Some applications of an implicit formula for the maximum of a Gaussian random field

Joint work with Mario Wschebor

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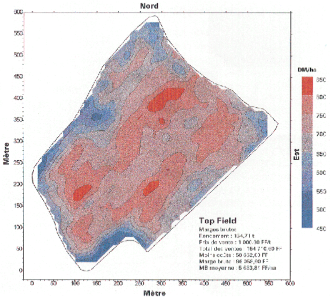
1. The distribution of the maximum
   - The implicit formula
2. The regularity of the density
3. Non-asymptotic bounds
4. Second order study

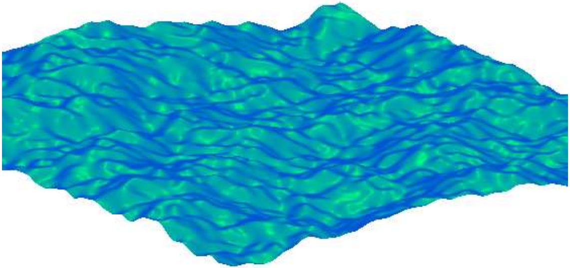
Signal + noise model

Spatial Statistics often uses " signal + noise model", for example : Representation of the yield per unit by GPS harvester .



Is there only noise or some region with higher fertility ? ? A good statistics is the maximum (of the absolute value)

Gaussian sea modeling



General results

Let $X(t)$ a real-valued (often Gaussian) random field and

$$F_M(x) = P(M_T \leq x)$$

the distribution function of its maximum.
The computation of $F_M(x)$ by means of a closed formula is known only in a very restricted number of cases : Brownian (Bridge), Ornstein-Ulhenbeck etc...

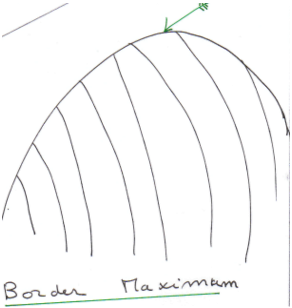
General inequalities Borell, Sudakov, Tsirelson are fundamental for the mathematical theory but numerically weak

An example in the simplest case The Brownian motion where some parameters are known

u	true values of $\mathbb{P}(M_W > u)$	Borell's b. mean	Borell's b. median
2	0.045	0.4855	0.2077
3	0.0027	0.0885	0.0347
4	6.33 10^{-5}	5.93 10^{-3}	1.98 10^{-3}
5	5.73 10^{-7}	1.46 10^{-4}	4.32 10^{-5}

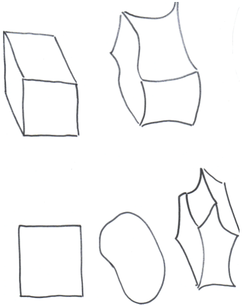
The implicit formula

Consider a realization with $M > u$, then necessarily there exists a local maximum or a border maximum above u



Border maximum : maximum in relative topology that is located on the border. Can be local or global

We consider parameter sets that are union of manifolds of dimension 1 to d + additional conditions



In fact results are simpler (and stronger) in term of the density $p_M(x)$ of the maximum.
Bounds for the distribution are obtained by integration.

Theorem

Let $M = \max_{t \in S} X(t)$. Under assumptions above , the distribution of M has the density

$$p_M(x) = \sum_{t \in S_0} \mathrm{E}\big(\mathbb{I}_{A_x} \big| X(t) = x \big) p_{X(t)}(x)$$

$$+ \sum_{j=1}^{d} \int_{S_j} \mathrm{E}\big(|\det(X_j''(t))| \, \mathbb{I}_{A_x} \big| X(t) = x, X_j'(t) = 0 \big) p_{X(t), X_j'(t)}(x, 0) \sigma_j(dt), \tag{1}$$

where $A_x = \{M \le x\}$.

What can we do with this implicit formula ?

The formula (1) is only implicit : M appears unfortunately on both sides.

Moreover terms like the expectation of the modulus of the determinant is hard to compute.

