

# Global Offensive Alliances in Graphs and Random Graphs

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## Abstract

A global offensive alliance in a graph  $G = (V, E)$  is a subset  $S$  of  $V$  such that for every vertex  $v$  not in  $S$  at least half of the vertices in the closed neighborhood of  $v$  are in  $S$ . The cardinality of a minimum size global offensive alliance in  $G$  is called the global offensive alliance number of  $G$ . We give an upper bound on the global (strong) offensive alliance number of a graph in terms of its degree sequence. We also study global offensive alliances of random graphs. In particular, it is proved that if  $p(\log n)^{1/2} \rightarrow \infty$  then with high probability  $G(n, p)$  has a global offensive alliance of size at most  $cn$  if  $c > 1/2$  and no global offensive alliance of size at most  $cn$  if  $c < 1/2$ .

**Keywords:** Global offensive alliances, degree sequence, upper bounds, random graph.

## 1 Introduction

The study of alliances in graphs was first introduced by Hedetniemi, Hedetniemi and Kristiansen [9]. They introduced the concepts of defensive and

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offensive alliances, global offensive and global defensive alliances and studied alliance numbers of a class of graphs such as cycles, wheels, grids and complete graphs. Haynes et al. [7] studied the global defensive alliance numbers of different classes of graphs. They gave lower bounds for general graphs, bipartite graphs and trees, and upper bounds for general graphs and trees. Rodriguez-Velazquez and Sigarreta [14] studied the defensive alliance number and the global defensive alliance number of line graphs. A characterization of trees with equal domination and global strong defensive alliance numbers was given by Haynes, Hedetniemi and Henning [8]. Offensive  $k$ -alliances were introduced in [4].

Offensive alliances were first studied by Favaron et. al [5], where they derived some bounds on the offensive alliance number. Rodriguez-Velazquez and Sigarreta [11] gave bounds for offensive and global offensive alliance numbers in terms of the algebraic connectivity, the spectral radius, and the Laplacian spectral radius of a graph. They also gave bounds on the global offensive alliance number of cubic graphs in [12] and the global offensive alliance number for general graphs in [13]. Some bounds on the global offensive alliances were given in [6]. Balakrishnan et al. [2] studied the complexity of global alliances. They showed that the decision problems for global defensive and global offensive alliances are both NP-complete for general graphs.

This paper further studies the global offensive alliance number of a graph. We start with the notation and definitions

Given a simple graph  $G = (V, E)$  and a vertex  $v \in V$ , the *open neighborhood* of  $v$ ,  $N(v)$ , is defined as  $N(v) = \{u : uv \in E\}$ . The *closed neighborhood* of  $v$ , denoted by  $N[v]$ , is  $N[v] = N(v) \cup \{v\}$ .

**Definition 1.1.** *A set  $S \subset V$  is a global offensive alliance if for every  $v \in V - S$ ,  $|N[v] \cap S| \geq |N[v] - S|$ .*

**Definition 1.2.** *A global offensive alliance  $S$  is called a global strong offensive alliance if for every  $v \in V - S$ ,  $|N[v] \cap S| > |N[v] - S|$ .*

**Definition 1.3.** *The global (strong) offensive alliance number of  $G$  is the cardinality of a minimum size global (strong) offensive alliance in  $G$ , and is denoted by  $\gamma_o(G)$  ( $\gamma_\delta(G)$ ). A minimum size global offensive alliance is called a  $\gamma_o(G)$ -set.*

In this paper, we study the global (strong) offensive alliance number of general graphs. We give an upper bound on the global (strong) offensive

alliance number of general graphs. Additionally, we study the global (strong) offensive alliance number of random graphs.

The rest of the paper is organized as follows. In Section 2, we give an upper bound on the global (strong) offensive alliance number of a general graph in terms of its order and degree sequence. Using this bound, we obtain a second upper bound on the global (strong) offensive alliance number in terms of the minimum degree of the graph. In Section 3, we study the global (strong) offensive alliance number of the random graph  $G(n, p)$ .

