Stochastic Calculus w.r.t. Gaussian Processes

Journées MAS 2014

Toulouse, 27-29 août 2014.

Outline of the presentation

1. Two particular Gaussian processes, Fractional and multifractional Brownian motion
   - Fractional and multifractional Brownian motions
   - Non semimartingales versus integration
2. Stochastic integral w.r.t. G in the White Noise Theory sense
   - Background on White Noise Theory
   - Stochastic Integral with respect to G
   - Comparison with Malliavin calculus or divergence integral
3. Miscellaneous formulas & some open problems
   - Miscellaneous formulas
     - Itô forumulas
     - Tanaka formula
   - Weighted and non weighted local times of G
   - Some open problems

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Multifractional Brownian Motion

Fractional Brownian motion (fBm)

A gaussian process more flexible than standard Brownian motion (A. Kolmogorov, 1949)

Definition

Let $H \in (0,1)$ be a real constant. A process $B^H := (B^H_t ; t \in \mathbb{R}_+)$ is an fBm if it is centred, Gaussian, with covariance function given by:

$$\mathbb{E}[B^H_t B^H_s] = 1/2(t^{2H} + s^{2H} - |t-s|^{2H}). \ .$$

Properties

The process B^H verifies

- $B^H_0 = 0$, *a.s.*
- *For all $t \geqslant s \geqslant 0$, $B^H_t - B^H_s$ follows the law $\mathcal{N}(0, (t-s)^{2H})$.*
- *The trajectoiries de B^H are contiuous.*

Properties of fractional Brownian Motion

Properties

If B^H is a fBm, it verifies the following assertions:

- $X_t = \frac{1}{a^H} B^H_{at}$ *with* $a > 0$ *is un fBm (self-similarity);* $B^{1/2}$ *is a sBm.*
- *For* $H > 1/2$, $B^H_{t+h} - B^H_t$ *et* $B^H_{t+2h} - B^H_{t+h}$ *are positively corelated and* B^H *has a long terme dependance.*
- *For* $H < 1/2$, $B^H_{t+h} - B^H_t$ *et* $B^H_{t+2h} - B^H_{t+h}$ *are negatively corelated..*
- *A.s, in all point* t_0 *of* $\mathbb{R}_+$, *the regularity of* B^H *is constant and equal to* H.

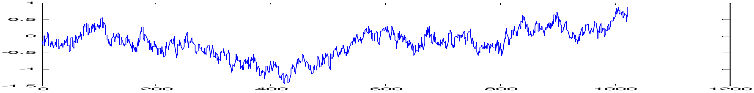
Because sBm and fBm are not differentiable, a good measure of their regularity, as a process, is the local Hölder exponent which is defined at every point t_0, by:

Definition

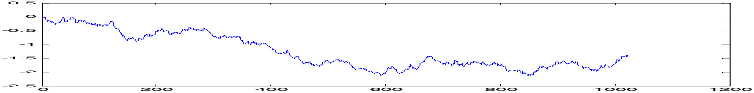
$$\alpha_{B^H}(t_0) := sup\{\alpha : \limsup_{\rho \to 0} \sup_{(s,t) \in B(t_0,\rho)^2} \frac{|B^H_t - B^H_s|}{|t-s|^\alpha} < +\infty\} = H.$$

Trajectories with different regularity (Fraclab)

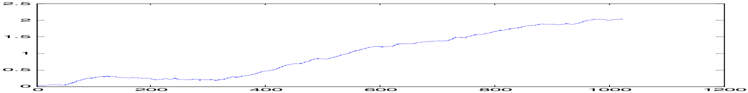
fBm with $H = 0.3$



fBm with $H = 0.5$



fBm with $H = 0.8$



Drawbacks of fBm

• long range dependence versus regularity of trajectories: model the increments present long range dependence only if $H > 1/2$.

• the regularity of trajectories remains the same along the time (equal to H)....

Multifractional Brownian Motion

What is Multifractional Brownian motion (mBm)?

A Gaussian process more flexible than the fBm: Lévy Véhel, Peltier (1995); Benassi, Jaffard and Roux (1997)

We here give the more recent definition of mBm given in[A].

Definition (Fractional Gaussian field)

A two parameters Gaussian process $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ *is said to be a fractional Gaussian field if, for every* $H\in(0,1)$*, the process* $(\mathbf{B}(t,H))_{t\in\mathbb{R}}$ *is a fractional Brownian motion.*

A multifractional Brownian motion is simply a "path" traced on a fractional Gaussian field.

