A self contained proof of the Rice formula for random fields *

Jean-Marc Azaïs †, azais@cict.fr   Mario Wschebor ‡, wschebor@cmat.edu.uy

December 22, 2006

AMS subject classification: Primary 60G70 Secondary 60G15
Key words and phrases: Gaussian fields, Rice Formula,

After an elementary proof of the Area Formula, we give a proof of Rice Formula for the expectation of the number of roots of a random system of equations. We provide a complete proof which is new and quite elementary, and in any case shorter than previous ones (see for example [1]).

Similar formulae hold true for higher order factorial moments of the number of roots (Theorem 2). Theorem 3 provides a formula for the expectation of the total weight, when random weights are put in each root.

1 The Area formula

We begin with a proof of the so-called Area formula, under conditions that will be sufficient for our main purpose. One can find this formula in its full generality in Federer [3] Th 3.2.5

For any function f, we denote $N_u^f(T)$ the number of roots of the equation $f(t) = u$ that belong to the subset T of the domain of f.

Proposition 1 (Area formula) *Let f be a $\mathcal{C}^1$ function defined on an open subset U of $\mathbb{R}^d$ taking values in $\mathbb{R}^d$. Assume that the set of critical values of f has zero Lebesgue measure. Let g: $\mathbb{R}^d \to \mathbb{R}$ be continuous and bounded. Then*

$$\int_{\mathbb{R}^d} g(u) N_u^f(T) du = \int_T |\det(f'(t))| g(f(t)) dt. \tag{1}$$

for any Borel subset T of U, whenever the integral in the right-hand side is well defined.

Proof. Notice first that, due to standard extension arguments, it suffices to prove (1) for non-negative g and for T a compact parallelotope contained in U. Second, if T is a compact parallelotope, since f is $\mathcal{C}^1$, the set of boundary values of f, that is, $f(\partial T)$ has Lebesgue measure zero.

We next define an auxiliary function $\delta(u)$ for $u \in \mathbb{R}^d$ in the following way:

• If u is neither a critical value nor a boundary value and $n := N_u^f(T)$ is non zero, we denote by $x^{(1)}, \dots, x^{(n)}$ the roots of $f(x) = u$ belonging to T. Using the local inversion theorem, we know that there exists some $\delta > 0$ and n neighborhoods $U_1, ..., U_n$ of of $x^{(1)}, \dots, x^{(n)}$ such that:

*This work was supported by ECOS program U03E01.

†Laboratoire de Statistique et Probabilités. UMR-CNRS C5583 Université Paul Sabatier. 118, route de Narbonne. 31062 Toulouse Cedex 4. France.

‡Centro de Matemática. Facultad de Ciencias. Universidad de la República. Calle Igua 4225. 11400 Montevideo. Uruguay.

1. f is a $\mathcal{C}^1$ diffeomorphism $U_i \to B(u;\delta)$, the ball centered at u with radius δ.
2. $U_1, ..., U_n$ are pairwise disjoint and included in T.
3. if $t \notin \bigcup_{i=1}^n U_i$, then $f(t) \notin B(u;\delta)$.

The compactness implies that n is finite.

In this case, we define

$$\delta(u) := \sup\{\delta > 0 : (1), (2), (3) \text{ hold true for all } \delta' \leq \delta\}.$$

• If u is a critical value or a boundary value we set $\delta(u) := 0$.
• If $N_u^f(T) = 0$, we put

$$\delta(u) := \sup\{\delta > 0 : f(T) \cap B(u;\delta) = \emptyset\}.$$

It is clear that in this case $\delta(u) > 0$.

The function $\delta(u)$ is Lipschitz. In fact, let u be a value of f which is not a critical value nor a boundary value, if u' belongs to $B(u;\delta(u))$, then $B(u';\delta(u) - \|u' - u\|) \subset B(u;\delta(u))$ and as a consequence $\delta(u') \geq \delta(u) - \|u' - u\|$. Exchanging the roles of u and u', we get

$$|\delta(u') - \delta(u)| \leq \|u - u'\|.$$

The Lipschitz condition is easily checked in the other two cases.

