Mapping planar graphs into the Coxeter graph

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Abstract

We conjecture that every planar graph of odd-girth at least 11 admits a homomorphism to the Coxeter graph. Supporting this conjecture, we prove that every planar graph of odd-girth at least 17 admits a homomorphism to the Coxeter graph.

Keywords: homomorphisms, planar graphs, projective cubes, Coxeter graph.

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1. Introduction

In this paper, considering a question of the second author [5], we conjecture that:

Conjecture 1. *Every planar graph of odd-girth at least* 11 *admits a homomorphism to the Coxeter graph.*

Supporting this conjecture we then prove that:

Theorem 2. *Every planar graph of odd-girth at least* 17 *admits a homomorphism to the Coxeter graph.*

We start with notation and motivation. For standard terminology of graph theory we simply refer to [2]. The length of a shortest odd-cycle of a non-bipartite graph is called *odd-girth*. The class of planar graphs of odd-girth at least $2k+1$ will be denoted by $\mathcal{P}_{2k+1}$. A *plane graph* is a planar graph with a given planar embedding. A *homomorphism* of a graph G to another graph H is a mapping $\varphi : V(G) \to V(H)$ which preserves adjacency. If there exists a homomorphism of G to H, we write $G \to H$. The image of G under φ is called a *homomorphic image* of G. Given a class $\mathcal{C}$ of graphs and a graph H, if every graph in $\mathcal{C}$ admits a homomorphism to H we write $\mathcal{C} \preceq H$ and we say H *bounds* $\mathcal{C}$.

The *projective cube* of dimension d, denoted PC(d), is the Cayley graph $(\mathbb{Z}_2^d, \{e_1, e_2, \ldots, e_d, J\})$ where e_i's are the standard basis and J is the all 1-vector. The projective cube PC(d) is isomorphic to the graph obtained from the hypercube of dimension $d+1$ by identifying antipodal vertices. It is easy to verify that PC($2k+1$) is bipartite. In contrast, PC($2k$) is of chromatic number 4, while its shortest odd-cycle is of length $2k+1$, see [4] for a proof. PC(2) is K_4, PC(3) is $K_{4,4}$ and PC(4) is the well-known Clebsch graph.

Given n and k such that $n \geq 2k$, the Kneser graph $K(n,k)$ is defined to be a graph whose vertices are k-subsets of an n-set where two such vertices are adjacent if they have no intersection. The Kneser graph $K(2k+1,k)$ is an induced subgraph of $\mathrm{PC}(2k)$ (see [5]).

The existence of a homomorphism from a class of graphs to a projective cube is of special importance. Generally, it captures a certain packing problem (see [6]). In particular, we have the following conjecture which extends the four-color theorem:

Conjecture 3 ([4]). *The class $\mathcal{P}_{2k+1}$ is bounded by* $\mathrm{PC}(2k)$.

Note that for $k=1$, the conjecture is equivalent to the four-color theorem. Conjecture 3 can be seen as an optimization for the following result of J. Nešetřil and P. Ossona De Mendez:

Theorem 4 ([8]). *Let $\mathcal{F}=\{F_1,F_2,\dots,F_r\}$ be a finite set of connected graphs. Let $\mathcal{M}$ be a minor-closed family of graphs and $\mathcal{M}_{\mathcal{F}}$ be the subclass consisting of those members of $\mathcal{M}$ which admit no homomorphism from a member of $\mathcal{F}$. Then there is a graph $H_{\mathcal{F}}$ which admits homomorphism from no member of $\mathcal{F}$, but bounds $\mathcal{M}_{\mathcal{F}}$.*

The bounds $H_{\mathcal{F}}$ built in the known proofs of this theorem in most cases are far from being optimal (in terms of order of the bound). The question of finding an optimal $H_{\mathcal{F}}$ captures some of the most well-known problems in graph theory. For instance, the simplest case of $\mathcal{F}=\{K_n\}$ with $\mathcal{M}$ being the class of K_n-minor free graphs captures the Hadwiger conjecture. For $\mathcal{F}=\{C_{2k-1}\}$ and $\mathcal{M}$ being the class of planar graphs, we have $\mathcal{M}_{\mathcal{F}}=\mathcal{P}_{2k+1}$. In this case, it was recently shown in [7] that $H_{\mathcal{F}}$ must have at least 2^{2k} vertices. Thus, if Conjecture 3 holds, then $\mathrm{PC}(2k)$ is an optimal bound. This conjecture is shown, in [4], to be equivalent to a special case of a conjecture of P. Seymour (in [9]) on determining the edge-chromatic number of planar multigraphs.

For $r>k$, since $\mathcal{P}_{2r+1}$ is included in $\mathcal{P}_{2k+1}$, if $\mathrm{PC}(2k)$ bounds $\mathcal{P}_{2k+1}$, then it also bounds $\mathcal{P}_{2r+1}$. However, in this case we believe that a proper subgraph of $\mathrm{PC}(2k)$ would suffice to bound $\mathcal{P}_{2r+1}$.

Problem 5 ([5]). *Given $r>k$, what are the optimal subgraphs of* $\mathrm{PC}(2k)$ *which bound $\mathcal{P}_{2r+1}$?*

It is shown in [5] that Problem 5 captures several interesting theories. In particular, if $K(2k+1,k)$ is an answer for $r=k+1$, it would determine the fractional chromatic number of $\mathcal{P}_{2r+1}$. In this regard, while the case $r=2$ and $k=1$ is implied by Grötzsch's theorem, the best result for $r=3$ and $k=2$ is that of [1] where it is proved that $\mathcal{P}_9$ is bounded by $K(5,2)$. Note that $K(5,2)$ is the well-known Petersen graph. For $r\geq 2k$ it is claimed by the Jeager-Zhang conjecture that C_{2k+1} is the optimal answer for Problem 5. This case would determine the circular chromatic number of $\mathcal{P}_{4k+1}$. While Grötzsch's theorem is a special case here, the best result for general k is that of X. Zhu [10] who proved that $\mathcal{P}_{8k-3}$ is bounded by C_{2k+1}.

The first case not covered by any of these theorems and conjectures is $k=3$ and $r=5$. For this case we introduce Conjecture 1. Following [2], we will use a definition of the Coxeter graph based on the Fano plane.

