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Minimal mass blow up solutions for a double power nonlinear Schrödinger equation

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Abstract. We consider a nonlinear Schrödinger equation with double power nonlinearity

$$i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u + \epsilon |u|^{p-1} u = 0, \quad \epsilon \in \{-1, 0, 1\}, \quad 1 < p < 1 + \frac{4}{d}$$

in $\mathbb{R}^d$ ($d = 1, 2, 3$). Classical variational arguments ensure that $H^1(\mathbb{R}^d)$ data with $\|u_0\|_2 < \|Q\|_2$ lead to global in time solutions, where Q is the ground state of the mass critical problem ($\epsilon = 0$). We are interested by the threshold dynamic $\|u_0\|_2 = \|Q\|_2$ and in particular by the existence of finite time blow up minimal solutions. For $\epsilon = 0$, such an object exists thanks to the explicit conformal symmetry, and is in fact unique from the seminal work [22]. For $\epsilon = -1$, simple variational arguments ensure that minimal mass data lead to global in time solutions. We investigate in this paper the case $\epsilon = 1$, exhibiting a new class of minimal blow up solutions with blow up rates deeply affected by the double power nonlinearity. The analysis adapts the recent approach [31] for the construction of minimal blow up elements.

1. Introduction

We consider the following double power nonlinear Schrödinger equation in $\mathbb{R}^d$

$$\text{(NLS)} \quad \begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u + \epsilon |u|^{p-1} u = 0, \\ u_{|t=0} = u_0, \end{cases} \qquad 1 < p < 1 + \frac{4}{d}, \quad \epsilon \in \{-1, 0, 1\}.$$

This model corresponds to a subcritical perturbation of the classical mass critical problem $\epsilon = 0$ which rules out the scaling symmetry of the problem. It is well-known (see e.g [6] and the references therein) that for any $u_0 \in H^1(\mathbb{R}^d)$, there exists a unique maximal solution $u \in \mathcal{C}((-T_\star, T^\star), H^1(\mathbb{R}^d)) \cap \mathcal{C}^1((-T_\star, T^\star), H^{-1}(\mathbb{R}^d))$ of

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(NLS). Moreover, the mass (i.e. L^2 norm) and energy E of the solution are conserved by the flow, where:

$$E(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2+\frac{4}{d}}\|u\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} - \epsilon\frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

Moreover, there holds the blow up criterion:

$$T^\star < +\infty \quad \text{implies} \quad \lim_{t\uparrow T^\star}\|\nabla u(t)\|_2 = +\infty. \tag{1.1}$$

In this paper, we are interested in the derivation of a sharp global existence criterion for (NLS) in connection with the existence of *minimal mass blow up solutions* of (NLS).

1.1. The mass critical problem

Let us briefly recall the structure of the mass critical problem $\epsilon = 0$. In this case, the scaling symmetry

$$u_\lambda(t,x) = \lambda^{\frac{d}{2}}u(\lambda^2 t, \lambda x)$$

acts on the set of solutions and leaves the mass invariant

$$\|u_\lambda(t,\cdot)\|_2 = \|u(\lambda^2 t,\cdot)\|_2.$$

From variational arguments [32], the unique ([3, 14]) ground state solution to

$$-\Delta Q + Q - |Q|^{\frac{4}{d}}Q = 0, \quad Q \in H^1(\mathbb{R}^d), \quad Q > 0, \quad Q \text{ radial}$$

attains the best constant in the Gagliardo-Nirenberg inequality

$$\|u\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} \le C\|u\|_2^{\frac{4}{d}}\|\nabla u\|_2^2,$$

so that for all $u \in H^1(\mathbb{R}^d)$ we have

$$E_{\text{crit}}(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2+\frac{4}{d}}\|u\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} \ge \frac{1}{2}\|\nabla u\|_2^2\left[1 - \left(\frac{\|u\|_2}{\|Q\|_2}\right)^{\frac{4}{d}}\right]. \tag{1.2}$$

Together with the conservation of mass and energy and the blow up criterion (1.1), this implies the global existence of all solutions with data $\|u_0\|_2 < \|Q\|_2$. In fact, there holds scattering, see [10] and references therein.

At the threshold $\|u_0\|_2 = \|Q\|_2$, the pseudo-conformal symmetry

$$\frac{1}{|t|^{\frac{d}{2}}}u\left(\frac{1}{t},\frac{x}{t}\right)e^{i\frac{|x|^2}{4t}} \tag{1.3}$$

applied to the solitary wave solution $u(t,x) = Q(x)e^{it}$ yields the existence of the following explicit minimal blow up solution

$$S(t,x) = \frac{1}{|t|^{\frac{d}{2}}}Q\left(\frac{x}{|t|}\right)e^{-i\frac{|x|^2}{4|t|}}e^{\frac{i}{|t|}}, \quad \|S(t)\|_2 = \|Q\|_2, \quad \|\nabla S(t)\|_2 \underset{t\sim 0^-}{\sim} \frac{1}{|t|}. \tag{1.4}$$

