Some applications of Rice formulas to waves

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Abstract

We use Rice's formulas in order to compute the moments of some level functionals which are linked to problems in oceanography and optics. For instance, we consider the number of specular points in one or two dimensions, the number of twinkles, the distribution of normal angle of level curves and the number or the length of dislocations in random wavefronts. We compute expectations and in some cases, also second moments of such functionals. Moments of order greater than one are more involved, but one needs them whenever one wants to perform statistical inference on some parameters in the model or to test the model itself. In some cases we are able to use these computations to obtain a Central Limit Theorem.

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1 Introduction

Many problems in applied mathematics require to estimate the number of points, the length, the volume and so on, of the level sets of a random function $W(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^d$, so that one needs to compute the value of certain functionals of the probability distribution of the size of the random set

$$\mathcal{C}_A^W(\mathbf{u}, \omega) := \{\mathbf{x} \in A : W(\mathbf{x}, \omega) = \mathbf{u}\},$$

for some given $\mathbf{u}$.

Let us mention some examples which illustrate this general situation:

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- The number of times that a random process $\{X(t) : t \in \mathbb{R}\}$ crosses the level u:

$$N_A^X(u) = \#\{s \in A : X(s) = u\}.$$

Generally speaking, the probability distribution of the random variable $N_A^X(u)$ is unknown, even for the simplest models of the underlying process. However, there exist some formulas to compute $\mathbb{E}(N_A^X)$ and also higher order moments.

- A particular case is the number of specular points of a random curve or a random surface.

Consider first the case of a random curve. We take cartesian coordinates Oxz in the plane. A light source placed at $(0, h_1)$ emits a ray that is reflected at the point $(x, W(x))$ of the curve and the reflected ray is registered by an observer placed at $(0, h_2)$.

Using the equality between the angles of incidence and reflexion with respect to the normal vector to the curve - i.e. $N(x) = (-W'(x), 1)$ - an elementary computation gives:

$$W'(x) = \frac{\alpha_2 r_1 - \alpha_1 r_2}{x(r_2 - r_1)} \tag{1}$$

where $\alpha_i := h_i - W(x)$ and $r_i := \sqrt{x^2 + \alpha_i^2}$, i=1,2.

The points $(x, W(x))$ of the curve such that x is a solution of (1) are called "specular points". We denote by $SP_1(A)$ the number of specular points such that $x \in A$, for each Borel subset A of the real line. One of our aims in this paper is to study the probability distribution of $SP_1(A)$.

- The following approximation, which turns out to be very accurate in practice for ocean waves, was introduced long ago by Longuet-Higgins (see [13] and [14]):

Suppose that h_1 and h_2 are big with respect to $W(x)$ and x, then $r_i = \alpha_i + x^2/(2\alpha_i) + O(h_i^{-3})$. Then, (1) can be approximated by

$$W'(x) \simeq \frac{x}{2}\frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2} \simeq \frac{x}{2}\frac{h_1 + h_2}{h_1 h_2} = kx, \tag{2}$$

where

$$k := \frac{1}{2}\Big(\frac{1}{h_1} + \frac{1}{h_2}\Big).$$

Denote $Y(x) := W'(x) - kx$ and $SP_2(A)$ the number of roots of $Y(x)$ belonging to the set A, an approximation of $SP_1(A)$ under this asymptotic. The first part of Section 3 below will be devoted to obtain some results on the distribution of the random variable $SP_2(\mathbb{R})$.

- Consider now the same problem as above, but adding a time variable t, that is, W becomes a random function parameterized by the pair (x,t). We denote $W_x, W_t, W_{xt}, \dots$ the partial derivatives of W.

  We use the Longuet-Higgins approximation (2), so that the approximate specular points at time t are $(x, W(x,t))$ where

$$W_x(x,t) = kx.$$

  Generally speaking, this equation defines a finite number of points which move with time. The implicit function theorem, when it can be applied, shows that the x-coordinate of a specular point moves at speed

$$\frac{dx}{dt} = -\frac{W_{xt}}{W_{xx} - k}.$$

  The right-hand side diverges whenever $W_{xx} - k = 0$, in which case a flash appears and the point is called a "twinkle". We are interested in the (random) number of flashes lying in a set A of space and in an interval $[0,T]$ of time. If we put:

$$\mathbf{Y}(x,t) := \begin{pmatrix} W_x(x,t) - kx \\ W_{xx}(x,t) - k \end{pmatrix}. \tag{3}$$

  then, the number of twinkles is:

$$\mathcal{TW}(A,T) := \sharp\{(x,t) \in A \times [0,T] : \mathbf{Y}(x,t) = 0\}$$

- Let $W : Q \subset \mathbb{R}^d \to \mathbb{R}^{d'}$ with $d > d'$ be a random field and let us define the level set

$$\mathcal{C}_Q^W(\mathbf{u}) = \{\mathbf{x} \in Q : W(\mathbf{x}) = \mathbf{u}\}.$$

  Under certain general conditions this set is a $(d-d')$-dimensional manifold but in any case, its $(d-d')$-dimensional Hausdorff measure is well defined. We denote this measure by $\sigma_{d-d'}$. Our interest will be to compute the mean of the $\sigma_{d-d'}$-measure of this level set i.e. $\mathbb{E}[\sigma_{d-d'}(\mathcal{C}_Q^W(\mathbf{u}))]$ as well as its higher moments. It will be also of interest to compute:

$$\mathbb{E}[\int_{\mathcal{C}_Q^W(\mathbf{u})} Y(s)d\sigma_{d-d'}(s)].$$

  where $Y(s)$ is some random field defined on the level set. Cabaña [7], Wschebor [19] ($d' = 1$) Azaïs and Wschebor [4] and, in a weak form, Zähle [20] have studied these types of formulas. See Theorems 5 and 6.

- Another interesting problem is the study of phase singularities, dislocations of random wavefronts. They correspond to lines of darkness, in light propagation, or threads of silence in sound [6]. In a mathematical framework they can be define as the loci of points where the amplitude of waves vanishes. If we represent the wave as

$$W(\mathbf{x},t)=\xi(\mathbf{x},t)+i\eta(\mathbf{x},t), \quad \text{where } \mathbf{x}\in\mathbb{R}^d$$

where ξ, η are independent homogenous Gaussian random fields the dislocations are the intersection of the two random surfaces $\xi(\mathbf{x},t)=0$, $\eta(\mathbf{x},t)=0$. We consider a fixed time, for instance $t=0$. In the case $d=2$ we will study the expectation of the following random variable

$$\#\{\mathbf{x}\in S:\ \xi(\mathbf{x},0)=\eta(\mathbf{x},0)=0\}.$$

In the case $d=3$ one important quantity is the length of the level curve

$$\mathcal{L}\{\mathbf{x}\in S:\ \xi(\mathbf{x},0)=\eta(\mathbf{x},0)=0\}.$$

All these situations are related to integral geometry. For a general treatment of the basic theory, the classical reference is Federer's "Geometric Measure Theory" [9].

The aims of this paper are: 1) to re-formulate some known results in a modern language or in the standard form of probability theory; 2) to prove new results, such as computations in the exact models, variance computations in cases in which only first moments have been known, thus improving the statistical methods and 3) in some case, obtain Central Limit Theorems.

The structure of the paper is the following: In Section 2 we review without proofs some formulas for the moments of the relevant random variables. In Section 3 we study expectation, variance and asymptotic behavior of specular points. Section 4 is devoted to the study of the distribution of the normal to the level curve. Section 5 presents three numerical applications. Finally, in Section 6 we study dislocations of wavefronts following a paper by Berry & Dennis [6].

Some additional notation and hypotheses

λ_d is Lebesgue measure in $\mathbb{R}^d$, $\sigma_{d'}(B)$ the d'-dimensional Hausdorff measure of a Borel set B and M^T the transpose of a matrix M. (const) is a positive constant whose value may change from one occurrence to another.
If not otherwise stated, all random fields are assumed to be Gaussian and centered.

2 Rice formulas

We give here a quick account of Rice formulas, which allow to express the expectation and the higher moments of the size of level sets of random fields by means of some integral formulas. The simplest case occurs when both the dimension of the domain and the range are equal to 1, for which the first results date back to Rice [17] (see also Cramér and Leadbetter's book [8]). When the dimension of the domain and the range are equal but bigger than 1, the formula for the expectation is due to Adler [1] for stationary random fields. For a general treatment of this subject, the interested reader is referred to the book [4], Chapters 3 and 6, where one can find proofs and details.

Theorem 1 (Expectation of the number of crossings, $d = d' = 1$) *Let $\mathcal{W} = \{W(t) : t \in I\}$, I an interval in the real line, be a Gaussian process having $\mathcal{C}^1$-paths. Assume that $\mathrm{Var}(W(t)) \neq 0$ for every $t \in I$.*
Then:

$$\mathbb{E}\big(N_I^W(u)\big) = \int_I \mathbb{E}\big(|W'(t)|\big|W(t) = u\big)p_{W(t)}(u)dt. \tag{4}$$

Theorem 2 (Higher moments of the number of crossings, $d = d' = 1$) *Let $m \geq 2$ be an integer. Assume that $\mathcal{W}$ satisfies the hypotheses of Theorem 1 and moreover, for any choice of pairwise different parameter values $t_1, ..., t_m \in I$ the joint distribution of the k-random vector $(W(t_1), ..., W(t_m))$ has a density (which amounts to saying that its variance matrix is non-singular). Then:*

$$\mathbb{E}\big(N_I^W(u)(N_I^W(u)-1)...(N_I^W(u)-m+1)\big)$$
$$= \int_{I^m} \mathbb{E}\Big(\prod_{j=1}^{m} |W'(t_j)|\big|W(t_1) = ... = W(t_m) = u\Big)p_{W(t_1),...,W(t_m)}(u, ..., u)dt_1...dt_m. \tag{5}$$

Under certain conditions, the formulas in Theorems 1 and 2 can be extended to non-Gaussian processes.

Theorem 3 (Expectation, $d = d' > 1$) *Let $W : A \subset \mathbb{R}^d \to \mathbb{R}^d$ be a Gaussian random field, A an open set of $\mathbb{R}^d$, $\mathbf{u}$ a fixed point in $\mathbb{R}^{d'}$. Assume that*

- *the sample paths of W are continuously differentiable*
- *for each $\mathbf{t} \in A$ the distribution of $W(\mathbf{t})$ does not degenerate*
- $\mathbb{P}(\{\exists \mathbf{t} \in A : W(\mathbf{t}) = \mathbf{u}, \det(W'(\mathbf{t})) = 0\}) = 0$

Then for every Borel set B included in A

$$\mathbb{E}\big(N_B^W(\mathbf{u})\big) = \int_B \mathbb{E}[|\det W'(\mathbf{t})|\big|W(\mathbf{t}) = \mathbf{u}]p_{W(\mathbf{t})}(\mathbf{u})d\mathbf{t}.$$

If B is compact, both sides are finite.

The next proposition provides sufficient conditions (which are mild) for the third hypothesis in the above theorem to be verified (see again [4], Proposition 6.5).

Proposition 1 *Under the same conditions of the above theorem one has*

$$\mathbb{P}(\{\exists \mathbf{t} \in A : W(\mathbf{t}) = \mathbf{u}\,, \det(W'(\mathbf{t})) = 0\}) = 0$$

if

- $p_{X(\mathbf{t})}(\mathbf{x}) \leq C$ *for all* $\mathbf{x}$ *in some neighborhood of* $\mathbf{u}$*,*
- *at least one of the two following conditions is satisfied*
  a) the trajectories of W *are twice continuously differentiable*
  b)
  $$\alpha(\delta) = \sup_{x \in V(\mathbf{u})} \mathbb{P}\{|\det W'(\mathbf{t})| < \delta \big| W(\mathbf{t}) = \mathbf{x}\} \to 0$$
  as $\delta \to 0$ *where* $V(\mathbf{u})$ *is some neighborhood of* $\mathbf{u}$*.*

Theorem 4 (m-th factorial moment $d = d' > 1$) *Let* $m \geq 2$ *be an integer. Assume the same hypotheses as in Theorem 3 except for (iii) that is replaced by*

(iii') for $\mathbf{t}_1, ..., \mathbf{t}_m \in A$ *distinct values of the parameter, the distribution of*

$$\big(W(\mathbf{t}_1), ..., W(\mathbf{t}_m)\big)$$

does not degenerate in $(\mathbb{R}^d)^m$*.*

Then for every Borel set B contained in A, one has

$$\mathbb{E}\left[\big(N_B^W(\mathbf{u})\big)\big(N_B^W(\mathbf{u}) - 1\big)...\big(N_B^W(\mathbf{u}) - m + 1\big)\right] = \int_{B^m} \mathbb{E}\Big(\prod_{j=1}^{m} |\det\big(W'(\mathbf{t}_j)\big)| \big| W(\mathbf{t}_1) = ... = W(\mathbf{t}_m) = u\Big) p_{W(\mathbf{t}_1),...,W(\mathbf{t}_m)}(\mathbf{u}, ..., \mathbf{u}) d\mathbf{t}_1 ... d\mathbf{t}_m, \quad (6)$$

where both sides may be infinite.

When $d > d'$ we have the following formula :

Theorem 5 (Expectation of the geometric measure of the level set. $d > d'$) *Let* $W : A \to \mathbb{R}^{d'}$ *be a Gaussian random field,* A *an open subset of* $\mathbb{R}^d$*,* $d > d'$ *and* $\mathbf{u} \in \mathbb{R}^{d'}$ *a fixed point. Assume that:*

- *Almost surely the function* $\mathbf{t} \rightsquigarrow W(\mathbf{t})$ *is of class* $\mathcal{C}^1$*.*
- *For each* $\mathbf{t} \in A$*,* $W(\mathbf{t})$ *has a non-degenerate distribution.*
- $\mathbb{P}\{\exists \mathbf{t} \in A, W(\mathbf{t}) = \mathbf{u}, W'(\mathbf{t})$ *does not have full rank*$\} = 0$

Then, for every Borel set B contained in A, one has

$$\mathbb{E}\left(\sigma_{d-d'}(W,B)\right) = \int_B \mathbb{E}\left(\left[\det\left(W'(\mathbf{t})(W'(\mathbf{t}))^T\right)\right]^{1/2}\big|W(\mathbf{t}) = \mathbf{u}\right)\ p_{W(\mathbf{t})}(\mathbf{u})d\mathbf{t}. \tag{7}$$

If B is compact, both sides in (7) are finite.

