On the Malliavin differentiability of BSDEs

Thibaut Mastrolia
Joint work with Dylan Possamaï and Anthony Réveillac

CEREMADE
Université Paris Dauphine

Journées MAS, Toulouse
August 27, 2014

Elements of BSDEs: an example using martingale representation Theorem

Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ a probability space. Let ξ a square integrable $\mathcal{F}_T$-r.v. and $Y := (Y_t)_{[0,T]}$ an adapted process such that $Y_T = \xi$.

- $Y_t = \mathbb{E}[\xi|\mathcal{F}_t]$.

Elements of BSDEs: an example using martingale representation Theorem

Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ a probability space. Let ξ a square integrable $\mathcal{F}_T$-r.v. and $Y := (Y_t)_{[0,T]}$ an adapted process such that $Y_T = \xi$.

- $Y_t = \mathbb{E}\left[\xi|\mathcal{F}_t\right]$.
- Then, there exists a square integrable adapted process Z such that

$$Y_t = \mathbb{E}[\xi] + \int_0^t Z_s dW_s.$$

Elements of BSDEs: an example using martingale representation Theorem

Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ a probability space. Let ξ a square integrable $\mathcal{F}_T$-r.v. and $Y := (Y_t)_{[0,T]}$ an adapted process such that $Y_T = \xi$.

- $Y_t = \mathbb{E}\left[\xi|\mathcal{F}_t\right]$.
- Then, there exists a square integrable adapted process Z such that

$$Y_t = \mathbb{E}[\xi] + \int_0^t Z_s dW_s.$$

Hence,

$$Y_t = \xi - \int_t^T Z_s dW_s, \quad t \in [0,T].$$

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \ t \in [0, T]. \tag{1}$$

▷ The data:

Elements of BSDEs

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \; t \in [0, T]. \tag{1}$$

- The data:
  - ξ, the terminal condition, a $\mathcal{F}_T$-measurable r.v. such that $\mathbb{E}[|\xi|^2] < \infty$,
  - $f : [0, T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$, the generator, such that $\mathbb{E}\left[\int_0^T |f(s, 0, 0)|^2 ds\right] < \infty$.

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \; t \in [0, T]. \qquad (1)$$

- The data:
  - ξ, the terminal condition, a $\mathcal{F}_T$-measurable r.v. such that $\mathbb{E}[|\xi|^2] < \infty$,
  - $f : [0, T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$, the generator, such that $\mathbb{E}\left[\int_0^T |f(s, 0, 0)|^2 ds\right] < \infty$.
- A solution is a pair (Y, Z) of **adapted** processes *regular enough*.

Elements of BSDEs

Theorem (Pardoux and Peng, 1990)

If f is Lipschitz in its space variables, then there exists a unique solution (Y, Z) to BSDE (1) such that

$$\underbrace{\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right] < \infty,}_{\mathbb{S}^2} \quad \underbrace{\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty}_{\mathbb{H}^2}.$$

The Markovian case: semi-linear Feynman Kac's Formula

$$\begin{cases} \partial_t v(t,x) + b(t,x)Dv(t,x) + \frac{1}{2}|\sigma(t,x)|^2 D^2 v(t,x) = f(t,x,v(t,x),(\sigma Dv)(t,x)) \\ v(T,\cdot) = g(\cdot). \end{cases}$$

The Markovian case: semi-linear Feynman Kac's Formula

$$\begin{cases} \partial_t v(t,x) + b(t,x)Dv(t,x) + \frac{1}{2}|\sigma(t,x)|^2 D^2 v(t,x) = f(t,x,v(t,x),(\sigma Dv)(t,x)) \\ v(T,\cdot) = g(\cdot). \end{cases}$$

$$"\Leftrightarrow"$$

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dW_s; \quad X_t^{t,x} = x. \\ dY_s^{t,x} = f(t,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds - Z_s^{t,x}dW_s; \quad Y_T^{t,x} = g(X_T^{t,x}). \end{cases}$$