Definition (Multifractional Brownian motion)

Let $h:\mathbb{R}\to(0,1)$ *be a deterministic measurable function and* $\mathbf{B}:=(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ *be a fractional Gaussian field. The Gaussian process* $B^h:=(\mathbf{B}(t,h(t)))_{t\in\mathbb{R}}$ *is called a mBm with functional parameter h.*

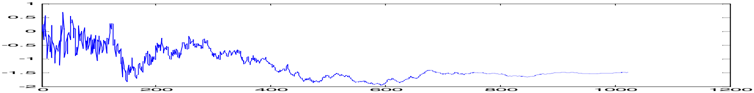
Example

$$\mathbf{B}(t,H):=\frac{1}{c_{h(t)}}\int_{\mathbb{R}}\frac{e^{itu}-1}{|u|^{h(t)+1/2}}\widetilde{\mathbb{W}}(du).$$

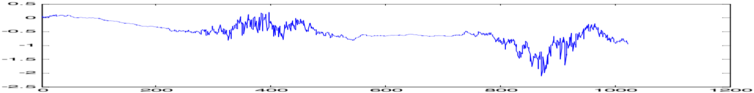
[A] Lebovits, J. and Lévy Véhel, J. and Herbin, E. 2014.

Graphic Representations of mBm $\mathbf{B}$ for several functions h obtained thanks to the software Fraclab

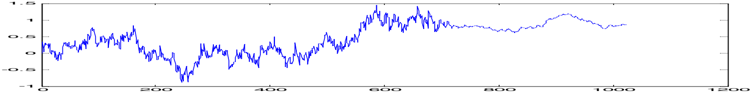
mBm with $h(t) := 0.1 + 0.8t$



mBm with $h(t) := 0.5 + 0.3\sin(4\pi t)$



mBm with $h(t) := 0.3 + 0.3(1 + \exp^{(-100(t-0.7))})^{-1}$



Gaussian process for which one wants to define a stochastic calculus

More generally, for every Gaussian process $G = (G_t)_{t\in\mathbb{R}_+}$ which can be written under the form:

$$G_t := \int_{\mathbb{R}} g_t(u)\ dB_u =< ., g_t >,$$

where $g_t \in L^2(\mathbb{R}_+)$.

How can one define a stochastic integral, and a stochastic calculus with respect to G?

Outline of the presentation

The stochastic calculus developed for continuous semi-martingales can not be applied for mBm

- Since fBm, mBm and more generally G, are not semimartingales, we can not use standard stochastic calculus for them.

$\Longrightarrow$ We then have to develop a different (new) stochastic calculus and hence new methods....

- How can one construct an integral with respect to fBm, mBm and G (especially when G is not a semimartingale)?

What do we want to do with an integral *w.r.t.* G?

To be able to solve S.D.Es that come from, e.g.

- finance
- physics
- Medicine
- Geology

Several approaches in order to obtain a stochastic calculus with respect to fractional Brownian motion (fBm)

Probabilistic approaches	**Deterministic approaches**
Malliavin Calculus Decreusefond, Üstunel, Alos, Mazet, Nualart...	Fractional Integration Zähle, Feyel & de la Pradelle
White Noise Theory Hida,Kuo,Elliott,Bender,Sulem...	Rough Path theory Coutin, Nourdin, Gubinelli
Enlargment of filtrations Jeulin, Yor	Extended integral via regularization Russo & Vallois

Approachs in order to obtain a stochastic calculus with respect to multifractional Brownian motion (mBm)

In the probabilistic approaches:

- The one provided by[B] using the divergence type integral (Malliavin Calculus), which is valid for Voltera processes.
- Only for mBm, (see[C]).

[B] Alòs, E. and Mazet, O. and Nualart, D. 2001.
[C] J. Lebovits and Lévy Véhel, J. 2014; Lebovits, J. and Lévy Véhel, J. and Herbin, E. 2014.

Outline of the presentation

Background on White Noise Theory

$(\Omega, \mathcal{F}, \mu) = (\mathscr{S}'(\mathbb{R}), \mathcal{B}(\mathscr{S}'(\mathbb{R})), \mu)$, where μ is the unique probability measure such that for all $f \in L^2_{\mathbb{R}}(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, the map

$$\begin{array}{rccc} <.,f>_{\mathscr{S}'(\mathbb{R}),L^2_{\mathbb{R}}(\mathbb{R})}: & (\Omega,\mathscr{F}) & \to & (\mathbb{R},\mathscr{B}(\mathbb{R})) \\ & \omega & \mapsto & <\omega,f>_{\mathscr{S}'(\mathbb{R}),L^2_{\mathbb{R}}(\mathbb{R})} \end{array}$$

is a real r.v following the law $\mathcal{N}(0, \|f\|^2_{L^2_{\mathbb{R}}(\mathbb{R})})$. For every n in $\mathbb{N}$, define the n^{th} Hermite function by:

$$e_n(x) := (-1)^n \pi^{-1/4} (2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2});$$

$(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$

Thorem-defintion (Test functions and Hida distributions spaces)

There exist two topological spaces noted $(\mathcal{S})$ and $(\mathcal{S})^*$ such that we have

- $$\boxed{(\mathcal{S}) \subset (L^2) := L^2(\Omega, \mathcal{F}, \mu) \subset (\mathcal{S})^*}$$
- *$(\mathcal{S})^*$ is the dual space of $(\mathcal{S})$ and we will note $<\, , \,>$ the duality bracket between $(\mathcal{S})^*$ and $(\mathcal{S})$.*
- *If Φ belongs to (L^2) then we have the equality* $\ll \Phi, \varphi \gg \; = < \Phi, \varphi >_{(L^2)} = \mathbb{E}[\Phi \; \varphi]$.