The same kind of result holds true for integrals over the level set, as stated in the next theorem.

Theorem 6 (Expected integral on the level set) *Let W be a random field that verifies the hypotheses of Theorem 5. Assume that for each $\mathbf{t} \in A$ one has another random field $Y^{\mathbf{t}} : V \to \mathbb{R}^n$, where V is some topological space, verifying the following conditions:*

- *$Y^{\mathbf{t}}(v)$ is a measurable function of $(\omega, \mathbf{t}, v)$ and almost surely, $(\mathbf{t}, v) \rightsquigarrow Y^{\mathbf{t}}(v)$ is continuous.*
- *For each $\mathbf{t} \in A$ the random process $(\mathbf{s}, v) \to \big(W(\mathbf{s}), Y^{\mathbf{t}}(v)\big)$ defined on $W \times V$ is Gaussian.*

Moreover, assume that $g : A \times \mathcal{C}(V, \mathbb{R}^n) \to \mathbb{R}$ is a bounded function, which is continuous when one puts on $\mathcal{C}(V, \mathbb{R}^n)$ the topology of uniform convergence on compact sets. Then, for each compact subset B of A, one has

$$\mathbb{E}\Big(\int_{B\cap W^{-1}(\mathbf{u})} g(t, Y^{\mathbf{t}})\sigma_{d-d'}(W, d\mathbf{t})\Big)$$
$$= \int_B \mathbb{E}\big([\det(W'(t)(W'(t))^T)]^{1/2} g(\mathbf{t}, Y^{\mathbf{t}})\big|Z(\mathbf{t}) = \mathbf{u}\big).p_{Z(\mathbf{t})}(\mathbf{u})d\mathbf{t}. \tag{8}$$

3 Specular points and twinkles

3.1 Number of roots

Let $W(\mathbf{t}) : \mathbb{R}^d \to \mathbb{R}^d$ be a zero mean stationary Gaussian field. If W satisfies the conditions of Theorem 3 one has:

$$\mathbb{E}\big(N_A^W(\mathbf{u})\big) = |A|\mathbb{E}[|\det(W'(\mathbf{0}))|]p_{W(\mathbf{0})}(\mathbf{u}).$$

where $|A|$ denotes the Lebesgue measure of A.

For $d = 1$, $N_A^W(u)$ is the number of crossings of the level u and the formula becomes

$$\mathbb{E}\big(N_{[0,T]}^W(u)\big) = \frac{T}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\, e^{-\frac{u^2}{2\lambda_0}}, \tag{9}$$

where

$$\lambda_i = \int_0^\infty \lambda^i d\mu(\lambda) \qquad i = 0, 2, 4, \ldots,$$

μ being the spectral mesure of W.

Formula (9) is in fact the one S.O. Rice wrote in the 40's see [17].

3.2 Number of specular points

We consider first the one-dimensional static case with the longuet-Higgins approximation (2) for the number of specular points, that is:

$$SP_2(I) = \#\{x \in I : Y(x) = W'(x) - kx = 0\}$$

We assume that the Gaussian process $\{W(x) : x \in \mathbb{R}\}$ has $\mathcal{C}^2$ paths and $\mathrm{Var}(W'(x))$ is constant equal to, say, $v^2 > 0$. (This condition can always be obtained by means of an appropriate non-random time change, the "unit speed transformation") . Then Theorem 1 applies and

$$\begin{aligned} \mathbb{E}(SP_2(I)) = \int_I \mathbb{E}(|Y'(x)| \big| Y(x) = 0) p_{Y(x)}(0) dx &= \int_I \mathbb{E}(|Y'(x)|) \frac{1}{v} \varphi(\frac{kx}{v}) dx \\ &= \int_I G(-k, \sigma(x)) \frac{1}{v} \varphi(\frac{kx}{v}) dv, \quad (10) \end{aligned}$$

where $\sigma^2(x)$ is the variance of $W''(x)$ and $G(\mu, \sigma) := \mathbb{E}(|Z|)$, Z with distribution $N(\mu, \sigma^2)$.

For the second equality in (10), in which we have erased the condition in the conditional expectation, take into account that since $\mathrm{Var}(W'(x))$ is constant, for each x the random variables $W'(x)$ and $W''(x)$ are independent (differentiate under the expectation sign and use the basic properties of the Gaussian distribution).

An elementary computation gives:

$$G(\mu, \sigma) = \mu[2\Phi(\mu/\sigma) - 1] + 2\sigma\varphi(\mu/\sigma),$$

where $\varphi(.)$ and $\Phi(.)$ are respectively the density and the cumulative distribution functions of the standard Gaussian distribution.

When the process $W(x)$ is also stationary, $v^2 = \lambda_2$ and $\sigma^2(x)$ is constant equal to λ_4. If we look at the total number of specular points over the whole line, we get

$$\mathbb{E}(SP_2(\mathbb{R})) = \frac{G(k, \sqrt{\lambda_4})}{k} \quad (11)$$

which is the result given by [14] (part II, formula (2.14) page 846). Note that this quantity is an increasing function of $\frac{\sqrt{\lambda_4)}}{k}$.

Since in the longuet-Higgins approximation $k \approx 0$, one can write a Taylor expansion having the form:

$$\mathbb{E}(SP_2(\mathbb{R})) \simeq \sqrt{\frac{2\lambda_4}{\pi}} \frac{1}{k} \Big(1 + \frac{1}{2} \frac{k^2}{\lambda_4} + \frac{1}{24} \frac{k^4}{\lambda_4^2} + \ldots\Big) \quad (12)$$

Let us turn to **the variance** of the number of specular points, under some additional restrictions. First of all, we assume for this computation that the given process $\{W(x) : x \in \mathbb{R}\}$ is stationary with covariance function $\mathbb{E}(W(x)W(y)) = \Gamma(x-y)$. Γ is assumed to have enough regularity as to perform the computations below, the precise requirements on it being given in the statement of Theorem 7.

Putting for short $S = SP_2(\mathbb{R})$, we have:

$$\mathrm{Var}(S) = \mathbb{E}(S(S-1)) + \mathbb{E}(S) - [\mathbb{E}(S)]^2 \tag{13}$$

The first term can be computed using Theorem 2:

$$\mathbb{E}(S(S-1)) = \int\int_{\mathbb{R}^2} \mathbb{E}\big(|W''(x)-k||W''(y)-k| \,\big|\, W'(x) = kx, W'(y) = ky\big) \\ .p_{W'(x),W'(y)}(kx,ky)\ dxdy \tag{14}$$

where

$$p_{W'(x),W'(y)}(kx,ky) = \frac{1}{2\pi\sqrt{\lambda_2^2 - \Gamma''^2(x-y)}} \exp\Big[-\frac{1}{2}\frac{k^2(\lambda_2 x^2 + 2\Gamma''^2(x-y)xy + \lambda_2 y^2)}{\lambda_2^2 - \Gamma''^2(x-y)}\Big], \tag{15}$$

under the additional condition that the density (15) does not degenerate for $x \neq y$.

For the conditional expectation in (14) we perform a Gaussian regression of $W''(x)$ (resp. $W''(y)$) on the pair $(W'(x), W'(y))$. Putting $z = x-y$, we obtain:

$$\begin{aligned} W''(x) &= \theta_y(x) + a_y(x)W'(x) + b_y(x)W'(y) \\ a_y(x) &= -\frac{\Gamma'''(z)\Gamma''(z)}{\lambda_2^2 - \Gamma''^2(z)} \\ b_y(x) &= -\frac{\lambda_2\Gamma'''(z)}{\lambda_2^2 - \Gamma''^2(z)}, \end{aligned} \tag{16}$$

where $\theta_y(x)$ is Gaussian centered, independent of $(W'(x), W'(y))$. The regression of $W''(y)$ is obtained by permuting x and y.

The conditional expectation in (14) can now be rewritten as an unconditional expectation:

$$\mathbb{E}\Big\{\Big|\theta_y(x) - k\Gamma'''(z)\big[1 + \frac{\Gamma''(z)x + \lambda_2 y}{\lambda_2^2 - \Gamma''^2(z)}\big]\Big|\Big|\theta_x(y) - k\Gamma'''(-z)\big[1 + \frac{\Gamma''(-z)y + \lambda_2 x}{\lambda_2^2 - \Gamma''^2(z)}\big]\Big|\Big\} \tag{17}$$

Notice that the singularity on the diagonal $x = y$ is removable, since a Taylor expansion shows that for $z \approx 0$:

$$\Gamma'''(z)\big[1 + \frac{\Gamma''(z)x + \lambda_2 y}{\lambda_2^2 - \Gamma''^2(z)}\big] = \frac{1}{2}\frac{\lambda_4}{\lambda_2} x\big(z + O(z^3)\big). \tag{18}$$

One can check that

$$\sigma^2(z) = \mathbb{E}\big((\theta_y(x))^2\big) = \mathbb{E}\big((\theta_x(y))^2\big) = \lambda_4 - \frac{\lambda_2\Gamma'''^2(z)}{\lambda_2^2 - \Gamma''^2(z)} \tag{19}$$

and

$$\mathbb{E}\big(\theta_y(x)\theta_x(y)\big) = \Gamma^{(4)}(z) + \frac{\Gamma'''^2(z)\Gamma''(z)}{\lambda_2^2 - \Gamma''^2(z)}. \tag{20}$$

Moreover, if $\lambda_6 < +\infty$, performing a Taylor expansion one can show that as $z \approx 0$ one has

$$\sigma^2(z) \approx \frac{1}{4}\frac{\lambda_2\lambda_6 - \lambda_4^2}{\lambda_2} z^2 \tag{21}$$

and it follows that the singularity at the diagonal of the integrand in the right-hand side of (14) is also removable.

We will make use of the following auxiliary statement that we state as a lemma for further reference.The proof requires some calculations, but is elementary and we skip it. The value of $H(\rho;0,0)$ below can be found for example in [8], p. 211-212.

Lemma 1 *Let*

$$H(\rho;\mu,\nu) = \mathbb{E}(|\xi+\mu||\eta+\nu|)$$

where the pair (ξ,η) *is centered Gaussian,* $\mathbb{E}(\xi^2) = \mathbb{E}(\eta^2) = 1, \quad \mathbb{E}(\xi\eta) = \rho.$

Then,

$$H(\rho;\mu,\nu) = H(\rho;0,0) + R_2(\rho;\mu,\nu)$$

where

$$H(\rho;0,0) = \frac{2}{\pi}\sqrt{1-\rho^2} + \frac{2\rho}{\pi}\arctan\frac{\rho}{\sqrt{1-\rho^2}},$$

and

$$|R_2(\rho;\mu,\nu)| \le 3(\mu^2+\nu^2)$$

if $\mu^2+\nu^2 \le 1$ *and* $0 \le \rho \le 1$*.*

In the next theorem we compute the equivalent of the variance of the number of specular points, under certain hypotheses on the random process and with the longuet-Higgins asymptotic. This result is new and useful for estimation purposes since it implies that, as $k \to 0$, the coefficient of variation of the random variable S tends to zero at a known speed. Moreover, it will also appear in a natural way when normalizing S to obtain a Central Limit Theorem.

Theorem 7 *Assume that the centered Gaussian stationary process* $\mathcal{W} = \{W(x) : x \in \mathbb{R}\}$ *is* $\delta-$*dependent, that is,* $\Gamma(z) = 0$ *if* $|z| > \delta$*, and that it has* $\mathcal{C}^4$*-paths. Then, as* $k \to 0$ *we have:*

$$\mathrm{Var}(S) = \theta\frac{1}{k} + O(1). \tag{22}$$

where

$$\theta = \Big(\frac{J}{\sqrt{2}} + \sqrt{\frac{2\lambda_4}{\pi}} - \frac{2\delta\lambda_4}{\sqrt{\pi^3\lambda_2}}\Big),$$

$$J = \int_{-\delta}^{+\delta} \frac{\sigma^2(z) H\big(\rho(z); 0, 0\big)}{\sqrt{2\pi(\lambda_2 + \Gamma''(z))}} dz, \tag{23}$$

the functions H and $\sigma^2(z)$ have already been defined above, and

$$\rho(z) = \frac{1}{\sigma^2(z)} \Big[\Gamma^{(4)}(z) + \frac{\Gamma'''(z)^2 \Gamma''(z)}{\lambda_2^2 - \Gamma''^2(z)} \Big].$$

Remarks on the statement.

- The assumption that the paths of the process are of class $\mathcal{C}^4$ imply that $\lambda_8 < \infty$. This is well-known for Gaussian stationary processes (see for example [8]).
- Notice that since the process is δ-dependent, it is also δ'-dependent for any $\delta' > \delta$. It is easy to verify that when computing with such a δ' instead of δ one gets the same value for θ.
- One can replace the δ-dependence by some weaker mixing condition, such as

  $$\left|\Gamma^{(i)}(z)\right| \ \leq \ (const)(1 + |z|)^{-\alpha} \quad (0 \leq i \leq 4)$$

  for some $\alpha > 1$, in which case the value of θ should be replaced by:

  $$\theta = \sqrt{\frac{2\lambda_4}{\pi}} + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \Big[\frac{\sigma^2(z) H(\rho(z); 0, 0)}{2\sqrt{\lambda_2 + \Gamma''(z)}} - \frac{1}{\pi} \frac{\lambda_4}{\sqrt{\lambda_2}} \Big] dz.$$

  The proof of this extension can be performed following the same lines as the one we give below, with some additional computations.