The Markovian case: semi-linear Feynman Kac's Formula

$$\begin{cases} \partial_t v(t,x) + b(t,x)Dv(t,x) + \frac{1}{2}|\sigma(t,x)|^2 D^2 v(t,x) = f(t,x,v(t,x),(\sigma Dv)(t,x)) \\ v(T,\cdot) = g(\cdot). \end{cases}$$

$$"\Leftrightarrow"$$

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dW_s; \quad X_t^{t,x} = x. \\ dY_s^{t,x} = f(t,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds - Z_s^{t,x}dW_s; \quad Y_T^{t,x} = g(X_T^{t,x}). \end{cases}$$

$$v(t,x) = Y_t^{t,x}, \quad Dv(t,x) = Z_t^{t,x} \quad (v \in \mathcal{C}^{1,2})$$

The Markovian case: semi-linear Feynman Kac's Formula

$$\begin{cases} \partial_t v(t,x) + b(t,x) Dv(t,x) + \frac{1}{2}|\sigma(t,x)|^2 D^2 v(t,x) = f(t,x,v(t,x),(\sigma Dv)(t,x)) \\ v(T,\cdot) = g(\cdot). \end{cases}$$

$$" \Leftrightarrow "$$

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dW_s; \quad X_t^{t,x} = x. \\ dY_s^{t,x} = f(t,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds - Z_s^{t,x}dW_s; \quad Y_T^{t,x} = g(X_T^{t,x}). \end{cases}$$

$$v(t,x) = Y_t^{t,x}, \quad Dv(t,x) = Z_t^{t,x} \quad (v \in \mathcal{C}^{1,2})$$

Rk.: $f \equiv 0 \Longrightarrow Y_s^{t,x} = \mathbb{E}[g(X_T^{t,x})|\mathcal{F}_s]$.

Our motivation

Investigate existence of densities for solutions to BSDEs.

$\hookrightarrow$ Antonelli and Kohatsu-Higa (2005), Aboura and Bourguin (2012), M., Possamaï and Réveillac (2014).

Our motivation

Investigate existence of densities for solutions to BSDEs.

↪ Antonelli and Kohatsu-Higa (2005), Aboura and Bourguin (2012), M., Possamaï and Réveillac (2014).

↪ Using Bouleau-Hirsch Criterion
→ Malliavin differentiability of BSDEs.

The problem

We denote by $\mathbb{D}^{1,2}$ the closure of the space of cylindrical functions with respect to the Sobolev norm $\|\cdot\|_{1,2}$:

$$\|F\|_{1,2} := \mathbb{E}[|F|^2] + \mathbb{E}\left[\int_0^T |D_t F|^2 dt\right].$$

The problem

We denote by $\mathbb{D}^{1,2}$ the closure of the space of cylindrical functions with respect to the Sobolev norm $\|\cdot\|_{1,2}$:

$$\|F\|_{1,2} := \mathbb{E}[|F|^2] + \mathbb{E}\left[\int_0^T |D_t F|^2 dt\right].$$

Problem: Conditions on ξ and f which ensure that $Y_t \in \mathbb{D}^{1,2}$ and "$Z_t \in \mathbb{D}^{1,2}$".

The problem

We denote by $\mathbb{D}^{1,2}$ the closure of the space of cylindrical functions with respect to the Sobolev norm $\|\cdot\|_{1,2}$:

$$\|F\|_{1,2} := \mathbb{E}[|F|^2] + \mathbb{E}\left[\int_0^T |D_t F|^2 dt\right].$$

Problem: Conditions on ξ and f which ensure that $Y_t \in \mathbb{D}^{1,2}$ and "$Z_t \in \mathbb{D}^{1,2}$".

Intuition: $\xi \in \mathbb{D}^{1,2}$ and $f : \omega \mapsto f(t, \omega, y, z) \in \mathbb{D}^{1,2}$ are the minimal conditions.

