Mathematical Modeling of Plasma Opening Switches

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Abstract
This article investigates Plasma Opening Switches with the help of an implicit code. The plasma is
described using a collisional bifluid model, where the low ratio of electronic and ionic masses allows us
to drop inertial terms from the electronic momentum equation. This asymptotic model filters plasma
oscillations from the equations and the scheme is stable for time steps exceeding the electron plasma
period. However, an implicit coupling of fluid and Maxwell’s equations is considered because of the
presence of a damped mode, whose characteristic time scale restricts the time steps of explicit schemes
to small values compared to the simulation requirements. The stability of the scheme is illustrated by
simulating certain operating phases of the Plasma Erosion Opening Switch.

Key words: Plasma opening switch, implicit methods, Maxwell’s equations, drift-diffusion model, boundary
conditions, particle methods, collisional plasma.

1 Introduction

In the last two decades, there has been considerable interest in the development of fast opening switches.
Plasma Opening Switches can conduct high power currents and open in a few nanoseconds in the case of
low density switches (17).
The device consists of two concentric electrodes short-circuited by means of a plasma bridge. During the
conduction phase, a current is transmitted via the plasma through the switch area. The opening of the
device is due to plasma interactions with an electromagnetic wave. As the magnetic field penetrates the
plasma, the electron flow is altered by a variety of processes leading to the switch opening.

Theoretical models all have certain assumptions in common. For instance, separation charge effects are
determinant in explaining the opening mechanisms (24; 17). Other phenomena such as space-charge-limited
emission of electrons at the cathode are also taken into account.

Among the various models considered to represent the plasma in a POS, use shall be made of a bi-fluid
model, for electrons and ions. This model is able to describe charge separation effects, and consequently,
allows the simulation of both conduction and opening phases of the POS in operation. This model is coupled
with Maxwell’s equations which describe the variations in electromagnetic field. Although the device presents
an axisymmetric geometry, the model considered here is Cartesian.

Numerical methods are detailed in the remainder of this paper. Maxwell’s equation are finite-differenced
on a grid, and the conservation laws associated with the fluids are discretized using particles.

Time-differencing is a determinant task for the efficiency of the scheme. An explicit discretization is stable
only for time steps and mesh intervals that resolve all time and space scales. The main constraints for the
time step are associated with two physical phenomena, the first being the electromagnetic wave propagation
(at the speed of light), the second due to the electronic plasma period. In the case of high density plasma
opening switches, with longer conduction and opening times (see for instance (4; 5)), the latter can restrict
the time steps to seven orders of magnitude lower than the time scale of the simulated phenomena.
In order to filter these oscillations, the inertial terms are dropped in the electronic momentum equation. This model is referred to as the Energy-Transport Model (7; 8). In the context of an isothermal plasma, it reduces to the drift-diffusion model. In addition to filtering electron oscillations, the drift-diffusion model assumes a zero electron larmor radius, allowing for coarser mesh intervals.

However with an explicit discretization, the time step has to resolve the dielectric relaxation, a damped mode, whose characteristic time scale induces severe limitations. Therefore we consider an implicit coupling of Maxwell’s and electronic fluid equations (see the table 1 for the discretization of the main modes described by the equations).

<table>
<thead>
<tr>
<th>Dielectric relaxation</th>
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<tr>
<td>Electron plasma frequency</td>
<td>F</td>
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<td>Electron cyclotron frequency</td>
<td>F</td>
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<tr>
<td>Electromagnetic wave propagation</td>
<td>E</td>
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Table 1: Fastest time scales involved in the switch dynamic.

Recent works deal with implicit discretizations. Mottez et al (19) propose a guiding centre implicit scheme for the coupling of massless electronic fluid equations with Maxwell’s system. This approach is valid in the context of strongly magnetized collisionless plasmas. Lapenta et al (14) developed an implicit coupling of the drift-diffusion model with the Poisson equation (for an electrostatic field). Here the full set of Maxwell’s equations is regarded and we generalize the work of Lapenta to an electromagnetic field and for collisional plasmas. Moreover, our approach allows the simulation of not uniformly magnetized plasmas and gives an extension of the guiding centre scheme of Mottez. To derive our implicit scheme, we shall make use of two classical approaches: the implicit moment method introduced by Mason (15; 3), and the direct method developed by Langdon et al (13; 11). The approach developed by Mason for inertial fluids (see (16)) is a useful guide for deriving an implicit moment method for a “massless” description of the electronic fluid. A simplification arises in that the inertial terms of the electronic fluid are dropped. The expression of the current density is simplified, an algebraic relation involving the implicit field is obtained, instead of an evolution equation, and, as a byproduct, the hand-coding necessary to form the implicit equation becomes a less tedious task. With this implicit discretization, the scheme’s stability is only constrained by the Courant condition associated to the electromagnetic wave propagation (at the speed of light) and on the other hand by the velocity of the fastest electrons.

The paper is organized as follows. Section 2 aims to present the model dealt with here. Boundary conditions required for the POS simulation are addressed and discussed in detail. In particular, they take into account space-charge-limited currents (Child-Langmuir law). Discretization of the conservation laws associated with the plasma is detailed in section 3. Time and space discretization of Maxwell’s equations can be found in section 4. The Yee scheme is considered and the components of the electromagnetic field are computed on a staggered grid. We propose a discretization of a transparent boundary condition. Implicit coupling of fields and particles is explained. A correction of the electrostatic part of the electric field must be applied in order to ensure consistency of the scheme with the Maxwell-Gauss equation. This point is considered in section 5. Finally, section 6 is devoted to demonstrative calculations concerning the PEOS operation. Sheath formation and near electrode fast magnetic diffusion are studied in detail.

2 The model

Consider a planar model. Electrons and ions move in the \((x, y)\) directions. The \(z\) component of the electric field is assumed to vanish whereas the magnetic field is oriented along the \(z\)-axis.