Proof of the Theorem: We use the notations and computations preceding the statement of the theorem.
Divide the integral on the right-hand side of (14) into two parts, according as $|x - y| > \delta$ or $|x - y| \leq \delta$, i.e.

$$\mathbb{E}(S(S-1)) = \int\int_{|x-y|>\delta} ... + \int\int_{|x-y|\leq\delta} ... = I_1 + I_2. \tag{24}$$

In the first term, the $\delta-$dependence of the process implies that one can factorize the conditional expectation and the density in the integrand. Taking into account that for each $x \in \mathbb{R}$, the random variables $W''(x)$ and $W'(x)$ are independent, we obtain for I_1:

$$I_1 = \int\int_{|x-y|>\delta} \mathbb{E}\big(|W''(x) - k|\big)\mathbb{E}\big(|W''(y) - k|\big) p_{W'(x)}(kx) p_{W'(y)}(ky) dx dy.$$

On the other hand, we know that $W'(x)$ (resp. $W''(x)$) is centered normal with variance λ_2 (resp. λ_4). Hence:

$$I_1 = \big[G(k, \sqrt{\lambda_4})\big]^2 \int\int_{|x-y|>\delta} \frac{1}{2\pi\lambda_2} \exp\Big[-\frac{1}{2} \frac{k^2(x^2 + y^2)}{\lambda_2} \Big] dx dy,$$

To compute the integral on the right-hand side, notice that the integral over the whole x, y plane is equal to $1/k^2$ so that it suffices to compute the integral over the set $|x-y| \leq \delta$. Changing variables, this last one is equal to

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} dx \int_{x-\delta}^{x+\delta} \frac{1}{2\pi\lambda_2} \exp\Big[-\frac{1}{2}\frac{k^2(x^2+y^2)}{\lambda_2}\Big] dy \\
&= \frac{1}{2\pi k^2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2} du \int_{u-\frac{k\delta}{\sqrt{\lambda_2}}}^{u+\frac{k\delta}{\sqrt{\lambda_2}}} e^{-\frac{1}{2}v^2} dv \\
&= \frac{\delta}{k\sqrt{\lambda_2 \pi}} + O(1),
\end{aligned}
$$

where the last term is bounded if k is bounded (in fact, remember that we are considering an approximation in which $k \approx 0$). So, we can conclude that:

$$
\int\int_{|x-y|>\delta} \frac{1}{2\pi\lambda_2} \exp\Big[-\frac{1}{2}\frac{k^2(x^2+y^2)}{\lambda_2}\Big] dxdy = \frac{1}{k^2} - \frac{\delta}{k\sqrt{\lambda_2\pi}} + O(1)
$$

Replacing in the formula for I_1 and performing a Taylor expansion, we get:

$$
I_1 = \frac{2\lambda_4}{\pi}\Big[\frac{1}{k^2} - \frac{\delta}{k\sqrt{\lambda_2\pi}} + O(1)\Big]. \tag{25}
$$

Let us now turn to I_2.

Using Lemma 1 and the equivalences (18) and (21), whenever $|z| = |x-y| \leq \delta$, the integrand on the right-hand side of (14) is bounded by

$$
(const)\big[H(\rho(z); 0, 0) + k^2(x^2+y^2)\big].
$$

We divide the integral I_2 into two parts:

First, on the set $\{(x, y) : |x| \leq 2\delta, |x-y| \leq \delta\}$ the integral is clearly bounded by some constant.

Second, we consider the integral on the set $\{(x, y) : x > 2\delta, |x-y| \leq \delta\}$. (The symmetric case, replacing $x > 2\delta$ by $x < -2\delta$ is similar,that is the reason for the factor 2 in what follows).

We have (recall that $z = x - y$):

$$
\begin{aligned}
I_2 = O(1) + 2\int\int_{|x-y|\leq\delta, x>2\delta} &\sigma^2(z)\Big[H\big(\rho(z); 0, 0\big) + R_2\big(\rho(z); \mu, \nu\big)\Big] \\
&\times \frac{1}{2\pi\sqrt{\lambda_2^2 - \Gamma''^2(z)}} \exp\Big[-\frac{1}{2}\frac{k^2(\lambda_2 x^2 + 2\Gamma''(x-y)xy + \lambda_2 y^2)}{\lambda_2^2 - \Gamma''^2(x-y)}\Big] dxdy
\end{aligned}
$$

which can be rewritten as:

$$
\begin{aligned}
I_2 = O(1) + 2\int_{-\delta}^{\delta} \sigma^2(z)\Big[H\big(\rho(z);0,0\big) + R_2\big(\rho(z);\mu,\nu\big)\Big] \\
\times \frac{1}{\sqrt{2\pi(\lambda_2+\Gamma''(z))}} \exp\Big[-\frac{1}{2}\frac{k^2z^2}{\lambda_2-\Gamma''(z)}\big(\frac{\lambda_2}{\lambda_2+\Gamma''(z)}-\frac{1}{2}\big)\Big]dz \\
\times \int_{2\delta}^{+\infty} \frac{1}{\sqrt{2\pi(\lambda_2-\Gamma''(z))}} \exp\Big[-k^2\frac{(x-z/2)^2}{\lambda_2-\Gamma''(z))}\Big]dx
\end{aligned}
$$

In the inner integral we perform the change of variables

$$\tau = \frac{\sqrt{2}k(x-z/2)}{\sqrt{\lambda_2-\Gamma''(z)}}$$

so that it becomes:

$$\frac{1}{k\sqrt{2}}\int_{\tau_0}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\big(-\frac{1}{2}\tau^2\big)d\tau \;\;=\;\; \frac{1}{2\sqrt{2}}\frac{1}{k} + O(1) \tag{26}$$

where $\tau_0 = 2\sqrt{2}k(2\delta - z/2)/\sqrt{\lambda_2-\Gamma''(z)}$.

Notice that $O(1)$ in (26) is uniformly bounded, independently of k and z, since the hypotheses on the process imply that $\lambda_2 - \Gamma''(z)$ is bounded below by a positive number, for all z.

We can now replace in the expression for I_2 and we obtain

$$I_2 = O(1) + \frac{J}{k\sqrt{2}}. \tag{27}$$

To finish, put together (27) with (25), (24), (13) and (12). ∎

Corollary 1 *Under the conditions of Theorem 7, as $k \to 0$:*

$$\frac{\sqrt{\mathrm{Var}(S)}}{\mathbb{E}(S)} \approx \sqrt{\theta k}.$$

The proof follows immediately from the Theorem and the value of the expectation.

The computations made in this section are in close relation with the two results of Theorem 4 in Kratz and León [12]. In this paper the random variable $SP_2(I)$ is expanded in the Wiener-Hermite Chaos. The aforementioned expansion yields the same formula for the expectation and allows obtaining also a formula for the variance. However, this expansion is difficult to manipulate in order to get the result of Theorem 7.

Let us now turn to the Central Limit Theorem.

Theorem 8 *Assume that the process $\mathcal{W}$ satisfies the hypotheses of Theorem 7. In addition, we will assume that the fourth moment of the number of approximate specular points on an interval having length equal to* 1 *is bounded uniformly in* k, *that is*

$$\mathbb{E}\big(\big[SP_2([0,1])\big]^4\big) \leq (const) \tag{28}$$

Then, as $k \to 0$,

$$\frac{S - \sqrt{\frac{2\lambda_4}{\pi}}\frac{1}{k}}{\sqrt{\theta/k}} \Rightarrow \quad N(0,1),$$

where $\Rightarrow$ *denotes convergence in distribution.*

Remark.

One can give conditions for the added hypothesis (28) to hold true, which require some additional regularity for the process. Even though they are not nice, they are not costly from the point of view of physical models. For example, either one of the following conditions imply (28):

- The paths $x \rightsquigarrow W(x)$ are of class $\mathcal{C}^{11}$. (Use Theorem 3.6 of [4] with $m = 4$, applied to the random process $\{W'(x) : x \in \mathbb{R}\}$. See also [16]).
- The paths $x \rightsquigarrow W(x)$ are of class $\mathcal{C}^9$ and the support of the spectral measure has an accumulation point: apply Exercice 3.4 of [4] to get the non-degeneracy condition, Proposition 5.10 of [4] and Rice formula (Theorem 2) to get that the fourth moment of the number of zeros of $W''(x)$ is bounded.

Proof of the Theorem. Let α and β be real numbers satisfying the conditions $1/2 < \alpha < 1$, $\alpha + \beta > 1$, $2\alpha + \beta < 2$. It suffices to prove the convergence as k takes values on a sequence of positive numbers tending to 0. To keep in mind that the parameter is k, we use the notation

$$S(k) := S = SP_2(\mathbb{R})$$

Choose k small enough, so that $k^{-\alpha} > 2$ and define the sets of disjoint intervals, for $j = 0, \pm 1, \dots, \pm[k^{-\beta}]$:

$$U_j^k = \big((j-1)[k^{-\alpha}]\delta + \delta/2, j[k^{-\alpha}]\delta - \delta/2\big),$$
$$I_j^k = \big[j[k^{-\alpha}]\delta - \delta/2, j[k^{-\alpha}]\delta + \delta/2\big].$$

[.] denotes integer part.

Notice that each interval U_j^k has length $[k^{-\alpha}]\delta - \delta$ and that two neighboring intervals U_j^k are separated by an interval of length δ. So, the δ-dependence of the process implies that the random variables $SP_2(U_j^k)$, $j = 0, \pm 1, \dots, \pm[k^{-\beta}]$

are independent. A similar argument applies to $SP_2(I_j^k),\ j=0,\pm1,\dots,\pm[k^{-\beta}]$.

We denote:

$$T(k)=\sum_{|j|\le[k^{-\beta}]} SP_2(U_j^k),$$

Denote

$$V_k=\big(\operatorname{Var}(S(k))\big)^{-1/2}\approx\sqrt{k/\theta}$$

where the equivalence is due to Theorem 7.

We give the proof in two steps, which easily imply the statement. In the first one, we prove that

$$V_k[S(k)-T(k)]$$

tends to 0 in the L^2 of the underlying probability space.

In the second step we prove that

$$V_kT(k)$$

is asymptotically standard normal.

Step 1. We prove first that $V_k[S(k)-T(k)]$ tends to 0 in L^1. Since it is non-negative, it suffices to show that its expectation tends to zero. We have:

$$S(k)-T(k)=\sum_{|j|<[k^{-\beta}]} SP_2(I_j^k)+Z_1+Z_2$$

where

$$Z_1=SP_2\big(-\infty,-[k^{-\beta}].[k^{-\alpha}]\delta+\delta/2\big),$$

$$Z_2=SP_2\big([k^{-\beta}].[k^{-\alpha}]\delta-\delta/2,+\infty)\big).$$

Using the fact that $\mathbb{E}\big(SP_2^k(I)\big)\le(const)\int_I\varphi(kx/\sqrt{\lambda_2})dx$, one can show that

$$V_k\mathbb{E}(S(k)-T(k))\le(const)k^{1/2}\Big[\sum_{\ell=0}^{+\infty}\varphi\big(\frac{\ell[k^{-\alpha}]k\delta}{\sqrt{\lambda_2}}\big)+\int_{[k^{-\alpha}][k^{-\beta}]\delta}^{+\infty}\varphi(kx/\sqrt{\lambda_2})dx\Big].$$

which tends to zero as a consequence of the choice of α and β.

It suffices to prove that $V_k^2\operatorname{Var}\big(S(k)-T(k)\big)\to0$ as $k\to0$. Using independence:

$$\begin{aligned}\operatorname{Var}\big(S(k)-T(k)\big)&=\sum_{|j|<[k^{-\beta}]}\operatorname{Var}\big(SP_2(I_j^k)\big)+\operatorname{Var}(Z_1)+\operatorname{Var}(Z_2)\\&\le\sum_{|j|<[k^{-\beta}]}\mathbb{E}\big(SP_2(I_j^k)(SP_2(I_j^k)-1)\big)\\&+\mathbb{E}(Z_1(Z_1-1))+\mathbb{E}(Z_2(Z_2-1))+\mathbb{E}\big(S(k)-T(k)\big).\end{aligned}\tag{29}$$

We already know that $V_k^2\ \mathbb{E}\big(S(k)-T(k)\big)\to 0$. Using the hypotheses of the theorem, since each I_j^k can be covered by a fixed number of intervals of size one, we know that $\mathbb{E}\big(SP_2(I_j^k)(SP_2(I_j^k)-1)\big)$ is bounded by a constant which does not depend on k and j. We can write

$$V_k^2 \sum_{|j|<[k^{-\beta}]} \mathbb{E}\big(SP_2(I_j^k)(SP_2(I_j^k)-1)\big) \le (const)k^{1-\beta}$$

which tends to zero because of the choice of β. The remaining two terms can be bounded by calculations similar to those of the proof of Theorem 7.

Step 2. $T(k)$ is a sum of independent but not equi-distributed random variables. To prove it satisfies a Central Limit Theorem, we use a Lyapunov condition based of fourth moments. Set:

$$M_j^m := \mathbb{E}\Big\{\big[SP_2(U_j^k)-\mathbb{E}\big(SP_2(U_j^k)\big)\big]^m\Big\}$$

For the Lyapunov condition it suffices to verify that

$$\Sigma^{-4} \sum_{|j|\le[k^{-\beta}]} M_j^4 \to 0 \quad \text{as} \quad k\to 0, \tag{30}$$

where

$$\Sigma^2 := \sum_{|j|\le[k^{-\beta}]} M_j^2.$$

To prove (30), let us partition each interval U_j^k into $p=[k^{-\alpha}]-1$ intervals $I_1,...I_p$ of equal size δ. We have

$$\mathbb{E}\big(SP_1+\cdots+SP_p\big)^4 = \sum_{1\le i_1,i_2,i_3,i_4\le p} \mathbb{E}\big(SP_{i_1}SP_{i_2}SP_{i_3}SP_{i_4}\big), \tag{31}$$

where SP_i stands for $SP_2(I_i)-\mathbb{E}\big(SP_2(I_i)\big)$ Since the size of all the intervals is equal to δ and given the finiteness of fourth moments in the hypothesis, it follows that $\mathbb{E}\big(SP_{i_1}SP_{i_2}SP_{i_3}SP_{i_4}\big)$ is bounded.

