(s) = O_p(1)\ ;\ \beta_{11}(s) = O(1)$$

Finally we get:

$$\mathrm{E}\Big(\big|\det\big(X^{t\prime\prime}(s) - \beta^{t\prime\prime}(s)x\big)\big|\,\xi_v\ \mathbb{1}_{A_x^{-\delta}}\Big) = O(\rho^{-2(d-1)}$$

Since $(\beta^t(s)$ is bounded we see that the integrand is $O(\rho^{1-d})$ which ensures convergence of I_1^δ as $\delta \downarrow 0$. One easily check that the bound for the integrand is uniform in t.

We consider now the limit of I_2^δ as $\delta \downarrow 0$. It is enough to prove that for each $x \in \mathcal{R}$ the expression

$$(-1)^{d-1}\int_u^{u+h} dx \int_{S(t,\delta)} \sigma(ds)\beta^t(s)$$
$$\mathrm{E}\big[\det\big(\tilde{X}^{t\prime\prime}(s)\big)\xi_v\ \mathbb{1}_{A_x^\delta}/X^t(s) = \beta^t(s)x, \tilde{X}^{t\prime}(s) = \tilde{\beta}^{t\prime}(s)x\big]p_{X^t(s),\tilde{X}^{t\prime}(s)}\big(\beta^t(s)x, \tilde{\beta}^{t\prime}(s)x\big). \tag{54}$$

converges boundedly as $\delta \downarrow 0$. Make in (54), the change of variable $s = t + \delta w$, $w \in S^{d-1}$, and it becomes

$$(-1)^{d-1}\int_u^{u+h} dx \int_{S^{d-1}} \sigma(dw)\beta^t(t+\delta w)$$
$$\mathrm{E}\big[\delta^{2(d-1)}\det\big(\tilde{X}^{t\prime\prime}(t+\delta w)\big)\xi_v\ \mathbb{1}_{A_x^\delta}/X^t(t+\delta w) = \beta^t(t+\delta w)x, \delta\tilde{X}^{t\prime}(t+\delta w) = \delta\tilde{\beta}^{t\prime}(t+\delta w)x\big]$$
$$p_{X^t(t+\delta w),\delta\tilde{X}^{t\prime}(t+\delta w)}\big(\beta^t(t+\delta w)x, \delta\tilde{\beta}^{t\prime}(t+\delta w)x\big), \tag{55}$$

where we have used that

$$P_{X,Y}(x,u) = \delta^{d-1}P_{X,\delta Y}(x,\delta u)$$

if (X,Y) is a random vector in $\mathcal{R} \times \mathcal{R}^{d-1}$ and $\delta > 0$.

Now consider the following decomposition in block of the matrix $\underline{X}^{t\prime\prime}(t)$ written in a basis with first vector equal to $w \in S^{d-1}$

- $\underline{X}^{t\prime\prime}_N(w)$ is the second derivative of $\underline{X}^t$ in the direction w : $w^T \underline{X}^{t\prime\prime} w$.
- $\underline{X}^{t\prime\prime}_T(w)$ is the $(d-1) \times (d-1)$ matrix that consists of the second derivatives of $\underline{X}^t$ in the direction that are orthogonal to w.
- $\underline{X}^{t\prime\prime}_{NT}(w)$ is the $(d-1)$ vector that consist of the cross second derivative $\underline{X}^t$, one in the direction w one in the $d-1$ direction orthogonal to w.

We have that the expression of $\underline{X}^{t\prime\prime}(t)$ in the new basis is

$$\begin{pmatrix} \underline{X}^{t\prime\prime}_N(w) & \left(\underline{X}^{t\prime\prime}_{NT}(w)\right)^T \\ \underline{X}^{t\prime\prime}_{NT}(w) & \underline{X}^{t\prime\prime}_T(w) \end{pmatrix}$$

We make the same decomposition with obvious notations for $\underline{\beta}^{t\prime\prime}(t)$.