All the above quantities are assumed to depend only on the \(x, y\) coordinates. Note that this assumption implies a divergence free magnetic field.
2.1 Plasma and field representation

The plasma consists of two species: electrons and ions. A fluid description is used for each species. Ion density $n_i$ and velocity $u_i$ satisfy the following conservation system:

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i u_i) = 0,$$

(1)

$$\frac{\partial}{\partial t} (m_i n_i u_i) + \nabla \cdot (m_i n_i u_i \otimes u_i) + \nabla (n_i T_i) - Z e n_i (E + u_i \times B) = R_i,$$

(2)

where $m_i$ is the ionic mass, $T_i$ the temperature, $e$ the elementary charge and $Z$ the ion charge state. $R_i$ is the momentum relaxation term.

Electrons satisfy an analogous system except that the inertial terms are dropped (since the electronic mass $m_e$ is much smaller than the ionic one).

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e v_e) = 0,$$

(3)

$$\nabla (n_e T_e) + e n_e (E + v_e \times B) = R_e.$$

(4)

Here $(m_e, n_e, v_e, T_e)$ are the electron mass, density, velocity and temperature. $R_e$ is the electronic momentum relaxation term. Conservation of the total momentum implies the relation: $R_e = -R_i$, and we shall take $R_e$ in the following form:

$$R_e = -R_i = -m_e n_e \tau^{-1} (u_i - v_e),$$

(5)

where $\tau$ is the momentum relaxation time (6).

These equations are coupled with the Maxwell’s equations which describe the variations in electromagnetic field:

$$\frac{\partial E}{\partial t} - c^2 \nabla \times B = -\frac{J}{\varepsilon_0},$$

(6)

$$\frac{\partial B}{\partial t} + \nabla \times E = 0,$$

(7)

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0},$$

(8)

where $c$ is the speed of light $\rho = e(\text{Z } n_i - n_e)$ the charge density and $J = e(\text{Z } n_i u_i - n_e v_e)$ the current density.

2.2 Boundary conditions

2.2.1 Particles

Ions: They are injected into the device at the $\Gamma_1$ part of the boundary. Their flow rate per unit of surface is given by $D_i = n_i u_i$. This fixes the injection of ions. On the other parts of the boundary ($\Gamma_{\text{in}}, \Gamma_{\text{out}}, \Gamma_a, \Gamma_c$), outgoing ions are absorbed.
Electrons: Explanations will be given for the injection of electrons at $\Gamma_1$ with the same density and flow rate as ions, the injected plasma being neutral. On the other hand, electrons are also emitted at the cathode $\Gamma_c$ (primary emission) in the case of a sufficiently high electric field. This mechanism will also be described (see section 5.3). Analogously to ions, outgoing electrons are absorbed.

### 2.2.2 Electromagnetic field

Electrodes ($\Gamma_a, \Gamma_c$) are perfect conductors. The injection area ($\Gamma_1$) is also considered as a perfect conductor which is required to observe the fast magnetic diffusion (18). The boundary conditions applied on ($\Gamma_a, \Gamma_c, \Gamma_1$) read:

$$E \times n = 0, \quad B \cdot n = 0,$$

where $n$ is the outward normal.

Equation (9) implies a homogeneous Dirichlet condition for the tangential component of the electric field ($E_z$). Equation (10) is still satisfied since the magnetic field is perpendicular to the plane containing $n$.

$\Gamma_{in}$ and $\Gamma_{out}$ do not represent material boundaries but artificial ones.

Since the electromagnetic wave travels from the left ($\Gamma_{in}$) to the right ($\Gamma_{out}$) of the device, boundary conditions at $\Gamma_{in}$ and $\Gamma_{out}$ account respectively for the arrival (and reflection) of the incident wave, and the escape of the transmitted part of this wave. We shall use the Silver-Müller boundary condition for the outgoing wave:

$$\left(E - c B \times n\right) \times n = 0 \quad \text{on} \quad \Gamma_{out}. \quad (11)$$

Let ($E_{inc}, B_{inc}$) be the electric and magnetic component of the incident wave, a transparent boundary condition is also considered for the reflected part of the wave:

$$\left[\left(E - E_{inc}\right) - c \left(B - B_{inc}\right) \times n\right] \times n = 0 \quad \text{on} \quad \Gamma_{in}. \quad (12)$$

### 3 Spatial and time differencing of the fluid equations

Spatial and time differencing of the conservation laws associated with each species (electrons and ions) are of particular interest here.

The convective part of these equations is discretized using particles. We shall first focus on the particle approximations for each fluid. The last part of this section is devoted to the differencing of spatial derivatives using a particle-in-cell scheme.

#### 3.1 Particle approximation of the conservation laws

Let us first consider the following model equation:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (v\phi) = 0. \quad (13)$$
To get a particle approximation of the transport equation (13), let us take a set of moving particles \((x_p(t), \omega_p(t))_{p \in \mathcal{P}}\) indexed by \(p \in \mathcal{P}\), where \(x_p(t)\) is the position of the particle \(p\) and \(\omega_p(t)\) its weight.

Particles are conventionally moved along the characteristic curves (21) of the field \(v\), with weight modification in order to take into account the deformation due to the transport field \(v\):

\[
\frac{d}{dt} x_p(t) = v(x_p(t), t),
\]

\[
\frac{d}{dt} \omega_p(t) = \omega_p(t) \nabla \cdot (v(x_p(t), t)).
\] (14)

Introducing the transport operator \(\mathcal{L}_v\):

\[
\phi \rightarrow \mathcal{L}_v(\phi) = \frac{\partial \phi}{\partial t} + \nabla \cdot (v \phi),
\]

we obtain the following property:

\[
\frac{d}{dt} (\omega_p(t) \varphi_p(t)) = \omega_p(t) \mathcal{L}_v(\varphi_p(t)),
\] (15)

where \(\omega_p\) is the solution of equation (14) and \(\varphi_p(t) = \varphi(x_p(t), t)\).

