Random Walks in Random Environment

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Random Walks in Random Environment on $\mathbb{Z}^d$

Let $(e_1, \dots, e_d)$ be the canonical base of $\mathbb{Z}^d$, and $e_{j+d} = -e_j$.
$(e_1, \dots, e_{2d})$ is the set of unit vectors of $\mathbb{Z}^d$.
The set of environments is the set of (weakly) elliptic transition probabilities

$$\Omega = \{(\omega(x, x + e_i)) \in (0,1)^{\mathbb{Z}^d \times \{1,\dots,2d\}}, \ \forall x \in \mathbb{Z}^d, \sum_{i=1}^{2d} \omega(x, x + e_i) = 1\}.$$

For $\omega \in \Omega$, the law of the Markov chain in environment ω, starting from x, is denoted by P_x^ω:

$$P_x^\omega(X_{n+1} = y + e_i | X_n = y) = \omega(y, y + e_i),$$

We define the law $\mathbb{P}$ on the environment Ω as follows: At each point x in $\mathbb{Z}^d$, we choose independently the transition probabilities

$$(\omega(x, x+e_i))_{i=1,\dots,2d}$$

according to the same law μ; μ is a law on the set

$$\{(\omega_1, \dots, \omega_{2d}) \in (0,1]^{2d}, \quad \sum_{i=1}^{2d} \omega_i = 1\}.$$

The annealed law is

$$P_x(\cdot) = \mathbb{E}(P_x^\omega(\cdot))$$

Definition

The RWRE *is uniformly elliptic* if there exists $c \in (0,1)$ such that a.s.

$$\omega(x, x+e_i) > c, \forall x \text{ and } i = 1, \dots, 2d.$$

Otherwise we say it is *weakly elliptic*.

We will consider dimension $d \geq 2$, where fundamental questions are open. Mainly three directions in which we know more about the RWRE.

- In the ballistic regime ((T) condition of Sznitman).
- For a small isotropic perturbation of the SRW (Bricmont-Kupianen, Sznitman Zeitouni, Bolthausen Zeitouni).
- For Dirichlet environments.

Basic Definitions

- The RWRE is *transient in the direction* $\ell \in \mathbb{R}^d \setminus \{0\}$ if
$$\lim_{n\to\infty} X_n \cdot \ell = +\infty.$$
- The RWRE is ballistic if there exists $v \neq 0$ such that
$$\lim_{n\to\infty} \frac{X_n}{n} = v.$$
- The RWRE satisfies an annealed CLT if
$$\frac{X_{\lfloor nt\rfloor} - \lfloor nt\rfloor v}{\sqrt{nt}}$$
converges under P_0 to a non-degenerate Brownian motion. It satisfies an quenched CLT if it converges under P_0^ω for a.s. ω.

Important conjecture : In dimension $d \geq 2$, if the RWRE is uniformly elliptic, directional transience and ballisticity are equivalent.

It is wrong for weakly elliptic RWRE, due to the presence of finite size traps.

If the RWRE is transient in the direction $\ell \in \mathbb{R}^d \setminus \{0\}$, we denote by

$$(\tau_i)_{i\geq 1},$$

the renewal times in the direction ℓ (cf Sznitman, Zerner).
They have the property that

$$X_{\tau_i+t}\cdot\ell \geq X_{\tau_i}\cdot\ell \text{ for all } t\geq 0$$

and

$$X_t\cdot\ell \leq X_{\tau_i}\cdot\ell \text{ for all } t<\tau_i.$$

Theorem (Sznitman-Zerner 99, Sznitman 00)

If τ_1 is integrable, the RWRE is ballistic. If it is square integrable then the RWRE satisfies an annealed CLT

Theorem (Rassoul-Agha, Seppäläinen, 09 (Berger, Zeitouni, 09))

If $\mathbb{E}(\tau_1^{p_0})<\infty$ for some $p_0>176d$, then the RWRE satisfies a Quenched CLT.