## 2 Global Offensive Alliances in Graphs

In this section we give an upper bound on  $\gamma_o(G)$  for any graph  $G$ . Our result derives an upper bound on  $\gamma_o(G)$  in terms of the degree sequence of the graph  $G$ . The method of the proof is probabilistic. All the required probabilistic tools can be found in [1]. Note that  $\exp(x)$  is the exponential function  $e^x$ .

**Theorem 2.1.** *Let  $G = (V, E)$  be a graph of order  $n$ . Let  $\deg(v)$  denote the degree of vertex  $v$ . Then for all  $1/2 > \alpha > 0$ ,*

$$\gamma_o(G) \leq \left(\frac{1}{2} + \alpha\right)n + \left(\frac{1}{2} - \alpha\right) \sum_{v \in V} \exp\left(-\frac{\alpha^2}{1 + 2\alpha} \cdot \deg(v)\right)$$

*Proof.* We put every vertex  $v \in V$  in a set  $S$  with probability  $p$ , independently. The value of  $p$  will be determined later. The random set  $S$  is going to be part of the global offensive alliance. For every vertex  $v \in V$ , let  $X_v$  denote the number of vertices in the neighborhood of  $v$  that are in  $S$ . Let  $Y = \left\{v \in V : v \notin S \text{ and } X_v \leq \left\lfloor \frac{\deg(v)}{2} \right\rfloor\right\}$ . Clearly,  $S \cup Y$  is a global offensive alliance. Note that  $\mathbb{E}[|S|] = np$ . Now, we estimate  $\mathbb{E}[|Y|]$ .

It is not hard to see that  $X_v$  is a Binomial( $\deg(v), p$ ) random variable. We use the Chernoff Bound(see, for example, Alon and Spencer [1]) to bound  $\mathbb{P}[X_v \leq \frac{\deg(v)}{2}]$ . The Chernoff Bound states that for any  $a > 0$  and random variable  $X$  that has binomial distribution with probability  $p$  and mean  $pn$ ,

$$\mathbb{P}[X - pn < -a] < e^{-a^2/2pn}. \tag{1}$$

Set  $a = \epsilon pn$ , where  $\epsilon = 1 - \frac{1}{2p}$ . Then, by the Chernoff Bound,

$$\begin{aligned} \mathbb{P}\left[X_v \leq \frac{\deg(v)}{2}\right] &= \mathbb{P}[X_v \leq (1 - \epsilon)p(\deg(v))] \\ &< e^{-\epsilon^2 \deg(v)p/2} \\ &= e^{-(1-\frac{1}{2p})^2 \deg(v)p/2}. \end{aligned}$$

Chernoff's bound holds whenever  $\epsilon > 0$ , or equivalently when  $p > \frac{1}{2}$ .

Now,

$$\begin{aligned} \mathbb{P}[v \in Y] &= \mathbb{P}[\{v \notin S\} \cap \{X_v \leq \deg(v)/2\}] \\ &= \mathbb{P}[v \notin S] \mathbb{P}[X_v \leq \deg(v)/2] \\ &\leq (1 - p)e^{-(1-\frac{1}{2p})^2 \deg(v)p/2}, \end{aligned}$$

by independence. By linearity of expectation, we get that

$$\mathbb{E}[|Y|] \leq \sum_{v \in V} (1 - p)e^{-(1-\frac{1}{2p})^2 \deg(v)p/2}.$$

Now, we have that

$$\mathbb{E}[|S \cup Y|] \leq np + \sum_{v \in V} (1 - p)e^{-(1-\frac{1}{2p})^2 \deg(v)p/2}. \quad (2)$$

Therefore, there exists a global offensive alliance in  $G$  of size at most

$$np + \sum_{v \in V} (1 - p)e^{-(1-\frac{1}{2p})^2 \deg(v)p/2}. \quad (3)$$

Thus, we have that

$$\gamma_o(G) \leq np + \sum_{v \in V} (1 - p)e^{-(1-\frac{1}{2p})^2 \deg(v)p/2}. \quad (4)$$

The only constraint we have on  $p$  is that  $p > \frac{1}{2}$ . We set  $p = \frac{1}{2} + \alpha$  for any  $1/2 > \alpha > 0$ . This completes the proof.  $\square$

A similar result can be derived for the global strong offensive alliance number of a graph.