### 3.1.1 Ions

Particles associated with the ionic fluid are indexed by \(l, l \in \mathcal{P}_i\).

The conservation of the ionic density (1) is rewritten as:

\[
\mathcal{L}_{u_{i,l}}(n_{i,l}) = 0.
\]

Thanks to (15), the following is obtained:

\[
\frac{d}{dt}(\omega_{n,l}) = 0.
\] (16)

We shall use the notation \(Q_l = \omega_l(t)n_{i,l}(t)\) which is the charge represented by the particle \(l\). The conservation of the density implies that for each particle the quantity \(Q_l\) remains constant.

Using the same notations (2) is rewritten as:

\[
m_i \mathcal{L}_{n_i,u_i}(u_i) + \nabla (n_i E_i) - Z n_i e (E + u_i \times B) = -m_e n_e \tau^{-1} (u_i - v_e),
\]

multiplying by \(\omega_i\) and thanks to (15) and (16), we obtain for \(l \in \mathcal{P}_i\):

\[
\frac{d}{dt} (u_{i,l}) = \frac{1}{m_i} \left[-\frac{\nabla (n_i T_i)}{n_i} (x_i) + Z e \left(E(x_i) + u_{i,l} \times B(x_i)\right)\right] - m_e \frac{n_e (x_i)}{n_{i,l}} \tau^{-1} (u_{i,l} - v_e(x_i)).
\]

The equations (1) and (2) are equivalent to the classical PIC system of o.d.e. (10; 1):

\[
\begin{cases}
(a) \quad \frac{d}{dt} x_i(t) = u_{i,l}(t), \\
(b) \quad \frac{d}{dt} u_{i,l}(t) = \frac{1}{m_i} \left[-\frac{\nabla (n_i T_i)}{n_i} (x_i) + Z e \left(E(x_i) + u_{i,l} \times B(x_i)\right)\right] - m_e \frac{n_e (x_i)}{n_{i,l}} \tau^{-1} (u_{i,l} - v_e(x_i)).
\end{cases}
\] (17)

The discretization in time is achieved using a leap-frog scheme:

\[
\begin{align*}
x_i^{n+1} & = x_i^n + \Delta t \frac{u_{i,l}^{n+\frac{1}{2}}}{m_i}, \\
u_{i,l}^{n+\frac{1}{2}} & = u_{i,l}^{n-\frac{1}{2}} + \Delta t \left[\frac{1}{m_i} \left[-\frac{\nabla (n_i T_i)}{n_{i,l}} n_i^n + Z e \left(E^n(x_i^n) + u_{i,l}^n \times B^n(x_i^n)\right)\right] - m_e \frac{n_e (x_i^n)}{n_{i,l}} \tau^{-1} (u_{i,l}^n - v_e^n(x_i^n))\right].
\end{align*}
\] (18)
3.1.2 Particle approximation of the electronic fluid

Particles associated with the electronic fluid are indexed by \( k \in \mathcal{P}_e \).

The momentum equation is used to express the velocity for the electronic fluid:

\[
\begin{align*}
\left[ \nabla \left( \frac{n_e T_e}{n_e} \right)(x_k) + e \left( E(x_k) + v_{e,k} \times B(x_k) \right) \right] &- m_e \tau^{-1} (v_{e,k} - u_i(x_k)) = 0.
\end{align*}
\]  

(19)

Introducing the electronic mobility \( \mu = \frac{e \tau}{m_e} \) we obtain:

\[
\begin{align*}
v_{e}(t) = F_e(B) \left[ -\mu \left( E + \frac{\nabla (n_e T_e)}{e n_e} \right) + u_i \right],
\end{align*}
\]  

(20)

where:

\[
F_e(B) = \frac{1}{1 + (\mu B_z)^2} \begin{pmatrix} 1 & -\mu B_z \\ \mu B_z & 1 \end{pmatrix}.
\]  

(21)

The above expression for electron velocity is valid assuming that the relaxation term (or magnetic field) does not vanish. This hypothesis might be false in areas where one species is absent. In this case, the equation (19) gives the velocity of electrons when the magnetic component of the field is not zero. Thus, electronic velocity is given by:

\[
v_e = \frac{E \times B}{B^2}.
\]  

(22)

The above formula describes the electron motion in the guiding centre approximation used in (19). Note that our model assumes the electron Larmor radius to be zero since the inertial terms are dropped from the electron momentum equation.

We consider the following time differencing:

\[
\begin{align*}
x_k^{n+1} &= x_k^n + \Delta t v_{e,k}^{n+\frac{1}{2}}, \\
v_{e,k}^{n+\frac{1}{2}} &= F_e \left( B^{n+\frac{1}{2}}(x_k^{n+\frac{1}{2}}) \right) \left[ -\left( E^{n+\frac{1}{2}}(x_k^{n+\frac{1}{2}}) \right) \\
&\quad + \frac{\nabla (n_e^{n+\frac{1}{2}} T_e^{n+\frac{1}{2}})}{e n_e^{n+\frac{1}{2}}}(x_k^{n+\frac{1}{2}}) \right] + \frac{1}{\mu} u_i^{n+\frac{1}{2}}(x_k^{n+\frac{1}{2}}). \tag{23}
\end{align*}
\]

3.2 The Particle-In-Cell scheme

An interpolation-assignment scheme is used to interpolate grid quantities at particle positions, and to weight particle quantities on the grid.