On the other hand, notice that the number of terms which do not vanish in the sum of the right-hand side of (31) is $\mathcal{O}(p^2)$. In fact, if one of the indices in (i_1,i_2,i_3,i_4) differs more than 1 from all the other, then $\mathbb{E}\big(SP_{i_1}SP_{i_2}SP_{i_3}SP_{i_4}\big)$ vanishes. Hence,

$$\mathbb{E}\Big[SP_2(U_j^k)-\mathbb{E}\big(SP_2(U_j^k)\big)\Big]^4 \le (const)k^{-2\alpha}$$

so that $\sum_{|j|\le[k^{-\beta}]} M_j^4 = \mathcal{O}(k^{-2\alpha}k^{-\beta})$. The inequality $2\alpha+\beta<2$ implies Lyapunov condition. ∎

3.3 Number of specular points without approximation

We turn now to the computation of the expectation of the number of specular points $SP_1(I)$ defined by (1). This number of specular points is equal to the number of zeros of the process

$$Z(x) := W'(x) - m_1(x, W(x)) = 0,$$

where

$$m_1(x, w) = \frac{x^2 - (h_1 - w)(h_2 - w) + \sqrt{[x^2 + (h_1 - w)^2][x^2 + (h_2 - w)^2]}}{x(h_1 + h_2 - 2w)}.$$

Assume that the process $\{W(x) : x \in \mathbb{R}\}$ is Gaussian, centered, stationary, with $\lambda_0 = 1$. The process $Z(t)$ is not Gaussian and we must use a generalization of Theorem 1, namely Theorem 3.2 of [4] to get

$$\mathbb{E}\big(SP_1([a, b])\big) = \int_a^b dx \int_{-\infty}^{+\infty} \mathbb{E}\big(|Z'(x)|\big|Z(x) = 0, W(x) = w\big) \\ .\frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} \frac{1}{\sqrt{2\pi\lambda_2}} e^{-\frac{m_1^2(x,w)}{2\lambda_2}} dw. \quad (32)$$

For the conditional expectation in (32), notice that

$$Z'(x) = W''(x) - \frac{\partial m_1}{\partial x}(x, W(x)) - \frac{\partial m_1}{\partial w}(x, W(x))W'(x),$$

so that under the condition,

$$Z'(x) = W''(x) - K(x, w), \text{ where } K(x, w) = \frac{\partial m_1}{\partial x}(x, w)) + \frac{\partial m_1}{\partial w}(x, w))m_1(x, w).$$

Using that for each x, $W''(x)$ and $W'(x)$ are independent random variables and performing a Gaussian regression of $W''(x)$ on $W(x)$, we can write (32) in the form:

$$\mathbb{E}\big(SP_1([a, b])\big) \\ = \int_a^b dx \int_{-\infty}^{+\infty} \mathbb{E}\big(|\zeta - \lambda_2 w - K(x, w)|\big) \frac{1}{2\pi\sqrt{\lambda_2}} \exp\Big(-\frac{1}{2}(w^2 + \frac{m_1^2(x, w)}{\lambda_2})\Big) dw. \quad (33)$$

where ζ is centered Gaussian with variance $\lambda_4 - \lambda_2^2$. Formula (33) can still be rewritten as:

$$\mathbb{E}\big(SP_1([a, b])\big) \\ = \frac{1}{2\pi}\sqrt{\frac{\lambda_4 - \lambda_2^2}{\lambda_2}} \int_a^b dx \int_{-\infty}^{+\infty} G(m, 1) \exp\Big(-\frac{1}{2}(w^2 + \frac{m_1^2(x, w)}{\lambda_2})\Big) dw, \quad (34)$$

where

$$m = m(x,w) = \frac{\lambda_2 w + K(x,w)}{\sqrt{\lambda_4 - \lambda_2^2}}.$$

Notice that in (34), the integral is convergent as $a \to -\infty,\ b \to +\infty$ and that this formula is well-adapted to numerical approximation.

3.4 Number of twinkles

We give a proof of a result stated in [14] (part III pages 852-853).

We consider $\mathbf{Y}(x,t)$ defined by (3) and we limit ourselves to the case in which $W(x,t)$ is centered and stationary. If $\mathbf{Y}$ satisfies the conditions of Theorem 3, by stationarity we get

$$\mathbb{E}\big(\mathcal{TW}(I,T)\big) = T\int_I \mathbb{E}\big(|\det \mathbf{Y}'(x,t)|\big|\mathbf{Y}(x,t)=0\big)p_{\mathbf{Y}(x,t)}(0)dx. \qquad (35)$$

Since W_{xx} and W_x are independent with respective variances

$$\lambda_{40} = \int_{-\infty}^{+\infty} \xi^4 \mu(d\xi, d\tau) \qquad \lambda_{20} = \int_{-\infty}^{+\infty} \xi^2 \mu(d\xi, d\tau),$$

where μ is the spectral measure of the stationary random field $W(x,t)$. The density in (35) satisfies

$$p_{\mathbf{Y}(x,t)}(0) = (\lambda_{20})^{-1/2}\varphi\big(kx(\lambda_{20})^{-1/2}\big)(\lambda_{40})^{-1/2}\varphi\big(k(\lambda_{40})^{-1/2}\big).$$

On the other hand

$$\mathbf{Y}'(x,t) = \begin{pmatrix} W_{xx}(x,t) - k & W_{xt}(x,t) \\ W_{xxx}(x,t) & W_{xxt}(x,t) \end{pmatrix}.$$

Under the condition $\mathbf{Y}(x,t) = 0$, one has

$$|\det(\mathbf{Y}'(x,t))| = |W_{xt}(x,t)W_{xxx}(x,t)|.$$

Computing the regression it turns out that the conditional distribution of the pair $(W_{xt}(x,t), W_{xxx}(x,t))$ under the same condition, is the one of two independent centered gaussian random variables, with the following parameters:

$$\text{expectation } \frac{\lambda_{31}}{\lambda_{40}}k \text{ and variance } \lambda_{22} - \frac{\lambda_{31}^2}{\lambda_{40}}, \text{ for the first coordinate} \qquad (36)$$

$$\text{expectation } \frac{\lambda_{40}}{\lambda_{20}}kx \text{ and variance } \lambda_{60} - \frac{\lambda_{40}^2}{\lambda_{20}}, \text{ for the second coordinate} \qquad (37)$$

It follows that:

$$\mathbb{E}\Big(|\det(\mathbf{Y}'(x,t))|\big|\mathbf{Y}(x,t)=0\Big) = G\Big(\frac{\lambda_{31}}{\lambda_{40}}k, \sqrt{\lambda_{22} - \frac{\lambda_{31}^2}{\lambda_{40}}}\Big).G\Big(\frac{\lambda_{40}}{\lambda_{20}}kx, \sqrt{\lambda_{60} - \frac{\lambda_{40}^2}{\lambda_{20}}}\Big)$$

Summing up:

$$\frac{1}{T}\mathbb{E}\big(\mathcal{TW}(\mathbb{R},T)\big) =$$

$$\frac{1}{\sqrt{\lambda_{40}}}\varphi\left(\frac{k}{\sqrt{\lambda_{40}}}\right) G\Big(\frac{\lambda_{31}}{\lambda_{40}}k, \sqrt{\lambda_{22}-\frac{\lambda_{31}^2}{\lambda_{40}}}\Big) \int_{\mathbb{R}} G\Big(\frac{\lambda_{40}}{\lambda_{20}}kx, \sqrt{\lambda_{60}-\frac{\lambda_{40}^2}{\lambda_{20}}}\Big) \frac{1}{\sqrt{\lambda_{20}}}\varphi\left(\frac{kx}{\sqrt{\lambda_{20}}}\right)$$

$$= \sqrt{\frac{2}{\pi}}\varphi\left(\frac{k}{\sqrt{\lambda_{40}}}\right) G\Big(\frac{\lambda_{31}}{\lambda_{40}}k, \sqrt{\lambda_{22}-\frac{\lambda_{31}^2}{\lambda_{40}}}\Big) \frac{1}{k} \frac{\sqrt{\widetilde{\lambda}_6\lambda_{20}+\lambda_{40}}}{\sqrt{\lambda_{20}\lambda_{40}}} \frac{\sqrt{\widetilde{\lambda}_6}}{\widetilde{\lambda}_6+\lambda_{40}^2} \tag{38}$$

setting $\widetilde{\lambda}_6 := \lambda_{60} - \frac{\lambda_{40}^2}{\lambda_{20}}$ This result is equivalent to formula (4.7) of [14] (part III page 853).

3.5 Specular points in two dimensions

We consider at fixed time a random surface depending on two space variables x and y. The source of light is placed at $(0,0,h_1)$ and the observer is at $(0,0,h_2)$. The point (x,y) is a specular point if the normal vector $n(x,y) = (-W_x, -W_y, 1)$ to the surface at (x,y) satisfies the following two conditions:

- the angles with the incident ray $I = (-x, -y, h_1 - W)$ and the reflected ray $R = (-x, -y, h_2 - W)$ are equal (for short the argument (x,y) has been removed),
- it belongs to the plane generated by I and R.

Setting $\alpha_i = h_i - W$ and $r_i = \sqrt{x^2+y^2+\alpha_i}$, $i = 1,2$, as in the one-parameter case we have:

$$\begin{aligned} W_x &= \frac{x}{x^2+y^2}\frac{\alpha_2 r_1 - \alpha_1 r_2}{r_2 - r_1}, \\ W_y &= \frac{y}{x^2+y^2}\frac{\alpha_2 r_1 - \alpha_1 r_2}{r_2 - r_1}. \end{aligned} \tag{39}$$

When h_1 and h_2 are large, the system above can be approximated by

$$\begin{aligned} W_x &= kx \\ W_y &= ky, \end{aligned} \tag{40}$$

under the same conditions as in dimension 1.

Next, we compute the expectation of $SP_2(Q)$, the number of approximate specular points in the sense of (40) that are in a domain Q. In the remaining of this paragraph we limit our attention to this approximation and to the case in which $\{W(x,y) : (x,y) \in \mathbb{R}^2\}$ is a centered Gaussian stationary random field.

Let us define:

$$\mathbf{Y}(x,y) := \begin{pmatrix} W_x(x,y) - kx \\ W_y(x,y) - ky \end{pmatrix}. \tag{41}$$

Under very general conditions, for example on the spectral measure of $\{W(x,y) : x, y \in \mathbb{R}\}$ the random field $\{Y(x,y) : x, y \in \mathbb{R}\}$ satisfies the conditions of Theorem 3, and we can write:

$$\mathbb{E}\big(SP_2(Q)\big) = \int_Q \mathbb{E}\big(|\det \mathbf{Y}'(x,y)|\big) p_{\mathbf{Y}(x,y)}(\mathbf{0}) \; dxdy, \tag{42}$$

since for fixed (x,y) the random matrix $Y'(x,y)$ and the random vector $Y(x,y)$ are independent, so that the condition in the conditional expectation can be erased.

The density in the right hand side of (42) has the expression

$$\begin{aligned} p_{\mathbf{Y}(x,y)}(\mathbf{0}) &= p_{(W_x,W_y)}(kx,ky) \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{\lambda_{20}\lambda_{02} - \lambda_{11}^2}} \exp\Big[- \frac{k^2}{2(\lambda_{20}\lambda_{02} - \lambda_{11}^2)} \big(\lambda_{02}x^2 - 2\lambda_{11}xy + \lambda_{20}y^2 \big) \Big]. \end{aligned} \tag{43}$$

To compute the expectation of the absolute value of the determinant in the right hand side of (42), which does not depend on x, y, we use the method of [6]. Set $\Delta := \det \mathbf{Y}'(x,y) = (W_{xx} - k)(W_{yy} - k) - W_{xy}^2$.

We have

$$\mathbb{E}(|\Delta|) = \mathbb{E}\left[\frac{2}{\pi} \int_0^{+\infty} \frac{1 - \cos(\Delta t)}{t^2} dt \right]. \tag{44}$$

Define

$$h(t) := \mathbb{E}\left[\exp\big(it[(W_{xx} - k)(W_{yy} - k) - W_{xy}^2] \big) \right].$$

Then

$$\mathbb{E}(|\Delta|) = \frac{2}{\pi} \Big(\int_0^{+\infty} \frac{1 - \mathfrak{Re}[h(t)]}{t^2} dt \Big). \tag{45}$$

To compute $h(t)$ we define

$$A = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and Σ the variance matrix of $W_{xx}, W_{yy}, W_{x,y}$

$$\Sigma := \begin{pmatrix} \lambda_{40} & \lambda_{22} & \lambda_{31} \\ \lambda_{22} & \lambda_{04} & \lambda_{13} \\ \lambda_{31} & \lambda_{13} & \lambda_{22} \end{pmatrix}.$$

Let$\Sigma^{1/2}A\Sigma^{1/2} = P\ diag(\Delta_1, \Delta_2, \Delta_3)P^T$ where P is orthogonal. Then by a diagonalization argument

$$h(t) = e^{itk^2}$$
$$\mathbb{E}\Big(\exp\big[it\big((\Delta_1 Z_1^2 - k(s_{11}+s_{21})Z_1) + (\Delta_2 Z_2^2 - k(s_{12}+s_{22})Z_2) + (\Delta_3 Z_3^2 - k(s_{13}+s_{23})Z_3)\big)\big]\Big) \tag{46}$$

where (Z_1, Z_2, Z_3) is standard normal and s_{ij} are the entries of $\Sigma^{1/2}P^T$.

One can check that if ξ is a standard normal variable and τ, μ are real constants, $\tau > 0$:

$$\mathbb{E}\big(e^{i\tau(\xi+\mu)^2}\big) = (1-2i\tau)^{-1/2} e^{\frac{i\tau\mu^2}{(1-2i\tau)}} = \frac{1}{(1+4\tau^2)^{1/4}} \exp\Big[\frac{-2\tau}{1+4\tau^2} + i\big(\varphi + \frac{\tau\mu^2}{1+4\tau^2}\big)\Big],$$

where
$$\varphi = \frac{1}{2}\arctan(2\tau), \quad 0 < \varphi < \pi/4.$$

Replacing in (46), we obtain for $\mathfrak{Re}[h(t)]$ the formula:

$$\mathfrak{Re}[h(t)] = \Big[\prod_{j=1}^{3} \frac{d_j(t,k)}{\sqrt{1+4\Delta_j^2 t^2}}\Big] \cos\Big(\sum_{j=1}^{3} \big(\varphi_j(t) + k^2 t \psi_j(t)\big)\Big) \tag{47}$$

where, for $j = 1, 2, 3$:

- $d_j(t,k) = \exp\Big[-\dfrac{k^2t^2}{2}\ \dfrac{(s_{1j}+s_{2j})^2}{1+4\Delta_j^2 t^2}\Big]$,
- $\varphi_j(t) = \dfrac{1}{2}\arctan(2\Delta_j t), \quad 0 < \varphi_j < \pi/4$,
- $\psi_j(t) = \dfrac{1}{3} - t^2 \dfrac{(s_{1j}+s_{2j})^2\Delta_j}{1+4\Delta_j^2 t^2}$.

