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★**Level sets and extrema of random processes and fields.**
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This book presents modern developments on the following two subjects: understanding the properties of level sets of a given random field $X = (X_t,\, t \in T)$ and analysis and computation of the distribution function of the random variable $M_T = \sup_{t\in T} X(t)$, provided that X is real-valued.

Chapter 1 of the book contains a number of fundamental classical results on stochastic processes, for example, Kolmogorov's consistency theorem and the 0-1 law for Gaussian processes, but a particular emphasis is placed on sufficient conditions for continuity, Hölder continuity and differentiability of trajectories of stochastic processes. Most of the results on path regularity are not restricted to the Gaussian case, and many apply to the multiparameter (i.e. random field) setting. The last section of this chapter contains Bulinskaya's sufficient condition for a one-parameter process not to have almost surely critical points in a given level set, plus an extension of Ylvisaker's theorem in the Gaussian case. Specifically, it is shown here that when the mean of the Gaussian process is bounded from below and its variance is bounded away from zero, the supremum of the process over a given fixed parameter set has probability distribution equal to the sum of an atom at infinity and a (possibly degenerate) probability measure on the reals with a locally bounded density. The end-of-chapter exercises include derivation of regularity properties of the paths of fractional Brownian motion and Brownian local time.

Chapter 2 opens with the proof of the latest (2002) refinement of the Slepian inequalities, due to W. V. Li and Q. M. Shao [Probab. Theory Related Fields **122** (2002), no. 4, 494–508; MR1902188 (2003b:60034)], where the difference between the cumulative distribution functions of two centered Gaussian n-dimensional vectors (with variances normalized to one and arbitrary $n \geq 2$) both evaluated at a given point $a \in \mathbb{R}^n$ is bounded above by the following sum:

$$\frac{1}{2\pi}\sum_{1\leq i<j\leq n}(\arcsin r_{ij}^X - \arcsin r_{ij}^Y)^+ \exp\Big(-\frac{a_i^2+a_j^2}{2(1+\rho_{ij})}\Big),$$

where r_{ij}^X and r_{ij}^Y are covariances between X_i and X_j and between Y_i and Y_j, respectively, and $\rho_{ij} = \max(|r_{ij}^X|, |r_{ij}^Y|)$. Two more related comparison lemmas are stated. One of these is the well-known Sudakov-Fernique inequality showing that if variances of arbitrary increments of a Gaussian process X are less than or equal to variances of similar increments of a Gaussian process Y then the mean of the supremum of X is less than or equal to the mean of the supremum of Y, provided that the two Gaussian processes are separable centered with almost surely bounded paths. Next the authors present the proof due to C. Borell of Ehrhard's inequality [C. R. Math. Acad. Sci. Paris **337** (2003), no. 10, 663–666; MR2030108 (2004k:60102)] valid for general Borel subsets of $\mathbb{R}^n$ (with no restrictions on the convexity of those sets). Namely, let γ_n be the standard Gaussian probability measure on $\mathbb{R}^n$. Then for any pair A and B of Borel sets in $\mathbb{R}^n$ and all $\lambda \in$

$(0, 1)$, the following inequality holds:

$$\Phi^{-1}(\gamma_n(\lambda A + (1-\lambda)B)) \geq \lambda\Phi^{-1}(\gamma_n(A)) + (1-\lambda)\Phi^{-1}(\gamma_n(B)).$$

The authors then derive a version of a Gaussian isoperimetric inequality and use it to prove the Borell-Sudakov-Tsirelson inequality, which gives an exponential bound for $P(|M_T - \mu(M_T)| > x)$, where M_T is the supremum of a Gaussian process over $[0, T]$ and $\mu(M_T)$ is the median of distribution of M_T. The next inequality for the tails of the distribution of the supremum is similar but involves the mean of M_T rather than the median and is due to Ibragimov, who proved the inequality using stochastic analysis tools. Chapter 2 concludes with the proof of Dudley's inequality, which establishes an upper bound on the mean of the supremum (of a possibly non-Gaussian process) in terms of an integral of a square-root of the logarithm of covering numbers.

Chapter 3 is entirely devoted to the treatment of Rice formulas for one-parameter processes and centers on integral representations of moments of the number of (up- and down-) crossings for both Gaussian and non-Gaussian processes having continuously differentiable sample paths. Formal proofs of these results are preceded by nice intuitive discussions, whereas at the end of the chapter the authors suggest a number of useful exercises.