There exists a hierarchy of interpolation-assignment schemes which present different noise properties (see (12; 2)), but to our knowledge, the Cloud-In-Cell scheme is the most widely used, and this is the choice we have made. The shape function of the CIC scheme is the hat function (piecewise linear) and will be denoted in the remainder of this paper by \( h \).

Considering a node coordinate \( X \), we note \( H \) the volume of the cell surrounding this node. The current density is then defined on the grid at location \( X \) by:

\[
J^m_{\alpha}(X) = \frac{1}{H} \sum_{p \in \mathcal{P}_\alpha} Q_p v^m_p h(X - x^m_p),
\]  

(24)

where \( Q_p \) is the charge and \( v_p \) the velocity associated to the simulation particles indexed by \( p \) (\( \mathcal{P}_\alpha \) stands for the set of particles). The charge density is defined on the grid by substituting \( v_p \) by 1 in equation (24).

Quantities calculated on the grid are interpolated on particles using the same shape function \( h \), for instance the electronic velocity computed on the grid can be interpolated on particles using (25)

\[
\begin{align*}
v^m_{e,p} = \sum_{X \in \text{Grid}} V^m_e(X) h(X - x^m_p).
\end{align*}
\]  

(25)
The pressure term ($\nabla n_i T_i$) involved in the computation of the ionic velocity can be computed on the grid as follows: the density is weighted on grid corners from particle positions $x^n$, taking the gradient via finite differences, the result of computations are finally interpolated back to particles. The same process is carried out for the electronic pressure term.

An analogous approach is used to compute momentum relaxation for each species. For this purpose, the ionic velocity is required on the grid, to compute the electronic relaxation term. In practice, it is computed as the ratio of ionic current and charge density (see (3)). Besides, electronic velocity is directly defined on the grid using equations (20) & (21).

In the remainder of this paper, we shall use lower case symbols for particle quantities defined on particles and upper case symbols for these quantities expressed on the grid. For instance, we may consider $v_e$ for the electronic velocity defined on particles, and $V_e$ when regarded as a grid quantity.

4 Time and space discretization of Maxwell’s equations

At this point, two approaches can be adopted, either a Coulomb-Gauge formulation, or a direct discretization of Maxwell’s equation.

The former approach consists in introducing a scalar potential $\phi$ giving the electrostatic component of the electric field ($\nabla \phi$), a vector potential $A$ that provides both magnetic field ($\nabla \times A$) and solenoidal component of the electric field ($\partial_t A$). This formulation is investigated in (12; 3). Maxwell’s equations are expressed as an elliptic equation for the electrostatic potential, and a wave equation for the magnetic vector potential. This formulation allows the constraints on the divergence of both electric and magnetic field to be correctly accounted for.

For the particular geometric configuration under consideration, the magnetic field is divergence-free. Therefore, it is sufficient to ensure the constraint on the electrostatic part of the electric field. A direct discretization of the set of Maxwell’s equations has been chosen as it seems to be more straightforward.

4.1 Space differencing

Maxwell’s equations are supplemented with perfect conductor and Silver-Müller boundary conditions. These equations do not provide boundary conditions for every unknown. For instance, the perfect conductor condition on horizontal boundaries reads $E_x = 0$. There is no condition on $E_y$ neither on $B_z$. Hence these unknowns are not computed on horizontal boundary nodes. Similar difficulties occur at vertical boundaries for $E_x$.

The Yee scheme has been used to deal with this problem, where unknowns are defined on a staggered grid as depicted in figure 3.

![Figure 3: Localization of physical quantities on the staggered grid](image-url)
To integrate Ampere’s law on a time step, it is necessary to calculate the values of the x and y components of the current density, on $\otimes$ nodes and $\oplus$ nodes respectively.

Langdon et al (11) suggest these values first be assigned at $\Box$ nodes and then centered averages can be used to find the corresponding values at the $\otimes$ nodes and $\oplus$ nodes. Since this induces a smoothing of the current density, and in order to avoid self force (see (2)), the same smoothing is also introduced when computing the value of the field at $\Box$ nodes by averaging the $\otimes$ node and $\oplus$ node values.

This is half expensive as a straight weighting to the relevant locations. Indeed, the above approach requires the counting of particles on both $\otimes$-centered and $\oplus$-centered cells, whereas the first approach only needs the $\Box$-centered cells to be processed.

We were first tempted was to use the less time-consuming approach (assignment and interpolation scheme with averages), but we noticed that for some particular applications, this scheme produced oscillations which altered the qualitative behavior of the device. Section 5.2 is devoted to the study of this item.

### 4.2 Time differencing

We propose the following time scheme for Maxwell’s equations :

\[
\begin{align*}
B^{n+\frac{1}{2}} &= B^{n-\frac{1}{2}} - \Delta t \nabla \times E^n, \\
E^{n+1} &= E^n + c^2 \Delta t \nabla \times B^{n+\frac{1}{2}} - \frac{\Delta t}{\varepsilon_0} J^{n+\frac{1}{2}},
\end{align*}
\]

(26)

where : $J^{n+\frac{1}{2}} = J^{n+\frac{1}{2}}_i + J^{n+\frac{1}{2}}_c$.

Here the tilde quantities are accumulated from particles by means of equation (24).

Consider now the particle-field system at time $t^n = n \Delta t$. In accordance with the considered time differencing, we assume that the following quantities are computed : $E^n, B^{n-\frac{1}{2}}, x^n, u^{n-\frac{1}{2}}$.

Advanced positions for ions are readily obtained by pushing particles from $x^n$ to $x^{n+1}$. Stored values of the electric and magnetic field allows for the computation of the velocity $u^{n+\frac{1}{2}}_i$. Ions are first pushed on a half time step to $x^{n+\frac{1}{2}}$, to allow current density $\tilde{J}^{n+\frac{1}{2}}_i$ weighting. Then they are pushed from $x^{n+\frac{1}{2}}$ to $x^{n+1}$. We refer to (2; 12) for a detailed account of ion and field equations coupling.