From [22], minimal blow up elements are *classified* in $H^1(\mathbb{R}^d)$ in the following sense

$$\|u(t)\|_2 = \|Q\|_2 \quad \text{and} \quad T^* < +\infty \quad \text{imply} \quad u \equiv S$$

up to the symmetries of the flow. Note that the minimal blow up dynamic (1.4) can be extended to the super critical mass case $\|u_0\|_2 > \|Q\|_2$ (see [5]) and that it corresponds to an unstable threshold dynamics between global in time scattering solutions and finite time blow up solutions in the *stable* blow up regime

$$\|\nabla u(t)\|_2 \underset{t \sim T^*}{\sim} \sqrt{\frac{\log|\log|T^* - t||}{T^* - t}}. \tag{1.5}$$

We refer to [26] and references therein for an overview of the existing literature for the L^2 critical blow up problem.

1.2. The case $\epsilon = -1$

Let us now consider the case of a defocusing perturbation. First, there are no solitary waves with subcritical mass $\|u_0\|_2 < \|Q\|_2$ from a standard Pohozaev integration by parts argument. At the threshold, we claim:

Lemma 1.1 (Global existence at threshold for $\epsilon = -1$). *Let $\epsilon = -1$. Let $u_0 \in H^1(\mathbb{R}^d)$ with $\|u_0\|_2 = \|Q\|_2$, then the solution of* (NLS) *is global and bounded in $H^1(\mathbb{R}^d)$.*

The proof follows from standard concentration compactness argument, see Appendix A. The global existence criterion of Lemma 1.1 is sharp in the sense that for all $\alpha^* > 0$, we can build an $H^1(\mathbb{R}^d)$ finite time blow up solution to (1.6) with $\|u_0\|_2 = \|Q\|_2 + \alpha^*$ and blow up speed given by the log-log law (1.5). This is a consequence of the strong structural stability of the log log regime and the proof would follow the lines of [28, 29, 30].

Note that, as stated in the next Lemma (see Appendix A for the proof), the usual virial argument also provides us with blowing up solutions with mass arbitrary close to (but larger than) the critical mass. We have however no further information on the blow-up behavior of these solutions.

Lemma 1.2. *Let $\epsilon = -1$. For any $\delta > 0$, there exists $u_0 \in H^1(\mathbb{R}^d)$ such that $\|u_0\|_2 = \|Q\|_2 + \delta$ and the solution u of* (NLS) *blows up in finite time.*

1.3. The case $\epsilon = 1$

We now turn to the case $\epsilon = 1$ for the rest of the paper, i.e. we consider the model

$$i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u + |u|^{p-1} u = 0 \quad \text{where} \quad 1 < p < 1 + \frac{4}{d}. \tag{1.6}$$

First, from mass and energy conservation, using (1.2) and (B.2), $H^1(\mathbb{R}^d)$ solutions with $\|u_0\|_2 < \|Q\|_2$ are global and bounded in $H^1(\mathbb{R}^d)$. However, large time scattering is not true in general, even for small L^2 solutions, since there exist arbitrarily small solitary waves.

Lemma 1.3 (Small solitary waves). *For all $M \in (0, \|Q\|_2)$, there exists $\omega(M) > 0$ and a Schwartz radially symmetric solution of*

$$\Delta Q_M - \omega(M) Q_M + Q_M^{1+\frac{4}{d}} + Q_M^p = 0, \quad \|Q_M\|_2 = M.$$

The proof follows from classical variational methods, see Appendix B.

The main result of this paper is the existence of a minimal mass blow up solution for (1.6), in contrast with the defocusing case $\epsilon = -1$.

Theorem 1.4 (Existence of a minimal blow up element). *Let $d = 1, 2, 3$ and $1 < p < 1 + \frac{4}{d}$. Then for all energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric Cauchy data $u(t_0) \in H^1(\mathbb{R}^d)$ with*

$$\|u(t_0)\|_2 = \|Q\|_2, \quad E(u(t_0)) = E_0,$$

such that the corresponding solution $u(t)$ of (1.6) blows up at time $T^* = 0$ with speed:

$$\|\nabla u(t)\|_2 = \frac{C(p) + o_{t\uparrow 0}(1)}{|t|^\sigma} \tag{1.7}$$

for some universal constants

$$\sigma = \frac{4}{4 + d(p-1)} \in \left(\tfrac{1}{2}, 1\right), \quad C(p) > 0.$$

Comments on the result.

1. On the existence of minimal elements. Since the pioneering work [22], it has long been believed that the existence of a minimal blow up bubble was related to the exceptional pseudo conformal symmetry (1.3), or at least to the existence of a sufficiently sharp approximation of it, see [2, 16]. However, a new methodology to construct minimal mass elements for a inhomogeneous (NLS) problem, *non perturbative* of critical (NLS), was developed in [31], and later successfully applied to problems without any sort of pseudo conformal symmetry, [4, 12, 20]. More generally, the heart of the matter is to be able to compute the trajectory of the solution on the soliton manifold, see [13, 18] for related problems for two solitary waves motion. The present paper adapts this approach which relies on the *direct* computation of the blow up speed and the control of non dispersive bubbles as in [15].

