On the Malliavin differentiability of BSDEs

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Joint work with Dylan Possamaï and Anthony Réveillac

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Let $(\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ a probability space. Let $\xi$ a square integrable $\mathcal{F}_T$-r.v. and $Y := (Y_t)_{[0,T]}$ an adapted process such that $Y_T = \xi$.

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- Then, there exists a square integrable adapted process $Z$ such that

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- Then, there exists a square integrable adapted process \(Z\) such that

\[
Y_t = \mathbb{E}[\xi] + \int_0^t Z_s dW_s.
\]

Hence,

\[
Y_t = \xi - \int_t^T Z_s dW_s, \quad t \in [0, T].
\]
Elements of BSDEs

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \] (1)

▷ The data:

- \( \xi \): the terminal condition, an \( F_T \)-measurable r.v. such that \( E[|\xi|^2] < \infty \).
- \( f: [0, T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R} \): the generator, such that \( E[\int_0^T |f(s, 0, 0)|^2 ds] < \infty \).

A solution is a pair \((Y, Z)\) of adapted processes regular enough.
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\[ \triangleright \text{A solution is a pair } (Y, Z) \text{ of } \textbf{adapted} \text{ processes } \textit{regular enough}. \]
Theorem (Pardoux and Peng, 1990)

If $f$ is Lipschitz in its space variables, then there exists a unique solution $(Y, Z)$ to BSDE (1) such that

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^2 \right] < \infty, \quad \mathbb{E} \left[ \int_0^T |Z_s|^2 \, ds \right] < \infty.
$$
\[ \begin{align*}
\partial_t v(t,x) + b(t,x) Dv(t,x) + \frac{1}{2} |\sigma(t,x)|^2 D^2 v(t,x) &= f(t,x,v(t,x), (\sigma Dv)(t,x)) \\
\quad v(T, \cdot) &= g(\cdot).
\end{align*} \]
The Markovian case: semi-linear Feynman Kac’s Formula

\[ \begin{aligned}
&\frac{\partial_t}{\partial t} v(t, x) + b(t, x) Dv(t, x) + \frac{1}{2} |\sigma(t, x)|^2 D^2 v(t, x) = f(t, x, v(t, x), (\sigma Dv)(t, x)) \\
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⇔

\[ \begin{aligned}
&dX_{s}^{t, x} = b(s, X_{s}^{t, x}) ds + \sigma(s, X_{s}^{t, x}) dW_s; \quad X_{t}^{t, x} = x. \\
&dY_{s}^{t, x} = f(t, X_{s}^{t, x}, Y_{s}^{t, x}, Z_{s}^{t, x}) ds - Z_{s}^{t, x} dW_s; \quad Y_{T}^{t, x} = g(X_{T}^{t, x}).
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The Markovian case: semi-linear Feynman Kac’s Formula

\[
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\partial_t v(t, x) + b(t, x)Dv(t, x) + \frac{1}{2}|\sigma(t, x)|^2 D^2 v(t, x) = f(t, x, v(t, x), (\sigma Dv)(t, x)) \\
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v(t, x) = Y_{t}^{t,x}, \quad Dv(t, x) = Z_{t}^{t,x} \quad (v \in C^{1,2})
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\end{cases}
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\[v(t, x) = Y_{t}^{t, x}, \quad Dv(t, x) = Z_{t}^{t, x} \quad (v \in C^{1,2})\]

Rk.: \( f \equiv 0 \implies Y_{s}^{t, x} = \mathbb{E}[g(X_{T}^{t, x})|\mathcal{F}_s]. \)
Our motivation

Investigate existence of densities for solutions to BSDEs.

\[ \rightarrow \text{Antonelli and Kohatsu-Higa (2005), Aboura and Bourguin (2012), M., Possamaï and Réveillac (2014).} \]
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Using Bouleau-Hirsch Criterion

Malliavin differentiability of BSDEs.
We denote by $D^{1,2}$ the closure of the space of cylindrical functions with respect to the Sobolev norm $\| \cdot \|_{1,2}$:

$$\|F\|_{1,2} := \mathbb{E}[|F|^2] + \mathbb{E}\left[ \int_0^T |D_t F|^2 dt \right].$$
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Problem: Conditions on $\xi$ and $f$ which ensure that $Y_t \in \mathbb{D}^{1,2}$ and $Z_t \in \mathbb{D}^{1,2}$.
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**Intuition:** $\xi \in \mathbb{D}^{1,2}$ and $f : \omega \mapsto f(t, \omega, y, z) \in \mathbb{D}^{1,2}$ are the minimal conditions.
We consider the Forward BSDE:

\[ \begin{aligned}
X_t &= X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s \\
Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T]
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The Markovian case: Pardoux, Peng

