Sparsity and homomorphisms of graphs and digraphs

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joint work

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Chromatic number and sparse graphs

Theorem (Erdős 1959, Canad. J. Math.)

$\forall\ g, k\ \exists$ *graph* G *s.t.* $girth(G) \geq g$ *and* $\chi(G) \geq k$.

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Theorem (Erdős 1959, Canad. J. Math.)

$\forall$ g, k $\exists$ *graph G s.t.* $girth(G) \geq g$ *and* $\chi(G) \geq k$.

Remark: Bollobas and Sauer (1976 Canad. J. Math.) showed that G can be taken to be *uniquely* k-colorable.

Coloring and homomorphisms

Definition

A homomorphism from graph G to H is a mapping $\phi : V(G) \to V(H)$ that preserves adjacencies.

Proposition

G is k-colorable if and only if $G \to K_k$.

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- Instead of K_k look at arbitrary graph H.
- Clearly, $\exists$ G (of arbitrary girth) s.t. $G \nrightarrow H$.
- Question: Does there exist graph G^* "diluted" from G s.t. $G^* \nrightarrow H$?

"Diluting" G

Idea: G and H given. Suppose $G \nrightarrow H$. Does there exist a sparse graph G^* s.t.

$$G^* \to G$$

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Theorem (Zhu 1996 J. Graph Theory)

G and H graphs, and $G \nrightarrow H$. Then $\forall g \; \exists \; G^*$ with: $girth(G^*) \geq g$, $G^* \to G$ and $G^* \nrightarrow H$.

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Remark: Set $G = K_r$ and $H = K_{r-1}$ to recover Erdős' theorem.

Digraphs

Digraphs here will have no loops and no multiple arcs but digons are allowed.

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We write $D \to_{ac} C$

Graph Homomorphisms and Acyclic digraph homomorphisms

Fact: Let G and H be graphs, D and C the bidirected digraphs of G and H, respectively. Then

$$G \to H \Leftrightarrow D \to_{ac} C.$$

Analog of Zhu's theorem

Theorem (H, Kayll, Mohar, Rafferty, 2012 Canad. J. Math)

D and C digraphs, and $D \not\to_{ac} C$. Then $\forall g \; \exists \; D^*$ with:
$girth(D^*) \geq g$, $D^* \to_{ac} D$ and $D^* \not\to_{ac} C$.

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Graph (digraph) H is a core if it is uniquely H-colorable.

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Remark: Setting $H = K_k$ gives Bollobas-Sauer .

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Remark: Has applications on coloring of digraphs and digraph circular chromatic number.

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Theorem

Let $1 \leq d \leq k$ be relative prime integers. Then $\forall g$, $\exists$ digraph D of girth at least g and $\chi_c(D) = \frac{k}{d}$.

Thank You