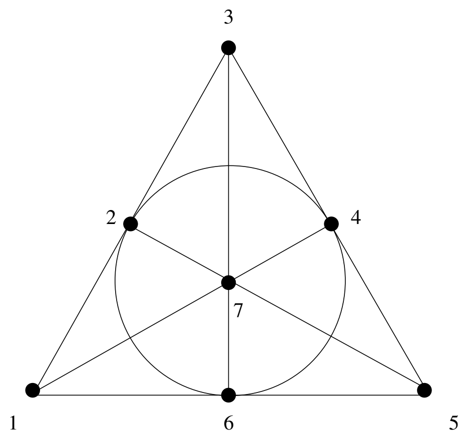

Figure 1: Fano plane

Given a set U of size 7, a Fano plane is a set of seven 3-subsets of U such that each pair of elements from U appears exactly in one 3-subset. It can be checked that there is a unique such collection up to isomorphism. This collection then satisfies the axioms of finite geometry and triples would be called lines. Throughout this paper we will use the labeling of Figure 1 to denote the Fano plane.

The Coxeter graph, denoted by Cox, is a subgraph of $K(7,3)$ obtained by deleting the vertices corresponding to the lines of the Fano plane. Therefore, Cox is an induced subgraph of PC(6). Hence, we propose that Cox is an answer for the case $k=3$ and $r=5$ of Problem 5.

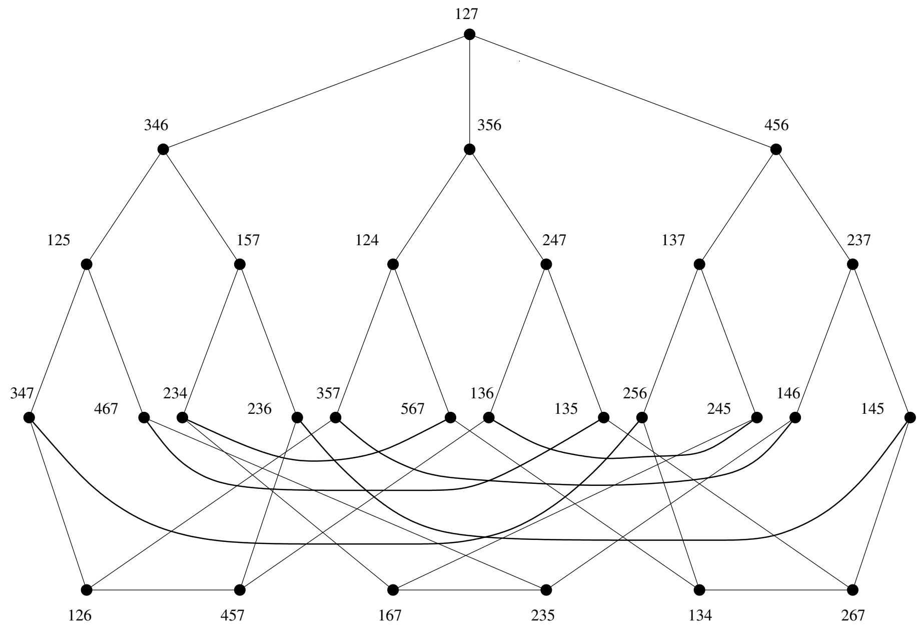

Figure 2: A representation of the Coxeter graph

Since C_7 is a subgraph of Cox, the result of Zhu [10] implies that Cox also bounds $\mathcal{P}_{21}$. The main result of this paper is to provide an improvement in this direction by proving Theorem 2.

Before proceeding further, we would like to mention the following interesting interpretation of a homomorphism to Cox. Given a graph G, a *Fano-coloring* of G is to associate to each vertex of G a triplet of elements of Fano satisfying the following two conditions:

(i) each triplet is in general position, i.e., it is not a line in Fano;

(ii) adjacent vertices have no element in common.

It simply follows from the definition that Fano-coloring is equivalent to homomorphism to Cox.

We will need the next corollary implied by successive applications of the Folding lemma of W. Klostermeyer and C. Q. Zhang (see [3]).

Lemma 6 (Folding lemma). *Let G be a plane graph of odd-girth $2k+1$. If $C = v_0v_1 \dots v_{r-1}v_0$ is a facial cycle of G with $r \neq 2k+1$, then there exists an $i \in \{0, 1, \dots, r-1\}$ such that the planar graph G' obtained from G by identifying v_{i-1} and v_{i+1} $(\operatorname{mod} r)$ is of odd-girth $2k+1$.*

Corollary 7. *Given a 2-connected planar graph G of odd-girth at least $2k+1$, there is a homomorphic image G' of G such that G' is a plane graph of odd-girth $2k+1$, and moreover every face of G' is a $(2k+1)$-cycle.*

Our proof of Theorem 2 is based on the discharging technique. Assuming that there is an element of $\mathcal{P}_{17}$ not mapping to Cox, we choose X to be such an element with the smallest value of $|V(X)| + |E(X)|$. Hence, X is simple and no proper homomorphic image of X is in $\mathcal{P}_{17}$. Since Cox is a vertex-transitive graph, X is 2-connected. Hence, Corollary 7 implies that X has a plane embedding whose faces are all 17-cycles. We fix such an embedding and denote it also by X.

The paper is organized as follows: in the next section we collect a list of properties of the Coxeter graph. In the following section we provide a list of small reducible trees. Finally, in the last section we use discharging technique to obtain a contradiction. Some larger configurations that show up during our discharging process are shown to be reducible in this section.

Some more notation we will use are as follows. Given a graph G, a vertex of degree d is called a d-vertex. Analogously, a d^+-vertex is a vertex whose degree is d or more. A path of length l is called an *l-path.* Any path $P_k = x_1x_2 \dots x_k$ is called an x_1-x_k path and the vertices $x_2, \dots, x_{k-1}$ are called the *internal* vertices of P_k. Note that P_k has length $k-1$. A *thread* in X is a maximal path $P = ux_1x_2 \dots x_nv$ where all the internal vertices $x_1, x_2, \dots x_n$ are 2-vertices of X. We will also say that P is a u-v thread. Distinct vertices x and y are said to be *weakly adjacent* if there exists a thread in X containing both of them. Given a 3^+-vertex x, the number of 2-vertices weakly adjacent to x is denoted by $d_{\text{weak}}(x)$. The *distance* $d_G(x, y)$ in G of two vertices x, y is the length of a shortest x-y path in G. Whenever the underlying graph is clear from the context, we will write $d(x, y)$ instead of $d_G(x, y)$. The greatest distance between any two vertices in G is the *diameter* of G. Given a positive integer i and a vertex x of G, $N_i(x)$ denotes the set of *i-th neighbors* of x, i.e., the set of vertices at distance exactly i from x. When $i = 1$, we simply write $N(x)$. For $U \subseteq V(G)$ we write $N_i(U) = \bigcup_{x \in U} N_i(x)$.

2. Coxeter graph and Cox-coloring

The Coxeter graph is well-known for its highly symmetric structure. There are many symmetric representations of it, but we will use the representation of Figure 2. Note that the labeling in Figure 2 is based on the labeling of the Fano plane given in Figure 1. The main properties of this graph we will need are collected in the following lemma.