The Markovian case: Pardoux, Peng

We consider the Forward BSDE:

$$\begin{cases} X_t &= X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T] \end{cases}$$

The Markovian case: Pardoux, Peng

We consider the Forward BSDE:

$$\begin{cases} X_t &= X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T] \end{cases}$$

Theorem (Pardoux, Peng 1992)

If g is differentiable ($\xi = g(X_T) \in \mathbb{D}^{1,2}$), f is $\mathcal{C}_b^1$ in its space variables then $Y_t \in \mathbb{D}^{1,2}$ and $Z_t \in \mathbb{D}^{1,2}$.

The Markovian case: Pardoux, Peng

We consider the Forward BSDE:

$$\begin{cases} X_t &= X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T] \end{cases}$$

Theorem (Pardoux, Peng 1992)

If g is differentiable ($\xi = g(X_T) \in \mathbb{D}^{1,2}$), f is $\mathcal{C}_b^1$ in its space variables then $Y_t \in \mathbb{D}^{1,2}$ and $Z_t \in \mathbb{D}^{1,2}$.

- What about the non Markovian case?

Consider BSDE (1)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

Theorem (El Karoui, Peng, Quenez 1997)

Assume that $\xi \in \mathbb{D}^{1,2}$, f is Lipschitz in (y, z), $\omega \mapsto f(t, \omega, y, z)$ is in $\mathbb{D}^{1,2}$ and

then, $Y_t \in \mathbb{D}^{1,2}$ and $Z_t \in \mathbb{D}^{1,2}$.

The general case: El Karoui, Peng, Quenez

Consider BSDE (1)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

Theorem (El Karoui, Peng, Quenez 1997)

Assume that $\xi \in \mathbb{D}^{1,2}$, f is Lipschitz in (y, z), $\omega \mapsto f(t, \omega, y, z)$ is in $\mathbb{D}^{1,2}$ and

- $\mathbb{E}[\xi^4] < \infty$,

then, $Y_t \in \mathbb{D}^{1,2}$ and $Z_t \in \mathbb{D}^{1,2}$.

The general case: El Karoui, Peng, Quenez

Consider BSDE (1)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

Theorem (El Karoui, Peng, Quenez 1997)

Assume that $\xi \in \mathbb{D}^{1,2}$, f is Lipschitz in (y, z), $\omega \mapsto f(t, \omega, y, z)$ is in $\mathbb{D}^{1,2}$ and

- $\mathbb{E}[\xi^4] < \infty$,
- *For all $\theta \in [0, T]$, there exists $(K_t^\theta)_{t\in[0,T]}$ regular enough such that for all $(y_1, y_2, z_1, z_2) \in \mathbb{R}^4$* $|D_\theta f(t, \omega, y_1, z_1) - D_\theta f(t, \omega, y_2, z_2)| \leq K_t^\theta(\omega)(|y_1 - y_2| + |z_1 - z_2|)$,

then, $Y_t \in \mathbb{D}^{1,2}$ and $Z_t \in \mathbb{D}^{1,2}$.

Idea of the proof: Picard iteration. Consider the following approached BSDE

$$Y_t^n = \xi + \int_t^T f(s, Y_s^{n-1}, Z_s^{n-1}) ds - \int_t^T Z_s^n dW_s.$$

We know

$$(Y^n, Z^n) \underset{n \to \infty}{\overset{\mathbb{S}^2 \times \mathbb{H}^2}{\to}} (Y, Z) \text{ the unique solution of BSDE (1).}$$

Besides $Y_t^n \in \mathbb{D}^{1,2}$ then $\int_0^t Z_s^n dW_s \in \mathbb{D}^{1,2} \underset{\text{Pardoux, Peng}}{\Longrightarrow} Z_t^n \in \mathbb{D}^{1,2}$.