Let now $\mathcal{F}$ be a real-valued monotone continuous function defined on $\mathbb{R}^+$ such that

$$\begin{aligned} &\mathcal{F} \equiv 0 \text{ on } [0, 1/2], \\ &\mathcal{F} \equiv 1 \text{ on } [1 + \infty). \end{aligned} \tag{2}$$

Let $\delta(u) > 0$ and $0 < \delta < \delta(u)$. Using the change of variable formula we have

$$\int_T |\det(f'(t))| \; \mathbb{1}_{\|f(t)-u\|<\delta} dt = \sum_{i=1}^n \int_{U_i} |\det(f'(t))| dt = V(\delta) n,$$

where $V(\delta)$ is the volume of the ball with radius δ in R^d. Thus, we have an exact counter for $N_u^f(T)$ when it is non-zero, which obviously holds true also when $N_u^f(T) = 0$ for $\delta < \delta(u)$

Let $g : \mathbb{R}^d \to \mathbb{R}$ continuous, bounded and non-negative and $\delta_0 > 0$. For every $\delta' < \delta_0/2$ we have:

$$\int_{\mathbb{R}^d} g(u) N_u^f(T) \mathcal{F}\left(\frac{\delta(u)}{\delta_0}\right) du = \int_{\mathbb{R}^d} g(u) \; \mathcal{F}\left(\frac{\delta(u)}{\delta_0}\right) \; du \; \frac{1}{V(\delta')} \int_T |\det(f'(t))| \; \mathbb{1}_{\|f(t)-u\|<\delta'} dt$$

Applying Fubini's Theorem we see that the expression above is equal to:

$$A_{\delta_0,\delta'} := \int_T \; |\det(f'(t))| \; dt \; \frac{1}{V(\delta')} \int_{B(f(t);\delta')} \mathcal{F}\left(\frac{\delta(u)}{\delta_0}\right) g(u) du.$$

$A_{\delta_0,\delta'}$ in fact does not depend on δ' so it is equal to its limit as $\delta' \to 0$ which is, because of the continuity of the function $u \leadsto \mathcal{F}\left(\frac{\delta(u)}{\delta_0}\right) g(u)$, equal to

$$\int_T |\det(f'(t))| \mathcal{F}\left(\frac{\delta(f(t))}{\delta_0}\right) g(f(t)) dt.$$

Let now δ_0 tend to zero and use monotone convergence. For the left-hand side, we take into account that the set of critical values and the set of boundary values have measure zero. For the right-hand side, we use the definition of $\mathcal{F}$, that the boundary of T has Lebesgue measure zero and the integrand is zero if t is a critical point of f. $\square$

Remarks:

1: By standard extension arguments the continuous function g can be replaced by the indicator function of a Borel set say B. Formula (1) can be rewritten as

$$\int_{\mathbb{R}^d} \sum_{t \in Z^{-1}(u)} h(t,u) du = \int_{\mathbb{R}^d} |\det(f'(t))| h((t, f(t)) dt, \tag{3}$$

where h is the function $(t,u) \rightsquigarrow \mathbb{1}_{t \in T} g(u)$. Again by a standard approximation argument (3) holds true for every bounded Borel function h such that the right-hand side of (3) is well-defined.

2:

2. Notice that the hypothesis that the set of critical values has zero Lebesgue measure is unnecessary, since this follows from the fact that f is C1. The statement above is sufficient to prove Rice formula, but the interested reader can prove this as an exercise. On the other hand, one can prove the result under weaker hypotheses: it suffices f to be Lipschitz ([3]).

2 Rice formulae

Theorem 1 (Rice formula) *Let $Z : U \to \mathbb{R}^d$ be a random field, U an open subset of $\mathbb{R}^d$ and $u \in \mathbb{R}^d$ a fixed point in the codomain. Assume that:*
(i) Z is Gaussian,
(ii) almost surely the function $t \rightsquigarrow Z(t)$ is of class $\mathcal{C}^1$,
(iii) for each $t \in U$, $Z(t)$ has a non degenerate distribution (i.e. $\mathrm{Var}\big(Z(t)\big) \succ 0$),
(iv) $\mathrm{P}\{\exists t \in U, Z(t) = u, \det\big(Z'(t)\big) = 0\} = 0$

Then, for every Borel set B contained in U, one has

$$\mathrm{E}\left(N_u^Z(B)\right) = \int_B \mathrm{E}\left(|\det(Z'(t))| / Z(t) = u\right) \; p_{Z(t)}(u) dt. \tag{4}$$

If B is compact both sides in (4) are finite.