Remark

We call $(\mathcal{S})$ the test function space and $(\mathcal{S})^*$ the Hida distributions space.

Definition (Convergence in $(\mathcal{S})^*$)

For every $\Phi_n := \sum_{k=0}^{+\infty} \; a_k^{(n)} < ., e_k >$, one says that $(\Phi_n)_{n \in \mathbb{N}}$ converge to $\Phi := \sum_{k=0}^{+\infty} \; a_k < ., e_k >$,

- in $(\mathcal{S})^*$ if $\exists p_0$ s.t. $\lim_{n \to +\infty} \sum_{k=0}^{+\infty} \frac{(a_k - a_k^{(n)})^2}{(2k+2)^{2p_0}} = 0$.
- in $(\mathcal{S})$ if $\forall p_0$, $\lim_{n \to +\infty} \sum_{k=0}^{+\infty} \; (a_k - a_k^{(n)})^2 (2k+2)^{2p_0} = 0$.

Definition (**stochastic distribution process**)

A measurable function $\Phi : I \to (\mathcal{S})^*$ is called a stochastic distribution process, or an $(\mathcal{S})^*$–process, or a Hida process.

Definition (**derivative in $(\mathcal{S})^*$**)

Let $t_0 \in I$. A stochastic distribution process $\Phi : I \to (\mathcal{S})^*$ is said to be differentiable at t_0 if the quantity $\lim_{r \to 0} \; r^{-1} \left(\Phi(t_0 + r) \; - \; \Phi(t_0)\right)$ exists in $(\mathcal{S})^*$. We note $\frac{d\Phi}{dt}(t_0)$ the $(\mathcal{S})^*$-derivative at t_0 of the stochastic distribution process Φ. Φ is said to be differentiable over I if it is differentiable at t_0 for every t_0 in I.

Definition (**integral in $(\mathcal{S})^*$**)

Assume that $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is weakly in $L^1(\mathbb{R}, dt)$, i.e assume that for all φ in $(\mathcal{S})$, the mapping $u \mapsto < \Phi(u), \; \varphi >$ from $\mathbb{R}$ to $\mathbb{R}$ belongs to $L^1(\mathbb{R}, dt)$. Then there exists an unique element in $(\mathcal{S})^*$, noted $\int_{\mathbb{R}} \Phi(u) du$ such that

$$\ll \int_{\mathbb{R}} \Phi(u) du, \varphi \gg = \int_{\mathbb{R}} \ll \Phi(u), \varphi \gg \; du \qquad \text{for all } \varphi \text{ in } \; (\mathcal{S}).$$

Assumptions $(\mathcal{A}_1)$ & $(\mathcal{A}_2)$

The Gaussian process $G := (< ., g_t >)_{t\in\mathbb{R}}$ being fixed, define

$$\begin{array}{cccc} g : & \mathbb{R} & \to & \mathscr{S}'(\mathbb{R}) \\ & t & \mapsto & g(t) := g_t \end{array}$$

In the sequel, we make the following assumption:

$(\mathcal{A}_1)$ The map g is differentiable on $\mathbb{R}$ (one notes $g'_t := g'(t)$).

$(\mathcal{A}_2)$ $\exists\ q \in \mathbb{N}^*$ *s.t.* the map $t \mapsto |g'_t|_{-q} \in L^1_{\text{loc}}(\mathbb{R})$,

where $|f|^2_{-q} := \sum_{k=0}^{+\infty} < f, e_k >^2 (2k+2)^{-2q}, \quad \forall (f,q) \in L^2(\mathbb{R}) \times \mathbb{N}$.

Remark

We will note and call Gaussian White Noise the process $(W_t^{(G)})_{t\in\mathbb{R}}$ defined by $W_t^{(G)} := < ., g'_t >$, where the equality holds in $(\mathcal{S})^*$. We will sometimes note $\frac{dG_t}{dt}$ instead of $W_t^{(G)}$.

Assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ are fulfilled for standard, fractional and multifractional Brownian motions. Indeed:

Example: Derivative of Brownian motion

Example (white noise)

The process defined by $B_t := < ., \mathbb{1}_{[0;t]} >_{\mathscr{S}'(\mathbb{R}), L^2_{\mathbb{R}}(\mathbb{R})}$ *is a Brownian motion and has the following expansion, in* $(\mathcal{S})^*$*:*

$$B_t = \sum_{k=0}^{+\infty} \Big(\int_0^t e_k(s)ds \Big) < ., e_k > .$$

It is natural to think to define the derivative of B *with respect to time, denoted* $(W_t)_{t\in[0,1]}$*, by setting:*

$$W_t := \sum_{k=0}^{+\infty} e_k(t) < ., e_k > .$$

The map $t \mapsto W_t$ *is the derivative, in sense of* $(\mathcal{S})^*$*, of the Bronwian motion. It is called white noise process and is sometimes denoted by* $\frac{dB}{dt}(t)$ *or* $\dot{B}_t$*.*

One can do the same for fBm & mBm

The operator M_H crucial to define both fBm and mBm

For all H in $(0,1)$, we define on the space $\mathcal{E}(\mathbb{R})$ of step functions the operator M_H by

$$\widehat{M_H(f)}(x) = x^{1/2-H}\,\widehat{f}\,(x), \text{ for } a.e.\ x \in \mathbb{R}.$$

The map $M_H : (\mathcal{E}(\mathbb{R}), <,>_H) \to (L^2_{\mathbb{R}}(\mathbb{R}), <,>_{L^2_{\mathbb{R}}(\mathbb{R})})$ is an isometry and can then be extended from $\mathcal{E}(\mathbb{R})$ to

$$L^2_H(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) \mid \widehat{u} \in L^1_{loc}(\mathbb{R}) \text{ and s.t } ||u||^2{}_{L^2_H(\mathbb{R})} < +\infty\},$$

where $\|u\|^2{}_H := \|u\|^2{}_{L^2_H(\mathbb{R})} := \beta_H^2 \int_{\mathbb{R}} |u|^{1-2H} |\widehat{f}\,(u)|^2 du.$

Example of fractional white noise

Example (fractional white noise)

Let $H \in (0,1)$, the process defined by $B^H_t := < ., M_H(\mathbb{1}_{[0;t]}) >_{\mathscr{S}'(\mathbb{R}), L^2_{\mathbb{R}}(\mathbb{R})}$ is a fBm and has the following expansion in $(\mathcal{S})^*$:

$$B^H_t = \sum_{k=0}^{+\infty} \left(\int_0^t M_H(e_k)(s)ds \right) < ., e_k > .$$

We define the fractional white noise W^H by

$$W^H_t := \sum_{k=0}^{+\infty} M_H(e_k)(t) < ., e_k > .$$

The map $t \mapsto W^H_t$ is the derivative, in sense of $(\mathcal{S})^*$, of the process B^H. We call it multifractional White Noise process and denote it sometimes by $\frac{dB^H}{dt}(t)$.

More generally, Gaussian white noise

Gaussian white Noise

Let $G := (< ., g_t >)_{t\in\mathbb{R}}$ be a Gaussian process that fulfills assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, G has the following expansion in $(\mathcal{S})^*$:

$$G_t = \sum_{k=0}^{+\infty} \left(< g_t, e_k >_{L^2(\mathbb{R})} \right) < ., e_k > .$$

We define the Gaussian white noise W^G by

$$W_t^{(G)} := \sum_{k=0}^{+\infty} \left(< g'_t, e_k >_{L^2(\mathbb{R})} \right) < ., e_k > .$$

The map $t \mapsto W_t^{(G)}$ is the Hida derivative, of the process G. We call it Gaussian White Noise process and denote it sometimes by $\frac{dG}{dt}(t)$.

Outline of the presentation

Recall

Definition (integral in $(\mathcal{S})^*$)

Assume that $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is weakly in $L^1(\mathbb{R}, dt)$, i.e assume that for all φ in $(\mathcal{S})$, the mapping $u \mapsto \ll \Phi(u), \varphi \gg$ from $\mathbb{R}$ to $\mathbb{R}$ belongs to $L^1(\mathbb{R}, dt)$. Then there exists an unique element in $(\mathcal{S})^*$, noted $\int_{\mathbb{R}} \Phi(u)du$ such that

$$\ll \int_{\mathbb{R}} \Phi(u)du, \varphi \gg = \int_{\mathbb{R}} \ll \Phi(u), \varphi \gg \; du \quad \text{for all } \varphi \text{ in } (\mathcal{S}).$$

Stochastic integral with respect to G

Definition (Wick-Itô integral w.r.t. Gaussian process)

Let $X : \mathbb{R} \to (\mathcal{S})^*$ be a process s.t. the process $t \mapsto X_t \diamond W_t^{(G)}$ is $(\mathcal{S})^*$-integrable on $\mathbb{R}$. The process X is then said to be dG-integrable on $\mathbb{R}$ or integrable on $\mathbb{R}$, with respect to the Gaussian process G. The integral on $\mathbb{R}$ of X with respect to G is defined by:

$$\int_{\mathbb{R}} X_s \, d^{\diamond} G_s := \int_{\mathbb{R}} X_s \diamond W_s^{(G)} \, ds.$$

For any Borel set I of $\mathbb{R}$, define $\int_I X_s \, d^{\diamond} G_s := \int_{\mathbb{R}} \mathbb{1}_I(s) \, X_s \, dG_s$.

