Limit theorems for nearly unstable Hawkes processes

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Journées MAS 2014

Thursday 28th August 2014
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Introduction

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Definition (Hawkes 1971)

A Hawkes process \( N \) is a point process on \( \mathbb{R}(+) \) of intensity:

\[
\lambda_t = \mu + \int_{-\infty}^{t} \phi(t - s) dN_s
\]

\[
= \mu + \sum_{J_i < t} \phi(t - J_i)
\]

where \( \mu \in \mathbb{R}_+^{*} \) is the exogenous intensity and \( \phi \) is a positive kernel supported in \( \mathbb{R}_+ \) which satisfies \( \int \phi < 1 \) and the \( J_i \) are the points of \( N \).
Basic properties

**Proposition (Hawkes 1971)**

*The process is well defined and admits a version with stationary increments under the stability condition:*

\[ |\phi| := \int \phi < 1. \]

**Proposition**

*The average intensity of a stationary Hawkes process is*

\[ E[\lambda_t] = \frac{\mu}{1 - |\phi|}. \]
Endogeneity of a Hawkes process.

- $\mu$ can be seen as the exogenous part of the intensity.
- $\lambda_t - \mu = \int_0^t \phi(t - s) dN_s$ as the endogenous part of the intensity.
- $\frac{\mathbb{E}[\lambda] - \mu}{\mathbb{E}[\lambda]} = |\phi|$ is thus a measure of the endogeneity of the process.
- $|\phi|$ close to one means that the process is very endogenous.
Proposition (Dayri et al. 2012)

The correlation of the $h$-increments of stationary Hawkes processes

\[ C^h_\tau = \text{Cov}(N_{t+\tau+h} - N_{t+\tau}, N_{t+h} - N_t) \]

can be computed:

\[ C^h_\tau = h\Lambda\left( g^h_\tau + (g^h \ast \psi)_{-\tau} + (g^h \ast \psi)_\tau + (g^h \ast \tilde{\psi} \ast \psi)_\tau \right) \]

where \( \Lambda = \mu/(1 - |\phi|) \), \( \psi = \sum_{k=1}^{+\infty} \phi^k \), \( \tilde{\psi}(x) = \psi(-x) \) and \( g^h_\tau = (1 - |\tau|/h)^+ \).

Proposition (Dayri et al. 2012)

Reversely, given an empirical correlation function, it is possible to numerically find a $\phi$ which fits it.
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Various applications

- Ecology (Hawkes, Oakes 1974).
- Seismology (Ogata 1998).
- Genomic analysis (Reynaud-Bouret, Schbath 2010).
- Sociology (Mohler et al. 2011).
Applications to finance

- Midquotes and transaction prices: Bowsher (07), Bauwens and Hautsch (04), Hewlett (06), Bacry, Delattre, Hoffmann, Muzy (13).
- Order books: Large (07).
- Financial contagion: Aït-Sahalia, Cacho-Diaz, Laeven (10).
- Credit Risk: Errais, Giesecke, Goldberg (10).
- Market activity.
Financial modelling of market activity

**Definition**

The order flow process is the cumulated number of market orders which arrived during the day.

- Hawkes processes are a natural way to reproduce the clusterization of this process.
- Nice branching interpretation (endogenous vs. exogenous orders).
- Tractability.
Clustering at low time scales

**Figure**: Cumulated number of trades as a function of time over 20 seconds (DAX 01/07/2013).
Intermediate scale

Figure: Cumulated number of trades as a function of time over 3 minutes (DAX 01/07/2013).
Persistence at high time scales

**Figure**: Cumulated number of trades as a function of time (green) over a trading day (DAX 01/07/2013).
As Poisson processes, at large time scales, Hawkes processes behave as deterministic processes. They thus cannot fit the data.