Theorem 2 *Let k, $k \geq 2$ be an integer. Assume the same hypotheses as in Theorem 1 except for (iii) that is replaced by*

(iii') for $t_1, ..., t_k \in U$ pairwise different values of the parameter, the distribution of

$$\big(Z(t_1), ..., Z(t_k)\big)$$

does not degenerate in $(\mathbb{R}^d)^k$. Then for every Borel set B contained in U, one has

$$\mathrm{E}\left[\big(N_u^Z(B)\big)\big(N_u^Z(B) - 1\big)...\big(N_u^Z(B) - k + 1\big)\right]$$
$$= \int_{B^k} \mathrm{E}\Big(\prod_{j=1}^{k} |\det\big(Z'(t_j)\big)| / Z(t_1) = ... = Z(t_k) = u\Big)$$
$$p_{Z(t_1),...,Z(t_k)}(u, ..., u) dt_1 ... dt_k, \tag{5}$$

where both members may be infinite.

Theorem 3 *Let Z be a random field that verifies the hypotheses of Theorem 1. Assume that for each $t \in U$ one has another random field $Y^t : W \to \mathbb{R}^{d'}$, where W is some topological space, verifying the following conditions:*

a) $Y^t(w)$ *is a measurable function of* (ω, t, w) *and almost surely,* $(t, w) \rightsquigarrow Y^t(w)$ *is continuous.*

b) *For each* $t \in U$ *the random process* $(s, w) \rightsquigarrow \big(Z(s), Y^t(w)\big)$ *defined on* $U \times W$ *is Gaussian.*

Moreover, assume that $g : U \times \mathcal{C}(W, \mathbb{R}^{d'}) \to \mathbb{R}$ *is a bounded function, which is continuous when one puts on* $\mathcal{C}(W, \mathbb{R}^{d'})$ *the topology of uniform convergence on compact sets. Then, for each compact subset* I *of* U*, one has*

$$\mathrm{E}\Big(\sum_{t \in I, Z(t)=u} g(t, Y^t)\Big) = \int_I \mathrm{E}\big(|\det(Z'(t)| g(t, Y^t)/Z(t) = u\big).p_{Z(t)}(u)dt. \qquad (6)$$

Sufficient conditions for (iv) are given in the following proposition

Proposition 2 *Let* $Z : U \to \mathbb{R}^d$ *,* U *a compact subset of* $\mathbb{R}^d$ *be a random field with paths of class* $\mathcal{C}^1$ *and* $u \in \mathbb{R}^d$*. Assume that*

- $p_{Z(t)}(x) \leq C$ *for all* $t \in U$ *and* x *in some neighborhood of* u*.*
- *at least one of the two following hypotheses is satisfied:*

  a) *a.s.* $t \rightsquigarrow Z(t)$ *is of class* $\mathcal{C}^2$

  b)
  $$\alpha(\delta) = \sup_{t \in U, x \in V(u)} \mathrm{P}\big\{|\det(Z'(t))| < \delta / Z(t) = x\big\} \to 0$$
  as $\delta \to 0$*, where* $V(u)$ *is some neighborhood of* u*.*

Then (iv) holds true.

Proof. If condition a) holds true, the result is Lemma 5 in Cucker and Wschebor [2].

To prove it under condition b), assume with no loss of generality that $I = [0,1]^d$ and that $u = 0$. Put $G_I = \big\{\exists t \in I, Z(t) = 0, \det\big(Z'(t)\big) = 0\big\}$ Choose $\varepsilon > 0$, $\eta > 0$; there exists a positive number M such that

$$\mathrm{P}(E_M) = \mathrm{P}\big\{\sup_{t \in I} \|Z'(t)\| > M\big\} \leq \varepsilon.$$

Denote by $\omega_{\det}$ the modulus of continuity of $|\det(X'(.))|$ and choose m large enough so that

$$\mathrm{P}(F_{m,\eta}) = \mathrm{P}\big\{\omega_{\det}(\frac{\sqrt{d}}{m}) \geq \eta\big\} \leq \varepsilon.$$