**Theorem 2.2.** *Let  $G = (V, E)$  be a graph of order  $n$ . Then for all  $1/2 > \alpha > 0$ ,*

$$\gamma_o(G) \leq \left(\frac{1}{2} + \alpha\right)n + \sum_{v \in V} \exp\left(-\frac{\alpha^2}{1 + 2\alpha} \cdot (\deg(v) + 1)\right).$$

*Proof.* The proof is in the same spirit as the proof of Theorem 2.1. We put every vertex  $v \in V$  in a set  $S$  with probability  $p$ , independently. For every vertex  $v \in V$ , let  $X_v$  denote the number of vertices of the closed neighborhood of  $v$  that are in  $S$ . Similarly to Theorem 2.1, let  $Y = \left\{v \in V : X_v < \lfloor \frac{\deg(v)+1}{2} \rfloor\right\}$ . Clearly,  $S \cup Y$  is a global strong offensive alliance. Now, we are going to estimate  $|Y|$ .

It is not hard to see that  $X_v$  is a *Binomial*( $\deg(v) + 1, p$ ) random variable and as before we will apply the Chernoff Bound to bound  $\mathbb{P}[X_v < \frac{\deg(v)+1}{2}]$ . Let  $\epsilon = 1 - \frac{1}{2p}$ . Then, by the Chernoff Bound,

$$\begin{aligned} \mathbb{P}\left[X_v < \frac{\deg(v) + 1}{2}\right] &= \mathbb{P}[X_v < (1 - \epsilon)p(\deg(v) + 1)] \\ &< e^{-\epsilon^2 p(\deg(v)+1)/2} \\ &= e^{-(1-\frac{1}{2p})^2 p(\deg(v)+1)/2}. \end{aligned}$$

As before, the bound holds whenever  $p > \frac{1}{2}$ . Now, we have that

$$\mathbb{E}[|S \cup Y|] \leq np + \sum_{v \in V} e^{-(1-\frac{1}{2p})^2 p(\deg(v)+1)/2}. \quad (5)$$

Therefore, there exists a global strong offensive alliance of size at most

$$np + \sum_{v \in V} e^{-(1-\frac{1}{2p})^2 p(\deg(v)+1)/2} \quad (6)$$

Thus, we have that

$$\gamma_o(G) \leq np + \sum_{v \in V} e^{-(1-\frac{1}{2p})^2 p(\deg(v)+1)/2} \quad (7)$$

Now, set  $p = \frac{1}{2} + \alpha$  where  $\alpha > 0$  is arbitrary. This completes the proof.  $\square$

If  $G$  is a regular graph, the expressions in Theorems 2.1 and 2.2 simplify and using simple calculus it is not hard to find the optimal value of  $\alpha$  that gives the best bound. In fact, we only require that the minimum degree of  $G$  be large.

**Corollary 2.3.** *Let  $G$  be a graph of minimum degree  $d \geq 2$ . Then*

$$\gamma_o(G) \leq \left( \frac{1}{2} + \left( \frac{\log d}{d} \right)^{1/2} + \frac{1}{2\sqrt{d}} - \frac{\sqrt{\log d}}{d} \right) n.$$

*Proof.* Let  $\alpha = \left( \frac{\log d}{d} \right)^{1/2}$  and apply Theorem 2.1. □

**Corollary 2.4.** *Let  $G$  be a graph of minimum degree  $d \geq 2$ . Then*

$$\gamma_o(G) \leq \left( \frac{1}{2} + \left( \frac{\log d}{d+1} \right)^{1/2} + \frac{1}{\sqrt{d}} \right) n.$$

*Proof.* Let  $\alpha = \left( \frac{\log d}{d+1} \right)^{1/2}$  and apply Theorem 2.2. □

Note that if the minimum degree  $d$  of a graph  $G$  tends to infinity, corollaries 2.3 and 2.4 imply that  $\gamma_o(G)$  and  $\gamma_\delta(G)$  approach to  $n/2$ .