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Theorem (Pardoux, Peng 1992)

If \( g \) is differentiable (\( \xi = g(X_T) \in D^{1,2} \)), \( f \) is \( C^1_b \) in its space variables then \( Y_t \in D^{1,2} \) and \( Z_t \in D^{1,2} \).
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What about the non Markovian case?
The general case: El Karoui, Peng, Quenez

Consider BSDE (1)

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s. \]

Theorem (El Karoui, Peng, Quenez 1997)

Assume that \( \xi \in \mathbb{D}^{1,2} \), \( f \) is Lipschitz in \((y, z)\), \( \omega \mapsto f(t, \omega, y, z) \) is in \( \mathbb{D}^{1,2} \) and

\[ \text{then}, \ Y_t \in \mathbb{D}^{1,2} \text{ and } Z_t \in \mathbb{D}^{1,2}. \]
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\[ \mathbb{E}[\xi^4] < \infty, \]

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\[ \mathbb{E}[\xi^4] < \infty, \]

\[ \text{For all } \theta \in [0, T], \text{ there exists } (K^\theta_t)_{t \in [0, T]} \text{ regular enough such that} \]

\[ \left| D_\theta f(t, \omega, y_1, z_1) - D_\theta f(t, \omega, y_2, z_2) \right| \leq K^\theta_t(\omega)(|y_1 - y_2| + |z_1 - z_2|), \]

then, \( Y_t \in \mathbb{D}^{1,2} \) and \( Z_t \in \mathbb{D}^{1,2} \).
Idea of the proof: Picard iteration. Consider the following approached BSDE

\[ Y^n_t = \xi + \int_t^T f(s, Y^{n-1}_s, Z^{n-1}_s)ds - \int_t^T Z^n_s dW_s. \]

We know 

\[(Y^n, Z^n) \xrightarrow{\mathbb{S}^2 \times \mathcal{H}^2} (Y, Z) \text{ the unique solution of BSDE (1)}. \]

Besides \(Y^n_t \in \mathbb{D}^{1,2}\) then \(\int_0^t Z^n_s dW_s \in \mathbb{D}^{1,2} \xrightarrow{\text{Pardoux, Peng}} Z^n_t \in \mathbb{D}^{1,2}.\)
The general case: El Karoui, Peng, Quenez

Taking the Malliavin derivative we obtain for all $0 \leq r \leq t \leq T$

$$D_r Y^n_t = D_r \xi + \int_t^T D_r f(s, Y^{n-1}_s, Z^{n-1}_s) + f_y(s, Y^{n-1}_s, Z^{n-1}_s) D_r Y^{n-1}_s$$

$$+ f_z(s, Y^{n-1}_s, Z^{n-1}_s) D_r Z^{n-1}_s ds - \int_t^T D_r Z^n_s dW_s.$$
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$$+ f_z(s, Y^{n-1}_s, Z^{n-1}_s) D_r Z^{n-1}_s \, ds - \int_t^T D_r Z^n_s \, dW_s.$$  

$$(DY^n_t, DZ^n_t) \xrightarrow{n \to \infty} (\tilde{Y}_t, \tilde{Z}_t), \text{ where}$$

$$\tilde{Y}^r_t = D_r \xi + \int_t^T D_r f(s, Y_s, Z_s) + f_y(s, Y_s, Z_s) \tilde{Y}^r_s$$

$$+ f_z(s, Y_s, Z_s) \tilde{Z}^r_s \, ds - \int_t^T \tilde{Z}^r_s \, dW_s.$$  \hfill (2)

Then, $Y_t, Z_t \in \mathbb{D}^{1,2}$.
Taking the Malliavin derivative we obtain for all $0 \leq r \leq t \leq T$

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D_r Y^n_t = D_r \xi + \int_t^T D_r f(s, Y_{s-1}^n, Z_{s-1}^n) + f_y(s, Y_{s-1}^n, Z_{s-1}^n) D_r Y_{s-1}^n \]  
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\]

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(DY^n_t, DZ^n_t) \xrightarrow{n \to \infty} (DY_t, DZ_t),
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+ f_z(s, Y_s, Z_s) D_r Z_s ds - \int_t^T D_r Z_s dW_s. \tag{2}
\]

Then, $Y_t, Z_t \in \mathbb{D}^{1,2}$ and its Malliavin derivatives $(D_r Y, D_r Z)$ is the solution of BSDE (2).
Problem of this proof: the choice of \((DY^n, DZ^n)\) to approach \((DY, DZ)\) is somehow arbitrary.
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Why \((Y^n_t, Z^n_t)\) should converge to \((Y_t, Z_t)\) in \(D^{1,2}\)?
Problem of this proof: the choice of \((DY^n, DZ^n)\) to approach \((DY, DZ)\) is somehow arbitrary.

