Estimation and joint testing of temporal and spatial patterns in Climate Change

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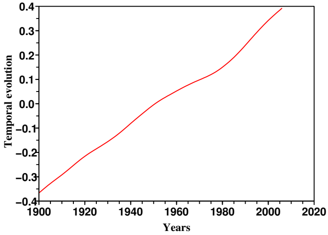
METEO FRANCE
Toujours un temps d'avance

INSTITUT de MATHEMATIQUES de TOULOUSE

Temperature

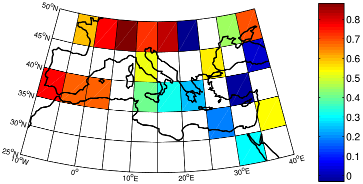
Time evolution

$\widehat{\mu}(.)$



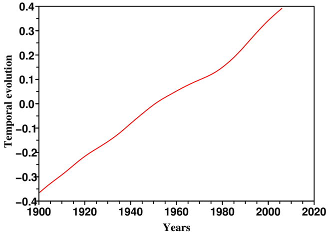
Spatial distribution

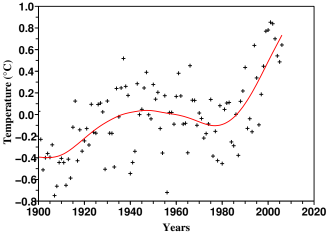
$\widehat{g}$



Temperature

Time evolution $\widehat{\mu}(.)$	Time evolution of the mean temperature over the domain





Statistical model

$y_{s,t}$ Is the measure (temperature) at space s and time (Year) t

$y_{s,t}$ is, for example, the mean summer minimal temperature. One measure per year

We assume independence between years (in fact an AR model is often used) and the model

$$y_{s,t} = m_s + g_s \, \mu(t) + \varepsilon_{s,t}, \qquad 1 \leq s \leq S, \quad 1 \leq t \leq T,$$

- m is the mean observation at each location,
- $g \in \mathbb{R}^S$, is the "spatial pattern"
- $\mu(.)$ is an unknown real function : the temporal pattern.
- data are Gaussian
- $$\mathrm{Cov}(\varepsilon_{s,t}, \varepsilon_{s',t}) = C_{s,s'}$$

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Many studies are conducted with a given "spatial pattern" from "in silico" experiment (ARPEGE CLIMAT) predicting the climate of 2100. Lead to Multivariate Analysis of variance test.

Unknown patterns

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Main motivations

Estimate g, μ ; Test H_0: “$g\mu(.) = 0$” vs H_1: “$g\mu(.) \neq 0$”.

Without smoothness assumption : no hope !

Forget, for a while the functional writing of μ.

$$y_{s,t} = m_s + g_s\,\mu_t + \varepsilon_{s,t}, \qquad 1 \le s \le S, \quad 1 \le t \le T,$$

with $\mathrm{Cov}(\varepsilon) = C$ unknown.

This model is **Not identifiable !** (in the sense that the likelihood is singular) .

As a matter of fact

- g is a deterministic evolution as a function of space
- C is a random evolution as a function of space.

that are redundant

Smoothness give identifiability

The estimation of $\mu(.)$ is performed using a spline penalty

$$\text{pen}(\mu(.)) = \int_0^1 \left(\mu^{(2)}(t)\right)^2 dt.$$

Remark : the second derivative can be replaced by higher derivatives.

We define penalized likelihood

$$pl(m, g, \mu(.), C) = l(m, g, \mu(.), C) + \rho \ \text{pen}(\mu(.)),$$

both for estimation and hypothesis testing.