Introducing these expressions in (45) and using (43) we obtain a new formula which has the form of a rather complicated integral. However, it is well adapted to numerical evaluation.

On the other hand, this formula allows us to compute the equivalent as $k \to 0$ of the expectation of the total number of specular points under the longuet-Higgins approximation. In fact, a first order expansion of the terms in the integrand gives a somewhat more accurate result, that we state as a theorem:

Theorem 9
$$\mathbb{E}\big(SP_2(\mathbb{R}^2)\big) = \frac{m_2}{k^2} + O(1) \tag{48}$$

where

$$m_2 = \int_0^{+\infty} \frac{1 - \left[\prod_{j=1}^{3}(1+4\Delta_j^2 t^2)\right]^{-1/2} \cos\left(\sum_{j=1}^{3}\varphi_j(t)\right)}{t^2} dt$$
$$= \int_0^{+\infty} \frac{1 - 2^{-3/2}\left[\prod_{j=1}^{3}\left(A_j\sqrt{1+A_j}\right)\right]\left(1 - B_1B_2 - B_2B_3 - B_3B_1\right)}{t^2} dt, \tag{49}$$

where

$$A_j = A_j(t) = \left(1+4\Delta_j^2 t^2\right)^{-1/2}, \quad B_j = B_j(t) = \sqrt{(1-A_j)/(1+A_j)}.$$

Notice that m_2 only depends on the eigenvalues $\Delta_1, \Delta_2, \Delta_3$ and is easily computed numerically.

In Flores and León [10] a different approach was followed in search of a formula for the expectation of the number of specular points in the two-dimensional case, but their result is only suitable for Montecarlo approximation.

We now consider the **variance** of the total number of specular points in two dimensions, looking for analogous results to the one-dimensional case (i.e. Theorem 7 and its Corollary 1), in view of their interest for statistical applications. It turns out that the computations become much more involved. The statements on variance and speed of convergence to zero of the coefficient of variation that we give below include only the order of the asymptotic behavior in the longuet-Higgins approximation, but not the constant. However, we still consider them to be useful. If one refines the computations one can give rough bounds on the generic constants in Theorem 10 and Corollary 2 on the basis of additional hypotheses on the random field.

We assume that the real-valued, centered, Gaussian stationary random field $\{W(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$ has paths of class C^3, the distribution of $W'(\mathbf{0})$ does not degenerate (that is $\mathrm{Var}(W'(\mathbf{0}))$ is invertible). Moreover, let us consider $W''(\mathbf{0})$, expressed in the reference system xOy of $\mathbb{R}^2$ as the 2×2 symmetric centered Gaussian random matrix:

$$W''(\mathbf{0}) = \begin{pmatrix} W_{xx}(\mathbf{0}) & W_{xy}(\mathbf{0}) \\ W_{xy}(\mathbf{0}) & W_{yy}(\mathbf{0}) \end{pmatrix}$$

The function

$$\mathbf{z} \rightsquigarrow \Delta(\mathbf{z}) = \det\left[\mathrm{Var}\left(W''(0)\mathbf{z}\right)\right],$$

defined on $\mathbf{z} = (z_1, z_2)^T \in \mathbb{R}^2$, is a non-negative homogeneous polynomial of degree 4 in the pair z_1, z_2. We will assume the non-degeneracy condition:

$$\min\{\Delta(\mathbf{z}) : \|\mathbf{z}\| = 1\} = \underline{\Delta} > 0. \tag{50}$$

Theorem 10 *Let us assume that $\{W(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$ satisfies the above condtions and that it is also δ-dependent, $\delta > 0$, that is, $\mathbb{E}\big(W(\mathbf{x})W(\mathbf{y})\big) = 0$ whenever*

$\|\mathbf{x}-\mathbf{y}\| > \delta$.

Then, for k small enough:

$$\operatorname{Var}\big(SP_2(\mathbb{R}^2)\big) \;\leq\; \frac{L}{k^2},$$

where L is a positive constant depending upon the law of the random field.

A direct consequence of Theorems 9 and 10 is the following:

Corollary 2 *Under the same hypotheses of Theorem 10, for k small enough, one has:*

$$\frac{\sqrt{\operatorname{Var}\big(SP_2(\mathbb{R}^2)\big)}}{\mathbb{E}\big(SP_2(\mathbb{R}^2)\big)} \;\leq\; L_1 k$$

where L_1 is a new positive constant.

Proof of Theorem 10. For short, let us denote $T = SP_2(\mathbb{R}^2)$. We have:

$$\operatorname{Var}(T) = \mathbb{E}(T(T-1)) + \mathbb{E}(T) - [\mathbb{E}(T)]^2 \tag{51}$$

We have already computed the equivalents as $k \to 0$ of the second and third term in the right-hand side of (51). Our task in what follows is to consider the first term.

The proof is performed along the same lines as the one of Theorem 7, but instead of applying Rice formula for the second factorial moment of the number of crossings of a one-parameter random process, we need Theorem 4 for dimension $d = 2$. We write the factorial moment of order $m = 2$ in the form:

$$\begin{aligned}
&\mathbb{E}(T(T-1)) \\
&= \int\int_{\mathbb{R}^2\times\mathbb{R}^2} \mathbb{E}\Big(|\det \mathbf{Y}'(\mathbf{x})||\det \mathbf{Y}'(\mathbf{y})| \;\Big|\; \mathbf{Y}(\mathbf{x}) = \mathbf{0}, \mathbf{Y}(\mathbf{y}) = \mathbf{0}\Big) p_{\mathbf{Y}(\mathbf{x}),\mathbf{Y}(\mathbf{y})}(\mathbf{0},\mathbf{0})\; d\mathbf{x}d\mathbf{y} \\
&= \int\int_{\|\mathbf{x}-\mathbf{y}\|>\delta} \ldots\, d\mathbf{x}d\mathbf{y} + \int\int_{\|\mathbf{x}-\mathbf{y}\|\leq\delta} \ldots\, d\mathbf{x}d\mathbf{y} = J_1 + J_2.
\end{aligned}$$

For J_1 we proceed as in the proof of Theorem 7, using the δ-dependence and the evaluations leading to the statement of Theorem 9. We obtain:

$$J_1 = \frac{m_2^2}{k^4} + \frac{O(1)}{k^2}. \tag{52}$$

Let us show that for small k,

$$J_2 = \frac{O(1)}{k^2}. \tag{53}$$

In view of (51), (48) and (52) this suffices to prove the theorem.

We do not perform all detailed computations. The key point consists in evaluating the behavior of the integrand that appears in J_2 near the diagonal $\mathbf{x} = \mathbf{y}$, where the density $p_{\mathbf{Y}(\mathbf{x}),\mathbf{Y}(\mathbf{y})}(\mathbf{0},\mathbf{0})$ degenerates and the conditional expectation tends to zero.

For the density, using the invariance under translations of the law of $W'(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2$, we have:

$$\begin{aligned} p_{\mathbf{Y}(\mathbf{x}),\mathbf{Y}(\mathbf{y})}(\mathbf{0},\mathbf{0}) &= p_{W'(\mathbf{x}),W'(\mathbf{y})}(k\mathbf{x},k\mathbf{y}) \\ &= p_{W'(\mathbf{0}),W'(\mathbf{y}-\mathbf{x})}(k\mathbf{x},k\mathbf{y}) \\ &= p_{W'(\mathbf{0}),[W'(\mathbf{y}-\mathbf{x})-W'(\mathbf{0})]}(k\mathbf{x},k(\mathbf{y}-\mathbf{x})). \end{aligned}$$

Perform the Taylor expansion, for small $\mathbf{z} = \mathbf{y} - \mathbf{x} \in \mathbb{R}^2$:

$$W'(\mathbf{z}) = W'(\mathbf{0}) + W''(\mathbf{0})\mathbf{z} + O(\|\mathbf{z}\|^2).$$

Using the non-degeneracy assumption (50) and the fact that $W'(\mathbf{0})$ and $W''(\mathbf{0})$ are independent, we can show that for $\mathbf{x}, \mathbf{z} \in \mathbb{R}^2$, $\|\mathbf{z}\| \le \delta$:

$$p_{\mathbf{Y}(\mathbf{x}),\mathbf{Y}(\mathbf{y})}(\mathbf{0},\mathbf{0}) \le \frac{C_1}{\|\mathbf{z}\|^2} \exp\big[-C_2 k^2 (\|\mathbf{x}\| - C_3)^2\big]$$

where C_1, C_2, C_3 are positive constants.

Let us consider the conditional expectation. For each pair $\mathbf{x}, \mathbf{y}$ of different points in $\mathbb{R}^2$, denote by τ the unit vector $(\mathbf{y}-\mathbf{x})/\|\mathbf{y}-\mathbf{x}\|$ and $\mathbf{n}$ a unit vector orthogonal to τ. We denote respectively by $\partial_\tau \mathbf{Y}, \partial_{\tau\tau}\mathbf{Y}, \partial_{\mathbf{n}}\mathbf{Y}$ the first and second partial derivatives of the random field in the directions given by τ and $\mathbf{n}$.

Under the condition

$$\mathbf{Y}(\mathbf{x}) = \mathbf{0}, \mathbf{Y}(\mathbf{y}) = \mathbf{0}$$

we have the following simple bound on the determinant, based upon its definition and Rolle's Theorem applied to the segment $[\mathbf{x},\mathbf{y}] = \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\}$:

$$\big|\det \mathbf{Y}'(\mathbf{x})\big| \le \|\partial_\tau \mathbf{Y}(\mathbf{x})\| \|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{x})\| \le \|\mathbf{y}-\mathbf{x}\| \sup_{\mathbf{s}\in[\mathbf{x},\mathbf{y}]} \|\partial_{\tau\tau}\mathbf{Y}(\mathbf{s})\| \|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{x})\| \quad (54)$$

So,

$$\begin{aligned} &\mathbb{E}\Big(|\det \mathbf{Y}'(\mathbf{x})||\det \mathbf{Y}'(\mathbf{y})| \;\Big|\; \mathbf{Y}(\mathbf{x}) = \mathbf{0}, \mathbf{Y}(\mathbf{y}) = \mathbf{0}\Big) \\ &\quad \le \|\mathbf{y}-\mathbf{x}\|^2 \mathbb{E}\Big[\sup_{\mathbf{s}\in[\mathbf{x},\mathbf{y}]} \|\partial_{\tau\tau}\mathbf{Y}(\mathbf{s})\|^2 \|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{x})\| \|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{y})\| \Big| W'(\mathbf{x}) = k\mathbf{x}, W'(\mathbf{y}) = k\mathbf{y}\Big] \\ &\quad = \|\mathbf{z}\|^2 \mathbb{E}\Big[\sup_{\mathbf{s}\in[\mathbf{0},\mathbf{z}]} \|\partial_{\tau\tau}\mathbf{Y}(\mathbf{s})\|^2 \|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{0})\| \|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{z})\| \Big| W'(\mathbf{0}) = k\mathbf{x}, \frac{W'(\mathbf{z}) - W'(\mathbf{0})}{\|\mathbf{z}\|} = k\tau\Big], \end{aligned}$$

where the last equality is again a consequence of the stationarity of the random field $\{W(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$.

At this point, we perform a Gaussian regression on the condition. For the condition, use again Taylor expansion, the non-degeneracy hypothesis and the independence of $W'(\mathbf{0})$ and $W''(\mathbf{0})$. Then, use the finiteness of the moments of the supremum of bounded Gaussian processes (see for example [4], Ch. 2), take into account that $\|z\| \leq \delta$ to get the inequality:

$$\mathbb{E}\Big(|\det \mathbf{Y}'(\mathbf{x})||\det \mathbf{Y}'(\mathbf{y})| \big| \ \mathbf{Y}(\mathbf{x}) = \mathbf{0}, \mathbf{Y}(\mathbf{y}) = \mathbf{0}\Big) \leq C_4 \ \|\mathbf{z}\|^2 \big(1 + k\|\mathbf{x}\|\big)^4 \quad (55)$$

where C_4 is a positive constant. Summing up, we have the following bound for J_2:

$$\begin{aligned} J_2 &\leq C_1 C_4 \ \pi\delta^2 \int_{\mathbb{R}^2} \big(1 + k\|\mathbf{x}\|\big)^4 \exp\big[- C_2 k^2 (\|\mathbf{x}\| - C_3)^2 \big] d\mathbf{x} \\ &= C_1 C_4 \ 2\pi^2 \delta^2 \int_0^{+\infty} \big(1 + k\rho\big)^4 \exp\big[- C_2 k^2 (\rho - C_3)^2 \big] \rho d\rho \end{aligned} \quad (56)$$

Performing the change of variables $w = k\rho$, (53) follows. ∎

4 The distribution of the normal to the level curve

Let us consider a modeling of the sea $W(x, y, t)$ as a function of two space variables and one time variable. Usual models are centered Gaussian stationary with a particular form of the spectral measure μ that we discuss briefly below. We denote the covariance by $\Gamma(x, y, t) = \mathbb{E}(W(0, 0, 0)W(x, y, t))$.

In practice, one is frequently confronted with the following situation: several pictures of the sea on time over an interval $[0, T]$ are stocked and some properties or magnitudes are observed. If the time T and the number of pictures are large, and if the process is ergodic in time, the frequency of pictures that satisfy a certain property will converge to the probability of this property to happen at a fixed time.