As for electrons, the calculation of $v^{n+\frac{1}{2}}_e$ requires $E^{n+\frac{1}{2}}$ which is an advanced value of the electric field. This means that field and electron coupling is in essence implicit, which introduces a difficulty stemming from the fact that, on the one hand, integration of Ampere’s law requires electronic current density to advance in time the electric field, and on the other hand construction of the current density would necessitate the advanced electric field.

We shall first prove that the explicit schemes are submitted to a drastic stability condition. For this purpose, a first order time discretization for Ampere’s law has been investigated. A second order scheme will be discussed in the remainder of this section.

Let us consider the following discretization :

\[
E^{n+1} = E^n - c^2 \Delta t \nabla \times B^{n+\frac{1}{2}} - \frac{\Delta t}{\varepsilon_0} J^n.
\]

(27)

For the sake of simplicity, we consider a warm plasma and ion motion is ignored. The expression of the current density is then given by :

\[
J^n = -e \tilde{N}^n_e F_e (B^n) (-\mu E^n).
\]

Substituting the current density expression in (27), we obtain, after some algebraic calculations, the following :

\[
E^{n+1} = \left( Id - \Delta t \tau (\omega_{pe}^n)^2 F_e (B^n) \right) E^n - c^2 \Delta t \nabla \times B^{n+\frac{1}{2}}.
\]

(28)

This scheme is stable only for time steps satisfying the dielectric relaxation constraint :

\[
\Delta t \tau \omega_{pe}^2 < 1.
\]

(29)
The “massless” description of the electronic fluid proves our first hypothesis: filtering plasma oscillations. The constraint \( \Delta \omega_{\text{pe}} < 1 \) does not have to be considered. Instead of this oscillatory phenomenon, we are faced with a damped mode whose characteristic time is the dielectric relaxation constant: \( \tau_\omega = 1/(\tau \omega_{\text{pe}}^2) \). This constraint induces severe limitations on explicit scheme time steps, since the dielectric time scale is much shorter than that observed in the operation of POS.

In order to design a scheme that remains stable for large \( \Delta t \)’s, it is necessary to carry out an implicit discretization of Ampere’s law. This operation might seem complex since an implicit discretization of the coupled system requires a simultaneous solution of field and particle equations. The difficulty is overcome by separating the advance of the fields from that of the particles.

### 4.2.1 A first order implicit scheme

In accordance with the implicit moment method (15; 22), fluid equations are introduced on the grid to predict source terms of Ampere’s law. To this end, the following expression for the electronic current density has been used:

\[
J_e^{n+\frac{1}{2}} = -eN_e^0 F_e(B^{n+\frac{1}{2}}) \left[ -\mu E^{n+1} + U_i^{n+\frac{1}{2}} \right],
\]

where \( N_e \) and \( V_e \) are grid quantities. Both are functions of the implicit electric field, but to obtain a linear equation for the advanced field, the electronic density remains explicit while the electric field is made implicit in the expression of the velocity. As noticed by Mason (15), this choice is sufficient to ensure stability of the scheme.

Ampere’s law is rewritten as:

\[
E^{n+1} = E^n + e^2 \Delta t \nabla \times B^{n+\frac{1}{2}} - \frac{\Delta t}{\varepsilon_0} \tilde{J}_i^{n+\frac{1}{2}} \nonumber
\]

\[+ \frac{\Delta t}{\varepsilon_0} \left[ eN_e^0 F_e(B^{n+\frac{1}{2}}) \left( -\mu E^{n+1} + U_i^{n+\frac{1}{2}} \right) \right]. \] (30)

Equation (30) can be rearranged as:

\[
M E^{n+1} = E^n - e^2 \Delta t \nabla \times B^{n+\frac{1}{2}} - \frac{\Delta t}{\varepsilon_0} \tilde{J}_i^{n+\frac{1}{2}} + eN_e^0 F_e(B^{n+\frac{1}{2}})U_i^{n+\frac{1}{2}} \] (31)

where \( M = \left[ I d + \Delta t \tau (\omega_{\text{pe}}^n)^2 F_e(B^{n+\frac{1}{2}}) \right] \) and \( I d \) is the unit diagonal matrix.

The matrix \( M \) is always invertible since \( \text{det}(M) = (1 + \Delta t \tau \omega_{\text{pe}}^2)^2 + \left( \mu B^{n+\frac{1}{2}} \Delta t \tau \omega_{\text{pe}}^2 \right)^2 > 0 \), and \( M^{-1} \) can be computed analytically, we have:

\[
M^{-1} = \frac{1}{\text{det}(M)} \begin{pmatrix} 1 + \Delta t \tau \omega_{\text{pe}}^2 & \mu B^{n+\frac{1}{2}} \Delta t \tau \omega_{\text{pe}}^2 \\ -\mu B^{n+\frac{1}{2}} \Delta t \tau \omega_{\text{pe}}^2 & 1 + \Delta t \tau \omega_{\text{pe}}^2 \end{pmatrix}. \] (32)

This implicit method has therefore the same numerical cost as an explicit scheme.

### 4.2.2 A second order implicit scheme

To design an accurate second order scheme, it is necessary to estimate the electronic current density and velocity up to the second order. Moreover, the interpolation of velocity at time \( (n+\frac{1}{2}) \Delta t \) requires particles at positions \( x^{n+\frac{1}{2}} \). For this purpose, particles are first pushed on a half time step using the explicit velocity \( v_e^n \):

\[
x^{n+\frac{1}{2}} = x^n + \frac{\Delta t}{2} v_e^n. \] (33)

Once the particles have been prepushed an explicit estimate of the electronic density can be weighted on the grid using equation (24). Maxwell’s equations are integrated in time, providing the electronic velocity \( v_e^{n+\frac{1}{2}} \), particles can now be pushed to \( x^{n+1} \) using:

\[
x^{n+1} = x^n + \Delta t v_e^{n+\frac{1}{2}}. \] (34)
The whole particle pusher (33) and (34) can be regarded as an accurate second order accurate Runge-Kutta method. This procedure is very similar to that used in the moment method developed for the Venus code (22).