Observe that the blow up speed (1.7) is quite surprising since it approaches the self similar blow up speed $|t|^{-\frac{1}{2}}$ as $p \to \left(1 + \frac{4}{d}\right)^-$.

2. Uniqueness. A delicate question investigated in [4, 20, 31] is the uniqueness of the minimal blow up element. Such a uniqueness statement should involve Galilean drifts since the Galilean symmetry applied to (1.6) is an L^2 isometry and automatically induces minimal elements with non trivial momentum. Uniqueness issues lie within the general question of classifying the compact elements of the flow in the Kenig-Merle road map [11]. A more limited question is to determine the global behavior of the minimal element for negative time, which is poorly understood in general.

3. Detailed structure of the singular bubble. The analysis provides the following detailed structure of the blow up bubble

$$u(t,x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} Q\left(\frac{x}{\lambda(t)}\right) e^{-i\sigma\frac{|x|^2}{4t}} e^{i\gamma(t)} + v(t,x) \tag{1.8}$$

where Q is the mass critical ground state, and

$$\lim_{t\to 0} \|v(t)\|_2 = 0, \quad \lambda(t) \sim C_p |t|^\sigma \quad \text{as } t \to 0^-,$$

for some constant $C_p > 0$. Note also that the dimension restriction $d \in \{1,2,3\}$ is for the sake of simplicity but not essential.

The construction of the minimal blow up element for (1.4) can be viewed as part of a larger program of understanding what kind of blow up speeds are possible for (NLS) type models. Let us repeat that log-log type solutions with super critical mass can be constructed for (NLS), but then the question becomes: do these examples illustrate all possible blow up types, at least near the ground state profile? The recent series of works [20, 21, 19] for the mass critical gKdV equation indicate that this is a delicate problem, and that the role played by the topology used to measure the perturbation is essential. More generally, symmetry breaking perturbations are very common in nonlinear analysis, and while they are expected to be lower order for generic stable blow up dynamics, our analysis shows that they can dramatically influence the structure of unstable threshold dynamics such as in our case minimal blow up bubbles.

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1.4. Notation

Let us collect the main notation used throughout the paper. For the sake of simplicity, we work in the radial setting only. The L^2 scalar product and L^q norm ($q \geq 1$) are denoted by

$$(u,v)_2 = \mathrm{Re}\left(\int_{\mathbb{R}^d} u(x)\bar{v}(x)dx\right), \quad \|u\|_q = \left(\int_{\mathbb{R}^d} |u|^q dx\right)^{\frac{1}{q}}.$$

We fix the notation:

$$f(z) = |z|^{\frac{4}{d}} z; \quad g(z) = |z|^{p-1} z; \quad F(z) = \frac{1}{\frac{4}{d}+2} |z|^{\frac{4}{d}+2}; \quad G(z) = \frac{1}{p+1} |z|^{p+1}.$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we denote the differential of these functions by df, dg, dF and dG. Let Λ be the generator of L^2-scaling, i.e.

$$\Lambda = \frac{d}{2} + y \cdot \nabla.$$

The linearized operator close to Q comes as a matrix

$$L_+ := -\Delta + 1 - \left(1 + \frac{4}{d}\right) Q^{\frac{4}{d}}, \qquad L_- := -\Delta + 1 - Q^{\frac{4}{d}}.$$

and the generalized kernel of

$$\begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}$$

is non-degenerate and spanned by the symmetries of the problem (see [14, 33] for the original results and [8] for a short proof). It is completely described in $H^1_{\mathrm{rad}}(\mathbb{R}^d)$ by the relations (we define ρ as the unique radial solution to $L_+\rho = |y|^2 Q$)

$$L_- Q = 0, \quad L_+ \Lambda Q = -2Q, \quad L_- |y|^2 Q = -4\Lambda Q, \quad L_+ \rho = |y|^2 Q. \tag{1.9}$$

Denote by $\mathcal{Y}$ the set of radially symmetric functions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\forall \alpha \in \mathbb{N}^d, \quad \exists C_\alpha,\ \kappa_\alpha > 0,\ \forall x \in \mathbb{R}^d, \quad |\partial^\alpha f(x)| \le C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x).$$

Recall that Q and its derivatives are exponentially decaying:

$$\forall x \in \mathbb{R}^d, \quad |\nabla Q(x)| + Q(x) \lesssim e^{-|x|}.$$

It follows from the kernel properties of L_+ and L_-, and from well-known properties of the Helmholtz kernel (see [1] for the properties of Helmholtz kernel (i.e. Bessel and Hankel functions) and [9, Appendix A] or proof of Lemma 3.2 in [27] for related arguments) that

$$\forall g \in \mathcal{Y},\ \exists f_+ \in \mathcal{Y},\ L_+ f_+ = g, \tag{1.10}$$

$$\forall g \in \mathcal{Y},\ (g, Q)_2 = 0,\ \exists f_- \in \mathcal{Y},\ L_- f_- = g. \tag{1.11}$$