Dimension 1

When the parameter t is a scalar variable that varies in $[0,1]$, the formula (1) reads

Theorem

Suppose that X is a Gaussian process with $\mathcal{C}^2$ paths and such for all $s,t,s\neq t\in[0,1]$, $X(s),X(t),X'(t)$ and $X(t),X'(t),X''(t)$ admit a joint density. Then M has a **continuous** *density p_M given for every u by*

$$p_M(u) = \mathbb{P}(M\le u\big|X(0)=u)p_{X(0)}(u) + \mathbb{P}(M\le u\big|X(1)=u)p_{X(1)}(u) + \int_0^1 \mathrm{E}\big(X''^-(t)\ \mathbf{1}_{M\le u}\big|X(t)=u,X'(t)=0\big)p_{X(t),X'(t)}(u,0)dt \quad (2)$$

Using **induction**

Theorem

Assume that $X(t)$ has $\mathcal{C}^{2k}$ sample paths and satisfies a non-degeneracy condition .
Then, F_M is of class $\mathcal{C}^k$ and its successive derivatives can be computed by induction using an extension of the preceding theorem.

This goes far beyond the general result given by Tsirelson's Theorem that who proved a general theorem (1975) on the density P_M for general processes

Proof

Differentiating, the "**Bad Guy**" is the term

$$\mathbb{P}\{M \leq u \big| X(t) = u, X'(t) = 0\}$$

We use **Regression** and **Desingularization** arguments

$$X(s) = b^t(s)X(t) + c^t(s)X'(t) + \frac{(s-t)^2}{2}X^t(s) \quad s \in [0,1] \quad s \neq t.$$

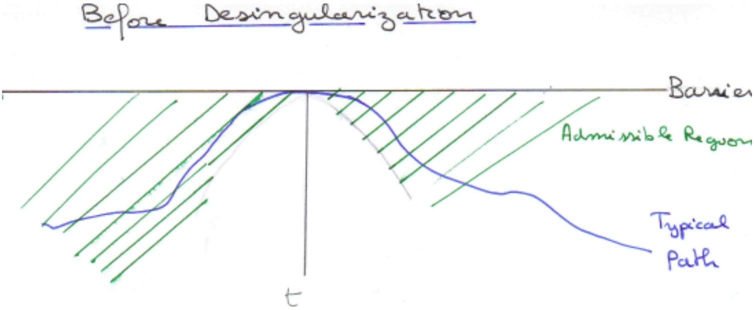
Under our hypotheses $X^t(s)$ is a "nice" process that admits a differentiable extension at zero.

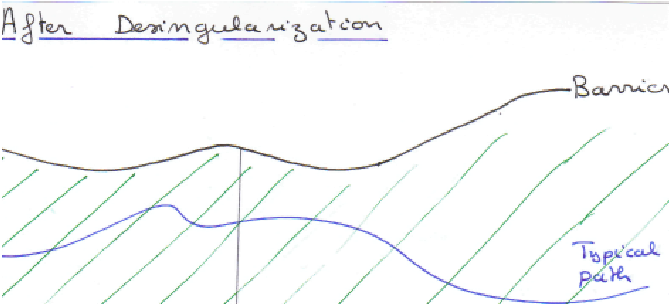
$$\{M \le u \big| X(t) = u, X'(t) = 0\}$$

can be translated as

$$X^t(s) \le b^t(s)u \quad \text{where } b^t(s) \text{ is some function}$$

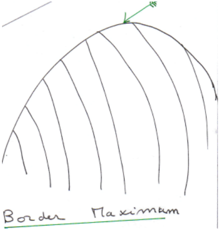
Induction begins using a generalization of the first theorem.




After Desingularization
Barrier
Typical path

In dimension greater than 1 the desingularization argument is not so nice and results are weaker but still stronger than those of Tsirelson's Theorem.