In our first attempt, the implicit field \( E^{n+1} \) was used to compute particle velocity, which is sufficient for a first order schemes, but, for a second order expression, an estimate of \( E^{n+\frac{1}{2}} \) is required. Denavit (9) studied multi-level schemes which provide unconditional linear stability. The electric field \( E^{n+\frac{1}{2}} \) is substituted by \( E^{n+\frac{1}{2}} = \alpha^{n+1} E^{n+1} + \alpha^n E^n + \alpha^n E^{n-1} \), for instance we may consider \( \alpha^{n+1} = 3/4, \alpha^n = 0, \alpha^n-1 = 1/4 \) or \( \alpha^{n+1} = 9/16, \alpha^n = 6/16, \alpha^n-1 = 1/16 \).

The calculations shown in the remainder of this paper are obtained using the first order scheme.

### 4.3 Discretization of the Silver-Müller boundary conditions

The transparent boundary condition for the outgoing wave (right boundary of the domain) reads : 

\[
E_y - c B_z = 0 \tag{10}
\]

This equation can be seen as a boundary condition for either the electric field or for the magnetic field. Our choice was to compute magnetic field on mesh points that are strictly included in the computational domain. With this choice, the vertical boundaries contain \( \bigotimes \) and \( \square \) nodes. Since \( E_y \) is defined on \( \bigotimes \) nodes, a boundary condition for this unknown is needed at vertical boundaries.

A difficulty arises in that quantities are computed on different meshes and at different times.

To get an accurate second order accurate discretization in space, the boundary condition is discretized at mesh points carrying \( B_z \) (\( \bigtriangleup \) nodes as depicted in figure 4), as a result \( E_y \) must be averaged in space.

A first order discretization in time reads : 

\[
E_y|_{N-\frac{1}{2}}^{n+1} = 2 c B_z|_{N-\frac{1}{2}}^{n+\frac{1}{2}} - E_y|_{N-1}^{n+1} \tag{37}
\]

Numerical calculations have proven that a centered time of \( E_y \) leads to an accurate second order accurate unstable formulation. The stability can be obtained by discretizing the boundary condition at time \( (n+1) \Delta t \).

\[
E_y|_{N-\frac{1}{2}}^{n+1} = E_y|_{N-1}^{n+1} - c \left( B_z|_{N_\frac{1}{2}}^{n+\frac{1}{2}} + B_z|_{N_{\frac{1}{2}}-\frac{1}{2}}^{n+\frac{1}{2}} \right) - E_y|_{N-1}^{n+1}.
\]

\( B_z|_{N_{\frac{1}{2}}}^{n+\frac{1}{2}} \) is constructed by means of Faraday’s law : 

\[
B_z|_{N_{\frac{1}{2}}}^{n+\frac{1}{2}} = B_z|_{N_{\frac{1}{2}}-\frac{1}{2}}^{n+\frac{1}{2}} - \Delta t \nabla \times E|^{n+1}.
\]

![Figure 4: Boundary discretization](image)

### 5 Longitudinal correction of the electric field

The solution of Ampere’s law fails in general to satisfy Maxwell’s Gauss equation :

\[
\nabla \cdot E|^{n+1} = \frac{\rho|^{n+1}}{\varepsilon_0}.
\]

This non consistent aspect is also observed in explicit codes and can be explained by the dissipation introduced with the CIC scheme. The electronic velocity, used on the grid, to form the predicted current involved in the advancement of the field, is not exactly the same as the one interpolated to particles. This non conservative aspect is corrected by an adjustment of the longitudinal part of the the current density (13) or electric field (13; 11).

The charge density provides information only on the electrostatic part of the electric field, in order not to affect its solenoidal part, \( E^{n+1} \) is a posteriori corrected via the gradient of a scalar potential \( \phi \).

\[
\nabla \cdot (E^{n+1} - \nabla \phi) = \frac{\rho|^{n+1}}{\varepsilon_0}.
\]

This leads to the following elliptic equation for \( \phi \) :

\[
-\Delta \phi = \frac{\rho|^{n+1}}{\varepsilon_0} - \nabla \cdot E^{n+1}.
\]
5.1 “Implicit” correction

In the context of implicit codes, the use of such a correction leads to unstable calculations or to unphysical large electrostatic field (11; 16). This is explain this by the fact that the correction is designed without any information on the plasma response. Indeed, the charge density $\rho^{n+1}$ is not modified back after the field $E^{n+1}$ is corrected.

In order to take this effect into account, we consider the motion of particles due to the field change. Using the continuity equation we obtain:

$$N^{n+1}_n = N^n_n - \Delta t \nabla \cdot \left[ N^{n+1/2}_n (v^{n+1/2}_n + \mu \alpha^{n+1} F_e (B^{n+1/2}) \nabla \phi) \right].$$

(37)

In the equation (37) the expression used for the velocity $v^{n+1/2}$ is the same as that used for the Ampere’s law integration, whereas the electric field includes the $-\nabla \phi$ correction.