The following tight bounds were obtained for  $\gamma_o(G)$  and  $\gamma_\delta(G)$  in [10]. For large minimum degree  $d$ , our results improve these bounds.

**Theorem 2.5** ([10]). *For every connected graph  $G$  of order  $n \geq 2$ ,  $\gamma_o(G) \leq \frac{2n}{3}$ . If the minimum degree of  $G$  is at least 2,  $\gamma_\delta(G) \leq \frac{5n}{6}$ .*

### 3 Global Offensive Alliances in Random Graphs

The *random graph*  $G(n, p)$  is the graph on  $n$  vertices where each possible edge is present with probability  $p$ , independently. It seems plausible that the random graph  $G(n, 1/2)$  should have a global offensive alliance number of approximately  $n/2$ . In this section, we provide some evidence that this is the case. The main result of this section is the following theorem.

**Theorem 3.1.** *Suppose that  $p(\log n)^{1/2} \rightarrow \infty$ . Then*

$$\mathbb{P}[\gamma_o(G(n, p)) = n/2 + o(n)] = 1 - o(1).$$

*Proof.* First, we will show that for a fixed constant  $c < 1/2$ ,  $\mathbb{P}[\gamma_o(G(n, p)) \leq cn] = o(1)$ . This will establish the lower bound.

Let  $A \subset V$  be a fixed set of size  $cn$ . We label the vertices of the set  $V - A$  as  $\{v_1, \dots, v_{n-cn}\}$ . We will compute the probability that  $A$  is a global offensive alliance. To this end, for a vertex  $v_i \in V - A$ , let  $B_i$  be the event that in the closed neighborhood of  $v_i$  there are at least as many vertices in  $A$  as in  $V - A$ . For  $n$  sufficiently large, we have

$$\begin{aligned} \mathbb{P}[\gamma_o(G(n, p)) \leq cn] &\leq \binom{n}{cn} \mathbb{P}[A \text{ is a global offensive alliance}] \\ &\leq 2^n \mathbb{P}[A \text{ is a global offensive alliance}] \\ &= 2^n \mathbb{P}[B_1 \cap B_2 \dots \cap B_{n-cn}] \\ &\leq 2^n \mathbb{P}[B_1 \cap B_2 \dots \cap B_{\log n}] \end{aligned}$$

Let  $E$  be the event that the set  $\{v_1, \dots, v_{\log n}\}$  is an independent set. Then clearly  $\mathbb{P}[B_1 \cap B_2 \dots \cap B_{\log n}] \leq \mathbb{P}[B_1 \cap B_2 \dots \cap B_{\log n} \mid E]$ . Now, it is easy to see that the events  $B_i \mid E$  are independent, and thus we have

$$\begin{aligned} \mathbb{P}[\gamma_o(G(n, p)) \leq cn] &\leq 2^n \mathbb{P}[B_1 \cap B_2 \dots \cap B_{\log n}] \\ &\leq 2^n \prod_{i=1}^{\log n} \mathbb{P}[B_i \mid E] \\ &\leq 2^n (\mathbb{P}[B_i \mid E])^{\log n} \end{aligned}$$

Now, we compute  $\mathbb{P}[B_1 \mid E]$ . Let  $X$  and  $Y$  be random variables with the following distributions:  $X \sim \text{Binomial}(cn, p)$  and  $Y \sim \text{Binomial}(n - cn - \log n, p)$ . Then clearly  $\mathbb{P}[B_1 \mid E] = \mathbb{P}[X > Y] = \mathbb{P}[X - Y > 0]$ .

To bound  $\mathbb{P}[X - Y > 0]$ , we let  $Z = X - Y$  and bound  $\mathbb{P}[Z > 0]$ . Clearly,  $\mathbb{E}[Z] = \mathbb{E}[X] - \mathbb{E}[Y] = ((2c - 1)n + \log n)p$ . Note that for  $n$  sufficiently large,  $\mathbb{E}[Z] < 0$ . We will use Azuma's inequality to show that  $Z$  is strongly concentrated around its expected value.