Let us illustrate this with the angle of the normal to the level curve at a point "chosen at random". We consider first the number of crossings of a level u by the process $W(\cdot, y, t)$ for fixed t and y, defined as

$$N_{[0,M_1]}^{W(\cdot,y,t)}(u) = \#\{x : 0 \leq x \leq M_1;\ W(x, y, t) = u\}.$$

We are interested in computing the total number of crossings per unit time when integrating over $y \in [0, M_2]$ i.e.

$$\frac{1}{T} \int_0^T dt \int_0^{M_2} N_{[0,M_1]}^{W(\cdot,y,t)}(u)\ dy. \quad (57)$$

If the ergodicity assumption in time holds true, we can conclude that a.s.:

$$\frac{1}{T}\int_0^T dt \int_0^{M_2} N_{[0,M_1]}^{W(\cdot,y,t))}(u)\, dy \to M_1\mathbb{E}\big(N_{[0,M_1]}^{W(\cdot,0,0))}(u)\big) = \frac{M_1 M_2}{\pi}\sqrt{\frac{\lambda_{200}}{\lambda_{000}}}e^{-\frac{1}{2}\frac{u^2}{\lambda_{000}}},$$

where

$$\lambda_{abc} = \iint_{\mathbb{R}^3} \lambda_x^a \lambda_y^b \lambda_t^c d\mu(\lambda_x, \lambda_y, \lambda_t)$$

are the spectral moments.

Hence, on the basis of the quantity (57) for large T, one can make inference about the value of certain parameters of the law of the random field. In this example these are the spectral moments λ_{200} and λ_{000}.

If two-dimensional level information is available, one can work differently because there exists an interesting relationship with Rice formula for level curves that we explain in what follows.

We can write ($\mathbf{x} = (x, y)$):

$$W'(\mathbf{x}, t) = ||W'(\mathbf{x}, t)||(\cos\Theta(\mathbf{x}, t), \sin\Theta(\mathbf{x}, t))^T.$$

Instead of using Theorem 1, we can use Theorem 6, to write

$$\begin{aligned}\mathbb{E}\big[\int_0^{M_2} N_{[0,M_1]}^{W(\cdot,y,0)}(u)\ dy\big] &= \mathbb{E}\big[\int_{\mathcal{C}_Q(0,u)} |\cos\Theta(\mathbf{x}, 0)|\ d\sigma_1\big] \\ &= \frac{\sigma_2(Q)}{\pi}\sqrt{\frac{\lambda_{200}}{\lambda_{000}}}\, e^{-\frac{u^2}{2\lambda_{000}}}, \qquad (58)\end{aligned}$$

where $Q = [0, M_1] \times [0, M_2]$. We have a similar formula when we consider sections of the set $[0, M_1] \times [0, M_2]$ in the other direction. In fact (58) can be generalized to obtain the Palm distribution of the angle Θ.

Set $h_{\theta_1,\theta_2} = \mathbb{1}_{[\theta_1\,,\,\theta_2]}$, and for $-\pi \le \theta_1 < \theta_2 \le \pi$ define

$$\begin{aligned}F(\theta_2) - F(\theta_1) :&= \mathbb{E}\Big(\sigma_1(\{\mathbf{x} \in Q :\ W(\mathbf{x}, 0) = u\,;\, \theta_1 \le \Theta(\mathbf{x}, s) \le \theta_2\}) \\ &= \mathbb{E}\Big(\int_{\mathcal{C}_Q(u,s)} h_{\theta_1,\theta_2}(\Theta(\mathbf{x}, s))d\sigma_1(\mathbf{x})ds\Big) \qquad (59) \\ &= \sigma_2(Q)\mathbb{E}[h_{\theta_1,\theta_2}(\frac{\partial_y W}{\partial_x W})((\partial_x W)^2 + (\partial_y W)^2)^{1/2}]\,\frac{\exp(-\frac{u^2}{2\lambda_{00}})}{\sqrt{2\pi\lambda_{000}}}.\end{aligned}$$

Denoting $\Delta = \lambda_{200}\lambda_{020} - \lambda_{110}$ and assuming $\sigma_2(Q) = 1$ for ease of notation, we readily obtain

$$
\begin{aligned}
&F(\theta_2) - F(\theta_1) \\
&= \frac{e^{-\frac{u^2}{2\lambda_{000}}}}{(2\pi)^{3/2}(\Delta)^{1/2}\sqrt{\lambda_{000}}} \int_{\mathbb{R}^2} h_{\theta_1,\theta_2}(\Theta)\sqrt{x^2+y^2} e^{-\frac{1}{2\Delta}(\lambda_{02}x^2 - 2\lambda_{11}xy + \lambda_{20}y^2)} dxdy \\
&\qquad\qquad = \frac{e^{-\frac{u^2}{2\lambda_{00}}}}{(2\pi)^{3/2}(\lambda_+\lambda_-)^{1/2}\sqrt{\lambda_{000}}} \\
&\int_0^\infty \int_{\theta_1}^{\theta_2} \rho^2 \exp(-\frac{\rho^2}{2\lambda_+\lambda_-}(\lambda_+ \cos^2(\varphi-\kappa) + \lambda_- \sin^2(\varphi - \kappa)))d\rho d\varphi
\end{aligned}
$$

where $\lambda_- \leq \lambda_+$ are the eigenvalues of the covariance matrix of the random vector $(\partial_x W(0,0,0), \partial_y W(0,0,0))$ and κ is the angle of the eigenvector associated to γ^+. Remarking that the exponent in the integrand can be written as $1/\lambda_-(1-\gamma^2 sin^2(\varphi-\kappa))$ with $\gamma^2 := 1 - \lambda_+/\lambda_-$ and that

$$
\int_0^{+\infty} \rho^2 \exp\Big(-\frac{H\rho^2}{2}\Big) = \sqrt{\frac{\pi}{2H}}
$$

it is easy to get that

$$
F(\theta_2) - F(\theta_1) = (const) \int_{\theta_1}^{\theta_2} \big(1 - \gamma^2 sin^2(\varphi-\kappa)\big)^{-1/2} d\varphi.
$$

From this relation we get the density $g(\varphi)$ of the Palm distribution, simply by dividing by the total mass:

$$
g(\varphi) = \frac{\big(1-\gamma^2\sin^2(\varphi-\kappa)\big)^{-1/2}}{\int_{-\pi}^{\pi} \big(1-\gamma^2\sin^2(\varphi-\kappa)\big)^{-1/2} d\varphi.} = \frac{\big(1-\gamma^2\sin^2(\varphi-\kappa)\big)^{-1/2}}{4\mathcal{K}(\gamma^2)}, \qquad (60)
$$

Here $\mathcal{K}$ is the complete elliptic integral of the first kind. This density characterizes the distribution of the angle of the normal at a point chosen "at random" on the level curve.

In the case of a random field which is isotropic in (x, y), we have $\lambda_{200} = \lambda_{020}$ and moreover $\lambda_{110} = 0$, so that g turns out to be the uniform density over the circle (Longuet-Higgins says that over the contour the "distribution" of the angle is uniform (cf. [15], pp. 348)).

Let now $\mathcal{W} = \{W(\mathbf{x}, t) : t \in \mathbb{R}^+, \mathbf{x} = (x, y) \in \mathbb{R}^2\}$ be a stationary zero mean Gaussian random field modeling the height of the sea waves. It has the following spectral representation:

$$
W(x, y, t) = \int_\Lambda e^{i(\lambda_1 x + \lambda_2 y + \omega t)} \sqrt{f(\lambda_1, \lambda_2, \omega)} dM(\lambda_1, \lambda_2, \omega),
$$

where Λ is the manifold $\{\lambda_1^2 + \lambda_2^2 = \omega^4\}$ (assuming that the acceleration of gravity g is equal to 1) and M is a random Gaussian orthogonal measure defined on Λ (see [13]). This leads to the following representation for the covariance function

$$\Gamma(x,y,t) = \int_{\Lambda} e^{i(\lambda_1 x + \lambda_2 y + \omega t)} f(\lambda_1, \lambda_2, \omega) \sigma_2(dV)$$
$$= \int_{-\infty}^{\infty} \int_0^{2\pi} e^{i(\omega^2 x \cos\varphi + \omega^2 y \sin\varphi + \omega t)} G(\varphi, \omega) d\varphi d\omega,$$

where, in the second equation, we made the change of variable $\lambda_1 = \omega^2 \cos\varphi$, $\lambda_2 = \omega^2 \sin\varphi$ and $G(\varphi,\omega) = f(\omega^2\cos\varphi, \omega^2\sin\varphi, \omega) 2\omega^3$. The function G is called the "directional spectral function". If G does not depend of φ the random field W is isotropic in x, y.

Let us turn to ergodicity. For a given subset Q of $\mathbb{R}^2$ and each t, let us define

$$\mathcal{A}_t = \sigma\{W(x,y,t) : \tau > t\,; (x,y) \in Q\}$$

and consider the σ-algebra of t-invariant events $\mathcal{A} = \bigcap \mathcal{A}_t$. We assume that for each pair (x,y), $\Gamma(x,y,t) \to 0$ as $t \to +\infty$. It is well-known that under this condition, the σ-algebra $\mathcal{A}$ is trivial, that is, it only contains events having probability zero or one (see for example [8], Ch. 7).

This has the following important consequence in our context. Assume further that the set Q has a smooth boundary and for simplicity, unit Lebesgue measure. Let us consider

$$Z(t) = \int_{\mathcal{C}_Q(u,t)} H\big(\mathbf{x}, t\big) d\sigma_1(\mathbf{x}),$$

where $H\big(\mathbf{x},t\big) = \mathcal{H}\big(W(\mathbf{x},t), \nabla W(\mathbf{x},t)\big)$, where $\nabla W = (W_x, W_y)$ denotes gradient in the space variables and $\mathcal{H}$ is some measurable function such that the integral is well-defined. This is exactly our case in (59). The process $\{Z(t) : t \in \mathbb{R}\}$ is strictly stationary, and in our case has a finite mean and is Riemann-integrable. By the Birkhoff-Khintchine ergodic theorem ([8] page 151), a.s. as $T \to +\infty$,

$$\frac{1}{T}\int_0^T Z(s) ds \to \mathbb{E}_{\mathcal{B}}[Z(0)],$$

where $\mathcal{B}$ is the σ-algebra of t-invariant events associated to the process $Z(t)$. Since for each t, $Z(t)$ is $\mathcal{A}_t$-measurable, it follows that $\mathcal{B} \subset \mathcal{A}$, so that $\mathbb{E}_{\mathcal{B}}[Z(0)] = \mathbb{E}[Z(0)]$. On the other hand, Rice's formula yields (take into account that stationarity of $\mathcal{W}$ implies that $W(\mathbf{0},0)$ and $\nabla W(\mathbf{0},0)$ are independent):

$$\mathbb{E}[Z(0)] = \mathbb{E}[\mathcal{H}\big(u, \nabla W(\mathbf{0},0)\big) ||\nabla W(\mathbf{0},0)||] p_{W(\mathbf{0},0)}(u).$$

We consider now the CLT. Let us define

$$\mathcal{Z}(t) = \frac{1}{t}\int_0^t \big[Z(s) - \mathbb{E}(Z(0))\big]\,ds,$$

In order to compute second moments, we use Rice formula for integrals over level sets (cf. Theorem 6), applied to the vector-valued random field

$$X(\mathbf{x}_1, \mathbf{x}_2, s_1, s_2) = (W(\mathbf{x}_1, s_1), W(\mathbf{x}_2, s_2))^T.$$

The level set can be written as:

$$\mathcal{C}_{Q^2}(u,u) = \{(\mathbf{x}_1, \mathbf{x}_2) \in Q\times Q:\ X(\mathbf{x}_1, \mathbf{x}_2, s_1, s_2) = (u,u)\} \qquad \text{for } 0 \le s_1 \le t,\ 0 \le s_2 \le t.$$

So, we get

$$\operatorname{Var}\mathcal{Z}(t) = \frac{2}{t}\int_0^t (1-\frac{s}{t}) I(u,s)ds,$$

where

$$I(u,s) = \int_{Q^2} \mathbb{E}\Big[H(\mathbf{x}_1,0)H(\mathbf{x}_2,s)\|\nabla W(\mathbf{x}_1,0)\|\|\nabla W(\mathbf{x}_2,s)\|\Big|\ W(\mathbf{x}_1,0)=u\,; W(\mathbf{x}_2,s)=u\Big]$$
$$\times\, p_{W(\mathbf{x}_1,0),W(\mathbf{x}_2,s)}(u,u)d\mathbf{x}_1 d\mathbf{x}_2 \ - \Big(\mathbb{E}[\mathcal{H}\big(u,\nabla W(\mathbf{0},0)\big)||\nabla W(\mathbf{0},0)||]p_{W(\mathbf{0},0)}(u)\Big)^2$$

Assuming that the given random field is time-δ-dependent, that is, $\Gamma(x,y,t) = 0\ \forall\ (x,y)$, whenever $t > \delta$, we readily obtain

$$t\operatorname{Var}\mathcal{Z}(t) \to 2\int_0^\delta I(u,s)ds := \sigma^2(u) \quad \text{as } t\to\infty.$$

Using now a variant of the Hoeffding-Robbins Theorem [11] for sums of δ-dependent random variables, we get the CLT:

$$\sqrt{t}\mathcal{Z}(t) \Rightarrow N(0,\sigma^2(u)).$$

5 Numerical computations

Validity of the approximation for the number of specular points

In the particular case of stationary processes we have compared the exact expectation given by (32) with the approximation (10).

In full generality the result depends on h_1, h_2, λ_4 and λ_2. After scaling, we can assume for example that $\lambda_2 = 1$.

The main result is that, when $h_1 \approx h_2$, the approximation (10) is very sharp. For example with the value $(100, 100, 3)$ for (h_1, h_2, λ_4), the expectation of the total number of specular points over $\mathbb{R}$ is 138.2; using the approximation (11) the result with the exact formula is around 2.10^{-2} larger but it is almost hidden by the precision of the computation of the integral.

If we consider the case $(90, 110, 3)$, the results are respectively 136.81 and 137.7.