Denoting $\Delta N^{n+1}_n = N^n_n - \Delta t \nabla \cdot \left( N^{n+1/2}_n v^{n+1/2}_n \right)$, and using (37), the Poisson equation (36) reads:

$$- \nabla \cdot \left[ I + \frac{\Delta t}{\tau^{n+1/2}_n} \alpha^{n+1} F_e (B^{n+1/2}) \right] \nabla \phi =$$

$$\frac{e}{\varepsilon_0} (N^{n+1}_i - N^{n+1}_n) - \nabla \cdot (E^{n+1}).$$

(38)

Equation (38) makes reference to the matrix $M$ introduced in equation (32). This correction only affects the longitudinal part of the advanced field, since it consists in substituting $M (E^{n+1} - \nabla \phi)$ to $M E^{n+1}$ in equation (31). Note that when the magnetic vanishes, equation (38) reduces to the elliptic equation used in (14) to compute the electrostatic field.

The Poisson equation (38) is supplemented with homogeneous Dirichlet boundary conditions in order not to alter the boundary conditions used for the discretisation of Ampere’s law.

The correction potential $\phi$ is carried out on the grid corners ($\Box$ nodes) in such a way that the corresponding electric field computed by centered differences is naturally defined on $\oplus$ and $\otimes$ nodes (respectively for $E_x$ and $E_y$). Equation (38) is discretized by a five point matrix equation and solved by a preconditioned QMR method (20).

5.2 Averages and oscillations

Simulations have been carried out which are very similar to the configurations appearing in the operating of the plasma erosion opening switch. We considered the simulation of an interface between a quasi-neutral area and a region deserted by electrons. Such a situation can appear near the cathode when a negative potential is applied to it. Electrons are pushed away from the cathode and a non-neutral region is created at the electrode surface. In a few cells, the charge density ranges from zero (in the quasi-neutral area) to its maximum value (depending on the plasma density). In such situations (each time a large gradient in electron density is simulated), the correction has led to the introduction of oscillations altering the physics of the phenomenon. These oscillations propagate from the interface and can be observed on both the electric field and density (see figures 5 and 6); they disappear when the correction is not applied.

Calculations were performed with a time step resolving all modes.

We have assumed this phenomenon to be due to numerical instabilities. To discuss these assumptions, a similar simulation has been carried out on a coarser grid with a mesh size twice as large. The amplitude of the observed oscillations remains constant but one can notice that the wave length is equal to the mesh size (figures 6 and 7). These oscillations are closely related to the grid parameters, and are certainly connected to numerical side effects.

The difficulties stemming from the longitudinal correction are in fact due to the dissipation introduced before assignment and after interpolation phases (see figure 8). As mentioned in section 4.1, all quantities are averaged to grid corners ($\Box$ nodes) before being interpolated and consequently, all quantities weighted to the grid ($\Box$ nodes) are then averaged to the appropriate locations ($\oplus$ and $\otimes$ nodes). As a result, the
electronic density used for the correction is not regularised (the potential is computed on □ nodes) while the density used for the Ampere’s law integration is averaged (through the interpolation from □ nodes to ⊕ and ⊗ nodes). This regularization is not responsible for damage in most calculations but it is so for our applications that deal with large gradient in electron density.

The assignment-interpolation scheme is now derived directly from the node where quantities are calculated or needed. The electronic density is weighted, on □ nodes to compute the electric field correction, and, on ⊗ and ⊕ nodes for Ampere’s law integration. The electric field computed on these nodes is also interpolated directly from these locations to particle positions. This new, more time consuming, scheme suppresses the oscillations discussed above.

As for the ions, the first assignment-interpolation scheme (with averages) is conserved and does not
produce oscillations.

5.3 Space charge limited emission

We consider that when the electrostatic field becomes high enough at the cathode, electrons are extracted from the surface of the electrode in a bipolar space-charged manner. Westermann (24) proposed a numerical implementation of the Child-Langmuir law in the context of diode experiment.

The Maxwell-Gauss equation is discretized on cells adjacent to the cathode. The normal component of the electric field (ie $E_y$) is assumed constant and equal to the threshold necessary for emission to occur.

For each cell the quantity $Q_{cell}$ which is homogeneous to a charge is computed and defined by :

$$Q_{cell} = e(N_i - N_e) - \varepsilon_0 \nabla \cdot E.$$

If $Q_{cell}$ is positive, an electron of charge $-Q_{cell}$ can be emitted in the cell and the Maxwell-Gauss equation is verified. When the electric field at the cathode is below the required value, the charge $Q_{cell}$ computed is likely to be negative, then no electron emission occurs.

6 Demonstrative calculations

Our interest lies in simulating some operating phases of the Plasma Erosion Opening Switch. PEOS are low density switches ( $10^{18} - 5 \times 10^{19}$ m$^{-3}$).

Figure 7: Electron fluid and electric field at time $t = 10.5$ ns with a coarse grid.

Figure 8: Visualization of the regularization introduced before interpolation and after assignment.
The operating of plasma opening switches can be conveniently divided into two parts: conduction and opening. During the conduction phase, a current is transmitted through the switch area via the plasma bridge which short-circuits the electrodes. The ionic component of the current is associated to the plasma injected from the anode toward the cathode. As mentioned above, the electronic component of the transmitted current is provided by space-charge-limited emission at the cathode.

The opening occurs when the electromagnetic field in the switch area is strong enough to alter significantly the current path.

Current conduction occurs through a non-neutral area termed “sheath”. The first simulation is aimed at investigating the formation of this region. The second simulation is aimed at studying a mechanism for both conduction and opening of low density switches.

For all calculations a carbon plasma \((C^+)\) is considered.

These simulations make use of the implicit discretization of both Ampere’s law and electrostatic correction. As a consequence, the plasma parameters are not responsible for limitations on the time step used during the computations, but we have to consider time steps \(\Delta t\) and mesh intervals \(\Delta x\) satisfying a Courant condition due to the propagation of the electromagnetic wave (at the speed of light \(c\)), and resolving the fastest electron motion:

\[
c \Delta t < \Delta x ; \quad v_e \Delta t < \Delta x.
\] (39)

The incident electromagnetic wave used for the following simulations are Transverse Electromagnetic Mode, characterized by their rising time (0.5 ns to 10 ns), amplitude \((10^8 \text{ m/s for the electric component and } 10^8/c \text{ for the magnetic component})\), and profile.

