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BOOK REVIEW

Jean-Marc Azaïs and Mario Wschebor: Level Sets and Extrema of Random Processes and Fields

Wiley, 2009. ISBN: 978-0-470-40933-6. 393 pages. £73.50

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The book of Azaïs and Wschebor is dedicated mainly, as the title says, to level sets and extrema of random fields and processes, but it contains also many other important results, for instance related to the use of random matrices in the study of the existence of real solutions for large systems of linear or polynomial equations, to statistical tests, and to genetics and oceanography.

It is a very original book, distinguished by its topic and its ability to make use of intuitive basic techniques, such as the Rice formula for instance. So we can say already that it is one of the most important books in probability theory published in the last twenty years. The approach to processes and fields described in this book is from my point of view just at the beginning of its applications and consequences even if the Rice formulas have been known for a long time. I will try to say why later.

Chapters 1 and 2 are concerned with most of the important results for Gaussian processes, such as those of Ylvisaker, Fernique, and Dudley for instance. Chapter 3 studies and often extends some seminal inequalities that are more or less directly linked to the Slepian lemma. They concern the tails of the distribution of maxima of Gaussian processes (Borel, Sudakov, Tsirelson). The presentation is very synthetic and useful for advanced lectures.

Chapter 3 starts with the basic facts on Rice formulas in the Gaussian case after a heuristic presentation.

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Rice formulas have been known in particular cases since the 1940s and were used first in signal theory and electronics.

The basic formula is as follows. Let u be a level, $\mathbb{X} = \{X(t);\ t \in I\}$, where $X(t)$ is a Gaussian process with C^1 trajectories, I an interval of the real line. It is supposed that no multidimensional distributions of X are degenerate, let $m \geq k$ be integers, $m^{[k]} = m(m-1)(m-k+1)$, and we are interested in the moments of the number $N_u = N_u(X, I)$ of crossings of the level u by the process $X(t)$ for $t \in I$. Then

$$E\big(N_u^{[k]}\big) = \int_{I^k} E\big(\big|X'(t_1)X'(t_2)\cdots X'(t_k)\big| \Big/ X(t_1) = u \cdots X(t_k) = u\big) p_{X(t_1)\cdots X(t_k)}\, dt_1 \cdots dt_k.$$

Of course, this formula is at the first glance complicated even in the Gaussian case, but it is very powerful. Roughly if $X^{(n)}(t)$ is the dyadic polygonal interpolation of $X(t)$, the proof uses the fact that $N_u(X^{(n)}, I) \to N_u(X, I)$ (in fact increases), and then it remains to prove the theorem in a simpler case, that of $X^{(n)}$.

The heuristic of this result is provided by the Kac lemma on C^1; if $\{t; t \in I, f(t) = u,\ f'(t) = 0\}$ is void then the number of crossings of the level u by f,

$$N_u(f, I) = \lim_{h \to 0} \frac{1}{2\delta} \int \mathbf{1}_{\{|f(t)-u|<\delta\}} \big|f'(t)\big|\, dt.$$

In fact Rice formulas are the only general tool to get results on the moments of N_u and at least for the two first moments these results are quite general even in the non-Gaussian case. For processes with irregular trajectories, nice asymptotic results can be obtain applying Rice formulas to a regularization of the trajectories by some operation as linear piece discretization, convolution, use of Green kernels, etc. The asymptotics are obtained for an adequate choice of the speed of the parameter of the regularization when it tends to zero. Details are given in the book only for the most important cases. This approach of irregular process by the pair regularization + Rice formulas is powerful.

Chapter 4 addresses statistical applications. Rice formulas are used to estimate the tails of the maxima of Gaussian processes or those of absolute values of the same processes and for moment evaluation. The problem of the evaluation of the distribution of maxima of Gaussian processes becomes ever more important in statistics, not only for tests.

Most of the situations require simulation and in these cases Rice formulas and series are now the main tool.

The first nice application concerns genetics and detection models in this field. A decisive step is to find the asymptotics and then to smooth the resulting process in order to have a tool of simulation. This kind of program, developed here in a genetic case is valid for many applications. Always linked to applications in genetics, the test of the order of a mixture Gaussian model is studied. The likelihood ratio has a complicated limit distribution and for the model is not identifiable, classical likelihood theory and χ^2 test are no more valid and so numerical studies are required for the likelihood test but also to see when it is more powerful than for instance tests based on moments.