Taking the Malliavin derivative we obtain for all $0 \leq r \leq t \leq T$

$$D_r Y_t^n = D_r \xi + \int_t^T D_r f(s, Y_s^{n-1}, Z_s^{n-1}) + f_y(s, Y_s^{n-1}, Z_s^{n-1}) D_r Y_s^{n-1} + f_z(s, Y_s^{n-1}, Z_s^{n-1}) D_r Z_s^{n-1} ds - \int_t^T D_r Z_s^n dW_s.$$

The general case: El Karoui, Peng, Quenez

Taking the Malliavin derivative we obtain for all $0 \leq r \leq t \leq T$

$$D_r Y_t^n = D_r \xi + \int_t^T D_r f(s, Y_s^{n-1}, Z_s^{n-1}) + f_y(s, Y_s^{n-1}, Z_s^{n-1}) D_r Y_s^{n-1} + f_z(s, Y_s^{n-1}, Z_s^{n-1}) D_r Z_s^{n-1} ds - \int_t^T D_r Z_s^n dW_s.$$

$(DY_t^n, DZ_t^n) \underset{n \to \infty}{\to} (\tilde{Y}_t, \tilde{Z}_t)$, where

$$\tilde{Y}_t^r = D_r \xi + \int_t^T D_r f(s, Y_s, Z_s) + f_y(s, Y_s, Z_s) \tilde{Y}_s^r + f_z(s, Y_s, Z_s) \tilde{Z}_s^r ds - \int_t^T \tilde{Z}_s^r dW_s. \quad (2)$$

Then, $Y_t, Z_t \in \mathbb{D}^{1,2}$.

Taking the Malliavin derivative we obtain for all $0 \leq r \leq t \leq T$

$$D_r Y_t^n = D_r \xi + \int_t^T D_r f(s, Y_s^{n-1}, Z_s^{n-1}) + f_y(s, Y_s^{n-1}, Z_s^{n-1}) D_r Y_s^{n-1} + f_z(s, Y_s^{n-1}, Z_s^{n-1}) D_r Z_s^{n-1} ds - \int_t^T D_r Z_s^n dW_s.$$

$(DY_t^n, DZ_t^n) \underset{n\to\infty}{\to} (DY_t, DZ_t)$, where

$$D_r Y_t = D_r \xi + \int_t^T D_r f(s, Y_s, Z_s) + f_y(s, Y_s, Z_s) D_r Y_s + f_z(s, Y_s, Z_s) D_r Z_s ds - \int_t^T D_r Z_s dW_s. \qquad (2)$$

Then, $Y_t, Z_t \in \mathbb{D}^{1,2}$ and its Malliavin derivatives $(D_r Y, D_r Z)$ is the solution of BSDE (2).

Problem of this proof: the choice of (DY^n, DZ^n) to approach (DY, DZ) is somehow arbitrary.

Problem of this proof: the choice of (DY^n, DZ^n) to approach (DY, DZ) is somehow arbitrary.

$\hookrightarrow$ Why (Y_t^n, Z_t^n) should converge to (Y_t, Z_t) in $\mathbb{D}^{1,2}$?

The general case: El Karoui, Peng, Quenez

Problem of this proof: the choice of (DY^n, DZ^n) to approach (DY, DZ) is somehow arbitrary.

$\hookrightarrow$ Why (Y_t^n, Z_t^n) should converge to (Y_t, Z_t) in $\mathbb{D}^{1,2}$?

Idea: Find a canonical sequence which approaches (DY, DZ).

Characterizations of the Malliavin-Sobolev space $\mathbb{D}^{1,2}$

Let $\Omega = \mathcal{C}([0,T])$.

- $H := \left\{ h : [0,T] \to \mathbb{R},\ \exists \dot{h} \in L^2([0,T]), h(t) = \int_0^t \dot{h}_s ds \right\}$.
- H is an Hilbert space $\langle h_1, h_2 \rangle_H = \langle \dot{h}_1, \dot{h}_2 \rangle_{L^2([0,T])}$.

Characterizations of the Malliavin-Sobolev space $\mathbb{D}^{1,2}$

Let $\Omega = \mathcal{C}([0,T])$.

- $H := \left\{ h : [0,T] \to \mathbb{R},\ \exists \dot{h} \in L^2([0,T]), h(t) = \int_0^t \dot{h}_s ds \right\}$.
- H is an Hilbert space $\langle h_1, h_2 \rangle_H = \langle \dot{h}_1, \dot{h}_2 \rangle_{L^2([0,T])}$.