It is also well known (see e.g. [23, 24, 31, 34]) that L_+ and L_- verify the following coercivity property: *there exists $\mu > 0$ such that for all $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1_{\mathrm{rad}}(\mathbb{R}^d)$,*

$$\langle L_+ \varepsilon_1, \varepsilon_1 \rangle + \langle L_- \varepsilon_2, \varepsilon_2 \rangle \ge \mu \|\varepsilon\|_{H^1}^2 - \frac{1}{\mu} \left((\varepsilon_1, Q)_2^2 + \left(\varepsilon_1, |y|^2 Q\right)_2^2 + (\varepsilon_2, \rho)_2^2 \right). \tag{1.12}$$

Throughout the paper, C denotes various positive constants whose exact values may vary from line to line but are of no importance in the analysis. When an inequality is true up to such a constant, we also use the notation $\lesssim$, $\gtrsim$ or $\approx$.

2. Construction of the blow-up profile

In this section, we define the blow-up profile which is relevant to construct the minimal mass solution – see Proposition 2.1 below.

2.1. Blow up profile

Let us start with some heuristic arguments justifying the construction. As usual in blow up contexts, we look for a solution of the following form, with rescaled variables (s, y):

$$u(t,x) = \frac{1}{\lambda^{\frac{d}{2}}(s)} w(s,y) e^{i\gamma(s) - i\frac{b(s)|y|^2}{4}}, \qquad \frac{ds}{dt} = \frac{1}{\lambda^2}, \qquad y = \frac{x}{\lambda(s)},$$

where the function w, and the time dependent parameters $\lambda > 0$, b and γ are to be determined satisfying the following equation

$$\begin{aligned} &iw_s + \Delta w - w + f(w) + \lambda^\alpha g(w) \\ &\quad - i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda w + (1-\gamma_s)w + (b_s + b^2)\frac{|y|^2}{4}w - b\left(b + \frac{\lambda_s}{\lambda}\right)\frac{|y|^2}{2}w = 0, \end{aligned} \tag{2.1}$$

where

$$\alpha = 2 - \frac{d(p-1)}{2} \in (0,2).$$

Since we look for blow up solutions, the parameter $\lambda(s)$ should converge to zero as $s \to \infty$. Therefore,

$$w(s,y) = Q(y), \qquad b + \frac{\lambda_s}{\lambda} = b_s + b^2 = 1 - \gamma_s = 0 \tag{2.2}$$

is a solution of (2.1) at the first order, i.e. when neglecting $\lambda^\alpha |w|^{p-1}w$. However, the first order error term $\lambda^\alpha Q^p$ cannot be neglected in the minimal mass blow up analysis (while it could be neglected easily in the log-log regime where $\lambda \sim e^{-e^{\frac{c}{b}}}$). Therefore, starting from Q, we need to look for a refined blow up ansatz. Actually, to close the analysis for any $\alpha \in (0,2)$, we need to remove error terms at any order of λ^α and b in the equation of w. It is important to note that in the process of constructing the approximate solution, we cannot exactly solve (2.1) since we need to introduce new terms in the equation (due to degrees of freedom necessary to construct the ansatz) that will modify the modulation equations in (2.2). These terms (gathered in the time dependent function $\theta(s)$ below) are responsible for the specific blow up law obtained in Theorem 1.4.

Fix $K \in \mathbb{N}$, $K \gg 1$ ($K > 20/\alpha$ is sufficient in the proof of Theorem 1.4), and

$$\Sigma_K = \{(j,k) \in \mathbb{N}^2 \mid j + k \leq K\}.$$

Proposition 2.1. *Let $\lambda(s) > 0$ and $b(s) \in \mathbb{R}$ be $\mathcal{C}^1$ functions of s such that $\lambda(s) + |b(s)| \ll 1$.*
(i) Existence of a blow up profile. *For any $(j,k) \in \Sigma_K$, there exist real-valued functions $P^+_{j,k} \in \mathcal{Y}$, $P^-_{j,k} \in \mathcal{Y}$ and $\beta_{j,k} \in \mathbb{R}$ such that $P(s,y) = \tilde{P}_K(y; b(s), \lambda(s))$, where $\tilde{P}_K$ is defined by*

$$\tilde{P}_K(y; b, \lambda) := Q(y) + \sum_{(j,k)\in\Sigma_K} \left(b^{2j}\lambda^{(k+1)\alpha}P^+_{j,k}(y) + ib^{2j+1}\lambda^{(k+1)\alpha}P^-_{j,k}(y)\right), \tag{2.3}$$

satisfies

$$i\partial_s P + \Delta P - P + f(P) + \lambda^\alpha g(P) + \theta \frac{|y|^2}{4} P = \Psi_K$$

where $\theta(s) = \tilde{\theta}(b(s), \lambda(s))$,

$$\tilde{\theta}(b, \lambda) = \sum_{(j,k)\in\Sigma_K} b^{2j} \lambda^{(k+1)\alpha} \beta_{j,k}$$

and

$$\sup_{y\in\mathbb{R}^d} \left(e^{\frac{|y|}{2}} \left(|\Psi_K(y)| + |\nabla \Psi_K(y)| \right) \right) \lesssim \lambda^\alpha \left(\left| b + \frac{\lambda_s}{\lambda} \right| + \left| b_s + b^2 - \theta \right| \right) + (|b|^2 + \lambda^\alpha)^{K+2}. \tag{2.4}$$