Chapter 4 starts with the application of Rice formulas to derive bounds for the tails of the distribution of the maximum of one-parameter Gaussian processes with continuously differentiable sample paths and, in the stationary case, to subsequently characterize the asymptotic behavior of $P(M_T > x)$ as $x \to \infty$. This chapter also contains two detailed examples of statistical applications of the distribution of the maximum to genetics and to the study of mixtures of Gaussian distributions. In the first case the problem is that of testing that a given putative gene has no influence on a given quantitative trait within the classical framework of a linear model with i.i.d. errors. In the second case the problem is that of testing

$$H_0\colon\ Y \sim N(\mu, \sigma^2)$$

versus

$$H_1\colon\ Y \sim pN(\mu_1, \sigma^2) + (1-p)N(\mu_2, \sigma^2),$$

first under the assumption that $\mu = \mu_1 = 0$ and $\mu_2 \in \mathbb{R}$ while $\sigma^2 = 1$ (which corresponds to a simple Gaussian mixture model), next under no additional assumptions on the means but $\sigma^2 = 1$ (i.e. test of one population versus two when variance is known), and finally with no additional assumptions on either means or variance (i.e. test of one population versus two when variance is unknown). Since the distribution of the likelihood ratio test (LRT) statistic is related to that of the maximum of a rather regular Gaussian process, the authors use the Rice formulas to address the question of whether the power of the LRT is influenced significantly by the size of the interval(s) in which the parameters live and whether the LRT is more powerful than the hypothesis tests based on moments (the answer to the latter question turns out to be negative).

The next chapter focuses on both theoretical and numerical analysis of the Rice series, which are representations of the distribution function of the maximum of a given stochastic process in terms of series of factorial moments of the number of up-crossings of the underlying process. The authors prove two key results. The first is applicable to both non-Gaussian and Gaussian cases but assumes that the underlying process X has C^∞ sample paths and establishes a general sufficient condition on the distribution of X and its derivatives such that the following Rice series representation of the cumulative distribution function F_{M_T} of the maximum of X in terms of factorial moments $\tilde{\nu}_m$ of the number of up-crossings of X of a given level u, starting below u at time 0, holds:

$$(*) \qquad 1 - F_{M_T} = P(X(0) > u) + \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\tilde{\nu}_m}{m!}.$$

Moreover, when the infinite series is truncated, the error bound for the resulting approximation is also given. The second key result shows that for a Gaussian centered and stationary process on $\mathbb{R}$ with covariance Γ such that $\Gamma(0) = 1$ and Γ has a Taylor expansion at zero which is absolutely convergent at $t = 2T$, the conditions of the above general Rice series theorem are satisfied and thus representation $(*)$ is valid. Much of the remainder of the chapter is devoted to efficient numerical computation of the factorial moments of up-crossings, which is important for applications of the Rice series. In particular it is shown that the Rice series approach is a priori better than the Monte Carlo method (in terms of comparison of the complexities of the computation of the distribution of the maximum) and, for standard error bounds, allows one to compute the desired distribution with just a few terms of the Rice series. Chapter 5 concludes with a modification of the general Rice series theorem discussed earlier to include continuous processes that do not have sufficiently differentiable paths, which is achieved by employing in the series the factorial moments of up-crossings of an ε-mollified version (with $\varepsilon > 0$) of the underlying process and then taking ε to 0.

Chapter 6 revisits the subject of Rice formulas but in a much richer multiparameter setting. The authors start by proving the area formula, then establish Rice formulas for the moments of multiparameter Gaussian random fields (from a domain in $\mathbb{R}^d$ to $\mathbb{R}^d$) having continuously differentiable trajectories, and also prove a closely related result on the expected number of weighted roots corresponding to a given level set. Next, Rice formulas for the expected number of local maxima and the expected number of critical points of a Gaussian random field with domain D are established, where D is a C^2-manifold (at first, the manifold has no additional structure, then the results are further specialized to the cases when D has a Riemannian metric and when D is embedded in a Euclidean space). Analogous results are subsequently also proved for the case of Gaussian random fields from $\mathbb{R}^d$ to $\mathbb{R}^{d'}$ but now $d > d'$.

Chapter 7 is devoted to the analysis of regularity of the distribution of the maximum of Gaussian random fields. The key result here is the representation formula for the density of the maximum of a Gaussian real-valued field with C^2-paths defined on an open set containing S, where S is a compact subset of $\mathbb{R}^d$ which can be written as the disjoint union of a finite number of orientable C^3 manifolds S_j of dimension j without boundary (where $j = 0, \ldots, d$). Moreover, under certain nondegeneracy conditions, this density of the maximum is shown to be continuous. On the other hand, restricting attention to the one-parameter case allows the authors to derive subtler results on the degree of smoothness of the distribution of the maximum. Namely, if a Gaussian process on $[0, 1]$ has paths in C^{2k} then the cumulative distribution function of the maximum is shown to be of class C^k.