Consider the partition of I into m^d small cubes with sides of length $1/m$. Let $C_{i_1 \dots i_d}$ be such a cube and $t_{i_1 \dots i_d}$ its centre ($1 \leq i_1, ..., i_d \leq m$). Then

$$\mathrm{P}(G_I) \leq \mathrm{P}(E_M) + \mathrm{P}(F_{m,\eta}) + \sum_{1 \leq i_1 \dots i_d \leq m} \mathrm{P}\Big(G_{C_{i_1 \dots i_d}} \cap E_M^c \cap F_{m,\eta}^c\Big) \qquad (7)$$

When the event in the term corresponding to $i_1 ... i_d$ of the last sum occurs, we have:

$$|Z_j(t_{i_1 \dots i_d})| \leq \frac{M}{m}\sqrt{d} \quad j = 1, ..., d$$

where Z_j denotes the j-th coordinate of Z, and:

$$\big|\det\big(Z'(t_{i_1 \dots i_d})\big)\big| < \eta.$$

So, if m is chosen sufficiently large so that $V(0)$ contains the ball centered at 0 with radius $\frac{M\sqrt{d}}{m}$, one has:

$$\mathrm{P}(G_I) \leq 2\epsilon + m^d(\frac{2M}{m}\sqrt{d})^d C\alpha(\eta)$$

Since ε and η are arbitrarily small, the result follows. □

Remark:

With the hypotheses of Theorem 1 it follows easily that if J is a subset of U, $\lambda_d(J) = 0$, then $\mathrm{P}\{N_u^Z(J) = 0\} = 1$ for each $u \in \mathbb{R}^d$.

Proof of Theorem 1

Let $\mathcal{F} : \mathbb{R}^+ \to [0, 1]$ be the function defined in (2), For m, n positive integers and $x \geq 0$, define:

$$\mathcal{F}_m(x) := \mathcal{F}(mx) \quad ; \quad G_n(x) := 1 - \mathcal{F}(x/n). \tag{8}$$

A standard extension argument says that it is enough to prove the theorem when B is a compact rectangle included in U. So we assume that this is the case. Let us introduce some more notations:

- $\Delta(t) := |\det(Z'(t))|$ $(t \in U)$
- For n, m positive integers and $u \in \mathbb{R}^d$:

$$C_u^m(B) := \sum_{s \in B : Z(s) = u} \mathcal{F}_m(\Delta(s)). \tag{9}$$

$$Q_u^{n,m}(B) := C_u^m(B) G_n(C_u^m(B)). \tag{10}$$

In (9) when the summation index set is empty, we put $C_u^m(B) = 0$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be continuous with compact support . We apply the area formula (3) for the function

$$h(t, u) = \mathcal{F}_m(\Delta(t)) G_n(C_u^m(B)) g(u) \ \mathbb{1}_{t \in B}$$

to get:

$$\int_{\mathbb{R}^d} g(u) Q_u^{n,m}(B) du = \int_B \Delta(t)\ \mathcal{F}_m(\Delta(t))\ G_n(C_{Z(t)}^m(B))\ g(Z(t)) dt.$$

Taking expectations on both sides :

$$\int_{\mathbb{R}^d} g(u)\ \mathrm{E}(Q_u^{n,m}(B)) du = \int_{\mathbb{R}^d} g(u) du \int_B \mathrm{E}\big[\Delta(t)\ \mathcal{F}_m(\Delta(t)) G_n(C_u^m(B)) / Z(t) = u\big]\ p_{Z(t)}(u) dt.$$

Since this equality holds for any g continuous with bounded support, it follows that

$$\mathrm{E}(Q_u^{n,m}(B)) = \int_B \mathrm{E}\big[\Delta(t) \mathcal{F}_m(\Delta(t) G_n(C_u^m(B)) / Z(t) = u\big] p_{Z(t)}(u) dt, \tag{11}$$

holds true for almost every $u \in \mathbb{R}^d$.