(ii) Rescaled blow up profile. *Let*

$$P_b(s, y) := P(s, y) e^{-i\frac{b(s)|y|^2}{4}}. \tag{2.5}$$

Then

$$\begin{aligned} i\partial_s P_b + \Delta P_b - P_b + f(P_b) + \lambda^\alpha g(P_b) - i\frac{\lambda_s}{\lambda}\Lambda P_b \\ = -i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda P_b + (b_s + b^2 - \theta)\frac{|y|^2}{4} P_b + \Psi_K e^{-i\frac{b|y|^2}{4}}. \end{aligned} \tag{2.6}$$

(iii) Mass and energy properties of the blow up profile. *Let*

$$P_{b,\lambda,\gamma}(s, y) = \frac{1}{\lambda^{\frac{d}{2}}} P_b\left(s, \frac{x}{\lambda}\right) e^{i\gamma}.$$

Then,

$$\left| \frac{d}{ds} \int_{\mathbb{R}^d} |P_{b,\lambda,\gamma}|^2 \right| \lesssim \lambda^\alpha \left(\left| b + \frac{\lambda_s}{\lambda} \right| + \left| b_s + b^2 - \theta \right| \right) + (|b|^2 + \lambda^\alpha)^{K+2}, \tag{2.7}$$

$$\left| \frac{d}{ds} E(P_{b,\lambda,\gamma}) \right| \lesssim \frac{1}{\lambda^2} \left(\left| b + \frac{\lambda_s}{\lambda} \right| + \left| b_s + b^2 - \theta \right| + (|b|^2 + \lambda^\alpha)^{K+2} \right). \tag{2.8}$$

Moreover, for any $(j, k) \in \Sigma_K$, *there exist* $\eta_{j,k} \in \mathbb{R}$ *such that*

$$\left| E(P_{b,\lambda,\gamma}) - \frac{\int_{\mathbb{R}^d} |y|^2 Q^2}{8} \mathcal{E}(b, \lambda) \right| \lesssim \frac{(b^2 + \lambda^\alpha)^{K+2}}{\lambda^2}, \tag{2.9}$$

where

$$\mathcal{E}(b, \lambda) = \frac{b^2}{\lambda^2} - \frac{2\beta}{2-\alpha}\lambda^{\alpha-2} + \lambda^{\alpha-2} \sum_{(j,k)\in\Sigma_K, j+k\geq 1} b^{2j} \lambda^{k\alpha} \eta_{j,k}. \tag{2.10}$$

See a similar construction of a blow up profile at any order of b in [27]. One sees in (2.6) the impact of the subcritical nonlinearity $g(u)$ on the blow up law $b_s + b^2 - \theta = 0$, which differs from the unperturbed equation $b_s + b^2 = 0$, and leads to leading order to $\lambda^\alpha \approx b^2$, see (2.19).

Proof of Proposition 2.1. Proof of (i). For time dependent functions $\lambda(s) > 0$, $b(s)$, we set

$$P = Q + \lambda^\alpha Z \quad \text{where} \quad Z = \sum_{(j,k)\in\Sigma_K} b^{2j}\lambda^{k\alpha}P^+_{j,k} + i\sum_{(j,k)\in\Sigma_K} b^{2j+1}\lambda^{k\alpha}P^-_{j,k},$$

$$\theta(s) = \sum_{(j,k)\in\Sigma_K} b^{2j}(s)\lambda^{(k+1)\alpha}(s)\beta_{j,k},$$

where $P^+_{j,k} \in \mathcal{Y}$, $P^-_{j,k} \in \mathcal{Y}$ and $\beta_{j,k}$ are to be determined. Set

$$\Psi_K = iP_s + \Delta P - P + |P|^{\frac{4}{d}}P + \lambda^\alpha|P|^{p-1}P + \theta\frac{|y|^2}{4}P.$$