Lemma 8. *The Coxeter graph satisfies the following:*

- *(i) It is distance-transitive.*
- *(ii) It is of diameter four.*
- *(iii) Its girth is seven.*
- *(iv) Given a vertex A, we have $|N(A)| = 3$, $|N_2(A)| = 6$, $|N_3(A)| = 12$ and $|N_4(A)| = 6$.*
- *(v) The independence number of* Cox *is* 12.
- *(vi) Let A and B be a pair of vertices in* Cox. *If $d(A, B) \leq 3$, then a 7-cycle passes through A and B. If $d(A, B) = 4$, then a 9-cycle passes through A and B.*
- *(vii) No homomorphic image of* Cox *is a proper subgraph of* Cox.
- *(viii) Given an edge A_1A_2, there exist exactly two vertices B_1 and B_2 such that $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$. Furthermore, B_1 and B_2 are adjacent vertices of* Cox.
- *(ix) Let $\mathcal{A}$ and $\mathcal{B}$ be two (not necessarily distinct) subsets of $V(\text{Cox})$ each of size at least 14. Then there are $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $AB \in E(\text{Cox})$.*
- *(x) For any two distinct vertices A and B of* Cox *we have $|N(A) \cap N(B)| \leq 1$.*
- *(xi) For any pair A and C of vertices of* Cox *we have $|N_2(A) \cap N(C)| \leq 2$ with equality only when $A \sim C$.*
- *(xii) For any pair A and C of vertices of* Cox *we have $|N_3(A) \cap N_3(C)| \geq 4$. Furthermore, when equality holds, there does not exist a vertex B in $N_2(A)$ and a vertex D in $N_2(C)$ such that $N_3(A) \cap N_3(C) \subseteq N(B) \cup N(D)$.*

Proof. The properties (i) through (v) are well known. We comment on the remaining seven.

- (vi) This is readily checked using the distance-transitivity of Cox.

- (vii) For contradiction, let ϕ be a homomorphism of Cox to a proper subgraph of itself. Then ϕ must identify at least two vertices, say A_1 and A_2. From (vi) follows the existence of an A_1-A_2 path P of odd length less than 7. Hence, the image of P under ϕ contains a closed odd walk of length strictly less than 7, contradicting (iii).

- $(viii)$ Since Cox is edge-transitive, without loss of generality, we may assume that $A_1 = 127$ and $A_2 = 346$. It is then implied that $\{B_1, B_2\} = \{134, 267\}$.

- (ix) Suppose some subsets $\mathcal{A}$ and $\mathcal{B}$ provide a counter-example, and let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$. We may assume each of $\mathcal{A}$ and $\mathcal{B}$ is of size 14. Note that by connectivity of Cox, $\mathcal{C}$ is not empty. Let $\mathcal{C}' = (\mathcal{A} \cup \mathcal{B})^c$. By our assumption, $\mathcal{C}$ is an independent set of Cox, thus $|\mathcal{C}| \leq 12$. Furthermore, for each vertex C in $\mathcal{C}$ all three neighbors of C are in $\mathcal{C}'$. Since Cox is 3-regular and $|\mathcal{C}'| = |\mathcal{C}|$, $\mathcal{C} \cup \mathcal{C}'$ induces a proper 3-regular subgraph of Cox, contradicting the connectivity of Cox.

(x) For otherwise, a 4-cycle would appear in Cox.

(xi) If A is not adjacent to C, then existence of two elements in $N_2(A) \cap N(C)$ would result in a cycle of length at most 6 which is a contradiction. If A is adjacent to C, then $N_2(A) \cap N(C) = N(C) \setminus \{A\}$.

(xii) Using the distance-transitivity of Cox, this is proved by considering the five possibilities for $d(A, C)$. If $d(A, C) = 0$ then $|N_3(A) \cap N_3(C)| = |N_3(A)| = 12$. If $d(A, C) = 1$, we may assume $A = 127$ and $C = 346$. Then $N_3(A) \cap N_3(C) = \{567, 135, 256, 145\}$ and it is readily checked that each of the second-neighbors of A has at most one neighbor among these four vertices, implying the last part of the statement. If $d(A, C) = 2$, we may assume $A = 127$ and $C = 125$. Then $N_3(A) \cap N_3(C) = \{234, 236, 357, 146\}$ and it is readily checked that the vertex 146 is respectively at distances $1, 3, 3$ from the vertices $357, 236, 234$. This clearly implies the last part of the statement. If $d(A, C) = 3$, we may assume $A = 127$ and $C = 347$. Then $N_3(A) \cap N_3(C) = \{236, 567, 136, 135, 245, 146\}$. Finally, if $d(A, C) = 4$, we may assume $A = 127$ and $C = 126$. Then $N_3(A) \cap N_3(C) = \{467, 567, 245, 145\}$ and each of the second-neighbors of A has at most one neighbor among these four vertices, implying the last part of the statement.

□

We may refer to a mapping of a graph H to the Coxeter graph as a Cox-coloring of H. A *partial* Cox-*coloring* of H is a mapping from a subset of vertices of H to vertices of the Coxeter graph which preserves adjacency among the mapped vertices. Let H be a graph, ϕ be a partial Cox-coloring of H and u be a vertex of H not colored yet. We define $\mathrm{ad}_{H,\phi}(u)$ to be the set of *admissible* colors for u, i.e., the set of distinct choices $A \in V(\mathrm{Cox})$ such that the assignment $\phi(u) = A$ is extendable to a Cox-coloring of H. When H and ϕ are clear from the context, we will simply write $\mathrm{ad}(u)$.

3. Reducible configurations

Recall that X is a minimal counter-example to Theorem 2. Given a subgraph T of X, let *boundary* of T, denoted $\mathrm{Bdr}(T)$, be the set of vertices of T which have at least one neighbor in $X - T$. Let the *interior* of T be $\mathrm{Int}(T) = T - \mathrm{Bdr}(T)$.

Let $X_T = X - \mathrm{Int}(T)$ be a subgraph of X induced by vertices not in $\mathrm{Int}(T)$. If $\mathrm{Int}(T)$ is an induced subgraph of X and at least one Cox-coloring of X_T can be extended to a Cox-coloring of X, then $(T, \mathrm{Bdr}(T))$ is called a *reducible* configuration. Each reducible configuration we will consider in this paper is a tree having all its leaf vertices as its boundary. Thus, we will simply use T to denote $(T, \mathrm{Bdr}(T))$.

We note that by the minimality, X cannot contain any reducible configuration.

In this section, we provide a list of ten reducible configurations, all of which are trees of small order. Sometimes to prove that a configuration is reducible, we will consider smaller configurations and prove that most of the local Cox-colorings on the boundary are extendable.