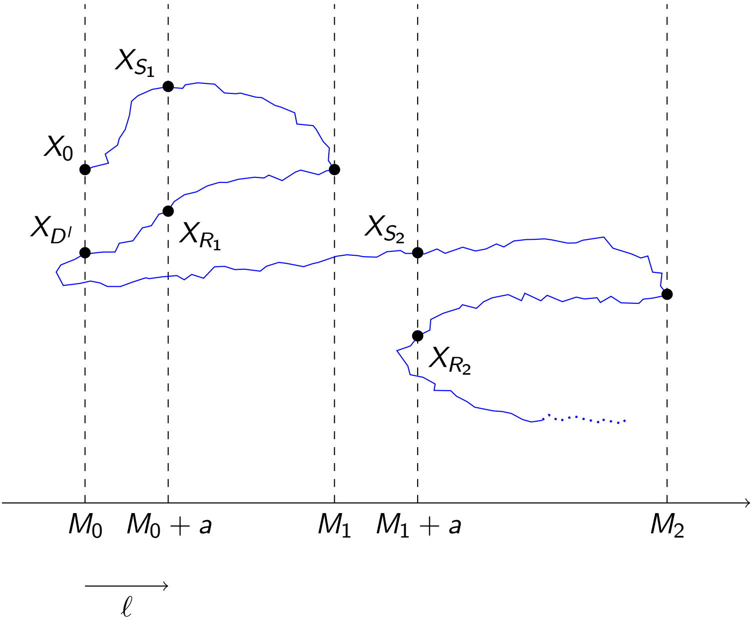

Figure : Construction of renewal times

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Definition (Sznitman 00)

Let $\gamma \in (0,1]$. The RWRE satisfies the $(T)_\gamma$ condition in the direction ℓ if it is transient in the direction ℓ and if there exists $c > 0$ such that

$$\mathbb{E}_0\left(\exp\left(c \sup_{0\le n\le\tau_1} \|X_n\|_2^\gamma\right)\right) < \infty.$$

We write $(T') = \cap_{\gamma\in(0,1)}(T)_\gamma$.

Theorem (Sznitman 00)

Let $d \geq 2$. If the RWRE is uniformly elliptic and satisfies (T') in the direction ℓ, then $\mathbb{E}(\tau_1^p) < \infty$ for all $p > 0$.

Important open question : is directional transience equivalent to (T')?

Theorem (Berger, Drewitz, Ramirez 12, Campos, Ramirez 12)

The (T') condition is equivalent to a polynomial condition $(P)_M$ for $M > 15d + 5$.

Kalikow's 0-1 law

Theorem

In any dimension d, for any $\ell \neq 0$,

$$\mathbb{P}(\{\lim X_n \cdot \ell = +\infty\} \cup \{\lim X_n \cdot \ell = -\infty\}) = 0 \text{ or } 1.$$

The following is an "annoying" open question!

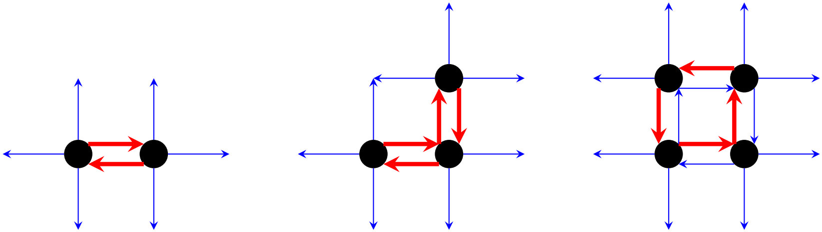
Conjecture : For all dimension d, for all $\ell \neq 0$

$$\mathbb{P}(\{\lim X_n \cdot \ell = +\infty\}) = 0 \text{ or } 1.$$

It is solved in dimension $d \leq 2$ (Zerner, Merkl, 2001), but it is still open in dimension $d \geq 3$.

The weak elliptic case

Weak ellipticity may create finite size traps.



Consequence : there exist directional transient RWRE with zero speed, i.e. such that $\mathbb{E}(\tau_1) = +\infty$ (cf Dirichlet RWRE).
In dimension $d \geq 2$, we expect that only finite traps matter.

We consider the following questions

- What ellipticity condition is needed to have $E_0((\tau_1)^p) < \infty$?
- What integrability condition is needed to have a Quenched CLT?

The special case of Dirichlet environments

- Fix some positive parameters $(\alpha_1, \ldots, \alpha_{2d})$, one for each direction of the space.
- The Dirichlet environment corresponds to the case where the law μ at one site is a Dirichlet law with parameters $\alpha_1, \ldots, \alpha_{2d}$,

More precisely, the $(\omega(x, e_i))_{i=1,\ldots,2d}$ are i.i.d. with density

$$\frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_{i=1}^n \Gamma(\alpha_i)} \left(\prod_{i=1}^n \omega_i^{\alpha_i - 1} \right) d\omega_1 \cdots d\omega_{n-1}.$$

Why Dirichlet laws?