Properties

- *The Wick-Itô integral of an $(\mathcal{S}^*)$-valued process, with respect to G is then an element of $(\mathcal{S})^*$.*
- *Let (a, b) in $\mathbb{R}^2$, $a < b$. Then $\int_a^b \, d^{\diamond} G_u = G_b - G_a$ almost surely.*
- *The Wick-Itô integration with respect to G is linear.*
- *Let $X : I \to (\mathcal{S}^*)$ be a dG-integrable process over I & assume $\int_I X(s) \, d^{\diamond} G_s$ belongs to (L^2). Then* $$\mathbb{E}\left[\int_I X(s) \, d^{\diamond} W_s^G\right] = 0.$$

Example I: A simple comptutation

Computation of $\int_0^T G_t \; dG_t$

Let $T > 0$and assume that $t \mapsto R_{t,t} := \mathrm{E}[G_t^2]$ is upper-bounded on $[0, T]$, then the following equality holds almost surely and in (L^2).

$$\int_0^T G_t \; d^{\diamond} G_t = \tfrac{1}{2} \left(G_T^2 - R_{T,T} \right)$$

Proof:

Existence of both sides, using S-transfrom.

$$S(\int_0^T G_t \; d^{\diamond} G_t)(\eta) = \int_0^T S(G_t)(\eta) \; S(W_t^{(G)})(\eta) \; dt = \int_0^T < g_t, \eta >< g_t', \eta > dt$$
$$= \tfrac{1}{2} (S(G_T)(\eta))^2 = S(\tfrac{1}{2} G_T^{\diamond 2})(\eta),$$

Wick product has replaced ordinary product.

Example II: A simple SDE, The Gaussian Wick exponential

The Gaussian Wick exponential

Let us consider the following Gaussian stochastic differential equation

$$(\mathcal{E}) \begin{cases} dX_t = \alpha(t)X_t \; dt + \beta(t)X_t \; dG_t \\ X_0 \in (\mathcal{S})^*, \end{cases}$$

where t belongs to $\mathbb{R}_+$ and where $\alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \to \mathbb{R}$ are two deterministic continuous functions. This equation is a shorthand notation for $X_t = X_0 + \int_0^t \alpha(s) \; X_s \; ds + \int_0^t \beta(s) \; X_s \; dG_s$, where the equality holds in $(\mathcal{S})^*$.

Theorem

The process $Z := (Z_t)_{t\in\mathbb{R}}$ defined by

$$Z_t := X_0 \diamond \exp^{\diamond}\left(\int_0^t \alpha(s) \; ds + \int_0^t \beta(s) \; dG_s\right), \qquad t \in \mathbb{R}_+, \tag{2.1}$$

is the unique solution, in $(\mathcal{S})^*$, of $(\mathcal{E})$.

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Comparison with Malliavin calculus or divergence integral I

Comparison

Our goal is now to compare the Wick-Itô integral with respect to G we just define to the divergence integral with respect to G, defined and studied in[a], on a compact set.

[a] Alòs, E. and Mazet, O. and Nualart, D. (2001). In: *Ann. Probab.* 29.2; Nualart, D. (2005). In: *Stochastic analysis: classical and quantum*.

We will denote $\int_0^T u_s \, \delta G_s$ the divergence integral with respect to G defined in[D]

[D] Alòs, E. and Mazet, O. and Nualart, D. (2001). In: *Ann. Probab.* 29.2; Nualart, D. (2005). In: *Stochastic analysis: classical and quantum*.

Comparison with Malliavin calculus or divergence integral II

Theorem: Comparison between Wick-Itô & divergence integral

Let u be a process in $L^2(\Omega, L^2([0,T]))$. If u belongs to the domain of the divergence of G, then u is Wick-Itô integrable on $[0,T]$ with respect to G. Moreover one has the equality

$$\int_0^T u_s \, \delta G_s = \int_0^T u_s \, dG_s.$$

Comparison with Malliavin calculus or divergence integral III

Remark

Note that $\int_0^T u_s \, \delta G_s = \int_0^T u_s \, dG_s$ is not true in general.
For example, $\int_0^T B_t^H \, dB_t^H$ exist and is equal to $\frac{1}{2}((B_T^H)^2 - T^{2H})$ for every $H \in (0,1)$ but $\int_0^T \; B_t^H \delta B_t^H$ does not even exist when $H < 1/4$.