Chapter 8 generally studies tails of the distribution of the maximum of a random field and is divided into two parts. In the first part the authors focus solely on the case of one-parameter Gaussian processes and analyze the asymptotic behavior of the successive derivatives of the distribution of the maximum as well as the tails of the distribution of the maximum of certain unbounded Gaussian processes. In the latter case the probability q that the supremum is finite is strictly less than one, and the aim is to understand the speed at which $P(M_T \le u)$ converges to q as u grows to $+\infty$. In the second part the authors establish bounds for the density of the maximum of a multiparameter Gaussian random field and subsequently analyze the asymptotic behavior of the maximum given by

$$P(M > u) = A(u)\exp(-u^2/(2\sigma^2)) + B(u),$$

where $A(u)$ is a known function with polynomially bounded growth as $u \to +\infty$, $\sigma^2 = \sup_t \mathrm{Var}(X(t))$, and $B(u)$ is an error bounded by a centered Gaussian density with variance smaller than σ^2.

Chapter 9 develops an efficient method, based on record times, for the numerical computation of the distribution of the maximum of one- and two-parameter Gaussian random fields. The authors first consider the parameter space $[0,1]$ and prove that if X is a Gaussian process with C^1-paths, then the maximum $M = \max\{X(t),\, t \in [0,1]\}$ has a distribution with tails of the form

$$(**)\quad P(M > u) = P(X(0) > u) + \int_0^1 E[(X'(t)^+)\mathbf{1}_{\{t\in\mathcal{R}\}}|X(t) = u]p_{X(t)}(u)dt,$$

where $p_{X(t)}(\cdot)$ is the probability density of $X(t)$ and $\mathcal{R}$ is the set of record times, i.e. $\mathcal{R} = \{t \in [0,1]\colon X(s) < X(t), \forall s \in [0,t)\}$. The latter result is derived from Rychlik's formula, which in turn is based on the idea that

$$P(M \ge u) = P(X(0) > u) + P(\exists t \in \mathcal{R}\colon X(t) = u) = P(X(0) > u) + E[\#\{t \in \mathcal{R}\colon X(t) = u\}],$$

since the number of record times t such that $X(t) = u$ is either 0 or 1. Then, upon using a discretization of the condition $\{X(s) < X(t), \forall s \in [0,t)\}$, one can use formula $(**)$ to obtain explicit upper bounds on $P(M > u)$:

$$P(X(0) > u) + \int_0^1 E\big[(X'(t)^+)\mathbf{1}_{\{X(0)<u,\dots,X(t(n-1)/n)<u\}}|X(t) = u\big]p_{X(t)}(u)dt.$$

On the other hand, a similar time discretization provides the trivial lower bound

$$P(M > u) \ge 1 - P(X(0) \le u, \dots, X((n-1)/n) \le u),$$

where (at least for n up to 100) the integrals in the above upper and lower bounds can be easily computed using the Matlab toolbox MAGP developed by Mercadier (2005). Subsequently this record method is adapted by the authors to deal with the case of a two-parameter Gaussian random field.

Chapter 10 presents asymptotic results for one-parameter stationary Gaussian processes on time intervals whose size tends to infinity. First, provided that the level u tends to infinity jointly with the size of the time interval so that the expectation of the number of up-crossings remains constant and under the assumption of some local regularity (given by Geman's condition) and some mixing (given by Berman's condition) of the underlying process, the Volkonskiĭ-Rozanov theorem [V. A.

Volkonskiĭ and Yu. A. Rozanov, Teor. Veroyatnost. i Primenen. **6** (1961), 202–215; MR0137141 (25 #597)] is proved, showing that the asymptotic distribution of the number of up-crossings is Poisson. The latter in turn implies that the suitably renormalized maximum of the process converges to a Gumbel distribution. On the other hand, when the level u is fixed, under certain conditions, the number of (up-)crossings is shown to satisfy a central limit theorem. In terms of extensions of these results to a multiparameter setting, the authors quote Piterbarg's theorem [V. I. Piterbarg, *Asymptotic methods in the theory of Gaussian processes and fields*, Translated from the Russian by V. V. Piterbarg, Amer. Math. Soc., Providence, RI, 1996; MR1361884 (97d:60044)] for a multiparameter analogue of the Volkonskiĭ-Rozanov theorem. The multiparameter extensions of the central limit type results for up-crossings are not directly developed in the book, but several useful references are provided.