**Theorem (Bacry et al. 2013)**

The sequence of renormalized Hawkes processes

\[ X_v^T = \frac{N_vT}{T} \]

is asymptotically deterministic, in the sense that the following convergence in \( L^2 \) holds:

\[ \sup_{v \in [0,1]} |X_v^T - \frac{\mu}{1 - |\phi|} v| \rightarrow 0. \]
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Formal framework

- Most estimation procedures applied to the financial order flow yield a parameter $|\phi|$ close to one. This is due to the persistence in the order flow.
- We want to study the long term behaviour of Hawkes processes close to criticality (whose kernel’s norm is close to one).
- Formally, we consider a sequence of Hawkes processes $(A_T N_{Tt}^T)_{t \geq 0}$ indexed by the observation scale $T$ of intensity $\mu$ and of kernel

$$\phi^T = a_T \phi$$

with $\int \phi = 1$ and $a_T \to 1$ but $a_T < 1$. 
Our asymptotic

Assumption

\[ T(1 - a_T) \xrightarrow{T \to +\infty} \lambda. \quad (3) \]

\[ \int_0^{+\infty} s\phi(s)ds = m < \infty. \quad (4) \]

\( \phi \) is differentiable with derivative \( \phi' \) such that

\[ \| \phi' \|_\infty < +\infty \text{ and } \| \phi' \|_1 < +\infty. \]

Finally, \( \| \psi^T \|_\infty \) is bounded.
The theorem

Let us denote $C^T_t = (1 - a_T) \lambda^T_{Tt}$.

**Theorem (Jaisson, Rosenbaum 2013)**

The sequence of renormalized Hawkes intensities $(C^T_t)$ converges in law, for the Skorohod topology, towards the law of the unique strong solution of the following Cox Ingersoll Ross stochastic differential equation on $[0, 1]$:

$$C_t = \int_0^t (\mu - C_s) \frac{\lambda}{m} ds + \frac{\sqrt{\lambda}}{m} \int_0^t \sqrt{C_s} dB_s.$$
The theorem

Furthermore, the sequence of renormalized Hawkes processes

\[ V_t^T = \frac{1 - a_T}{T} N_{tT} \]

converges in law, for the Skorohod topology, towards the process

\[ \int_0^t C_s ds, \quad t \in [0, 1]. \]
We have essentially shown that if one looks at a Hawkes process of kernel’s norm close to one at a time scale of the order $1/(1 - |\phi|)$ then one sees an integrated CIR process.

- At macroscopic time scales, the cumulated order flow is empirically proportional to the integrated variance:

$$V_t = \kappa \int_0^t \sigma_s^2 ds.$$ 

- In many usual frameworks, the macroscopic squared volatility is modelled as a CIR process.
The model

Bidimensional Hawkes process

We consider the model for the mid price of Bacry et al (11):

\[ P^T_t = N^{T+}_t - N^{T-}_t, \]

with \((N^{T+}, N^{T-})\) a bidimensional Hawkes process with intensity

\[
\begin{pmatrix}
\lambda^{T+}_t \\
\lambda^{T-}_t
\end{pmatrix} = \begin{pmatrix}
\mu \\
\mu
\end{pmatrix} + \int_0^t \begin{pmatrix}
\phi^T_1(t - s) & \phi^T_2(t - s) \\
\phi^T_2(t - s) & \phi^T_1(t - s)
\end{pmatrix} \begin{pmatrix}
dN^{T+}_s \\
dN^{T-}_s
\end{pmatrix}.
\]
Assumption

We assume

$$\phi^T_i(t) = a_T \phi_i(t),$$

where $\left(a_T\right)_{T\geq 0}$ is a sequence of positive numbers converging to one such that for all $T$, $a_T < 1$ and $\phi_1$ and $\phi_2$ such that

$$\int_0^{+\infty} \phi_1(s) + \phi_2(s) \, ds = 1 \text{ and } \int_0^{+\infty} s(\phi_1(s) + \phi_2(s)) \, ds = m.$$
Properties of the model

The preceding model takes into account the discreteness and the negative autocorrelation of the prices at the microstructure level.

Figure: Traded price as a function of time (Bund 01/07/2013).
Convergence to a Heston model

**Theorem**

Let \( \phi = \phi_1 - \phi_2 \). The renormalized process

\[
P^T_t = \frac{1}{T} (N^{T+}_{Tt} - N^{T-}_{Tt})
\]

converges in law, for the Skorohod topology, towards a Heston type process \( P \) on \([0, 1]\) defined by:

\[
\begin{align*}
    dC_t &= \left(\frac{2\mu}{\lambda} - C_t\right) \frac{\lambda}{m} dt + \frac{1}{m} \sqrt{C_t} dB^1_t \\
    dP_t &= \frac{1}{1 - \|\phi\|_1} \sqrt{C_t} dB^2_t
\end{align*}
\]

with \((B^1, B^2)\) a bidimensional Brownian motion.
Thank you for your attention!