Finally, the only thing one can say, in general, is that we have the dense inclusion $L^2(\Omega, L^2([0, T])) \cap \mathbb{D}\text{om}(\delta_G) \subset \Lambda$, where

- $\mathcal{H}_T := \overline{\text{span}\{\mathbb{1}_{[0,t]}, \; t \in [0, T]\}}^{(L^2)}$
- $\Lambda := \{u \in L^2(\Omega; \mathcal{H}_T); \; u$ is Wick-Itô integrable w.r.t. G & s.t.$\int_0^T u_s \; dG_s \in L^2(\Omega)\}$.

- $\mathbb{D}\text{om}(\delta_G) := \{u \in L^2(\Omega; \mathcal{H}_T); \;$ & s.t.$\int_0^T u_s \; \delta G_s$*exists* and $\in L^2(\Omega)\}$.

Conclusion on the comparison

Remark (Comparison with Itô integral)

- *When G is a Brownian motion (or even a Gaussian martingale), the Wick-Itô integral with respect to G is nothing but the classical Itô integral, provided X is Itô-integrable (which implies in particular that X is a previsible process).*

Remark

- *When G is a fractional (resp. multifractional) Brownian motion, the Wick-Itô integral with respect to G coincide with the fractional (resp. multifractional) Wick-Itô integral defined in*[a] *(resp. in*[b]*).*

---

[a] R. J. Elliott and J. van der Hoek (2003). In: *Mathematical Finance* 13(2); Biagini,F., AND Sulem, A., AND Øksendal, B. AND Wallner, N.N. (2004). In: *Proc. Royal Society, special issue on stochastic analysis and applications*; Bender,C. (2003). In: *Stochastic Processes and their Applications* 104; Bender, C. (2003). In: *Bernoulli* 9.6.

[b] J. Lebovits and Lévy Véhel, J. 2014; Lebovits, J. and Lévy Véhel, J. and Herbin, E. 2014; Lebovits, J. 2013.

Outline of the presentation

An Itô formula in (L^2)

Denote $t \mapsto R_t$ the variance function of G.

Theorem: Itô formula in (L^2)

Let $T > 0$ and f be a $C^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$ function. Furthermore, assume that $t \mapsto R_{t,t}$ is differentiable and that there are constants $C \geqslant 0$ and $\lambda < (4 \max_{t \in [0,T]} R_t)^{-1}$ such that for all (t,x) in $[0,T] \times \mathbb{R}$,

$$\max_{t \in [0,T]} \left\{ |f(t,x)|, \left|\frac{\partial f}{\partial t}(t,x)\right|, \left|\frac{\partial f}{\partial x}(t,x)\right|, \left|\frac{\partial^2 f}{\partial x^2}(t,x)\right| \right\} \leqslant C e^{\lambda x^2}.$$

Assume moreover that the map $t \mapsto R_t$ is both continuous and of bounded variations on $[0,T]$. Then, for all t in $[0,T]$, the following equality holds in (L^2):

$$f(T, G_T) = f(0,0) + \int_0^T \frac{\partial f}{\partial t}(t, G_t)\, dt + \int_0^T \frac{\partial f}{\partial x}(t, G_t)\, d^\diamond G_t + \frac{1}{2} \int_0^T \frac{\partial^2 f}{\partial x^2}(t, G_t)\, dR_t.$$

Remark

It is clear that one can extend the definition of integral in $(\mathcal{S})^*$ to the case where the measure considered is not the Lebesgue measure but a difference of two positive measures, denoted m. The definition of integral in $(\mathcal{S})^*$ wrt m is then:

Definition (**integral in** $(\mathcal{S})^*$ *wrt m*)

Assume that $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is weakly in $L^1(\mathbb{R}, m)$, i.e assume that for all φ in $(\mathcal{S})$, the mapping $u \mapsto < \Phi(u), \ \varphi >$ from $\mathbb{R}$ to $\mathbb{R}$ belongs to $L^1(\mathbb{R}, |m|)$. Then there exists an unique element in $(\mathcal{S})^*$, denoted $\int_{\mathbb{R}} \Phi(u) \ m(du)$ such that

$$< \int_{\mathbb{R}} \Phi(u) \ m(du), \varphi > = \int_{\mathbb{R}} < \Phi(u), \varphi > \ m(du) \qquad \text{for all } \varphi \text{ in } (\mathcal{S}).$$

Theorem: Tanaka formula for G

Let $T > 0$ be such that $[0, T] \subset$ and c be real number. Assume that the map $t \mapsto R_t$ is both continuous and of bounded variations on $[0, T]$ and such that:

(i) $t \mapsto R_t^{-1/2} \in L^1([0, T], dR_t)$,

(ii) $\exists\ q \in$ such that $t \mapsto |g_t'|_{-q}\ R_t^{-1/2} \in L^1([0, T], dt)$,

(iii) $\lambda(\mathcal{Z}_R^T) = \alpha_R(\mathcal{Z}_R^T) = 0$.