Chapter 11 deals with applications of Rice formulas to the study of some geometric characteristics of random sea surfaces. The random sea surface is modeled as a Gaussian stationary 3-parameter field which is the limit of the superposition of infinitely many elementary sea waves. Namely, if one considers a moving incompressible fluid in a domain of infinite depth, then the classical Euler equations, after some approximations, imply that the sea level $X(t,x,y)$, where t is time and (x,y) are spatial variables, satisfies

$$X(t,x,y) = f\cos(\lambda_t t + \lambda_x x + \lambda_y y + \theta),$$

where f and θ are the amplitude and phase, and the pulsations λ_t, λ_x and λ_y are some parameters satisfying the Airy relation $\lambda_x^2 + \lambda_y^2 = \frac{\lambda_t^2}{g}$, where g is the acceleration of gravity. If units are chosen so that $g = 1$ and if f and g are independent random variables with f having Rayleigh distribution and θ being uniform on $[0,\pi]$, then $X(t,x,y)$ is the Gaussian sine-cosine process of the form

$$X(t,x,y) = \xi_1 \sin(\lambda_t t + \lambda_x x + \lambda_y y) + \xi_2 \cos(\lambda_t t + \lambda_x x + \lambda_y y),$$

where ξ_1 and ξ_2 are independent standard normal random variables. The Rice formula is used to derive from the directional spectrum of the sea various properties of the distribution of such geometric characteristics like length of crests and velocities of contours. In addition, two non-Gaussian generalizations of the above Gaussian sea surface model are also briefly discussed.

Chapter 12 is devoted to the application of the Rice formula to the study of the number of real roots of a system of random equations, with a particular emphasis placed on large polynomial systems with random coefficients. The authors start by proving the Shub-Smale theorem [M. Shub and S. J. Smale, in *Computational algebraic geometry (Nice, 1992)*, 267–285, Progr. Math., 109, Birkhäuser Boston, Boston, MA, 1993; MR1230872 (94m:68086)] showing that if N^X equals the number of roots of the system of equations $X_i(t) = 0$ for all $i = 1,\dots,m$, where

$$X_i(t) := \sum_{j_1+\cdots+j_m \le d_i} a^{(i)}_{j_1,\dots,j_m} t_1^{j_1}\cdots t_m^{j_m},$$

with coefficients $\{a^{(i)}_{j_1,\dots,j_m}\colon i = 1,\dots,m;\ j_1+\cdots+j_m \le d_i\}$ being centered independent Gaussian random variables with variances $\mathrm{Var}(a^{(i)}_{j_1,\dots,j_m}) = \frac{d_i!}{j_1!\cdots j_m!(d_i-(j_1+\cdots+j_m))!}$, then $E(N^X) = \sqrt{d_1\cdots d_m}$. Next, assuming that $d_i = d$ for all $i = 1,\dots,m$, where $2 \le d \le d_0 < \infty$ for some constant d_0 independent of m, the authors establish the asymptotic behavior as $m \to \infty$ of the variance of $N^X/\sqrt{d^m}$. Namely, it is shown that for $d=2$ the asymptotic variance of $N^X/\sqrt{d^m}$ is $\frac{\log(m)}{2m}$, for $d=3$ the asymptotic variance is $\frac{3\log(m)}{2m^2}$, while for $d \geq 4$ the asymptotic variance is $\frac{K_d}{m^{3\wedge(d-2)}}$ for certain known constants K_d. Further extensions of the Shub-Smale result to other systems that are invariant under the orthogonal group of the underlying Euclidean space $\mathbb{R}^m$ and to certain systems with noncentered random coefficients are also developed.

The last chapter (Chapter 13) of the book is devoted to the application of the Rice formula to the study of condition numbers of random matrices. Condition numbers arise when one wants to understand how the solution $x \in \mathbb{R}^n$ of a linear system of equations $Ax = b$ is affected by perturbations in the input (A, b), in which case the condition number is defined as $k(A) = \|A\| \cdot \|A^{-1}\|$, where $\|A\|$ denotes the usual operator norm. The meaning of $k(A)$ is that of a bound for the amplification of the relative error between output and input when the input is small. This type of application is a new field aiming to further the understanding of algorithm complexity via the randomization of the problems that the algorithms are designed to solve.

The book is a very valuable addition to the literature on Gaussian processes, random fields and extreme value theory. It is well written and self-contained and presents a significant number of detailed and original applications to genomics, oceanography, the study of systems of random equations and condition numbers of random matrices. In comparison with another recent book [R. J. Adler and J. E. Taylor, *Random fields and geometry*, Springer, New York, 2007; MR2319516 (2008m:60090)] (with which it has some overlap in the material on the Rice formula and Rice series and on tails of the distribution of the maximum), this book has a distinct analytic rather than geometric flavor, making it more accessible to audiences with no background in differential geometry (albeit at the expense of omitting some beautiful results on the geometry of excursion sets, for example). Since the approaches adopted in these two books are very different and there is generally little overlap in the material, the two books complement each other well. Another valuable feature of the book under review, both from the self-study point of view and for its use as a textbook in graduate classes, is the inclusion of end-of-chapter exercises. The latter not only reinforce the material presented but also expose readers to a variety of new topics and ideas.

Reviewed by *Anna Amirdjanova*