The objective is to choose the unknown functions and parameters so that the error term Ψ_K is controlled as in (2.4). First,

$$\begin{aligned}
iP_s &= i\frac{\lambda_s}{\lambda}\sum_{(j,k)\in\Sigma_K}(k+1)\alpha b^{2j}\lambda^{(k+1)\alpha}P^+_{j,k} + ib_s\sum_{(j,k)\in\Sigma_K}2jb^{2j-1}\lambda^{(k+1)\alpha}P^+_{j,k}\\
&- \frac{\lambda_s}{\lambda}\sum_{(j,k)\in\Sigma_K}(k+1)\alpha b^{2j+1}\lambda^{(k+1)\alpha}P^-_{j,k} - b_s\sum_{(j,k)\in\Sigma_K}(2j+1)b^{2j}\lambda^{(k+1)\alpha}P^-_{j,k}\\
&= -i\sum_{(j,k)\in\Sigma_K}(k+1)\alpha b^{2j+1}\lambda^{(k+1)\alpha}P^+_{j,k}\\
&- i\left(b^2 - \sum_{(j',k')\in\Sigma_K} b^{2j'}\lambda^{(k'+1)\alpha}\beta_{j',k'}\right)\sum_{(j,k)\in\Sigma_K}2jb^{2j-1}\lambda^{(k+1)\alpha}P^+_{j,k}\\
&+ \sum_{(j,k)\in\Sigma_K}(k+1)\alpha b^{2(j+1)}\lambda^{(k+1)\alpha}P^-_{j,k}\\
&+ \left(b^2 - \sum_{(j',k')\in\Sigma_K} b^{2j'}\lambda^{(k'+1)\alpha}\beta_{j',k'}\right)\sum_{(j,k)\in\Sigma_K}(2j+1)b^{2j}\lambda^{(k+1)\alpha}P^-_{j,k} + \Psi^{P_s}
\end{aligned}$$

where

$$\begin{aligned}
\Psi^{P_s} &= \left(\frac{\lambda_s}{\lambda} + b\right)\sum_{(j,k)\in\Sigma_K}(k+1)\alpha b^{2j}\lambda^{(k+1)\alpha}\left(iP^+_{j,k} - bP^-_{j,k}\right)\\
&+ \left(b_s + b^2 - \theta\right)\sum_{(j,k)\in\Sigma_K} b^{2j-1}\lambda^{(k+1)\alpha}\left(2jiP^+_{j,k} - (2j+1)bP^-_{j,k}\right).
\end{aligned} \tag{2.11}$$

The purpose of such a decomposition is to express iP_s in terms of powers of b and λ^α (as P itself) plus an error term Ψ^{P_s} depending on the modulation laws $\frac{\lambda_s}{\lambda}+b$ and $b_s+b^2-\theta$ (rather than on $\frac{\lambda_s}{\lambda}$ and b_s) which will be controlled by modulation theory.

We rewrite

$$\begin{aligned} iP_s = &-i\sum_{(j,k)\in\Sigma_K}((k+1)\alpha+2j)\,b^{2j+1}\lambda^{(k+1)\alpha}P^+_{j,k}\\ &+i\sum_{j,k\geq 0}b^{2j+1}\lambda^{(k+1)\alpha}F^{P_s,-}_{j,k}+\sum_{j,k\geq 0}b^{2j}\lambda^{(k+1)\alpha}F^{P_s,+}_{j,k}+\Psi^{P_s},\end{aligned}$$

where for $j,k\geq 0$, $F^{P_s,\pm}_{j,k}$ is a polynomial with real coefficients in the $P^\pm_{j',k'}$ and $\beta_{j',k'}$ for $(j',k')\in\Sigma_K$ such that either $k'\leq k-1$ and $j'\leq j+1$ or $k'\leq k$ and $j'\leq j-1$. Only a finite number of these functions are nonzero.

Next, using $\Delta Q-Q+Q^{\frac4d+1}=0$, we get

$$\begin{aligned}\Delta P-P+|P|^{\frac4d}P=&-\sum_{(j,k)\in\Sigma_K}b^{2j}\lambda^{(k+1)\alpha}L_+P^+_{j,k}-i\sum_{(j,k)\in\Sigma_K}b^{2j+1}\lambda^{(k+1)\alpha}L_-P^-_{j,k}\\ &+f(Q+\lambda^\alpha Z)-f(Q)-\lambda^\alpha df(Q)Z.\end{aligned}$$

Let

$$\Psi^f=f(Q+\lambda^\alpha Z)-\sum_{k=0}^K\frac{1}{k!}d^kf(Q)(\lambda^\alpha Z,\dots,\lambda^\alpha Z),$$

so that for some real coefficients $c_{k,l}$,

$$\begin{aligned}f(Q+\lambda^\alpha Z)&-f(Q)-\lambda^\alpha df(Q)Z\\ &=\sum_{k=2}^K\lambda^{k\alpha}Q^{1+\frac4d-k}\sum_{l=0}^k c_{k,l}Z^l\bar Z^{k-l}+\Psi^f.\end{aligned}\tag{2.12}$$

Thus,

$$\begin{aligned}\Delta P-P+|P|^{\frac4d}P=&-\sum_{(j,k)\in\Sigma_K}b^{2j}\lambda^{(k+1)\alpha}L_+P^+_{j,k}-i\sum_{(j,k)\in\Sigma_K}b^{2j+1}\lambda^{(k+1)\alpha}L_-P^-_{j,k}\\ &+i\sum_{j\geq0,k\geq1}b^{2j+1}\lambda^{(k+1)\alpha}F^{f,-}_{j,k}+\sum_{j,k\geq0}b^{2j}\lambda^{(k+1)\alpha}F^{f,+}_{j,k}+\Psi^f.\end{aligned}$$

where for $j,k\geq 0$, $F^{f,\pm}_{j,k}$ is a polynomial with real coefficients in Q and the $P^\pm_{j',k'}$ for $(j',k')\in\Sigma_K$ such that $k'\leq k-1$ and $j'\leq j$.