Then, the following equality holds in (L^2):

$$|G_t - c| = |c| + \int_0^T \text{sign}(G_t - c)\ G_t + \int_0^T \delta_{\{c\}}(G_t)\ dR_t, \qquad (3.1)$$

where the function sign is defined on $\mathbb{R}$ by $\text{sign}(x) := \mathbb{1}_{\mathbb{R}_+^*}(x) - \mathbb{1}_{\mathbb{R}_-}(x)$ and where $\delta_{\{a\}}(G_t)$ is the stochastic distribution defined by:

$$\delta_{\{c\}}(G_t) := \frac{1}{\sqrt{2\pi R_t}} \sum_{k=0}^{+\infty} \frac{1}{k!\ R_t^k} < \delta_{\{c\}}, \xi_{t,k} > I_k\left(g_t^{\otimes k}\right),$$

$$\text{with}\quad \xi_{t,k}(x) := \pi^{1/4}(k!)^{1/2} R_t^{k/2}\ \exp\{-\tfrac{x^2}{4R_t}\} e_k(x/(\sqrt{2R_t})),$$

for all (x, H, k) in $\mathbb{R} \times (0, 1) \times \mathbb{N}$.

About Itô and Tanaka formula

The results given in the last thee slides generalize the results provided

- for fBm by C. Bender[E]
- for mBm by J.L & J.Lévy Véhel [F]

[E]Bender,C. (2003). In: *Stochastic Processes and their Applications* 104; Bender, C. (2003). In: *Bernoulli* 9.6.

[F]J. Lebovits and Lévy Véhel, J. (2014). In: *Stochastics An International Journal of Probability and Stochastic Processes* 86.1.

Outline of the presentation

1. Two particular Gaussian processes, Fractional and multifractional Brownian motion
   - Fractional and multifractional Brownian motions
   - Non semimartingales versus integration
2. Stochastic integral w.r.t. G in the White Noise Theory sense
   - Background on White Noise Theory
   - Stochastic Integral with respect to G
   - Comparison with Malliavin calculus or divergence integral
3. Miscellaneous formulas & some open problems
   - Miscellaneous formulas
     - Itô forumulas
     - Tanaka formula
   - Weighted and non weighted local times of G
   - Some open problems

Definition (non weighted local time of G)

The (non weighted) local time of G at a point $a \in \mathbb{R}$, up to time $T > 0$, denoted by $\ell_T^{(G)}(a)$, is defined by:

$$\ell_T^{(G)}(a) := \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \lambda(\{s \in [0, T];\ G_s \in (a - \varepsilon, a + \varepsilon)\}),$$

where λ denotes the Lebesgue measure on $\mathbb{R}$ and where the limit holds in $(\mathcal{S})^*$, when it exists.

Proposition

Let $T > 0$. Assume that the variance map $s \mapsto R_s$ is continuous on $[0, T]$ and s.t. $\lambda(\mathcal{Z}_R^T) = 0$, then:

1. *The map $s \mapsto \delta_a(G_s)$ is $(\mathcal{S})^*$-integrable on $[0, T]$ for every $a \in \mathbb{R}^*$. The map $s \mapsto \delta_0(G_s)$ is $(\mathcal{S})^*$-integrable on $[0, T]$ if $s \mapsto R_s^{-1/2}$ belongs to $L^1([0, T])$.*
2. *The following equality holds in $(\mathcal{S})^*$:*

$$\ell_T^{(G)}(a) = \int_0^T \delta_a(G_s)\, ds, \tag{3.2}$$

for every $a \in \mathbb{R}^*$; and, also for $a = 0$ if $s \mapsto R_s^{-1/2} \in L^1([0, T])$.

Thorem-defintion (weighted local time of G)

Let $T > 0$. Assume that the map $s \mapsto R_s$ is continuous and of bounded variations on $[0, T]$ and such that $\alpha_R()$ is equal to 0. Then:

1. *The map $s \mapsto \delta_a(G_s)$ is $(\mathcal{S})^*$- integrable on $[0, T]$ with respect to the measure α_R, for every $a \in \mathbb{R}^*$. The map $s \mapsto \delta_0(G_s)$ is $(\mathcal{S})^*$-integrable on $[0, T]$ with respect to the measure α_R, if $s \mapsto R_s^{-1/2}$ belongs to $L^1([0, T], dR_s)$.*

2. *For every $a \in \mathbb{R}$, when $s \mapsto \delta_a(G_s)$ is $(\mathcal{S})^*$-integrable on $[0, T]$ with respect to the measure α_R, one can define the weighted local time of G at point a, up to time T, denoted $\mathscr{L}_T^{(G)}(a)$, as being the $(\mathcal{S})^*$- process defined by setting:*

$$\mathscr{L}_T^{(G)}(a) := \int_0^T \delta_a(G_s)\, dR_s,$$

where the equality holds in $(\mathcal{S})^*$.

Weighted and non-weighted local times are (L^2) random variables

Denote $\mathscr{M}_b(\mathbb{R})$ the set of positive Borel functions defined on $\mathbb{R}$.