The implicit formula can be turned into a bound by replacing the condition $\{M \leq x\}$ by $\{X''_T(t)$ definite negative $\}$, and $\{X'_N(t)$ extended outwards$\}$



$$\mathrm{E}\big[|\det(X''(t)|\ \mathbb{1}_{X''(t)\text{ d. negative}}\big] \leq \frac{1}{2}\Big[\mathrm{E}\big|\det(X''(t))\big| + (-1)^d \mathrm{E}\big[\det(X''(t))\big]\Big]$$

The term $\mathrm{E}\big[\det(X''(t))\big]$ appears in the computation of the expectation of the Euler Characteristic

Lemma

(Adler)

$$\mathrm{E}\Big(\det(X''(t))/X(t) = x, X'(t) = 0\Big) = \det\big(\mathrm{Var}(X'(t))\big)H_d(x)$$

where $H_d(x)$ *is the* d*th Hermite polynomial and* $\Lambda := \mathrm{Var}(X'(t))$

The computation of expectations of modulus of quadratic forms and determinants has received some attention in the recent years.

- By Fourier method : Berry and Dennis (2000)
- using Fourier and other methods Li and Wei (2009)
- Using analytic theory of random matrices Fyodorov (2006)

making it possible, in some cases, to do the computations.

computation of $\overline{p}_M$

$$p_M(x) = \sum_{t\in S_0} \mathrm{E}\big(\mathbb{I}_{X'(t) EO} \big| X(t) = x\big) p_{X(t)}(x)$$

$$+ \sum_{j=1}^{d} \int_{S_j} \mathrm{E}\big(|\det(X_j''(t))| \; \mathbb{I}_{X_N'(t) EO} \big| X(t) = x, X_j'(t) = 0\big) p_{X(t),X_j'(t)}(x,0)\sigma_j(dt),$$

A key point is the following
If X is stationary and isotropic with covariance $\rho(\|t-s\|^2)$ normalized by $\mathrm{Var}(X(t)) = 1$ and $\mathrm{Var}(X'(t)) = Id$
Then under the condition $\{X(t) = x, X'(t) = 0\}$

$$X"(t) = \sqrt{8\rho"}G + \xi\sqrt{\rho" - \rho'^2}Id + xId$$

Where G is a GOE matrix (Gaussian Orthogonal Ensemble), and ξ a standard normal independent variable.

Theorem

Assume that the random field $\mathcal{X}$ is centered, Gaussian, stationary and isotropic and is "regular" Let S have polyhedral shape (the faces are flat) . Then,

$$\overline{p}(x) = \varphi(x)\left\{\sum_{t\in S_0}\widehat{\sigma}_0(t) + \sum_{j=1}^{d_0}\left[\left(\frac{|\rho'|}{\pi}\right)^{j/2}H_j(x) + R_j(x)\right]g_j\right\} \tag{3}$$

- $g_j = \int_{S_j}\widehat{\theta}_j(t)\sigma_j(dt)$, $\widehat{\theta}_j(t)$ *is the normalized solid angle of the cone of the extended outward directions at t in the normal space with the convention $\theta_d(t) = 1$.*
  For convex or other usual polyhedra $\widehat{\theta}_j(t)$ is constant for $t \in S_j$,
- H_j *is the j th(probabilistic) Hermite polynomial.*

Theorem

- $R_j(x) = \left(\frac{2\rho''}{\pi|\rho'|}\right)^{\frac{j}{2}} \frac{\Gamma((j+1)/2}{\pi} \int_{-\infty}^{+\infty} T_j(v) \exp\left(-\frac{y^2}{2}\right) dy$

$$v := -(2)^{-1/2}\left((1-\gamma^2)^{1/2}y - \gamma x\right) \text{ with } \gamma := |\rho'|(\rho'')^{-1/2},$$

$$T_j(v) := \left[\sum_{k=0}^{j-1} \frac{H_k^2(v)}{2^k k!}\right] e^{-v^2/2} - \frac{H_j(v)}{2^j(j-1)!} I_{j-1}(v),$$

where

$$I_n(v) = 2e^{-v^2/2} \sum_{k=0}^{[\frac{n-1}{2}]} 2^k \frac{(n-1)!!}{(n-1-2k)!!} H_{n-1-2k}(v)$$
$$+ \mathbf{I}_{\{n \text{ even}\}} 2^{\frac{n}{2}} (n-1)!! \sqrt{2\pi}(1-\Phi(x))$$