In the case $(100, 300, 3)$, the results differ significantly and Figure 1 displays the densities (32) and (10)

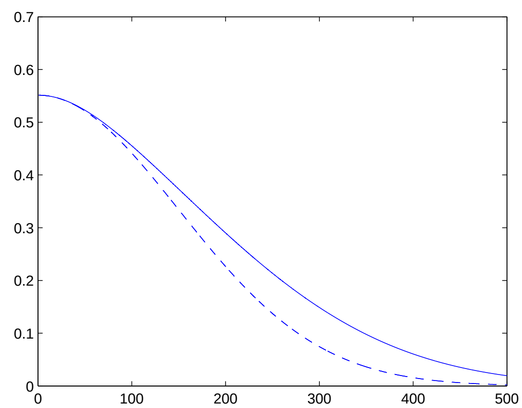

Figure 1: Intensity of specular points in the case $h_1 = 100, h_2 = 300, \lambda_4 = 3$. In solid line exact formula, in dashed line approximation (10)

Effect of anisotropy on the distribution of the angle of the normal to the curve

We show the values of the density given by (60) in the case of anisotropic processes $\gamma = 0.5$ and $\kappa = \pi/4$. Figure 2 displays the densities of the Palm distribution of the angle showing a large departure from the uniform distribution.

Specular points in dimension 2

We use a standard sea model with a Jonswap spectrum and spread function $\cos(2\theta)$. It corresponds to the default parameters of the Jonswap function of the toolbox WAFO [18]. The variance matrix of the gradient is equal to

$$10^{-4} \begin{pmatrix} 114 & 0 \\ 0 & 81 \end{pmatrix}$$

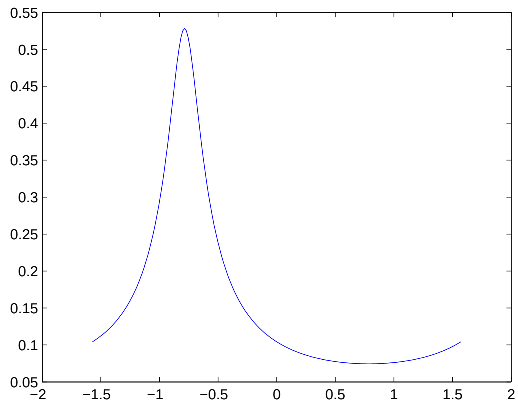

Figure 2: Density of the Palm distribution of the angle of the normal to the level curve in the case $\gamma = 0.5$ and $\kappa = \pi/4$

and the matrix Σ of Section 3.5 is

$$\Sigma = 10^{-4} \begin{pmatrix} 9 & 3 & 0 \\ 3 & 11 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The spectrum is presented in Figure 3

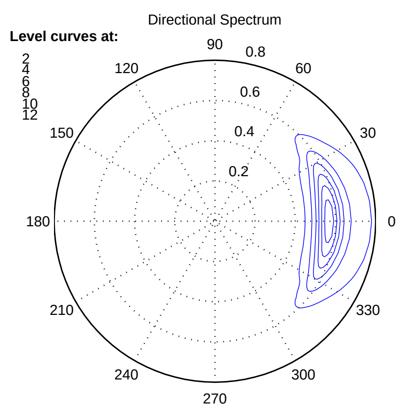

Figure 3: Directional Jonswap spectrum as obtained using the default options of Wafo

The integrand in (42) is displayed in Figure 4 as a function of the two space variables x, y. The value of the asymptotic parameter m_2 defining the expansion on the expectation of the numbers of specular points, see(48), is 2.52710^{-3}.

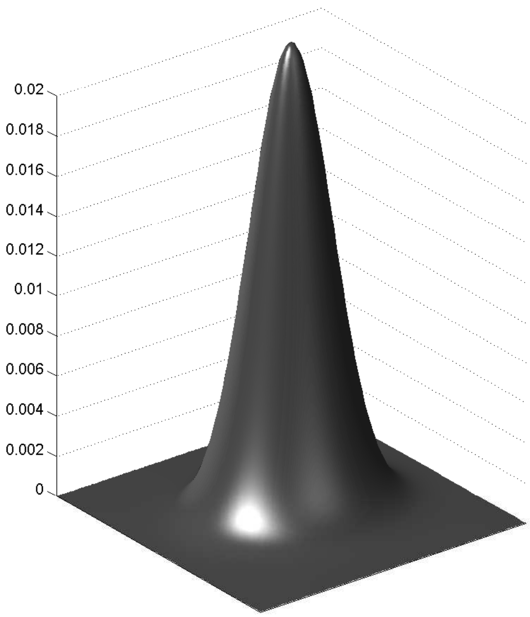

Figure 4: Intensity function of the specular points for the Jonswap spectrum

The Matlab programs used for these computations are available at

```
\protect\vrule width0pt\protect\href{http://www.math.univ-toulouse.fr/\string~azais/prog/pro
```

6 Application to dislocations of wavefronts

In this section we follow the article by Berry and Dennis [6]. As these authors, we are interested in dislocations of wavefronts. These are lines in space or points in the plane where the phase χ, of the complex scalar wave $\psi(\mathbf{x}, t) = \rho(\mathbf{x}, t)e^{i\chi(\mathbf{x},t)}$, is undefined, ($\mathbf{x} = (x_1, x_2)$) is a two dimensional space variable). With respect to light they are lines of darkness; with respect to sound, threads of silence.

It will be convenient to express ψ by means of its real and imaginary parts:

$$\psi(\mathbf{x},t) = \xi(\mathbf{x},t) + i\eta(\mathbf{x},t).$$

Thus the dislocations are the intersection of the two surfaces

$$\xi(\mathbf{x},t) = 0 \qquad \eta(\mathbf{x},t) = 0.$$

We assume an isotropic Gaussian model. This means that we will consider the wavefront as an isotropic Gaussian field

$$\psi(\mathbf{x},t) = \int_{\mathbb{R}^2} \exp\left(i[\langle \mathbf{k}\cdot\mathbf{x}\rangle - c|\mathbf{k}|t]\right)(\frac{\Pi(|\mathbf{k}|)}{|\mathbf{k}|})^{1/2} dW(\mathbf{k}),$$

where, $\mathbf{k} = (k_1,k_2)$, $|\mathbf{k}| = \sqrt{k_1^2+k_2^2}$, $\Pi(k)$ is the isotropic spectral density and $W = (W_1 + iW_2)$ is a standard complex orthogonal Gaussian measure on $\mathbb{R}^2$, with unit variance. Here we are interested only in $t = 0$ and we put $\xi(\mathbf{x}) := \xi(\mathbf{x},0)$ and $\eta(\mathbf{x}) := \eta(\mathbf{x},0)$.

We have, setting $k = |\mathbf{k}|$

$$\xi(\mathbf{x}) = \int_{\mathbb{R}^2} \cos(\langle \mathbf{k}\cdot\mathbf{x}\rangle)(\frac{\Pi(k)}{k})^{1/2} dW_1(\mathbf{k}) - \int_{\mathbb{R}^2} \sin(\langle \mathbf{k}\cdot\mathbf{x}\rangle)(\frac{\Pi(k)}{k})^{1/2} dW_2(\mathbf{k}) \quad (61)$$

and

$$\eta(\mathbf{x}) = \int_{\mathbb{R}^2} \cos(\langle \mathbf{k}\cdot\mathbf{x}\rangle)(\frac{\Pi(k)}{k})^{1/2} dW_2(\mathbf{k}) + \int_{\mathbb{R}^2} \sin(\langle \mathbf{k}\cdot\mathbf{x}\rangle)(\frac{\Pi(k)}{k})^{1/2} dW_1(\mathbf{k}) \quad (62)$$

The covariances are

$$\mathbb{E}\left[\xi(\mathbf{x})\xi(\mathbf{x}')\right] = \mathbb{E}\left[\eta(\mathbf{x})\eta(\mathbf{x}')\right] = \rho(|\mathbf{x}-\mathbf{x}'|) := \int_0^\infty J_0(k|\mathbf{x}-\mathbf{x}'|)\Pi(k)dk \quad (63)$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν. Moreover $\mathbb{E}\left[\xi(\mathbf{r}_1)\eta(\mathbf{r}_2)\right] = 0$.

Three dimensional model

In the case of a three dimensional Gaussian field, we have $\mathbf{x} = (x_1,x_2,x_3)$, $\mathbf{k} = (k_1,k_2,k_3)$,$k = |\mathbf{k}| = \sqrt{k_1^2+k_2^2+k_3^2}$ and

$$\psi(\mathbf{x}) = \int_{\mathbb{R}^3} \exp\left(i[\langle \mathbf{k}\cdot\mathbf{x}\rangle]\right)(\frac{\Pi(k)}{k^2})^{1/2} dW(\mathbf{k}).$$

In this case, we write the covariances in the form:

$$\mathbb{E}\left[\xi(\mathbf{r}_1)\xi(\mathbf{r}_2)\right] = 4\pi \int_0^\infty \frac{\sin(k|\mathbf{r}_1-\mathbf{r}_2|)}{k|\mathbf{r}_1-\mathbf{r}_2|}\,\Pi(k)dk. \quad (64)$$

The same formula holds true for the process η and also $\mathbb{E}\left[\xi(\mathbf{r}_1)\eta(\mathbf{r}_2)\right] = 0$ for any $\mathbf{r_1}, \mathbf{r_2}$, showing that the two coordinates are independent Gaussian fields .

6.1 Mean length of dislocation curves, mean number of dislocation points

Dimension 2: Let us denote $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$ a random field with values in $\mathbb{R}^2$, with coordinates $\xi(\mathbf{x}), \eta(\mathbf{x})$, which are two independent Gaussian stationary isotropic random fields with the same distribution. We are interested in the expectation of the number of dislocation points

$$d_2 := \mathbb{E}[\#\{\mathbf{x} \in S : \xi(\mathbf{x}) = \eta(\mathbf{x}) = 0\}],$$

where S is a subset of the parameter space having area equal to 1.

Without loss of generality we may assume that $\mathrm{Var}(\xi(\mathbf{x})) = \mathrm{Var}(\eta(\mathbf{x})) = 1$ and for the derivatives we set $\lambda_2 = \mathrm{Var}(\eta_i(\mathbf{x})) = \mathrm{Var}(\xi_i(\mathbf{x}))$, $i = 1, 2$. Then, using stationarity and the Rice formula (Theorem 3) we get

$$d_2 = \mathbb{E}[|\det(\mathbf{Z}'(\mathbf{x}))|/\mathbf{Z}(\mathbf{x}) = 0]p_{\mathbf{Z}(\mathbf{x})}(0),$$

The stationarity implies independence between $\mathbf{Z}(\mathbf{x})$ and $\mathbf{Z}'(\mathbf{x})$ so that the conditional expectation above is in fact an ordinary expectation. The entries of $\mathbf{Z}'(\mathbf{x})$ are four independent centered Gaussian variables with variance λ_2, so that, up to a factor, $|\det(\mathbf{Z}'(\mathbf{x}))|$ is the area of the parallellogram generated by two independent standard Gaussian variables in $\mathbb{R}^2$. One can easily show that the distribution of this volume is the product of independent square roots of a $\chi^2(2)$ and a $\chi^2(1)$ distributed random variables. An elementary calculation gives then: $\mathbb{E}[|\det(\mathbf{Z}'(\mathbf{x}))|] = \lambda_2$. Finally, we get

$$d_2 = \frac{1}{2\pi}\lambda_2$$

This quantity is equal to $\frac{K_2}{4\pi}$ in Berry and Dennis [6] notations, giving their formula (4.6).

Dimension 3: In the case, our aim is to compute

$$d_3 = \mathbb{E}[\mathcal{L}\{\mathbf{x} \in S : \xi(\mathbf{x}) = \eta(\mathbf{x}) = 0\}]$$

where S is a subset of $\mathbb{R}^3$ having volume equal to 1 and $\mathcal{L}$ is the length of the curve. Note that d_3 is denoted by d [6]. We use the same notations and remarks except that the form of the Rice's formula is (cf. Theorem 5)

$$d_3 = \frac{1}{2\pi}\mathbb{E}[(\det \mathbf{Z}'(\mathbf{x})\mathbf{Z}'(\mathbf{x})^T)^{1/2}].$$

Again

$$\mathbb{E}[(\det(\mathbf{Z}'(\mathbf{x})\mathbf{Z}'(\mathbf{x})^T)^{1/2}] = \lambda_2\mathbb{E}(V),$$

where V is the surface area of the parallelogram generated by two standard Gaussian variables in $\mathbb{R}^3$. A similar method to compute the expectation of this random area gives:

$$\mathbb{E}(V) = \mathbb{E}(\sqrt{\chi^2(3)}) \times \mathbb{E}(\sqrt{\chi^2(2)}) = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}} = 2$$

Leading eventually to

$$d_3 = \frac{\lambda_2}{\pi}.$$

In Berry and Dennis' notations [6] this last quantity is denoted by $\frac{k_2}{3\pi}$ giving their formula (4.5).

6.2 Variance

In this section we limit ourselves to dimension **2**. Let S be again a measurable subset of $\mathbb{R}^2$ having Lebesgue measure equal to 1. The computation of the variance of the number of dislocations points is performed using Theorem 4 to express

$$\mathbb{E}\big(N_S^{\mathbf{Z}}(\mathbf{0})\big(N_S^{\mathbf{Z}}(\mathbf{0})-1\big)\big) = \int_{S^2} A(\mathbf{s}_1,\mathbf{s}_2)d\mathbf{s}_1 d\mathbf{s}_2.$$

We assume that $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$ satisfies the hypotheses of Theorem 4 for $m = 2$. Then use

$$\mathrm{Var}\left(N_S^{\mathbf{Z}}(\mathbf{0})\right) = \mathbb{E}\big(N_S^{\mathbf{Z}}(\mathbf{0})\big(N_S^{\mathbf{Z}}(\mathbf{0})-1\big)\big) + d_2 - d_2^2.$$

Taking into account that the law of the random field is invariant under translations and orthogonal transformations of $\mathbb{R}^2$, we have

$$A(\mathbf{s}_1,\mathbf{s}_2) = A\big((0,0),(r,0)\big) = A(r) \quad \text{whith } r = \|\mathbf{s}_1 - \mathbf{s}_2\|,$$

The Rice's function $A(r))$, has two intuitive interpretations. First it can be viewed as

$$A(r) = \lim_{\epsilon\to 0} \frac{1}{\pi^2\epsilon^4}\mathbb{E}\big[N\big(B((0,0),\epsilon)\big) \times N\big(B((r,0),\epsilon)\big)\big].$$

Second it is the density of the Palm distribution (a generalization Horizontal window conditioning of [8]) of the number of zeroes of $\mathbf{Z}$ per unit of surface, locally around the point $(r,0)$ given that there is a zero at $(0,0)$.
$A(r)/d_2^2$ is called "correlation function" in [6].