Our first lemma is about paths. Given a u-v path P of length at most five we characterize all possible Cox-colorings of $\{u, v\}$ which are extendable to P.

Lemma 9. *Let P be a u-v path of length l, $l \leq 5$. Consider a partial* Cox*-coloring ϕ given by $\phi(u) = A$ and $\phi(v) = B$. Then, ϕ is extendable to P if and only if:*

(i) $l = 2$ and $d(A, B) \in \{0, 2\}$, or

(ii) $l = 3$ and $d(A, B) \in \{1, 3\}$, or

(iii) $l = 4$ and $d(A, B) \neq 1$, or

(iv) $l = 5$ and $A \neq B$.

The proof of this lemma follows from Lemma 8 (i), (iii), (vi), and the following general remark: Let P be a u-v path, and P' be a u'-v' path of lengths l and l', respectively. Then, the mapping $\psi(u) = u'$, $\psi(v) = v'$ is extendable to a mapping of P to P' if and only if $l \equiv l' \pmod 2$ and $l \geq l'$.

Proposition 10. *P_7 is a reducible configuration.*

Proof. Let u and v be the two end-vertices of P_7. Let ϕ be a Cox-coloring of $X - \mathrm{Int}(P_7)$, say $\phi(u) = A$ and $\phi(v) = B$. Choose a neighbor C of B distinct from A. Let v' be the neighbor of v in P_7. Extend ϕ to v' by setting $\phi(v') = C$. Then by Lemma 9 (iv), ϕ is extendable to a Cox-coloring of P_7. □

In other words, the maximum length of a thread in X is at most 5. It follows immediately that:

Corollary 11. *Given a vertex v of X we have $d_{\text{weak}}(v) \leq 4d(v)$.*

For paths of length five or six we will need to know the number of ways a Cox-coloring of the two end-vertices extends to the interior. This is achieved by the next two lemmas.

Lemma 12. *Let $P = xv_1v_2v_3v_4y$ be a 5-path. Let $\phi(x) = A$ and $\phi(y) = B$, with $B \neq A$, be a partial* Cox*-coloring. If $d(A, B) = 2$, then $|ad(v_1)| = |ad(v_2)| = 2$ with the two possible choices for v_2 being at distance three in* Cox*. Otherwise, $|ad(v_1)| = 3$ and $|ad(v_2)| \geq 4$.*

Proof. Since Cox is distance-transitive, the statement can be proven by considering the four possibilities for $d(A, B)$ and applying Lemma 9. If $d(A, B) = 1$, we may assume $B = 127$ and $A = 346$. Then $\mathrm{ad}(v_1) = N(A)$ and $\mathrm{ad}(v_2) = \{346, 356, 456, 347, 467, 234, 236\}$, hence $|\mathrm{ad}(v_1)| = 3$ and $|\mathrm{ad}(v_2)| = 7$. In case $d(A, B) = 2$, we may assume $B = 127$ and $A = 125$. Then $\mathrm{ad}(v_1) = \{347, 467\}$ and $\mathrm{ad}(v_2) = \{135, 256\}$, hence $|\mathrm{ad}(v_1)| = 2$ and $|\mathrm{ad}(v_2)| = 2$, with the two admissible colors for v_2 being at distance three in Cox. If $d(A, B) = 3$, we may assume $B = 127$ and $A = 347$. Then $\mathrm{ad}(v_1) = N(A)$ and $\mathrm{ad}(v_2) = \{346, 347, 467, 357\}$, hence $|\mathrm{ad}(v_1)| = 3$ and $|\mathrm{ad}(v_2)| = 4$. Finally, if $d(A, B) = 4$, we may assume $B = 127$ and $A = 126$. Then $\mathrm{ad}(v_1) = N(A)$ and $\mathrm{ad}(v_2) = \{236, 136, 256, 146\}$, hence $|\mathrm{ad}(v_1)| = 3$ and $|\mathrm{ad}(v_2)| = 4$. □

Lemma 13. *Let $P = xv_1v_2v_3v_4v_5y$ be a 6-path. Let $\phi(x) = A$ and $\phi(y) = B$ be a partial* Cox*-coloring. If $d(A, B) = 1$, then $|\mathrm{ad}(v_3)| = 4$, furthermore these four colors constitute the neighbors of an edge of* Cox*. If $d(A, B) \neq 1$, then $|ad(v_3)| \geq 8$.*

Proof. We again apply Lemma 9. First we consider the case of $d(A, B) = 1$. Since Cox is distance-transitive, we may assume without loss of generality that $A = 127$ and $B = 346$. In this case $\mathrm{ad}(v_3) = \{567, 135, 256, 145\}$. Note that these are the neighbors of the edge $A'B'$, where $A' = 134$ and $B' = 267$. We note that each of A' and B' is at distance 4 from both A and B. This property uniquely determines the edge $A'B'$.

If $A = B$, then each vertex in $N(A) \cup N_3(A)$ is an admissible color for v_3, and we have $|\mathrm{ad}(v_3)| = 15$. If $d(A, B) = 2$, since Cox is distance-transitive, we may assume $A = 127$ and $B = 125$. In this case $\mathrm{ad}(v_3) = \{346, 356, 456, 347, 467, 234, 236, 357, 146\}$. For the case of $d(A, B) = 3$ we may assume $A = 127$ and $B = 347$, thus $\mathrm{ad}(v_3) = \{456, 256, 135, 245, 567, 146, 236, 136\}$. Finally, if $d(A, B) = 4$ we may assume $A = 127$ and $B = 126$. In this case we have $\mathrm{ad}(v_3) = \{346, 356, 347, 357, 467, 567, 145, 245\}$. □

Observe that, as a consequence of the fact that X is 2-connected, any 2-vertex x in X has exactly two weakly adjacent 3^+-vertices. Thus, there exists a unique thread having x as an internal vertex.

Proposition 14. *Given distinct 3^+-vertices u and v of X, there exists at most one u-v thread.*

Proof. Suppose there are two such threads, say P and P', of lengths l and l', respectively. Since the length of each thread is at most 5, l and l' must have the same parity, otherwise there would be an odd cycle of length less than 17 in $P \cup P'$. Without loss of generality, we may assume that $l \geq l'$. But then there exists a homomorphism $P \to P'$ that leaves u and v fixed, and hence there exists a homomorphism $X \to X - E(P)$, contradicting the fact that no proper homomorphic image of X is in $\mathcal{P}_{17}$.

