Colorings and acyclic sets in planar graphs and digraphs

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Some Definitions

Acyclic digraph: digraph without directed cycles.

Digon: the directed cycle of length two.

An old conjecture

Conjecture (Albertson, Berman 1979)

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Fact: every n-vertex planar graph contains an induced forest of order at least $2n/5$.

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Theorem (Borodin 1979)

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$\exists$ a pair of color classes of total size at least $2n/5$.

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Approach: can vertices of every planar graph be colored with two colors such that each color class induces a forest?

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Vertex-arboricity, $a(G)$, of graph G: smallest k s.t. $V(G)$ can be k-colored with each color class inducing a forest.

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$\exists$ planar G with $a(G) = 3$ (Chartrand, Kronk, Wall (1968), Raspaud, Wang (2008))

Fact: $a(G) \leq 3$ if G is planar.

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- Theorem (Mohar, H., 2012)
  Every planar graph G has at least $2^{|V(G)|/9}$ *different 3-arboricities.*

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- Theorem (Mohar, H., 2012)
  Every planar graph G has at least $2^{|V(G)|/9}$ different 3-arboricities.
    - Generalizes to 3-list colorings.

Digraphs: a conjecture

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Albertson-Berman conjecture would imply $n/2$ (instead of $3n/5$).

Colorings in digraphs

The dichromatic number $\chi(D)$ of digraph D is the smallest k s.t. $V(D)$ can be partitioned into k sets $V_1, \ldots, V_k$ each of which induces an acyclic subdigraph.

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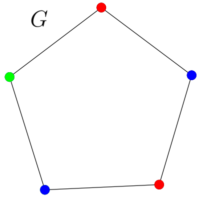

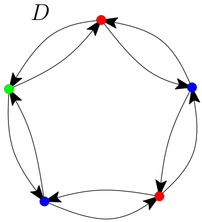

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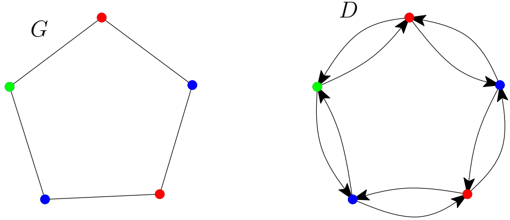

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Introduced by Victor Neumann-Lara in 1982.

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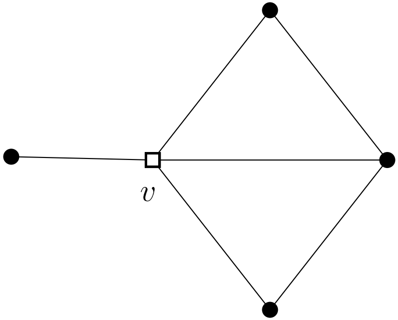
Theorem (Mohar, H., 2013)

Every planar digraph D of digirth at least five has $\chi(D) \leq 2$.

The proof

Idea: Discharging...but messy. Configurations are graphs, not digraphs.

A reducible configuration



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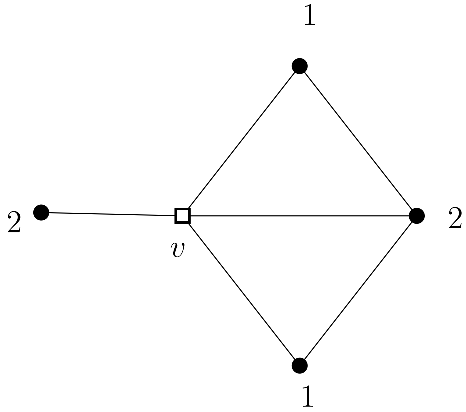

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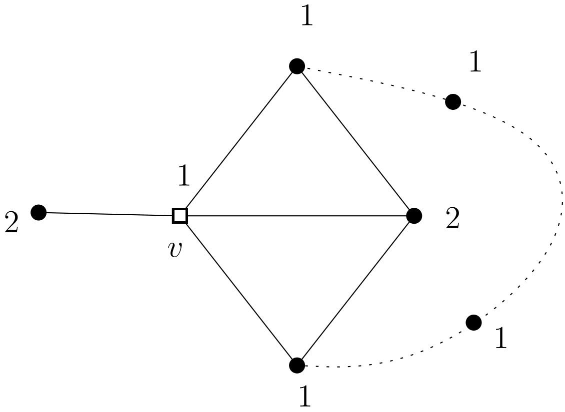

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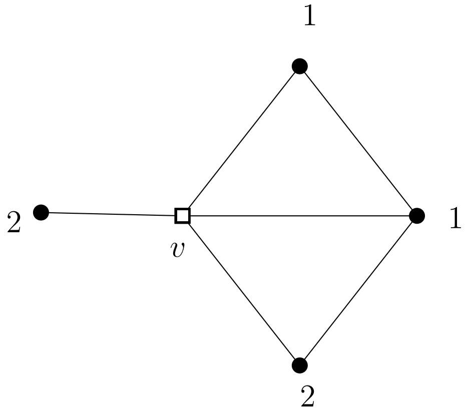

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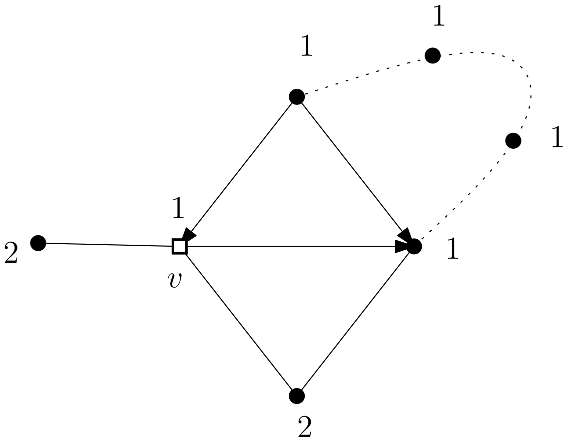

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Conjecture (McDiarmid, Mohar 2002)

Every oriented graph D *with maximum degree* Δ *has* $\chi(D) \leq C \cdot \frac{\Delta}{\log \Delta}$.

Thank You