Theorem: (Occupation time formula)

Let $T > 0$ and assume the variance map $s \mapsto R_s$ is continuous on $[0, T]$.

(i) Assume that $s \mapsto R_s^{-1/2} \in L^1([0, T])$ and that $\lambda_R(\mathcal{Z}_R^T) = 0$. If $\mathrm{E}\big[\int_{\mathbb{R}} \big| \int_0^T e^{i\xi G_s}\, ds \big|^2 \, d\xi \big] < +\infty$, then the map $a \mapsto \ell_T^{(G)}(a)$ belongs to $L^2(\lambda \otimes \mu)$, where λ denotes the Lebesgue measure. Moreover one has the following equality, valid for μ-a.e. ω in Ω,

$$\forall\ \Phi\ \in \mathscr{M}_b(\mathbb{R}),\ \int_0^T \Phi(G_s(\omega))\, ds = \int_{\mathbb{R}} \ell_T^{(G)}(y)(\omega)\ \Phi(y)\ dy.$$

(ii) Assume that $s \mapsto R_s$ is of bounded variation on $[0, T]$, and such that $s \mapsto R_s^{-1/2} \in L^1([0, T], dR_t)$. Assume moreover that $\alpha_R(\mathcal{Z}_R^T) = 0$. If $\mathrm{E}\big[\int_{\mathbb{R}} \big| \int_0^T e^{i\xi G_s}\, dR_s \big|^2 \, d\xi \big] < +\infty$, then $a \mapsto \mathscr{L}_T^{(G)}(a)$ belongs to $L^2(\lambda \otimes \mu)$. Moreover one has the following equality, valid for μ-a.e. ω in Ω,

$$\forall\ \Phi \in \mathscr{M}_b(\mathbb{R}),\ \int_0^T \Phi(G_s(\omega))\, dR_s = \int_{\mathbb{R}} \mathscr{L}_T^{(G)}(y)(\omega)\ \Phi(y)\ dy.$$

The previous results are, in particular valid, when one consider fBm or mBm.

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An open problem I

Consider the fractional Brownian motion B^H its non weighted and weighted local times: $\ell^H := (\ell^H_t)_{t\in[0,T]}$ and $\mathscr{L}^H := (\mathscr{L}^H_t)_{t\in[0,T]}$ defined by:

$$\ell^H_t := \int_0^T \delta_{\{a\}}(B^H_t)\, dt \quad \& \quad \mathscr{L}^H_t := \int_0^T \tfrac{d}{dt}[t^{2H}]\, \delta_{\{a\}}(B^H_t)\, dt$$

Question

What can one say about the behavior of process ℓ^H?

More precisely, following the Result of Yor 1983 [G]

Theorem Yor - 1983

$$\left\{\beta_t, \ell_t(a), \tfrac{\lambda^{1/2}}{2}(\ell_t(a/\lambda)-\ell_t(0))\,;(t,a)\in\mathbb{R}^2_+\right\} \xrightarrow[\lambda\to\infty]{\text{in law}} \left\{\beta_t, \ell_t(a), \mathbf{B}(\ell_t(0),a)\,;(t,a)\in\mathbb{R}^2_+\right\}$$

where $\beta = (\beta_t)_{t\in[0,T]}$ is a Brownian motion and $\ell = (\ell_t(a))_{t\in[0,T]}$ is the local time of B at point a until time t and $\mathbf{B}$ a Brownian sheet, independent of the Brownian motion β.

[G] **Yor83**.

An open problem II

How can we translate this result in the world of fractional Brownian motion?

Conjecture

$$\left\{\beta^H_t, \ell^H_t(a), \tfrac{\lambda^s}{2}(\ell^H_t(a/\lambda) - \ell^H_t(0)), \ ; \ (t,a) \in \mathbb{R}^2_+\right\} \xrightarrow[\lambda\to\infty]{\text{in law}} \left\{\beta^H_t, \ell^H_t(a), \mathbf{B}(\ell^H_t(0), a), \ ; \ (t,a) \in \mathbb{R}^2_+\right\}$$

where $s := \frac{1}{2}(\frac{1}{H} - 1)$, β^H is a fractional Brownian motion and $\ell^H = (\ell^H_t(a))_{t\in[0,T]}$ is the local time of B^H at point a until time t and $\mathbf{B}$ a Brownian sheet, independent of the fractional Brownian motion β^H.

An open problem III

In general, what can one say about the convergence in law of

$$\Big\{ G_t, \ell_t^{(G)}(a), \tfrac{\lambda^s}{2}(\ell_t^{(G)}(a/\lambda) - \ell_t^{(G)}(0))\ ; (t,a) \in \mathbb{R}_+^2 \Big\},$$

for a Gaussian process $G := (< ., g_t >)_{t\in\mathbb{R}}$ that fulfills assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$?

Thank you for your attention!