Central limit theorem for specular points

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1 Introduction

2 Expectation

3 Second moment

4 Central limit Theorem

Specular points

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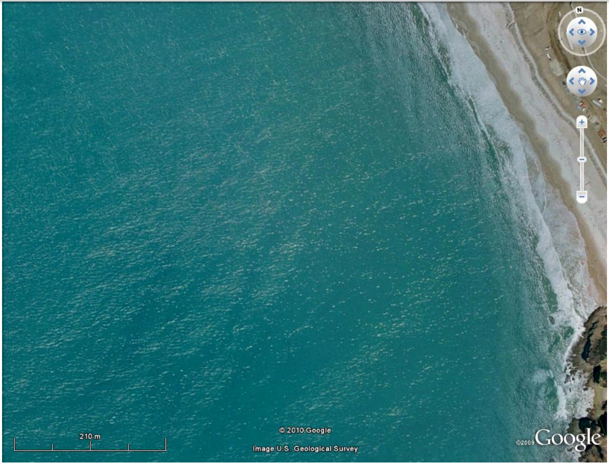
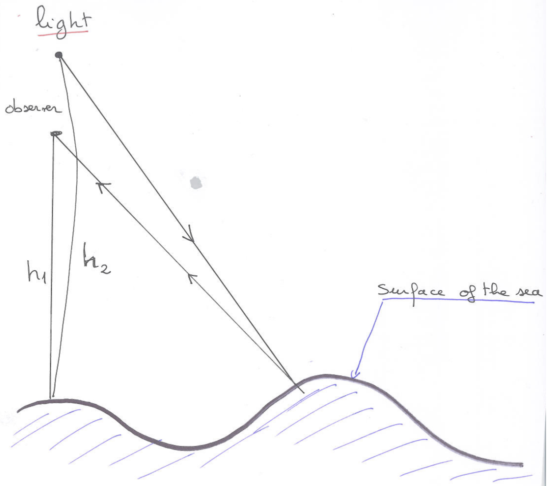


The cylindric model

Time is fixed and we consider one space variable only : x.

A source of light is placed at position $(0, h_1)$

The observer located at position $(0, h_1)$ may see the reflected light


light
observer
h_1
h_2
Surface of the sea

Characterization of specular points

The model for the elevation of the sea is a Gaussian stationary model $W(x)$ with regular paths (Bounded spectrum)
Reflexion rules imply that the point $(x, W(x))$ is a specular point if

$$W'(x) = \frac{\alpha_2 r_1 - \alpha_1 r_2}{x(r_2 - r_1)}$$

where $\alpha_i := h_i - W(x)$ and $r_i := \sqrt{x^2 + \alpha_i^2}, \quad i = 1, 2.$
$\rightarrow SP_1(A)$, A Borel set.

When h_1 and h_2 are big with respect to $W(x)$ and x, the equation can be approximated as

$$W'(x) = \frac{x}{2} \frac{h_1 + h_2}{h_1 h_2} \overset{d}{=} kx,$$

$\rightarrow SP_2(A)$,

1 Introduction

2 Expectation

3 Second moment

4 Central limit Theorem

Rice formula

Theorem (Azaïs-Wschebor, 2009 for example)

Let $X(t)$ a "good process" , N_u number of crossings of the level u on I.

$$\mathrm{E}(N_u) = \int_I \mathrm{E}\big(|X'(t)|\big|X(t) = u\big) p_{X(t)}(u)dt$$

$$\mathrm{E}(SP_2(I)) = \int_I \mathrm{E}\big(|W''(x) - k|\big|\ W'(x) = kx)\big) p_{W'(x)}(kx)\ dx$$

$$= G(-k, \sqrt{\lambda_4}) \frac{1}{\sqrt{\lambda_2}} \varphi(\frac{kx}{\sqrt{\lambda_2}}) dx,$$

$G(\mu, \sigma) := \mathrm{E}(|Z|)$, Z with distribution $N(\mu, \sigma^2)$.

Over the whole line, we get

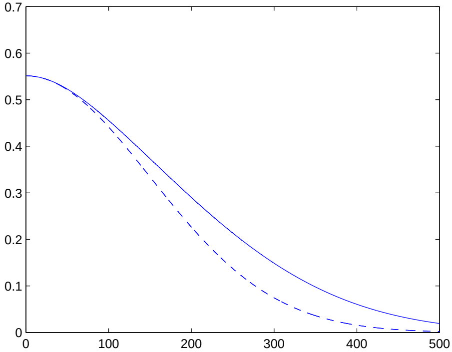
$$\mathrm{E}(SP_2(\mathbb{R})) = \frac{G(k, \sqrt{\lambda_4})}{k},$$

This was known by Longuet-Higgins (1957).

Quality of the approximation

$\mathrm{E}(SP_1)$ can be computed through a Rice formula(complicated result)
Comparison with $\mathrm{E}(SP_1)$ depends on h_1, h_2, λ_4 and λ_2, and after scaling, we can assume $\lambda_2 = 1$.