□

Therefore, for any pair of weakly adjacent 3^+-vertices u and v, there exists a unique u-v thread. We define $T_{k_1 k_2 \dots k_r}$ with $0 \leq k_1 \leq k_2 \leq \cdots \leq k_r$ to be a graph obtained from $K_{1,r}$ by subdividing the ut_i-edge k_i times, where t_i's are the leaf vertices and u is the central vertex of $K_{1,r}$. Given an r-vertex u, with $r \geq 3$, we will denote by $T(u)$ the union of all the threads in X which have u as an end-vertex. A direct consequence of Proposition 14 is that $T(u)$ is a $T_{k_1 k_2 \dots k_r}$ with $k_r \leq 4$. The next few lemmas are about the possibilities for $T(u)$ when u is of degree 3 or 4.

Lemma 15. *Let $T = T_{222}$. Then the partial* Cox*-coloring $\phi(t_i) = A_i$, $i = 1, 2, 3$ is extendable to T unless $\{A_1, A_2, A_3\}$ induces a P_3 in* Cox.

Proof. Consider the t_1-t_2 path P in T. Let v be the middle vertex of this path and let $\mathcal{A}$ be the set of colors whose assignment to v is extendable to P. We use the proof of Lemma 13 for the different values of $d(A_1, A_2)$. In three of these possibilities, to be precise, when $d(A_1, A_2) \neq 1, 2$, we have $N(\mathcal{A}) \cup N_3(\mathcal{A}) = V(\mathrm{Cox})$. Thus, in these cases any choice of A_3 is extendable.

If $d(A_1, A_2) = 1$, then $N(\mathcal{A}) \cup N_3(\mathcal{A}) = V(\mathrm{Cox}) \setminus N(\{A_1, A_2\}) \cup \{A_1, A_2\}$. Thus, in this case a choice of A_3 is extendable unless either $A_3 \sim A_1$ and $A_3 \neq A_2$ or $A_3 \sim A_2$ and $A_3 \neq A_1$.

Finally if $d(A_1, A_2) = 2$, then $N(\mathcal{A}) \cup N_3(\mathcal{A}) = V(\mathrm{Cox}) \setminus \{B\}$, where B is the common neighbor of A_1 and A_2.

□

Proposition 16. *The configurations T_{123} and T_{034} are reducible.*

Proof. We give a proof for T_{123}, the proof for T_{034} is similar. Let X' be the subgraph of X obtained by deleting the interior of the u-t_3 thread. By the minimality of X, there is a Cox-coloring of X'. Thus, by Lemma 9, we may consider a Cox-coloring ϕ of $X - \mathrm{Int}(T_{123})$ for which $\phi(t_1) \neq \phi(t_2)$. Now, by Lemma 12, there is an extension of ϕ to the t_1-t_2 path of T_{123} such that $\phi(u) \nsim \phi(t_3)$. By Lemma 9, this extends to the rest of T_{123}. □

Proposition 16 yields the following corollary.

Corollary 17. *If v is a 3-vertex in X, then $d_{\text{weak}}(v) \leq 6$. Furthermore, if $d_{\text{weak}}(v) = 6$, then $T(v)$ is one of the following trees: T_{024}, T_{033}, T_{114}, T_{222}.*

Proof. It is easily observed that the only maximal subtrees $T_{k_1k_2k_3}$ with central vertex v, containing neither of the reducible configurations P_7, T_{123}, T_{034} are T_{024}, T_{033}, T_{114}, T_{222}. □

Proposition 18. *The configurations T_{1334}, T_{2234}, T_{2333} are reducible.*

Proof. Let T be one of the three configurations, and let ϕ be a Cox-coloring of $X - \mathrm{Int}(T)$ with $\phi(t_1) = A_1, \phi(t_2) = A_2, \phi(t_3) = A_3, \phi(t_4) = A_4$.

First assume $T = T_{1334}$. Using Lemma 9, all we need is to find a choice of color which is at distance 0 or 2 from A_1 (7 choices), adjacent to neither A_2 nor A_3, and distinct from A_4. Thus, by Lemma 8 (*xi*), at least one member of $N_2(A_1)$ satisfies all four conditions.

For the case of $T = T_{2234}$, using Lemma 13, if $d(A_1, A_2) \neq 1$, then we have at least eight choices for u each of which is extendable on the u-t_1 and u-t_2 threads. By Lemma 9, at least four of these choices can be extended also to the u-t_3 and u-t_4 threads. If $d(A_1, A_2) = 1$, then by Lemma 13 there are exactly four choices for u each of which is extendable on the u-t_1 and u-t_2 threads, furthermore at most two of these four colors are in $N(A_3)$. Of the remaining two we have a choice distinct from A_4.

For the last case, i.e., $T = T_{2333}$, by Lemma 9, the number of choices for $\phi(u)$ which are extendable on the u-t_i threads, $i = 1, 2, 3, 4$ are 15, 25, 25 and 25, respectively. Since there are 28 vertices in the Coxeter graph, there are at least 6 choices for $\phi(u)$ each of which extends on all the four threads. □

Corollary 19. *If v is a 4-vertex in X, then $d_{\text{weak}}(v) \leq 12$. Furthermore, if $d_{\text{weak}}(v) = 12$, then $T(v)$ is T_{0444}. Otherwise, $d_{\text{weak}}(v) \leq 10$.*

The simple proof of this corollary is analogous to the one of the previous corollary, hence we leave it to the reader.

Let u and v be weakly adjacent 3-vertices. We now would like to investigate $T(u) \cup T(v)$ (see Figures 3 and 4 where the black vertices have degrees as depicted in the figures, whereas the white vertices have arbitrary degrees greater that 2).

Proposition 20. *The three trees in Figure 3 are reducible.*

Proof. Consider a partial Cox-coloring of each of the configurations by assigning colors A, B, C and D to the vertices on the boundary as shown in Figure 3. We will use these colors to denote the respective leaf vertices of the configuration. By Lemma 9, to color u so that it is extendable to the A-u and B-u paths of the given configuration there are at

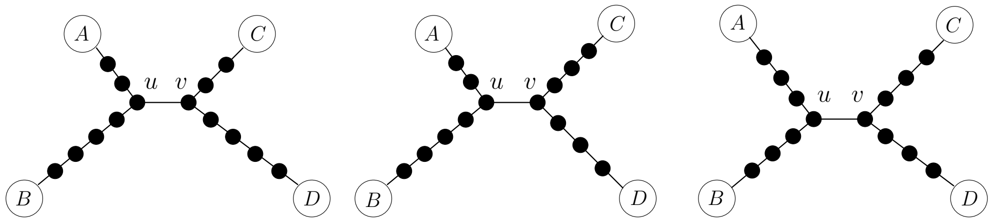

Figure 3: Reducible configurations of adjacent 3-vertices with a Cox-coloring of the boundary.

least 14 choices. Let $\mathcal{A}$ be set of those colors for u. Similarly, we have a set $\mathcal{B}$ of size at least 14 for v with respect to C and D. By Lemma 8 (ix), there are adjacent elements $S \in \mathcal{A}$ and $S' \in \mathcal{B}$. Assignment of S to u and S' to v is now extendable. □

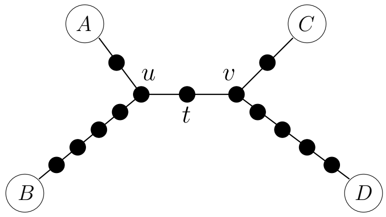

Figure 4: Reducible configuration with a Cox-coloring of the boundary.