Similarly,

$$\lambda^\alpha|P|^{p-1}P=i\sum_{j\geq0,k\geq1}b^{2j+1}\lambda^{(k+1)\alpha}F^{g,-}_{j,k}+\sum_{j\geq0,k\geq1}b^{2j}\lambda^{(k+1)\alpha}F^{g,+}_{j,k}+\Psi^g,$$

where

$$\Psi^g = \lambda^\alpha\left(|Q + \lambda^\alpha Z|^{p-1}(Q + \lambda^\alpha Z) - \sum_{k=0}^{K-1}\frac{1}{k!}d^k g(Q)(\lambda^\alpha Z,\ldots,\lambda^\alpha Z)\right),$$

and where for $j,k \geq 0$, $F^{g,\pm}_{j,k}$ is a polynomial with real coefficients in Q and the $P^\pm_{j',k'}$ for $(j',k') \in \Sigma_K$ such that $k' \leq k-1$ and $j' \leq j$.

Finally, we have

$$\begin{aligned}\theta\frac{|y|^2}{4}P = &\left(\sum_{(j,k)\in\Sigma_K} b^{2j}\lambda^{(k+1)\alpha}\beta_{j,k}\right)\frac{|y|^2}{4}Q \\ &+ i\sum_{j,k\geq 0} b^{2j+1}\lambda^{(k+1)\alpha}F^{\theta,-}_{j,k} + \sum_{j,k\geq 0} b^{2j}\lambda^{(k+1)\alpha}F^{\theta,+}_{j,k},\end{aligned}$$

where $F^{\theta,\pm}_{j,k}$ is a polynomial with real coefficients in Q, the $P^\pm_{j',k'}$ and the $\beta_{j',k'}$ for $(j',k') \in \Sigma_K$ such that $k' \leq k-1$ and $j' \leq j$.

Combining these computations, we obtain

$$\begin{aligned}\Psi_K = &-\sum_{(j,k)\in\Sigma_K} b^{2j}\lambda^{(k+1)\alpha}\left(L_+P^+_{j,k} - F^+_{j,k} - \beta_{j,k}|y|^2Q\right) \\ &- i\sum_{(j,k)\in\Sigma_K} b^{2j+1}\lambda^{(k+1)\alpha}\left(L_-P^-_{j,k} - F^-_{j,k} + ((k+1)\alpha + 2j)P^+_{j,k}\right) \\ &+ \Psi^{>K} + \Psi^{P_s} + \Psi^f + \Psi^g,\end{aligned}$$

where

$$F^\pm_{j,k} = F^{P_s,\pm}_{j,k} + F^{f,\pm}_{j,k} + F^{g,\pm}_{j,k} + F^{\theta,\pm}_{j,k},$$

and

$$\Psi^{>K} = \sum_{j,k>0,\ (j,k)\notin\Sigma_K} b^{2j}\lambda^{(k+1)\alpha}F^+_{j,k} + i\sum_{j,k>0,\ (j,k)\notin\Sigma_K} b^{2j+1}\lambda^{(k+1)\alpha}F^-_{j,k}.$$

Note that the series in the expression of $\Psi^{>K}$ contains only a finite number of terms. Now, for any $(j,k) \in \Sigma_K$, we want to choose recursively $P^\pm_{j,k} \in \mathcal{Y}$ and $\beta_{j,k}$ to solve the system

$$(S_{j,k}) \qquad \left\{\begin{array}{l} L_+P^+_{j,k} - F^+_{j,k} - \beta_{j,k}|y|^2Q = 0 \\ L_-P^-_{j,k} - F^-_{j,k} + ((k+1)\alpha + 2j)P^+_{j,k} = 0,\end{array}\right.$$

where $F^\pm_{j,k}$ are source terms, polynomial with real coefficients in previously determined $P^\pm_{j',k'}$ and $\beta_{j',k'}$. We argue by a suitable induction argument on the two parameters j and k. For $(j,k) = (0,0)$, we see that the system writes

$$\begin{aligned}&L_+P^+_{0,0} - Q^p - \beta_{0,0}|y|^2Q = 0\\ &L_-P^-_{0,0} + \alpha P^+_{0,0} = 0,\end{aligned}$$