**Lemma 3.2** (Azuma's Inequality). [3] *Let  $Z_1, \dots, Z_l$  be independent random variables with  $Z_k$  taking values in a set  $\Omega_k$ . Let  $f : \Omega = \Omega_1 \times \dots \times \Omega_l \rightarrow R$*

be a measurable function such that if  $\omega \in \Omega$  and  $\omega' \in \Omega$  differ only in their  $k$ th coordinate then  $|f(\omega) - f(\omega')| \leq c$ , for some positive constant  $c$ . Then the random variable  $Z = f(Z_1, \dots, Z_l)$  satisfies the following inequality for all  $t \geq 0$ ,

$$\mathbb{P}[Z \geq \mathbb{E}[Z] + t] \leq e^{-2t^2/lc^2}.$$

Clearly, the random variable  $Z = X - Y$  depends only on presence of the edges that could be the neighbors of  $v_1$ . Since we know that  $\{v_1, \dots, v_{\log n}\}$  is an independent set,  $Z$  depends on exactly  $n - \log n$  indicator random variables. Note that each indicator random variable could change  $Z$  by at most 2, and thus,  $c = 2$  in Azuma's inequality. Now, since  $p = \omega(1/\sqrt{\log n})$  and  $c < 1/2$ , we may choose a function  $w(n) \rightarrow \infty$  such that  $\frac{nw(n)}{\sqrt{\log n}} + \mathbb{E}[Z] < 0$ . Now, applying Azuma's inequality with  $t = \frac{nw(n)}{\sqrt{\log n}}$ , we have that

$$\mathbb{P}[Z > 0] \leq \mathbb{P}[Z \geq \mathbb{E}[Z] + t] \leq \exp(-nw(n)^2/4 \log n).$$

Now, it follows that

$$\mathbb{P}[\gamma_o(G(n, p)) \leq cn] \leq 2^n (\mathbb{P}[B_i | E])^{\log n} \leq 2^n \exp(-nw(n)^2/4) = o(1).$$

This establishes the lower bound. For the upper bound, assume that  $c > 1/2$ . We need to prove that  $\mathbb{P}[\gamma_o(G(n, p)) \leq cn] = 1 - o(1)$ . We will actually show something stronger:  $\mathbb{P}[\gamma_o(G(n, p)) \leq cn] = 1 - o(1)$ .

Given  $c > 1/2$ , let  $d$  be the smallest positive integer such that  $(\frac{\log d}{d+1})^{1/2} + \frac{1}{\sqrt{d}} \leq c - \frac{1}{2}$ . We first show that  $G(n, p)$  has minimum degree at least  $d$  with high probability. Let  $X_i$  be the number of vertices of degree  $i$  in  $G(n, p)$ . Then

$$\mathbb{E}[X_i] \leq n \binom{n-1}{i} (1-p)^{n-1-i} \leq n^{i+1} e^{-pn+p(i+1)}.$$

Since  $d$  is a constant and  $p(\log n)^{1/2} \rightarrow \infty$ , it follows that for all  $i \leq d$ ,  $\mathbb{E}[X_i] = o(1)$ . Thus,

$$\sum_{i=0}^d \mathbb{E}[X_i] = o(1).$$

Now, Markov's inequality yields that  $G(n, p)$  has no vertex of degree at most  $d$ . Therefore, with high probability,  $G(n, p)$  has minimum degree at least  $d$ . Now, Corollary 2.4 implies that, with probability  $1 - o(1)$ ,

$$\gamma_{\delta}(G(n, p)) \leq \left( \frac{1}{2} + \left( \frac{\log d}{d+1} \right)^{1/2} + \frac{1}{\sqrt{d}} \right) n \leq cn,$$

as required. □

Since a global strong offensive alliance in a graph  $G$  is also a global offensive alliance in  $G$ , Theorem 3.1 and its proof immediately imply the following.

**Corollary 3.3.** *Suppose that  $p(\log n)^{1/2} \rightarrow \infty$ . Then*

$$\mathbb{P}[\gamma_{\delta}(G(n, p)) = n/2 + o(n)] = 1 - o(1).$$

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