Proposition 21. *The tree in Figure 4 is reducible.*

Proof. Consider a Cox-coloring of the leaf vertices, as depicted in Figure 4. We will again use these colors to denote the respective leaf vertices. By Lemma 9, to color u so that it is extendable to the A-u path one can choose any vertex in $N_2(A)$. Any such choice, except possibly B, would be extendable on the B-u path and any of its neighbors would be admissible for t. Thus, for at least ten elements $A' \in N_3(A)$ the assignment of A' to t is extendable on the left. Furthermore, two third-neighbors of A are not admissible colors for t if and only if B is a second-neighbor of A, and then these particular two non-admissible colors belong to $N(B)$. Similarly, for at least ten choices of $C' \in N_3(C)$ the assignment of C' to t is extendable on the right with ten being the exact number of choices if $D \in N_2(C)$. We now apply Lemma 8 (xii) to find a color for t admissible from both sides. □

The next two configurations we consider are not reducible. But we show that, up to isomorphism, there is a unique Cox-coloring of the boundary which is not extendable to the interior. This implies, in particular, that if there exists a second choice for a color of one of the vertices on the boundary, then the coloring is extendable.

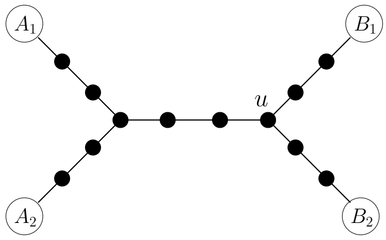

Figure 5: Configuration F_1 with a Cox-coloring of the boundary.

Proposition 22. *The partial* Cox*-coloring of the configuration F_1 given in Figure 5 is extendable to the whole configuration unless $d(A_1, A_2) = d(B_1, B_2) = 1$ and $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$.*

Proof. Consider the T_{222} configuration whose boundary consists of the vertex u and the two vertices colored with A_1 and A_2, respectively. If $d(A_1, A_2) \neq 1$, then by Lemma 15, there are at least 27 choices of color for u which is extendable to the interior of this T_{222}. On the other hand, by Lemma 13, there are at least 4 choices of color for u that is extendable to a Cox-coloring of the partially colored 6-path connecting B_1 and B_2. Thus, there are at least three common choices of color for u which is extendable on the whole configuration.

If $d(A_1, A_2) = 1$, then, again by Lemma 15, there are exactly four non-extendable choices of color for u for the considered T_{222} configuration. These particular four choices are the neighbors of A_1 and A_2 distinct from A_1 and A_2. If any of the other 24 choices is extendable on the B_1-B_2 path, then the coloring is extendable to the whole configuration. Otherwise, by the proof of Lemma 13, we have $d(B_1, B_2) = 1$ and, furthermore, $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$. □

Corollary 23. *For the configuration F_1 of Figure 5, if the given partial* Cox*-coloring is not extendable to the whole configuration, then A_1 is uniquely determined by A_2, B_1 and B_2.*

Proof. Note that given an edge A_1A_2, the property $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$ determines a unique edge in Cox, as shown in Lemma 8 (*viii*). □

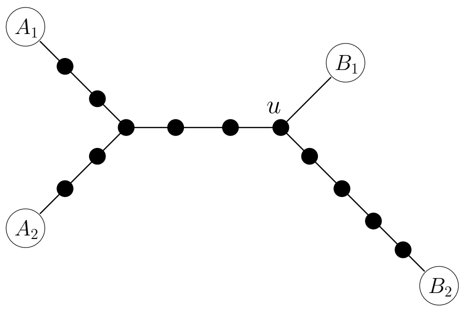

Figure 6: Configuration F_2 with a Cox-coloring of the boundary.

Proposition 24. *The partial* Cox*-coloring of the configuration F_2 given in Figure 6 is extendable to the whole configuration unless $d(A_1, A_2) = 1$ and $\{B_1, B_2\} = \{A_1, A_2\}$.*

Proof. The proof is similar to that of the previous proposition. Again we consider the possibilities on u. If $d(A_1, A_2) \neq 1$, then, by Lemma 15, there are at least 27 choices of color for u that would be extendable on the part connecting to A_1 and A_2. Of these 27 vertices in Cox, at least two are neighbors of B_1 and of these two, one is distinct from B_2. This color is an extendable choice.

If $d(A_1, A_2) = 1$, then the four neighbors of A_1 and A_2, distinct from A_1 and A_2, are the only choices for u that would make the coloring non-extendable on the left side. If $B_1 \notin \{A_1, A_2\}$, then there are at least two neighbors of B_1 whose assignments to u are extendable on the left side of u, and at least one of these two is different from B_2. Thus, we may assume without loss of generality that $B_1 = A_1$. Then A_2 is an extendable choice for u unless $B_2 = A_2$. □

Corollary 25. *If the partial* Cox*-coloring of the configuration F_2 given in Figure 6 is not extendable to the whole configuration, then A_1 is uniquely determined by A_2, B_1 and B_2.*

4. Discharging and further reducible configurations

Recall that X, our minimal counterexample, is a 2-connected plane graph whose faces are all 17-cycles.

Being a plane graph, for any choice of real numbers α and β satisfying the condition $\alpha + \beta = 3$, the Euler formula gives the following identity for the vertices and faces of X:

$$\sum_v (\alpha \cdot d(v) - 6) + \sum_f (\beta \cdot l(f) - 6) = -12, \tag{1}$$

where the second sum is taken over all faces of X with $l(f)$ denoting the length of face f. We remark that the identity (1) is just the well-known Euler formula $V - E + F = 2$ in disguise because of the identity $\sum_v d(v) = 2E = \sum_f l(f)$.

Thus, by setting $\beta = \frac{6}{17}$ and $\alpha = \frac{45}{17}$ the identity (1) reduces to

$$\sum_v (45 \cdot d(v) - 102) + \sum_f 0 = -204. \tag{2}$$

This would lead to the following initial charge on each vertex v of X:

$$w_0(v) = 45 \cdot d(v) - 102.$$

Note that $\sum_{v \in V(X)} w_0(v) = -204 < 0$, and that each 2-vertex has initial charge -12, each 3-vertex has initial charge 33, each 4-vertex has initial charge 78, etc. Note that X has no 1-vertices as it is 2-connected. Our aim is to redistribute the charges on the vertices so that at the final step, the charge on each vertex is non-negative. This contradiction would disprove the existence of X. We will accomplish this through two phases of discharging. In the first phase, we will take care of vertices of degree 2. Then, in the second phase, we design a discharging rule that would take care of all negatively charged vertices after the first phase. We will then show that each configuration which may lead to a vertex of negative charge is reducible. This would complete our proof.