Malliavin, Shigekawa

Kusuoka, Stroock

$F \in \mathbb{D}^{1,2} \quad \Longrightarrow \quad \exists\, \nabla F \in L^2(H),\ \forall h \in H$

$$\underbrace{\frac{F(\omega + \varepsilon h) - F(\omega)}{\varepsilon} \underset{\varepsilon \to 0}{\overset{\text{proba}}{\longrightarrow}} \langle \nabla F, h \rangle_H}_{\text{Stochastically Gâteaux Differentiable}}$$

$$\nabla_t F = \int_0^t D_r F dr$$

Characterizations of the Malliavin-Sobolev space $\mathbb{D}^{1,2}$

Let $\Omega = \mathcal{C}([0,T])$.

- $H := \left\{ h : [0,T] \to \mathbb{R},\ \exists \dot{h} \in L^2([0,T]), h(t) = \int_0^t \dot{h}_s ds \right\}$.
- H is an Hilbert space $\langle h_1, h_2 \rangle_H = \langle \dot{h}_1, \dot{h}_2 \rangle_{L^2([0,T])}$.

Malliavin, Shigekawa

Kusuoka, Stroock

$$F \in \mathbb{D}^{1,2} \quad \Longrightarrow \quad \exists\, \nabla F \in L^2(H),\ \forall h \in H$$

$$\underbrace{\frac{F(\omega + \varepsilon h) - F(\omega)}{\varepsilon} \overset{\text{proba}}{\underset{\varepsilon \to 0}{\longrightarrow}} \langle \nabla F, h \rangle_H}_{\text{Stochastically Gâteaux Differentiable}}$$

$$\nabla_t F = \int_0^t D_r F dr$$

$\Longleftarrow$?

Theorem (Sugita, 1985)

$F \in \mathbb{D}^{1,2} \Longleftrightarrow \exists\, \nabla F \in L^2(H)$ such that F is (SGD) and (RAC).

RAC: Ray Absolutely Continuous: property which holds true **for all** ω.

Problem: It is not convenient for BSDEs.

Theorem (M., Possamaï, Réveillac, 2014)

$F \in \mathbb{D}^{1,2} \Longleftrightarrow \exists \nabla F \in L^2(H)$ *and* $\exists q \in (1,2)$ *such that for all* $h \in H$

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\left|\frac{F(\cdot + \varepsilon h) - F(\cdot)}{\varepsilon} - \langle \nabla F, h\rangle_H\right|^q\right] = 0.$$

$\hookrightarrow$ May fail for q=2 (work in progress).

Application to BSDEs

We apply the previous result to $F = Y_t$

Theorem (M., Possamaï, Réveillac)

Assume that $\xi \in \mathbb{D}^{1,2}, \omega \to f(t,\omega,y,z) \in \mathbb{D}^{1,2}$ *and there exists* $p \in (1,2)$ *such that for all* $h \in H$

- $\mathbb{E}\left[\left(\int_0^T \left| \frac{f(t,\cdot+\varepsilon h, Y_t, Z_t) - f(t,\cdot,Y_t,Z_t)}{\varepsilon} - \langle Df(s,\cdot,Y_s,Z_s,\dot{h}\rangle_{L^2} \right| ds\right)^p\right] \to 0$
- $f_y(t,\omega+\varepsilon^n h, \alpha_t^n, \beta_t^n) - f_y(t,\omega,\alpha_t,\beta_t) \underset{n\to\infty}{\overset{proba}{\longrightarrow}} 0.$
  For every $(\alpha^n, \beta^n) \to (\alpha, \beta)$.

Then, $Y_t, Z_t \in \mathbb{D}^{1,2}$.

Examples

- In the Markovian case, our assumptions are automatically satisfied when f is differentiable in its space variables (and are more general than the existing results).

Examples

- In the Markovian case, our assumptions are automatically satisfied when f is differentiable in its space variables (and are more general than the existing results).
- In the quadratic case, we have the same kind of result using our characterization of the Malliavin-Sobolev space (extend the results obtained by Imkeller and dos Reis who deal with Markovian quadratic BSDEs).