Why \((Y^n_t, Z^n_t)\) should converge to \((Y_t, Z_t)\) in \(D^{1,2}\)?

Idea: Find a canonical sequence which approaches \((DY, DZ)\).
Let $\Omega = C([0, T])$.

- $H := \left\{ h : [0, T] \to \mathbb{R}, \exists \dot{h} \in L^2([0, T]), h(t) = \int_0^t \dot{h}_s ds \right\}$.
- $H$ is an Hilbert space $\langle h_1, h_2 \rangle_H = \langle \dot{h}_1, \dot{h}_2 \rangle_{L^2([0, T])}$.

Malliavin, Shigekawa Kusuoka, Stroock

If $F \in D^{1,2}$, then $\exists \nabla F \in L^2(H)$, $\forall h \in H$ $F(\omega + \varepsilon h) - F(\omega) \varepsilon \xrightarrow{\text{proba}} \varepsilon \rightarrow 0 \langle \nabla F, h \rangle_H$. Stochastically Gâteaux Differentiable

$\nabla_t F = \int_0^t D_r F dr$
Let $\Omega = C([0, T])$.

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### Malliavin, Shigekawa

- $F \in \mathbb{D}^{1,2}$
- $\nabla_t F = \int_0^t D_r F dr$

### Kusuoka, Stroock

- $\exists \nabla F \in L^2(H), \forall h \in H$
- $\frac{F(\omega + \varepsilon h) - F(\omega)}{\varepsilon} \xrightarrow{\text{proba}} \varepsilon \to 0 \langle \nabla F, h \rangle_H$

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**Malliavin, Shigekawa**

$F \in \mathbb{D}^{1,2} \implies \exists \nabla F \in L^2(H), \forall h \in H$

$\nabla_t F = \int_0^t D_r F dr$

**Kusuoka, Stroock**

$\begin{align*}
F(\omega + \varepsilon h) - F(\omega) & \xrightarrow{\text{proba}} \varepsilon \langle \nabla F, h \rangle_H \\
\varepsilon & \xrightarrow{\text{\varepsilon \to 0}} 0
\end{align*}$

\[\text{Stochastically Gâteaux Differentiable}\]
Theorem (Sugita, 1985)

\[ F \in D^{1,2} \iff \exists \nabla F \in L^2(H) \text{ such that } F \text{ is (SGD) and (RAC).} \]

RAC: Ray Absolutely Continuous: property which holds true for all \( \omega \).

Problem: It is not convenient for BSDEs.
Theorem (M., Possamaï, Réveillac, 2014)

\[ F \in \mathbb{D}^{1,2} \iff \exists \nabla F \in L^2(H) \text{ and } \exists q \in (1,2) \text{ such that for all } h \in H \]

\[ \lim_{\varepsilon \to 0} \mathbb{E} \left[ \left\| \frac{F(\cdot + \varepsilon h) - F(\cdot)}{\varepsilon} - \langle \nabla F, h \rangle_H \right\|^q \right] = 0. \]

\[ \hookrightarrow \text{ May fail for } q=2 \text{ (work in progress).} \]
We apply the previous result to $F = Y_t$

**Theorem (M., Possamaï, Réveillac)**

Assume that $\xi \in \mathbb{D}^{1,2}$, $\omega \rightarrow f(t, \omega, y, z) \in \mathbb{D}^{1,2}$ and there exists $p \in (1, 2)$ such that for all $h \in H$

\[
\mathbb{E} \left[ \left( \int_0^T \left| \frac{f(t, \cdot + \varepsilon h, Y_t, Z_t) - f(t, \cdot, Y_t, Z_t)}{\varepsilon} - \langle Df(s, \cdot, Y_s, Z_s, \dot{h}) \rangle_{L^2} \right| \, ds \right)^p \right] \rightarrow 0
\]

\[
f_y(t, \omega + \varepsilon^n h, \alpha^n_t, \beta^n_t) - f_y(t, \omega, \alpha_t, \beta_t) \xrightarrow{\text{proba}} 0, \text{ for every } (\alpha^n, \beta^n) \rightarrow (\alpha, \beta).
\]

Then, $Y_t, Z_t \in \mathbb{D}^{1,2}$. 
In the Markovian case, our assumptions are automatically satisfied when \( f \) is differentiable in its space variables (and are more general than the existing results).
Examples

- In the Markovian case, our assumptions are automatically satisfied when $f$ is differentiable in its space variables (and are more general than the existing results).

- In the quadratic case, we have the same kind of result using our characterization of the Malliavin-Sobolev space (extend the results obtained by Imkeller and dos Reis who deal with Markovian quadratic BSDEs).