4.1. First phase of discharging

Here we use the following discharging rule:

(R1) For each pair x, y of weakly adjacent vertices in X with $d(x) = 2$ and $d(y) \geq 3$, y sends charge of 6 to x.

Let $w_1(v)$ denote the new charge at each vertex v. If $d(v) = 2$ then v receives a total charge of 12 (6 from each of its weakly adjacent 3^+-vertices), hence $w_1(v) = 0$. If $d(v) = 3$, then by Corollary 17, we have $w_1(v) \geq -3$. Furthermore, if $d_{\text{weak}}(v) \neq 6$, then $w_1(v) \geq 3$. For $d(v) \geq 4$ we have $w_1(v) \geq 6$ by Corollaries 11 and 19.

A vertex v of X is called *poor* if $w_1(v) < 0$. As a consequence of Corollary 17, we have the following characterization of poor vertices.

Proposition 26. *A vertex v of X is poor if and only if $d(v) = 3$ and $d_{\text{weak}}(v) = 6$.*

Corollary 17 also implies that for each poor vertex v, $T(v)$ is one of the following trees: T_{024}, T_{033}, T_{114}, T_{222}. Our aim is to seek charge for v from its closest leaf vertices of $T(v)$. Given a 3^+-vertex $x \in V(X)$, we say x *supports* v if: (*i*) $w_1(x) > 0$, and (*ii*) x is a leaf vertex of $T(v)$ on a shortest thread of $T(v)$. Note that each such thread has length at most 3. Furthermore, observe that v may have more than one supporting vertex x.

4.2. Second phase of discharging

In this phase we try to increase the charge of all poor vertices. The discharging rule is as follows:

(R2) Whenever y supports a poor vertex x, then y gives charge of 3 to x if $d(x, y) = 1$, and charge of 1.5 to x if $d(x, y) \neq 1$.

Let $w_2(v)$ be the charge of an arbitrary vertex v after this phase. We will show that $w_2(v) \geq 0$, for every vertex v of X.

Observe that the charge of each 2-vertex v remains the same, i.e. $w_2(v) = 0$. If v is a 5^+-vertex, then by Corollary 11 we have $w_1(v) \geq w_0(v) - 24d(v) = 21d(v) - 102 \geq 3$. Furthermore, if v is a support for a vertex u, then the number of 2-vertices on the v-u thread is at most two. Thus, if v supports r vertices then $w_1(v) \geq 3 + 12r$. This implies that $w_2(v) \geq 3$. Now, assume v is a 4-vertex. By Corollary 19, unless $T(v) = T_{0444}$, we have $w_1(v) \geq 18$ and this clearly gives $w_2(v) \geq 6$. If $T(v) = T_{0444}$, then v supports at most one vertex, and therefore $w_2(v) \geq 3$.

We are left to consider the case of a 3-vertex v. If v is non-poor, we have $w_1(v) \geq 3$, and if, furthermore, $d_{\text{weak}}(v) \leq 4$, then $w_1(v) \geq 9$. Therefore, if v supports only one poor vertex, or if $d_{\text{weak}}(v) \leq 4$, then $w_2(v) \geq 0$ is assured. The remaining two possibilities for v are as follows: either (i) v is a poor vertex, or (ii) $d_{\text{weak}}(v) = 5$ and v supports at least two poor vertices.

We will complete our proof by showing that:

(i) If v is a poor vertex, then either v has an adjacent supporting vertex or it has at least two supporting vertices.

(ii) If $d(v) = 3$ and $d_{\text{weak}}(v) = 5$, then when applying (R2) v sends total charge of at most 3.

To prove (i), first we consider the case when v is adjacent to a 3^+-vertex, say x. We claim that x is the adjacent supporting vertex of v. To see this, suppose by contradiction x is poor. Then, the union $T(x) \cup T(v)$ must be one of the configurations of Figure 3. But these are reducible configurations as shown in Proposition 20.

Thus, we may assume that all neighbors of v are 2-vertices. Now consider the case when there is a 3^+-vertex, say x, at distance 2 from v. Then $T(v)$ must be T_{114}. Hence, there are two 3^+-vertices at distance 2 from v. It remains to prove that neither of these two vertices is poor. By contradiction, suppose x is a poor vertex. Then $T(x)$ itself must be T_{114}. Since each face of X is a 17-cycle, the union $T(x) \cup T(v)$ must be the configuration of Figure 4, which is shown to be reducible in Proposition 21.

Hence we may assume that $T(v)$ is T_{222}. We prove that at most one vertex in $N_3(v)$ is poor. If $x \in N_3(v)$ is a poor vertex, then $T(x)$ must be either T_{222} or T_{024}. Then, the union $F = T(v) \cup T(x)$ is, respectively, the configuration of Figure 5 or the configuration of Figure 6. Let y be another vertex in $N_3(v)$. If y is also a poor vertex, then in $X - \text{Int}(F)$ the vertex y is an internal vertex of an induced 5-path P. Thus, by Lemma 12, there are at least two choices for extending a Cox-coloring of $(X - \text{Int}(F)) - \text{Int}(P)$ to y, one of which is extendable to a Cox-coloring of X by Corollary 23 or Corollary 25.

To prove (ii), we begin by observing that $T(v)$ must be one of the configurations: T_{014}, T_{023}, T_{113}, T_{122}.

Assume first that $T(v)$ is T_{014} and v supports two poor vertices. There are only two such possible configurations, shown in Figure 7. In this figure, vertices in square are the poor vertices whose support is v. We claim that each configuration of Figure 7 is reducible. To prove this, consider a Cox-coloring of the leaf vertices (as depicted in Figure 7), and look first for the minimum number of possible colors for v if we were to extend the partial Cox-coloring by A and B from left until v. This number, which is shown on the left of v, is derived as follows. For the first configuration, by Lemma 9 we have that $\text{ad}(\square) = (N(A) \cup N_3(A)) \backslash \{B\}$. Using the vertex-transitivity of Cox, it is readily observed that the considered $\text{ad}(v)$ surely includes A, all 6 second-neighbors of A, at least 11 third-neighbors of A, and all 6 fourth-neighbors of A, giving the total of 24 choices. For the second configuration, Lemma 9 gives $\text{ad}(\square) = (N(A) \cup N(B))^c$. Hence, Lemma 8 (ix) implies that the considered $\text{ad}(v) = \{A, B\}^c$. Similarly, the minimum number of possible extensions of coloring by C and D to v, from right only, is given on the right of v. This number can be easily deduced from Lemma 9. Since the sum of the two numbers in each of the configurations is greater than the number of vertices of Cox, a good common choice for coloring v from both sides exists in each case.