Chapters 5 and 6 concern Rice formulas and more specifically Rice series which give an expansion of the distribution of the maximum for very general classes of processes with C^∞ trajectories in terms of conditional expectations not too complicated to control in some universal developments. This part is very useful in the Gaussian case and allows first moments evaluation but much more. Numerical computations of Rice series are given, allowing sharper results than the theoretical bounds already known and used as the Davies one. How to pass from C^∞ to C? This is one step of regularization methods by convolution using a smooth kernel.

This chapter is the last one only dedicated to the basic results for regularization methods for processes. Until this moment we do not write anything about the exercises which are at the end of every chapter. They are important first as complements of information, but more as useful tools to build a sequence of lectures or to enter completely in the subject.

Chapter 6 thus concerns fields, and Rice formulas. It is surely more difficult to read because of the necessary basic and a little more than basic knowledge of differential geometry. Once overpassing these difficulties which from my point of view can be left to a second or third reading, the next chapters are more easy to understand. Chapter 7 starts with the popular Tsirelson result on the distribution of the max of a countable number of Gaussian variables. Deep extensions are given, extracted from previous works of the authors when the trajectories have some regularity properties such as C^2.

Chapter 8 gives important results for the tail of the maximum, in the case of processes as in the case of isotropic fields. These results are obtained by a quite different approach than that of Taylor and Adler and Taylor. They are probably the most useful known for a numerical work as too briefly indicated in Chapter 9.

Chapter 10 contains mostly classical results for asymptotics when the length of observation interval tends to infinity.

Chapter 11 is a too short introduction to the application of some of the previous results to some characteristics of the geometry of sea waves as level curves and lengths of crests.

Chapter 12 and 13 are dedicated to a quite new and fascinating topic concerning the study of solutions of systems of random equations, that means equations with random coefficients. The first step is the study of the number of real roots of polynomial systems with Gaussian coefficients, and specifically systems with a large number of equations. This is a topic interesting for numerical analysis and complexity theory but also for its own deep problems. We are in the first years of investigation on this theory where the partial results are not necessarily linked. The basic model is the Shub–Smale one, where the polynomials X_i of m variables $t_1 \cdots t_m$ of global degree d_i have Gaussian centered coefficients with a specific form of the variance associated to the invariance of the distribution with respect to the orthogonal group on $\mathbb{R}^m$. A deep and much simpler proof of the Smale–Shub theorem on the expectation of the number of roots is given using Rice formulas for random Gaussian fields and new results from the authors concerning the variance are detailed in particular cases such as $d_i = d$ for every i.

The authors' work then addresses the case of correlated polynomials with invariance properties of the correlation always with respect to the orthogonal group. Many examples and almost new chapters are also included in this chapter. It should be too long to give an exhaustive list. There are variations about the Shub–Smale model. There is also the theory of the non-centered case where polynomials are in fact scattered into a deterministic part and a random one similar to that of the centered case. But here the problem can be thought of as a problem of perturbation: how much the random part modifies the number of solutions of the deterministic part. In fact this chapter opens an impressive number of questions! What is the role of the distribution of coefficients in the theory, what happens if the systems are not square when the number of equations is different from that of variables, what are the good asymptotics as $m \to \infty$?

The last chapter studies the solutions of linear systems $Ax = b$ and more specifically asks what happens if a known system is perturbed randomly. So the theory can be considered as a part of that of random matrices. One of the main objects is the condition number $\kappa(A)$ introduced by Turing. For some specific distributions with bounded support and symmetric matrices a series of first examples on the number of well or ill conditioned matrices is given. More accurate results are then given in the case of centered Gaussian matrices where bounds in probability for $\kappa(A)$ are given with respect to the dimension. Once more the use of Rice formulas is a crucial step in the proofs.

Once the reading is finished, we have different kinds of reactions. The first one is the evidence that the Rice formulas are a central tool of probabilities today and for the future. Of course their formalism is sometimes heavy, the non stochastic analysis which is partly its support is not popular. But no matter, important problems not developed in the book, for instance related to random matrices in the context of glass spins or statistics in signal theory, begin to be solved using Rice formulas.

The second impression is that the book enters in some topics too briefly for an evident reason: the size is upper bounded. Problems such as crossings, regularization of trajectories, links to effective measurements in physics are not detailed and the reader has to go to other publications of the authors or others on these topics. Let us wait for a second edition, bigger; or better: for a second volume!