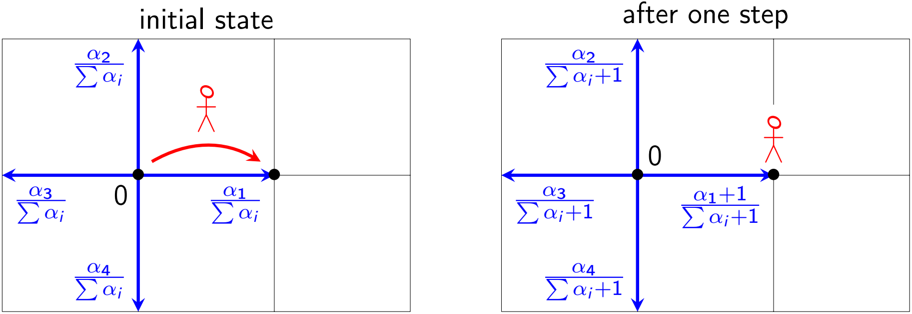
- There annealed law is the law of a Directed reinforced random walk.
- There is a remarkable property of invariance by time-reversal that gives specific properties.

Relation with Reinforced Random Walks

The annealed law of Random Walk in Dirichlet random environment is a **directed edge reinforced random walks** :

$$\mathbb{P}_x^{(\alpha)}(X_{n+1} = X_n + e_i | \sigma(X_k, k \leq n)) = \frac{\alpha_i + N_i(X_n, n)}{\sum_{k=1}^{2d} (\alpha_k + N_k(X_n, n))},$$

where $N_k(x, n)$ is the number of crossings of the directed edge $(x, x + e_k)$ before time n.

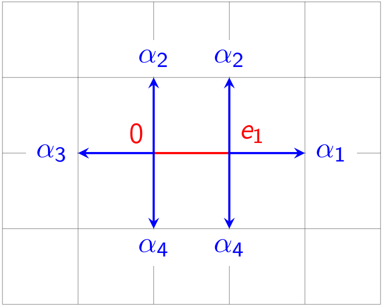


Finite traps in the Dirichlet case

Let

$$\kappa = 2(\sum_{i=1}^{2d} \alpha_i) - \max_{i=1,\dots,d} (\alpha_i + \alpha_{i+d}),$$

κ minimizes the weights of the edges exiting of a subset $\{0, e_i\}$.



In a finite box Λ, $E_0(T^p_{\Lambda^c}) < \infty$ iff $p < \kappa$.

RWDE on directed graphs

Consider a connected directed graph $G = (V, E)$ with finite degree. We denote by $\underline{e}$, $\overline{e}$ the tail and the head of the edge $e = (\underline{e}, \overline{e})$. We consider a set of positive weights $(\alpha_e)_{e \in E}$, $\alpha_e > 0$.

- The environment set is

$$\Omega = \{(\omega_e)_{e \in E} \in (0,1]^E, \text{ s. t. } \forall x \in V, \sum_{\underline{e}=x} \omega_e = 1\}.$$

  This is the set of possible transition probabilities on the graph G.

- The random Dirichlet environment is defined as follows : at each vertex x, pick independently the exit probabilities $(\omega_e)_{\underline{e}=x}$ according to a Dirichlet law $\mathcal{D}ir((\alpha_e)_{\underline{e}=x})$.

It defines the law $\mathbb{P}^{(\alpha)}$ on Ω.

Stability under time reversal

Let $G = (V, E)$ be a finite connected directed graph. Let div : $\mathbb{R}^E \mapsto \mathbb{R}^V$ be the divergence operator

$$\mathrm{div}(\theta)(x) = \sum_{\underline{e}=x} \theta(e) - \sum_{\overline{e}=x} \theta(e), \quad \forall \theta \in \mathbb{R}^E.$$

If $G = (V, E)$ we denote by $\check{G} = (V, \check{E})$ the reversed graph obtained by reversing all the edges. If ω is an environment we denoted by $\check{\omega}$ the time-reversed environment defined as usual

$$\check{\omega}_{\check{e}} = \pi(\underline{e})\omega_e \frac{1}{\pi(\overline{e})}.$$

where π is the invariant probability measure in the environment ω.

Lemma (S., 08)

Let $(\alpha_e)_{e\in E}$ be positive weights with null divergence. If (ω_e) is a Dirichlet random environment with distribution $\mathbb{P}^{(\alpha)}$ then $\check{\omega}$ is a Dirichlet random environment on $\check{G}$ with distribution $\mathbb{P}^{(\check{\alpha})}$, where $\check{\alpha}$ is defined by $\check{\alpha}_{\check{e}} = \alpha_e$.

Let $d_\alpha = \sum_{i=1,\dots,2d} \alpha_i e_i$.

Theorem (S. Tournier 09, Tournier 12)

The Dirichlet RWRE is transient in direction ℓ iff $d_\alpha \cdot \ell > 0$, and the asymptotic direction is proportional to d_α.

Theorem (S.10, Bouchet 12, Tournier 12)

Kalikow 0-1 law is true.