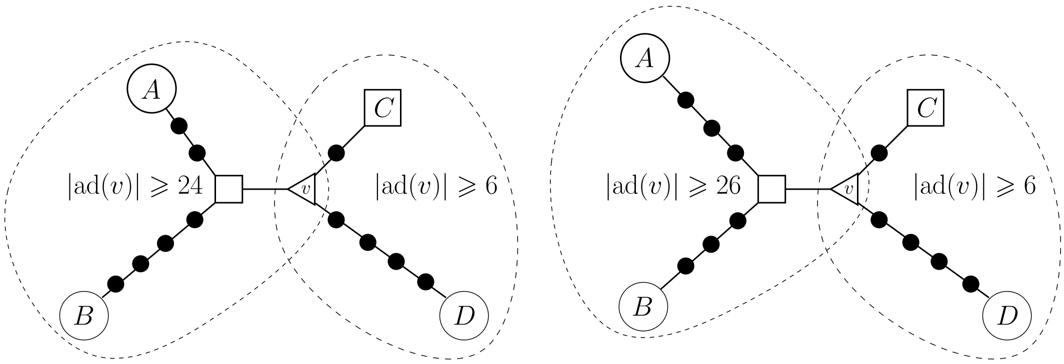

Figure 7: Local configurations of a center of T_{014} supporting two poor vertices.

For the case when $T(v)$ is isomorphic to T_{023}, using the Figure 8, a similar argument is applied.

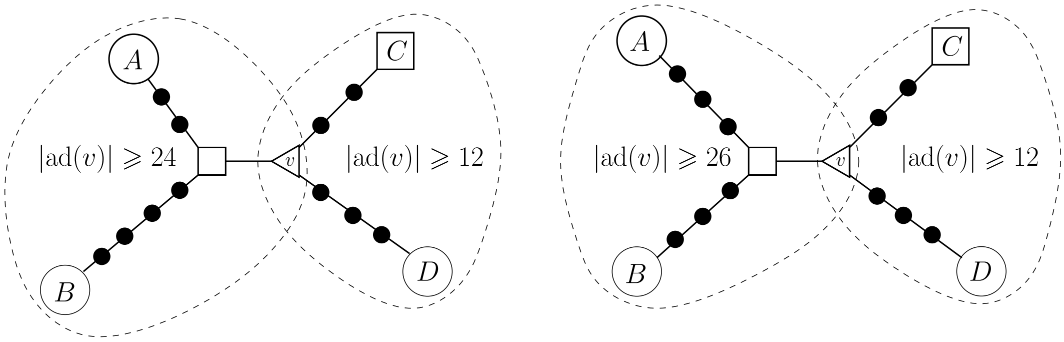

Figure 8: Local configurations of a center of T_{023} supporting two poor vertices.

If $T(v)$ is T_{113}, then v is a support of at most two poor vertices to each of which it may send charge of 1.5. Hence, in the second phase v gives as support at most charge of +3.

Finally, assume that $T(v)$ is T_{122}. If $w_2(v) < 0$, then by $(R2)$, v must have given charges to three weakly adjacent poor vertices. In this case, we have a unique local configuration, given in Figure 9. To prove its reducibility, consider a Cox-coloring of its leaf vertices. In this figure the minimum number of possible choices of colors for v extending partial Cox-colorings from the three different directions to v are as given in Figure 9. The first of these three numbers, namely the 23 choices for extending to v the partial coloring by A and B, is derived as follows. If $d(A, B) = 2$, the set of admissible colors for the relevant neighbor of v consists of the neighbors of A and the 10 third-neighbors of A that are not adjacent to B. Hence, it is readily checked (using the distance-transitivity of Cox) that the considered $|\mathrm{ad}(v)| = 23$. Otherwise $d(A, B) \neq 2$, and each third-neighbor of A is an admissible color for the relevant neighbor of v, which readily gives at least 24 colors in the considered $\mathrm{ad}(v)$. The remaining two numbers in the Figure 9 follow from Lemma 9 and Lemma 15, respectively. From these three minimum numbers for admissible choices of colors for v, it easily follows that there is a common choice.

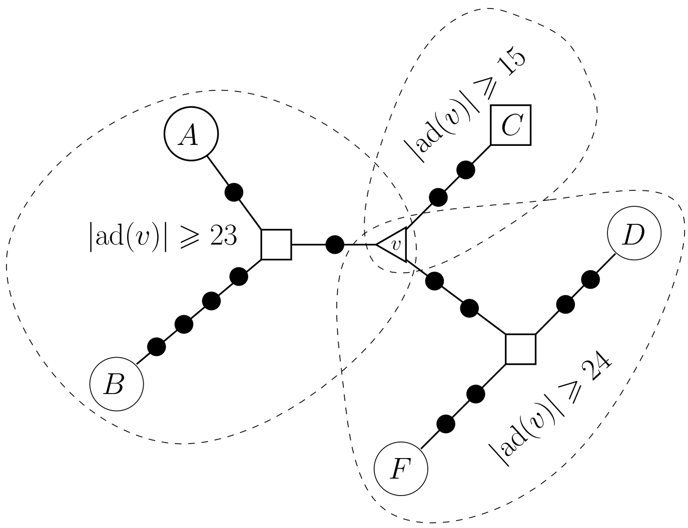

Figure 9: Local configuration of a center of T_{122} supporting three poor vertices.

5. Concluding remarks and further work

We have shown in this paper that one may use the existence of a combinatorial design to propose answer for special cases of the Problem 5. Our primary concern in this paper was the case $r = 5$ and $k = 3$ of this question and we proposed an answer using the Fano plane. At a 2011 summer workshop in Prague, Peter Cameron proposed a similar conjecture for the case of $r = 7$ and $k = 5$ based on the existence of a unique Steiner quintuple system of order 11.

The condition of odd-girth 17 was used only when applying Euler formula, indeed each of the 15 reducible configurations we used in our proof is a tree. Thus if X is a minimal graph which admits no homomorphism to Cox (i.e., every proper subgraph admits a homomorphism to Cox), then X does not contain any of these reducible configurations. We believe that with a larger set of reducible trees and together with cumbersome discharging steps we can improve the result for odd-girth 15. However, it seems that to prove the conjecture using the discharging technique, if possible at all, one has to consider reducible configurations that involve cycles.

We have considered Problem 5 for PC($2k$). However, using recent developments on signed graphs, a similar question could be asked for PC($2k-1$), see [6]. We will address this question in forth coming works.

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