Second order study

We go back to the implicit formula giving p_M and we study the bound $\overline{p}_M$ Let us look, for simplicity to the term of dimension $j = d$. When we remove $\mathbb{I}_{A_x}$ in

$$\mathrm{E}\Big(|\det(X''(t))| \ \mathbb{I}_{A_x} \Big| X(t) = x, X_j'(t) = 0 \Big)$$

it is easy to show that the logarithmic behavior (as defined below) is the same as

$$\mathbb{P}\Big(M \geq x \Big| X(t) = x, X_j'(t) = 0 \Big)$$

and using Borel Sudakov Tsirelson inequality we get

Theorem

Under conditions above + $\mathrm{Var}(X(t)) \equiv 1$ *Then*

$$\underline{\lim}_{x\to+\infty} -\frac{2}{x^2}\log\left[\widehat{p}_M(x) - p_M(x)\right] \geq 1 + \inf_{t\in S}\frac{1}{\sigma_t^2 + \overline{\lambda}(t)\kappa_t^2}$$

$$\sigma_t^2 := \sup_{s\in S\setminus\{t\}} \frac{\mathrm{Var}\big(X(s)/X(t), X'(t)\big)}{(1-r(s,t))^2}$$

and κ_t *is some geometrical characteristic et* $\Lambda_t = GEV(\Lambda(t))$

The right hand side is finite and > 1
$\widehat{p}_M$ stands for p_M^E, $\widehat{p}_M$ or the mean of the two quantities

Two examples of new results

Suppose that the process is stationary and isotropic with covariance $\rho(\|t-s\|^2)$ with normalization $\mathrm{Var}(X(t)) = 1$ $\mathrm{Var}(X'(t)) = Id$ and suppose that the parameter set is convex

if the covariance ρ is monotone

$$\underline{\lim}_{x\to+\infty} -\frac{2}{x^2}\log\left[\bar{p}_M(x) - p_M(x)\right] = 1 + \frac{1}{12\rho''(0)-1}$$

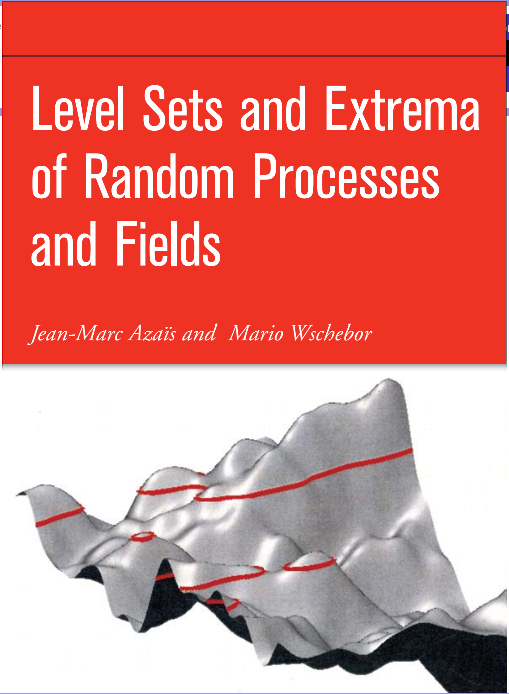
general case

$$\underline{\lim}_{x\to+\infty} -\frac{2}{x^2}\log\left[\widehat{p}_M(x) - p_M(x)\right] \geq 1 + 1/Z_\Delta$$

with

$$Z_\Delta := \sup_{z\in(0,\Delta]} \frac{1-\rho^2(z^2) - 4\rho'^2(z^2)z^2}{[1-\rho(z^2)]^2} + \max_{z\in(0,\Delta]} \frac{4\left[\rho'(z^2)^+ z\right]^2}{[1-\rho(z^2)]^2},$$

and Δ is the diameter of S.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