When $h_1 \approx h_2$, the approximation is very sharp. For example $h_1 = 90, h_2 = 110, \lambda_4 = 3$, the results are 136.81 and 137.7 for the whole line.



Intensity of specular points in the case $h_1 = 100, h_2 = 300, \lambda_4 = 3$. In solid line the exact formula SP_1, in dashed line the approximation SP_2

1 Introduction

2 Expectation

3 Second moment

4 Central limit Theorem

The Rice formula can be used to compute

$$\mathrm{Var}(S) = \mathrm{E}(S(S-1)) + \mathrm{E}(S) - [\mathrm{E}(S)]^2$$

with $S := SP_2(\mathbb{R})$ and

$$\begin{aligned} &\mathrm{E}(S(S-1)) \\ &= \int_{\mathbb{R}^2} \mathrm{E}\Big(|W''(x) - k||W''(y) - k| \Big| \; W'(x) = kx, W'(y) = ky \Big) \\ &\qquad\qquad\qquad\qquad \times \; p_{W'(x),W'(y)}(kx, ky) \; dxdy \end{aligned}$$

Theorem

Assume $W(x)$ *is* $\delta-$*dependent, and has* $\mathcal{C}^4$*-paths. Then, as* $k \to 0$ *we have:*

$$\mathrm{Var}(S) = \theta \frac{1}{k} + O(1),$$

$$\theta = \Big(\frac{J}{\sqrt{2}} + \sqrt{\frac{2\lambda_4}{\pi}} - \frac{2\delta\lambda_4}{\sqrt{\pi^3\lambda_2}} \Big),$$

$$J = \int_{-\delta}^{+\delta} \frac{\sigma^2(z) H\big(\rho(z); 0, 0\big)\big)}{\sqrt{2\pi(\lambda_2 + \Gamma''(z))}} dz.$$

H *and* $\sigma^2(z)$ *are some functions*

This implies that the coefficient of variation of S

$$\frac{\sqrt{\mathrm{Var}(S)}}{\mathrm{E}(S)} = O(\sqrt{k}) \text{ as } k \to 0.$$

So a natural question is to study the normalized variable

$$\frac{S - \mathrm{E}(S)}{\sqrt{\mathrm{Var}(S)}}$$

1 Introduction

2 Expectation

3 Second moment

4 Central limit Theorem

Theorem

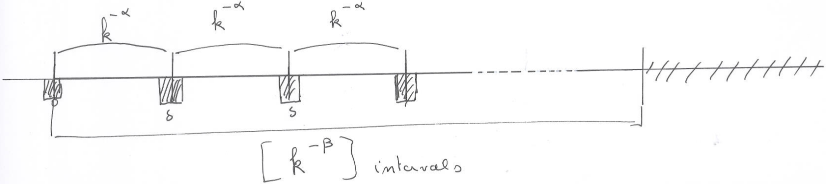
Assume δ dependence + uniform born of order 4 moment of the number of specular points on a bounded interval.

$$\mathrm{E}\left(\left[SP_2([a,a+1])\right]^4\right) \leq (const).$$

Then, as $k \to 0$,

$$\frac{S - \sqrt{\frac{2\lambda_4}{\pi}}\frac{1}{k}}{\sqrt{\theta/k}} \Rightarrow \quad N(0,1), \qquad \textit{in distribution.}$$

$k \to 0$!



The hatched part is peanuts.
The other is a sum on independent random variables but that are **not** equidistributed.
We used Lyapounov condition on the fourth moment.
Set

$$M_j^m := \mathrm{E}\Big\{\big[SP_2(U_j^k) - \mathrm{E}\big(SP_2(U_j^k)\big)\big]^m\Big\}.$$

we want

$$\Sigma^{-4} \sum_{|j| \leq [k^{-\beta}]} M_j^4 \to 0 \quad \text{as} \quad k \to 0,$$

where

$$\Sigma^2 := \sum_{|j| \leq [k^{-\beta}]} M_j^2.$$

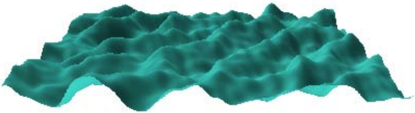
To prove it , we divide each interval U_j^k into $p = [k^{-\alpha}] - 1$ intervals $I_1, \ldots I_p$ of equal size δ. and we use the assumption.

Sufficient condition for having the fourth moment bounded

- The paths $x \rightsquigarrow W(x)$ are of class $\mathcal{C}^{11}$. (Use Theorem 3.6 of Azaïs Wschebor with $m = 4$, applied to the random process $\{W'(x) : x \in \mathbb{R}\}$).
- The paths $x \rightsquigarrow W(x)$ are of class $\mathcal{C}^{9}$ and the support of the spectral measure has an accumulation point: apply Ex.3.4, Proposition 5.10 and Th. 3.4. of Azaïs Wschebor to get that the fourth moment of the number of zeros of $W''(x)$ is bounded.

Generalizations

The condition of δ-dependence can be replaced by some mixing condition at the cost of some heavier presentation.
The studies on the first to moments has been generalized to two dimension specular points. Central Limit Theorem remains to be done.



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THANK-YOU