Theorem (Enriquez S. 06, Tournier 09)

The (T) condition is true if $\sum_{i=1}^{d} |\alpha_{e_i} - \alpha_{-e_i}| > 1$.

Theorem (S. 08)

If $d \geq 3$, the random walk in Dirichlet environment is transient for any choice of parameters.

Theorem (S. 10, Bouchet 12)

For $d \geq 3$
i) there exists an invariant measure viewed from the particle absolutely continuous with respect to $\mathbb{P}$ if and only if $\kappa > 1$.
ii) In the case $\kappa \leq 1$, there exists an absolutely continuous invariant measure for an accelerated process.
iii) If $d_\alpha \cdot \ell > 0$

- *The walk is ballistic if and only if $\kappa > 1$.*
- *If $\kappa \leq 1$, X_n is of order n^κ.*

- Let $\beta > 0$. The environment satisfies $(E')_\beta$ if there exists $(\beta_i)_{i=1,\ldots,2d}$, $\beta_i > 0$, such that

$$\kappa\left((\beta_i)_{i=1,\ldots,2d}\right) := 2\sum_{i=1}^{2d} \beta_i - \sup_i \left(\beta_i + \beta_{i+d}\right) > \beta$$

and such that for all $j = 1, \ldots, 2d$,

$$\mathbb{E}\left(e^{\sum_{i=1, i\neq j}^{2d} \beta_i \log \frac{1}{\omega(0,e_i)}}\right) < \infty.$$

Theorem (Bouchet, Ramirez, S., 13)

If (T') is satisfied (or weaker $(P)_M$) and if the environment satisfies $(E')_\beta$ then $\mathbb{E}(\tau_1^\beta) < \infty$.

- The condition is optimal for Dirichlet environments and for environments that have "polynomial tails".
- A strictly better condition has been given by Fribergh and Kious.

Quenched CLT under T_γ

Theorem (Rassoul-Agha, Seppäläinen, 09 (Berger, Zeitouni, 09))

Assume the RWRE is transient in the direction ℓ. If $\mathbb{E}(\tau_1^{p_0}) < \infty$ for some $p_0 > 176d$, then the RWRE satisfies a Quenched CLT.

Quenched CLT under T_γ

Theorem (Bouchet, dos Santos, S., 14)

Assume the RWRE is transient in the direction ℓ. If the condition (T_γ) is satisfied and if

$$\mathbb{E}(\tau_1^2(\log \tau_1)^m) < \infty,$$

for some $m > 1 + 1/\gamma$, then the RWRE satisfies a Quenched CLT.

- Under (T') the condition is $\mathbb{E}(\tau_1^2(\log \tau_1)^{2+\epsilon}) < \infty$.
- The condition (T_γ) seems very difficult to avoid to get an "optimal" integrability exponent.

Ideas of the proof

We use the method of Bolthausen and Sznitman. Let $T > 0$, and $F : C([0, T], \mathbb{R}) \to \mathbb{R}$ a bounded Lipschitz function such that $\forall f, g \in C([0, T], \mathbb{R})$,

$$|F(f) - F(g)| \leq \sup_{x \in [0,T]} \{f(x) - g(x)\}.$$

Let $W_{\cdot}^{(n)}$ be the polygonal interpolation of $\frac{k}{n} \to B^n(\frac{k}{n})$. The quenched CLT is implied by the following estimate

$$\mathrm{Var}_{\mathbb{P}}\left(E_{0,\omega}(F(W_{\cdot}^{(n)}))\right) = \left\| \mathbb{E}_0 \left[F(W^{(n)}) \middle| \omega \right] - \mathbb{E}_0 \left[F(W^{(n)}) \right] \right\|_2^2.$$

Let Q_n be the number of intersections of two independent random walks in the same environment ω before time n.

We assume $(T)_\gamma$. Let $m > 1 + 1/\gamma$ be such that $\mathbb{E}_0\left[\tau_1^2(\log \tau_1)^m\right] < +\infty$, then for all $\delta \in (0,1)$, there exists $C > 0$ such that: $\forall n \in \mathbb{N}$,

$$\left\|\mathbb{E}_0\left[F(W^{(n)})\middle|\omega\right] - \mathbb{E}_0\left[F(W^{(n)})\right]\right\|_2^2 \leq C\left(\log n^{-(m-1/\gamma)} + n^{-(1-\delta)}\mathbb{E}_0\left[Q_n\right]\right).$$

Assume $(T)_\gamma$. For all $0 < \eta < \frac{1}{2}$, there exists $0 < C_\eta < \infty$ depending only on η such that for all $n \geq 1$,

$$\mathbb{E}_0\left[Q_n\right] \leq C_\eta n^{1-\eta}.$$