To compute $A(r)$, we put $\xi_1, \xi_2, \eta_1\eta_2$ for the partial derivatives of ξ, η with respect to first and second coordinate.

and

$$\begin{aligned} A(r) &= \mathbb{E}\big[|\det \mathbf{Z}'(0,0)\det \mathbf{Z}'(r,0)|\big|\mathbf{Z}(0,0) = \mathbf{Z}(r,0) = \mathbf{0}_2\big]p_{\mathbf{Z}(0,0),\mathbf{Z}(r,0)}(\mathbf{0}_4) \\ &= \mathbb{E}\Big[\Big|\big(\xi_1\eta_2 - \xi_2\eta_1\big)(0,0)\big(\xi_1\eta_2 - \xi_2\eta_1\big)(r,0)\big)\Big|\Big|\mathbf{Z}(0,0) = \mathbf{Z}(r,0) = \mathbf{0}_2\Big] \\ &\qquad p_{\mathbf{Z}(0,0),\mathbf{Z}(r,0)}(\mathbf{0}_4) \end{aligned} \tag{65}$$

where $\mathbf{0}_p$ denotes the null vector in dimension p.

The density is easy to compute

$$p_{\mathbf{Z}(0,0),\mathbf{Z}(r,0)}(\mathbf{0}_4) = \frac{1}{(2\pi)^2(1-\rho^2(r))}, \text{ where } \rho(r) = \int_0^\infty J_0(kr)\Pi(k)dk.$$

We use now the same device as above to compute the conditional expectation of the modulus of the product of determinants, that is we write:

$$|w| = \frac{1}{\pi}\int_{-\infty}^{+\infty}(1-\cos(wt)t^{-2}dt. \tag{66}$$

and also the same notations as in [6]

$$\begin{cases} C := \rho(r) \\ E = \rho'(r) \\ H = -E/r \\ F = -\rho"(r) \\ F_0 = -\rho"(0) \end{cases}$$

The regression formulas imply that the conditional variance matrix of the vector

$$\mathbf{W} = \Big(\xi_1(\mathbf{0}), \xi_1(r,0), \xi_2(\mathbf{0}), \xi_2(r,0), \eta_1(\mathbf{0}), \eta_1(r,0), \eta_2(\mathbf{0}), \eta_2(r,0)\Big),$$

is given by

$$\Sigma = Diag\Big[\mathcal{A}, \mathcal{B}, \mathcal{A}, \mathcal{B}\Big]$$

with

$$\mathcal{A} = \begin{pmatrix} F_0 - \frac{E^2}{1-C^2} & F - \frac{E^2C}{1-C^2} \\ F - \frac{E^2C}{1-C^2} & F_0 - \frac{E^2}{1-C^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} F_0 & H \\ H & F_0 \end{pmatrix}$$

Using formula (66) the expectation we have to compute is equal to

$$\frac{1}{\pi^2}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 t_1^{-2} t_2^{-2}\Big[1 - \frac{1}{2}T(t_1,0) - \frac{1}{2}T(-t_1,0) - \frac{1}{2}T(0,t_2) - \frac{1}{2}T(0,-t_2) + \frac{1}{4}T(t_1,t_2) + \frac{1}{4}T(-t_1,t_2) + \frac{1}{4}T(t_1,-t_2) + \frac{1}{4}T(-t_1,-t_2)\Big] \tag{67}$$

where

$$T(t_1,t_2) = \mathbb{E}\big[\exp\big(i(w_1t_1 + w_2t_2)\big)\big]$$

with

$$w_1 = \xi_1(\mathbf{0})\eta_2(\mathbf{0}) - \eta_1(\mathbf{0})\xi_2(\mathbf{0}) = \mathbf{W}_1\mathbf{W}_7 - \mathbf{W}_3\mathbf{W}_5$$

$$w_2 = \xi_1(r,0)\eta_2(r,0) - \eta_1(r,0)\xi_2(r,0) = \mathbf{W}_2\mathbf{W}_8 - \mathbf{W}_4\mathbf{W}_6.$$

$T(t_1,t_2) = \mathbb{E}\big(\exp(i\mathbf{W}^T\mathcal{H}\mathbf{W})\big)$ where $\mathbf{W}$ has the distribution $N(0,\Sigma)$ and

$$\mathcal{H} = \begin{bmatrix} 0 & 0 & 0 & \mathcal{D} \\ 0 & 0 & -\mathcal{D} & 0 \\ 0 & -\mathcal{D} & 0 & 0 \\ \mathcal{D} & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{D} = \frac{1}{2}\begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}.$$

A standard diagonalization argument shows that

$$T(t_1,t_2) = \mathbb{E}\big(\exp(i\mathbf{W}^T\mathcal{H}\mathbf{W})\big) = \mathbb{E}\big(\exp(i\sum_{j=1}^{8}\lambda_j\xi_j^2)\big),$$

where the ξ_j's are independent with standard normal distribution and the λ_j are the eigenvalues of $\Sigma^{1/2}\mathcal{H}\Sigma^{1/2}$. Using the characteristic function of the $\chi^2(1)$ distribution:

$$\mathbb{E}\big(\exp(i\mathbf{W}^T\mathcal{H}\mathbf{W})\big) = \prod_{j=1}^{8}(1-2i\lambda_j)^{-1/2}. \tag{68}$$

Clearly

$$\Sigma^{1/2} = Diag\Big[\mathcal{A}^{1/2},\mathcal{B}^{1/2},\mathcal{A}^{1/2},\mathcal{B}^{1/2}\Big]$$

and

$$\Sigma^{1/2}\mathcal{H}\Sigma^{1/2} = \begin{bmatrix} 0 & 0 & 0 & \mathcal{M} \\ 0 & 0 & -\mathcal{M}^{\mathcal{T}} & 0 \\ 0 & -\mathcal{M} & 0 & 0 \\ \mathcal{M}^{\mathcal{T}} & 0 & 0 & 0 \end{bmatrix}$$

with $\mathcal{M} = \mathcal{A}^{1/2}\mathcal{D}\mathcal{B}^{1/2}$.

Let λ be an eigenvalue of $\Sigma^{1/2}\mathcal{H}\Sigma^{1/2}$ It is easy to check that λ^2 is an eigenvalue of $\mathcal{M}\mathcal{M}^T$. Respectively if λ_1^2 and λ_2^2 are the eigenvalues of $\mathcal{M}\mathcal{M}^T$, those of $\Sigma^{1/2}\mathcal{H}\Sigma^{1/2}$ are $\pm\lambda_1$(twice) and $\pm\lambda_2$ (twice).

Note that λ_1^2 and λ_2^2 are the eigenvalues of $\mathcal{M}\mathcal{M}^T = \mathcal{A}^{1/2}\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A}^{1/2}$ or equivalently, of $\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A}$. Using (68)

$$\mathbb{E}\big(\exp(i\mathbf{W}^T\mathcal{H}\mathbf{W})\big) = \big(1+4(\lambda_1^2+\lambda_2^2)+16\lambda_1^2\lambda_2^2\big)^{-1} = \big(1+4tr(\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A})+16\det(\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A})\big)^{-1}$$

where

$$\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A} = \frac{1}{4}\begin{bmatrix} t_1^2F_0(F_0-\frac{E^2}{1-C^2})+t_1t_2H(F-\frac{E^2C}{1-C^2}) & t_1^2F_0(F-\frac{E^2C}{1-C^2})+t_1t_2H(F_0-\frac{E^2}{1-C^2}) \\ t_1t_2H(F_0-\frac{E^2}{1-C^2})+t_2^2F_0(F-\frac{E^2C}{1-C^2}) & t_1t_2H(F-\frac{E^2C}{1-C^2})+t_2^2F_0(F_0-\frac{E^2}{1-C^2}) \end{bmatrix}$$

So,

$$4tr(\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A}) = (t_1^2+t_2^2)F_0(F_0-\frac{E^2}{1-C^2})+2t_1t_2H(F-\frac{E^2C}{1-C^2}) \tag{69}$$

$$16\det(\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A}) = t_1^2t_2^2\big[F_0^2-H^2\big]\big[(F_0-\frac{E^2}{1-C^2})^2-(F-\frac{E^2C}{1-C^2})^2\big] \tag{70}$$

giving

$$
\begin{aligned}
T(t_1,t_2) &= \mathbb{E}\Big(\exp(i\mathbf{W}^T\mathcal{H}\mathbf{W})\Big) \\
&= \Big(1+(t_1^2+t_2^2)F_0(F_0-\frac{E^2}{1-C^2})+2t_1t_2H(F-\frac{E^2C}{1-C^2}) \\
&\quad + t_1^2t_2^2\big[F_0^2-H^2\big]\big[(F_0-\frac{E^2}{1-C^2})^2-(F-\frac{E^2C}{1-C^2})^2\big]\Big)^{-1} \qquad (71)
\end{aligned}
$$

Performing the change of variable $t' = \sqrt{A_1}t$ with $A_1 = F_0(F_0 - \frac{E^2}{1-C^2})$ the integral (67) becomes

$$
\begin{aligned}
&\frac{A_1}{\pi^2}\int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{+\infty}dt_2t_1^{-2}t_2^{-2} \\
&\Big[1-\frac{1}{1+t_1^2}\frac{1}{1+t_2^2}+-\frac{1}{2}\Big\{\frac{1}{1+(t_1^2+t_2^2)-2A_2t_1t_2+t_1^2t_2^2Z}+\frac{1}{1+(t_1^2+t_2^2)+2A_2t_1t_2+t_1^2t_2^2Z}\Big\}\Big] \\
&= \frac{A_1}{\pi^2}\int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{+\infty}dt_2t_1^{-2}t_2^{-2} \\
&\Big[1-\frac{1}{1+t_1^2}-\frac{1}{1+t_2^2}+\frac{1+(t_1^2+t_2^2)+t_1^2t_2^2Z}{\Big(1+(t_1^2+t_2^2)+t_1^2t_2^2Z\Big)^2-4A_2^2t_1^2t_2^2}\Big] \qquad (72)
\end{aligned}
$$

where

$$
\begin{cases}
A_2 = \frac{H}{F_0}\frac{F(1-C^2)-E^2C}{F_0(1-C^2)-E^2} \\
Z = \frac{F_0^2-H^2}{F_0^2}\Big[1-(F-\frac{E^2C}{1-C^2})^2.(F_0-\frac{E^2}{1-C^2})^{-2}\Big].
\end{cases}
$$

In this form, and up to a sign change, this result is equivalent to Formula (4.43) of [6] (note that $A_2^2 = Y$ in [6]).

In order to compute the integral (72), first we obtain

$$
\int_{-\infty}^{\infty}\frac{1}{t_2^2}\big[1-\frac{1}{1+t_2^2}\big]dt_2 = \pi.
$$

We split the other term into two integrals, thus we have for the first one

$$
\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\big[\frac{1}{1+(t_1^2+t_2^2)-2A_2t_1t_2+t_1^2t_2^2Z}-\frac{1}{1+t_1^2}\big]dt_2 \\
&\quad = -\frac{1}{2(1+t_1^2)}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\frac{(1+t_1^2Z)t_2^2-2A_2t_1t_2}{1+t_1^2-2A_2t_1t_2+(1+t_1^2Z)t_2^2}dt_2 \\
&\quad = -\frac{1}{2(1+t_1^2)}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\frac{t_2^2-2Z_1t_1t_2}{t_2^2-2Z_1t_1t_2+Z_2}dt_2 = I_1,
\end{aligned}
$$

where $Z_2 = \frac{1+t_1^2}{1+Zt_1^2}$ and $Z_1 = \frac{A_2}{1+Zt_1^2}$.
Similarly for the second integral we get

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\Big[\frac{1}{1+(t_1^2+t_2^2)+2A_2t_1t_2+t_1^2t_2^2Z}-\frac{1}{1+t_1^2}\Big]dt_2$$

$$=-\frac{1}{2(1+t_1^2)}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\frac{t_2^2+2Z_1t_1t_2}{t_2^2+2Z_1t_1t_2+Z_2}\,dt_2=I_2$$

$$\begin{aligned}I_1+I_2&=-\frac{1}{2(1+t_1^2)}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\Big[\frac{t_2^2-2Z_1t_1t_2}{t_2^2-2Z_1t_1t_2+Z_2}+\frac{t_2^2+2Z_1t_1t_2}{t_2^2+2Z_1t_1t_2+Z_2}\Big]dt_2\\&=-\frac{1}{(1+t_1^2)}\int_{-\infty}^{\infty}\frac{t_2^2+(Z_2-4Z_1^2t_1^2)}{t_2^4+2(Z_2-2Z_1^2t_1^2)t_2^2+Z_2^2}\,dt_2\\&=-\frac{1}{(1+t_1^2)}\frac{\pi(Z_2-2Z_1^2t_1^2)}{Z_2\sqrt{(Z_2-Z_1^2t_1^2)}}.\end{aligned}$$

In the third line we have used the formula provided by the method of residues. In fact, if the polynomial X^2-SX+P with $P>0$ has not root in $[0,\infty)$, then

$$\int_{-\infty}^{\infty}\frac{t^2-\gamma}{t^4-St^2+P}\,dt=\frac{\pi}{\sqrt{P(-S+2\sqrt{P})}}(\sqrt{P}-\gamma).$$

In our case $\gamma=-(Z_2-4Z_1^2t_1^2)$, $S=-2(Z_2-2Z_1^2t_1^2)$ and $P=Z_2^2$.
Therefore we get

$$A(r)=\frac{A_1}{4\pi^3(1-C^2)}\int_{-\infty}^{\infty}\frac{1}{t_1^2}\Big[1-\frac{1}{(1+t_1^2)}\frac{(Z_2-2Z_1^2t_1^2)}{Z_2\sqrt{(Z_2-Z_1^2t_1^2)}}\Big]dt_1.$$

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