Let us prove that the left-hand side of (11) is a continuous function of u. Fix $u \in \mathbb{R}^d$. Outside the compact set

$$\{t \in B : \Delta(t) \geq 1/2m\},$$

the contribution to the sum (9) defining $C_v^m(B)$ is zero, for any $v \in \mathbb{R}^d$. Using the local inversion theorem, the number of points $t \in B$ such that $Z(t) = u$; $\Delta(t) \geq 1/2m$, say k, is finite. Notice that almost surely there is no such point in the boundary of B.

If k is non-zero, $Z(t)$ is locally invertible in k neighborhoods $V_1, \ldots, V_k \subset B$ around these k points. For v in some (random) neighborhood of u, there is exactly one root of $Z(s) = v$ in each $V_1, ..., V_k$ and the contribution to $C_v^m(B)$ of these points can be made arbitrarily close to

the one corresponding to $v = u$. Outside the union of $V_1, ..., V_k$, $Z(t) - u$ is bounded away from zero in B, so that the contribution to $C_v^m(B)$ vanishes if v is sufficiently close to u.
This shows that a.s., the function $v \rightsquigarrow Q_v^{n,m}$ is continuous at $v = u$. On the other hand, it is obvious from its definition that $Q_v^{n,m}(B) \leq n$ and an application of the Lebesgue dominated convergence theorem implies the continuity of $\mathrm{E}(Q_u^{n,m}(B))$ as a function of u.

Let us now write the regression formulae for fixed $t \in B$:

$$\begin{aligned} Z(s) &= a^t(s)Z(t) + Z^t(s) \\ Z'(s) &= (a^t)'(s)Z(t) + (Z^t)'(s), \end{aligned} \tag{12}$$

where "$'$" denotes the derivative with respect to s and the pair $\big(Z^t(s), (Z^t)'(s)\big)$ is independent from $Z(t)$ for all $s \in U$.
Then, we write the conditional expectation on the right-hand side of (11) as the unconditional expectation :

$$\mathrm{E}\big[\Delta_u^t(t)\mathcal{F}_m(\Delta_u^t(t))G_n(\widetilde{C}_u^m(B))\big], \tag{13}$$

where we use the notations

$$\begin{aligned} \Delta_u^t(s) &:= |\det(Z_u^t)'(s)| \\ Z_u^t(s) &:= a^t(s)u + Z^t(s) \\ \widetilde{C}_u^m(B) &:= \sum_{s \in B, Z_u^t(s) = u} \mathcal{F}_m\big(\Delta_u^t(s)\big). \end{aligned}$$

Now, observe that (11) implies that for almost every $u \in \mathbb{R}^d$ one has the inequality

$$\mathrm{E}(Q_u^{n,m}(B)) \leq \int_B \mathrm{E}\big[\Delta(t)/Z(t) = u\big]p_{Z(t)}(u)dt, \tag{14}$$

which is in fact true for all $u \in \mathbb{R}^d$ since both sides are continuous functions of u.

The remainder of the proof consists in proving the converse inequality. Let us fix n, m, u and t. Let K be the compact set

$$K := \{s \in B : \Delta_u^t(s) \geq 1/4m\}$$

If v varies in a sufficient small (random) neighborhood of u, the points outside K do not contribute to the sum defining $\widetilde{C}_v^m(B)$.

Let k the almost surely finite number of roots of $Z_u^t(s) = u$ lying in the set K. Assume that k does not vanish and denote these roots by $\bar{s}_1, \ldots, \bar{s}_k$. Consider the equation

$$Z_v^t(s) - v = 0. \tag{15}$$

in a neighborhood of each one of the pairs $s = \bar{s}_i$, $v = u$. Applying the Implicit Function Theorem, one can find k pairwise disjoint open sets sets $V_1, ...V_k$ such that if v is sufficiently close to u, equation (15) has exactly one root $s_i = s_i(v)$ in V_i , $1 = 1, \ldots, k$. These roots vary continuously with v and $s_i(u) = \bar{s}_i$. On the other hand on the compact set $K \backslash (V_1 \cup ... \cup V_k)$ the quantity $\|Z_u^t(s) - u\|$ is bounded away from zero so $\|Z_v^t(s) - v\|$ does not vanishes if v is sufficiently close to u. As a conclusion, we have that