Definition of the estimators

One defines the M.L. estimates:

$$(\widehat{m}, \widehat{g}, \widehat{\mu}(.), \widehat{C}) = \underset{\substack{(m,g,\mu(.),C) \\ g^T C^{-1} g = 1}}{\text{Argmin}} \; pl(m, g, \mu(.), C).$$

A convenient normalisation has been imposed to g

Case $S = 1$

Statistical model

$$y_t = m + \mu(x_t) + \varepsilon_t, \qquad 1 \leq t \leq T,$$

$$\widehat{\mu}(.) = \underset{\mu(.)}{\text{Argmin}} \sum_{t=1}^{T} \|y_t - \mu(x_t)\|^2 + \rho \int_0^1 \left(\mu''(t)\right)^2 dt.$$

Smoothing spline estimate (Wahba, 1990): (ρ known)

$$\mu(x) = \mu_1 + \mu_2\, x + \cdots + \mu_4 x^3 + \sum_{k=1}^{T} \mu_{4+k} (x - x_k)_+^3,$$

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Estimation of $(\mu_1, \ldots, \mu_{4+T})$ instead of $\widehat{\mu}(.)$.
$(\widehat{\mu}_1, \ldots, \widehat{\mu}_{4+T})$ can be computed explicitly.

$S > 1$, the model is identifiable

Proposition

$S \geq T + 2$, with probability one the penalized likelihood is bounded and continuous on a sphere. It has a maximum !!

Order 1 conditions

$$\widehat{m} = \frac{1}{T} Y \mathbb{1}_T, \tag{1}$$

$$\widehat{g} = \lambda.\mathrm{ev}_1\left(Y \Pi \Gamma_\rho \Pi Y^* \widehat{C}^{-1}\right), \quad \lambda \text{ given by } \widehat{g}^* \widehat{C}^{-1} \widehat{g} = 1, \tag{2}$$

$$\widehat{\mu} = \Gamma_\rho \Pi Y^* \widehat{C}^{-1} \widehat{g}, \tag{3}$$

$$\widehat{C} = \frac{1}{T} \left(Y\Pi - \widehat{g}\widehat{\mu}^*\right)\left(Y\Pi - \widehat{g}\widehat{\mu}^*\right)^*, \tag{4}$$

where

- $\widehat{\mu} = (\widehat{\mu}(x_1), \ldots, \widehat{\mu}(x_T))$,
- $\Pi = (I_T - \frac{1}{T} \mathbb{1}_T \mathbb{1}_T^*)$,
- Γ_ρ is the matrix of a scalar product in $\mathbb{R}^T$ of the vectors of the spline basis
- $\mathrm{ev}_1(M)$ the first eigenvector of the matrix M.

Computation of $(\widehat{g}, \widehat{\mu}, \widehat{C})$

An iterative algorithm can be constructed by successive solutions of Equations (1-4) gives $\widehat{g}_n, \widehat{\mu}_n, \widehat{C}_n$
Proof : The algorithm is monotone : the likeihood increases. By compactness, the likelihood converges. It remains to prove that the sequence of estimators cannot admit two accumulation points. This is true exept for Y in a negligible set.

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Proposition: convergence

With probability 1 For $T \geq S + 2$, With probability 1 the likelihood converges. If the sequence $(\widehat{g}_n, \widehat{\mu}_n, \widehat{C}_n)$ converges, it converges to a solution of (1-4)

Proof : The algorithm is monotone : the likeihood increases. By compactness, the likelihood converges. It remains to prove that the sequence of estimators cannot admit two accumulation points. This is true exept for Y in a negligible set.

Penalized Likelihood Ratio Test

One tests H_0: "$g\mu(.) = 0$" vs H_1: "$g\mu(.) \neq 0$",
based on the variable v (ρ being fixed):

$$v = \min_{H_1} \; pl(m, g, \mu(.), C) - \min_{H_0} \; pl(m, C).$$

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Let $\mathcal{D}_{m,C}$ the distribution of v under H_0.

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$\mathcal{D}$ is easy to simulate via bootstrap.

Case ρ not fixed

- Case $S = 1$: smoothing splines can be linked with Linear Mixed Model (Wang, 1998).
  Tests have been proposed without specifying a value for ρ (Crainiceanu et al., 2005)
- Case $S > 1$: a similar link still occurs, but the Model is no longer linear (the dependence in g is quadratic).
  Here is another way for testing H_0, potentially attractive.

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