Relations (49) to (52) imply that as $\delta \downarrow 0$

$$2^{(d-1)} p_{X^t(t+\delta w), \delta \tilde{X}^{t\prime}(t+\delta w)}\left(\beta^t(t+\delta w)x, \delta \tilde{\beta}^{t\prime}(t+\delta w)x\right) \to p_{\underline{X}^{t\prime\prime}_N(w), \underline{X}^{t\prime\prime}_{NT}(w)}\left(\underline{\beta}^{t\prime\prime}_N(w)x, \underline{\beta}^{t\prime\prime}_{NT}(w)x\right) \quad (56)$$

and (53) implies that

$$\frac{\delta^2}{2} \tilde{X}^{t\prime\prime}(t+\delta w) \to \underline{X}^{t\prime\prime}_T(w) \quad , \quad \frac{\delta^2}{2} \tilde{\beta}^{t\prime\prime}(t+\delta w) \to \underline{\beta}^{t\prime\prime}_T(w).$$

Noting that $\mathbb{I}_{A_x^\delta} \uparrow \mathbb{I}_{A_x}$ we get that as $\delta \downarrow 0$:

$$\frac{\delta^{2(d-1)}}{2^{(d-1)}} \mathrm{E}\big[\det\left(\tilde{X}^{t\prime\prime}(t+\delta w)\right)\xi_v \; \mathbb{I}_{A_x^\delta} / X^t(t+\delta w) = \beta^t(t+\delta w)x, \delta \tilde{X}^{t\prime}(t+\delta w) = \delta \tilde{\beta}^{t\prime}(t+\delta w)x\big]$$
$$\to \mathrm{E}\big[\det\left(\underline{X}^{t\prime\prime}_T(w) - \underline{\beta}^{t\prime\prime}_T(w)x\right)\xi_v \; \mathbb{I}_{A_x} / \underline{X}^{t\prime\prime}_N(w) = \underline{\beta}^{t\prime\prime}_N(w)x, \underline{X}^{t\prime\prime}_{NT}(w) = \underline{\beta}^{t\prime\prime}_{NT}(w)x\big].$$

Remarking that the integrand is uniformly bounded, we are ready to pass to the limit and get the result. □

6 Asymptotic expansion of $F'(u)$ for large u

Corollary 6.1 *Suppose that the process X satisfies the conditions of Theorem 4.2 and that in addition $\mathrm{E}(X_t) = 0$ and $\mathrm{Var}(X_t) = 1$.*

Then, as $u \to +\infty$ $F'(u)$ is equivalent to

$$\frac{u^d}{(2\pi)^{(d+1)/2}} e^{-u^2/2} \int_I \left(\det(\Lambda(t))\right)^{1/2} dt, \quad (57)$$

where $\Lambda(t)$ is the variance-covariance matrix of $X'(t)$.

Proof: Set $r(s,t) := \mathrm{E}\big(X(s), X(t)\big)$, and for $i, j = 1, d$,

$$r_{i;}(s,t) := \frac{\partial}{\partial s_i} r(s,t); \quad r_{ij;}(s,t) := \frac{\partial^2}{\partial s_i \partial s_j} r(s,t); \quad r_{i;j}(s,t) := \frac{\partial^2}{\partial s_i \partial t_j} r(s,t).$$

For every t, i and j

$$r_{i;}(t,t) = 0, \quad \Lambda_{ij}(t) = r_{i;j}(t,t) = -r_{ij;}(t,t).$$

Thus $X(t)$ and $X'(t)$ are independent. Regression formulae imply that

$$a_s^t = r(s,t), \quad \beta^t(s) = \frac{1 - r(t,s)}{n(s,t)}.$$

This implies that $\underline{\beta}^t(t) = \Lambda(t)$ and that the possible limit values of $\beta^t(s)$ as $s \to t$ are in the set $\{v^T \Lambda(t) v : v \in S^{d-1}\}$. Due to the non-degeneracy condition these quantities are minorized by a positive constant. On the other hand for $s \neq t$ $\beta^t(s) > 0$. This shows that for every $t \in I$ one has $\inf_{s \in I} \beta^t(s) > 0$. Since for every $t \in I$ the process X^t is bounded it follows that

$$a.s. \quad 1\!\!1_{A_u(X^t, \beta^t)} \to 1 \text{ as } u \to +\infty.$$

Also

$$\det\big(\underline{X}^{t\prime\prime}(t) - \underline{\beta}^{t\prime\prime}(t)u\big) \simeq (-1)^d \det\big(\Lambda(t)\big) u^d.$$

A dominated convergence argument shows that the first term in (35) is equivalent to

$$\int_I u^d \det(\Lambda^t)(2\pi)^{-1/2} e^{-u^2/2} (2\pi)^{-d/2} \big(\det(\Lambda^t)\big)^{-1/2} dt = \frac{u^d}{(2\pi)^{(d+1)/2}} e^{-u^2/2} \int_I \big(\det(\Lambda^t)\big)^{1/2} dt.$$

The same kind of argument shows that the second term is $O\big(u^{d-1} e^{-u^2/2}\big)$ which completes the proof. □

7 References

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