For the following calculations, electric fields are given in \(\text{V/m}\), magnetic field in Tesla, density in \(\text{m}^{-3}\), lengths in \(\text{m}\), time in \(\text{s}\), and velocity in \(\text{m/s}\).

### 6.1 Sheath formation

We consider a plasma of density \(2 \times 10^{19} \text{ m}^{-3}\) with no drift velocity. Initially the plasma situated in the switch area is neutral and the electromagnetic field is set to zero.

At time \(t = 0\) an electromagnetic wave (TEM) is injected into the device. The electric component of the incident wave rises to \(10^8 \text{ V/m}\) in 0.5 ns.

A \(150 \times 60\) mesh is used with four particles per cell. The time step verifies \(\Delta t > 10 \tau_D\). This simulation is aimed at validating stability of the scheme in implicit situations. With the arrival of the pulse, a high voltage is applied to the cathode, and the electronic plasma is drawn away from this electrode (figures 9(a), 9(c)). Carbon ions cannot react in this time scale and a non-neutral area is created at the cathode surface due to the presence of ions in the absence of electrons. The current associated with electron motion acts as a shield to protect the interior plasma from the rising \(B_z\) (figure 9). As time increases, the magnetic field penetrates the switch area along the cathode (figures 9(b), 9(d), 9(f)). This is also observed in (17) when similar simulation parameters are considered.

With the sheath formation, a high electrostatic field is observed in the non quasi-neutral area (figure 10), created by separation charge effects in a manner consistent with the Maxwell-Gauss equation. This high electric field tends to accelerate ions toward the cathode, which may erode the plasma. This account for one of the possible mechanisms responsible for the switch opening.

Note that for long incoming pulses, the transmitted wave has a rising time much shorter than for that of the incident (figure 11).

At the early stages, the transmitted current is seen to be carried by the plasma electrons. But the electric field at the cathode surface becomes rapidly high enough for space-charge-limited emission to occur.

### 6.2 PEOS operating: Conduction and Opening

For this second simulation, space-charge-limited-emission is considered when the electric field at the cathode exceeds a threshold of \(-3 \times 10^7 \text{ V/m}\) (17). We consider an initial plasma with a density of \(10^{18} \text{ m}^{-3}\)
and a $10^6$ m/s drift velocity. A neutral plasma is injected from the anode toward the cathode with the same velocity ($10^6$ m/s).

The magnetic field rises to $-0.5$ Tesla in ten nanoseconds.

The simulation is performed using a $150 \times 80$ grid, with 8 particles per cell for the electrons and 4 for the ions.

The stability of the scheme here is mainly constrained by particle velocity. For these initial conditions, time steps satisfy $\Delta t \approx 0.3 \tau_D$.

The depletion layer observed in the first simulation is formed near the cathode (figure 12(a)). Initially the only current flowing through the sheath is the ionic current associated to the injected plasma.

At early stages, the transmitted current is carried by plasma electrons in the manner depicted in figure 9. The electrostatic field (figure 12(b)) at the electrode surface rapidly becomes high enough for the cathode to become a space-charge-limited emitter.

Figure 9: Evolution of the magnetic field (Tesla) and electronic current module ($C m^{-2} s^{-1}$).
Figure 10: Particle positions and $E_y$ component of the electric field (V/m)

Figure 11: Pulse compression: Incident and outgoing magnetic waves (Tesla).

During the conduction phase, the ionic component of the transmitted current is supplied by plasma injection, and the electronic current flowing through the switch area is due to space-charge-limited emission produced at the cathode by the electric field. The incoming electric field has a nominal value much higher than the threshold required for emission to occur at the cathode surface. Much of the current, therefore, is carried mainly on the source (left) side of the switch. This emitted current prevents the electromagnetic...
field from penetrating the plasma. As time passes, emitted electrons are deflected by the magnetic field, and cross the switch area in a diagonal drift, entering the anode at increasingly longer depths (figure 14). The current flow is finally altered in such a way that electrons can not cross field lines, electrons no longer reach the anode, and the switch no longer transmits any current. This near electrode fast magnetic diffusion is also observed in (23) and analyzed in (18).

Note that during this simulation, particle emission produced by the electric field induces much higher electronic densities than initially, leading to a much smaller dielectric relaxation time than the time step used. Indeed, in some calculation regions, corresponding to the highest electronic current density locations, the time step is five to ten times larger than the dielectric relaxation constant. Relativistic effects have not proven to be of importance in these simulations. Electrons emitted in the sheath have large velocities but for the density ranges considered here, a relativistic description of these particles is not required.

7 Conclusions

We have presented an accurate second order scheme based on a “massless” description for the electronic plasma. The implicitness of the scheme has resulted in calculations that do not resolve both plasma period and dielectric relaxation modes, which can induce severe limitations on explicit schemes time steps.

The extra work represented by the implicitness of the scheme consists mainly in the reconstruction, each time step, of the matrix used to compute the electrostatic potential for the longitudinal correction of the
Figure 14: Electron paths

electric field.

Calculations performed on some phases of Plasma Opening Switches operating prove that the model considered is relevant to deal with these applications, and that a representation of the plasma by a large number of particles per cell is not required.

Moreover, this scheme is a useful tool for investigating high density switches whose opening mechanisms are quite different. These POS are characterized by plasma densities higher than $10^{20}$ m$^{-3}$ and characteristic time scales one or two orders of magnitude larger than the one considered by now. For these applications, ion motion has certainly to be considered.

This model is also suitable for application to strongly magnetized plasma, since the guiding center approximation is assumed.

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