$$\limsup_{v \to u} \widetilde{C}_v^m(B) \leq \widetilde{C}_u^m(B)$$

where the inequality arises from the fact that some of the points $s_i(v)$ may not belong to B and hence, don't contribute to the sum defining $\widetilde{C}_v^m(B)$. Now since (11) holds for a.e. u, one can find a sequence $\{u_N, N = 1, 2, \ldots\}$ converging to u such that (11) holds true for $u = u_N$ and all $N = 1, 2, \ldots$. Using the continuity -already proved- of $u \to \mathrm{E}(Q_u^{n,m}(B))$, Fatou's Lemma and the fact that G_n is non-increasing, we have :

$$\begin{aligned}
\mathrm{E}\big(Q_u^{n,m}(B)\big) &= \lim_{N\to+\infty} \mathrm{E}\big(Q_{u_N}^{n,m}(B)\big) \\
&= \lim_{N\to+\infty} \int_B \mathrm{E}\big[\Delta_{u_N}^t(t)\mathcal{F}_m(\Delta_{u_N}^t(t)G_n\big(\widetilde{C}_{u_N}^m(B)\big)\big]p_{Z(t)}(u_N)dt \\
&\geq \int_B \mathrm{E}\big[\Delta_u^t(t)\mathcal{F}_m(\Delta_u^t(t)G_n\big(\widetilde{C}_u^m(B)\big)\big]p_{Z(t)}(u)dt.
\end{aligned}$$

Since $\widetilde{C}_u^m(B)$ is a.s. finite, we can now pass to the limit as $n \to +\infty$, $m \to +\infty$ in that order and applying Beppo-Levi's Theorem, conclude the proof. □

Proof of Theorem 2:

For each $\delta > 0$, define the domain

$$D_{k,\delta}(B) = \{(t_1, \ldots, t_k) \in B^k,\ \|t_i - t_j\| \geq \delta \text{ if } i \neq j, i, j = 1, \ldots, k\}$$

and the process $\widetilde{Z}$

$$(t_1, \ldots, t_k) \in D_{k,\delta}(B) \rightsquigarrow \widetilde{Z}(t_1, \ldots, t_k) = \big(Z(t_1), \ldots, Z(t_k)\big).$$

It is clear that $\widetilde{Z}$ satisfies the hypotheses of Theorem 1 for every value $(u, \ldots, u) \in (\mathbb{R}^d)^k$. So,

$$\begin{aligned}
&\mathrm{E}\left[N_{(u,\ldots,u)}^{\widetilde{Z}}\big(D_{k,\delta}(B)\big)\right] \\
&\quad = \int_{D_{k,\delta}(B)} \mathrm{E}\Big(\prod_{j=1}^{k} |\det\big(Z'(t_j)\big)|/Z(t_1) = \ldots = Z(t_k) = u\Big)p_{Z(t_1),\ldots,Z(t_k)}(u, \ldots, u)dt_1 \ldots dt_k \quad (16)
\end{aligned}$$

To finish, let $\delta \downarrow 0$, note that

$$\big(N_u^Z(B)\big)\big(N_u^Z(B) - 1\big) \ldots \big(N_u^Z(B) - k + 1\big)$$

is the monotone limit of

$$N_{(u,\ldots,u)}^{\widetilde{Z}}\big(D_{k,\delta}(B)\big),$$

and that the diagonal $D_k(B) = \big\{(t_1, \ldots, t_k) \in B^k, t_i = t_j \text{ for some pair } i, j, i \neq j\big\}$ has zero Lebesgue measure in $(\mathbb{R}^d)^k$. □

Proof of Theorem 3:

The proof is essentially the same. It suffices to consider instead of $C_u^m(B)$ the quantity

$$C_u'^m(I) := \sum_{s\in I: Z(s)=u} \mathcal{F}_m(\Delta(s)).g_s(s, Y^s). \quad (17)$$

□

References

[1] Azaïs J-M. and Wschebor M (2005). On the Distribution of the Maximum of a Gaussian Field with d Parameters. *Annals Applied Probability*, 15 (1A), 254-278.

[2] Cucker, F. and Wschebor M. (2003). On the Expected Condition Number of Linear Programming Problems, *Numer. Math.*, 94, 419-478.

[3] Federer, H. (1969). *Geometric measure theory.* Springer-Verlag, New York