(the term Q^p in the first line is coming from Ψ^g). By (1.11), for any $\beta_{0,0} \in \mathbb{R}$, there exists a unique $P^+_{0,0} \in \mathcal{Y}$ so that $L_+P^+_{0,0} - Q^p - \beta_{0,0}|y|^2Q = 0$. We choose $\beta_{0,0} \in \mathbb{R}$ so that

$$\left(P^+_{0,0}, Q\right)_2 = -\frac{1}{2}\left(L_+P^+_{0,0}, \Lambda Q\right)_2 = -\frac{1}{2}\left(Q^p + \beta_{0,0}\frac{|y|^2}{4}Q, \Lambda Q\right)_2 = 0$$

(recall from (1.9) that $L_+\Lambda Q = -2Q$), which gives

$$\beta := \beta_{0,0} = -\frac{4\,(Q^p, \Lambda Q)_2}{(|y|^2Q, \Lambda Q)_2} = \frac{2d(p-1)}{p+1}\frac{\|Q\|^{p+1}_{p+1}}{\|yQ\|^2_2} > 0. \tag{2.13}$$

By (1.11), there exists $P^-_{0,0} \in \mathcal{Y}$ (unique up to the addition of cQ) such that $L_-P^-_{0,0} + \alpha P^+_{0,0} = 0$. Now, we assume that for some $(j_0, k_0) \in \Sigma_K$, the following assertion is true:

$H(j_0, k_0)$: *for all* $(j, k) \in \Sigma_K$ *such that either* $k < k_0$, *or* $k = k_0$ *and* $j < j_0$, *the system* $(S_{j,k})$ *has a solution* $(P^+_{j,k}, P^-_{j,k}, \beta_{j,k})$, $P^\pm_{j,k} \in \mathcal{Y}$.

In view of the definition of $F^\pm_{j_0,k_0}$, $H(j_0, k_0)$ implies in particular that $F^\pm_{j_0,k_0} \in \mathcal{Y}$. We now solve the system (S_{j_0,k_0}) as before. By (1.11), for any $\beta_{j_0,k_0} \in \mathbb{R}$, there exists a unique $P^+_{j_0,k_0} \in \mathcal{Y}$ so that $L_+P^+_{j_0,k_0} - F^+_{j_0,k_0} - \beta_{j_0,k_0}|y|^2Q = 0$. We uniquely choose $\beta_{j_0,k_0} \in \mathbb{R}$ so that

$$\left(-F^-_{j_0,k_0} + ((k_0+1)\alpha + 2j_0)P^+_{j_0,k_0}, Q\right)_2 = 0.$$

By (1.11), there exists $P^-_{j_0,k_0} \in \mathcal{Y}$ (unique up to the addition of cQ) such that $L_-P^-_{j_0,k_0} - F^-_{j_0,k_0} + ((k_0+1)\alpha + 2j_0)P^+_{j_0,k_0} = 0$. In particular, we have proved that if $j_0 < K$, then $H(j_0, k_0)$ implies $H(j_0+1, k_0)$, and $H(K, k_0)$ implies $H(1, k_0+1)$. This is enough to complete an induction argument on the two parameters (j, k). Therefore, system $(S_{j,k})$ is solved for all $(j, k) \in \Sigma_K$.

It remains to estimate Ψ_K and $\nabla\Psi_K$. It is straightforward to check that

$$\sup_{y\in\mathbb{R}^d}\left(e^{\frac{|y|}{2}}\left(|\Psi^{P_s}(y)| + |\nabla\Psi^{P_s}(y)|\right)\right) \lesssim \lambda^\alpha\left(\left|\frac{\lambda_s}{\lambda} + b\right| + \left|b_s + b^2 - \theta\right|\right).$$

Next, we claim

$$|\Psi^f| \lesssim \left(\lambda^{(K+2)\alpha} + \lambda^\alpha b^{2K+2}\right)Q. \tag{2.14}$$

Indeed, first, if y is such that $\left|\lambda^\alpha\frac{Z(y)}{Q(y)}\right| < \frac{1}{2}$ then the result follows from (2.1) and Taylor expansion of order $K+1$. Second, if on the contrary, $\left|\lambda^\alpha\frac{Z(y)}{Q(y)}\right| \geq \frac{1}{2}$, then, since $Z \in \mathcal{Y}$, we have, for such y,

$$Q(y) \leq 2\lambda^\alpha|Z(y)| \lesssim \lambda^\alpha(1+|y|^\kappa)Q(y)$$

and so $|y| \geq c\lambda^{\alpha/\kappa}$ (for some $c > 0$) and

$$Q(y) + |Z(y)| \lesssim e^{-\frac{c}{2}\lambda^{\alpha/\kappa}}.$$

This finishes the proof of (2.14).

The proofs of estimates for $\nabla\Psi_f$, Ψ_g and $\nabla\Psi_g$ are similar. Finally the following estimates for $\Psi^{>K}$ and $\nabla\Psi^{>K}$ are clear:

$$|\Psi^{>K}| + |\nabla\Psi^{>K}| \lesssim \left(\lambda^{(K+2)\alpha} + \lambda^\alpha |b|^{2K+2}\right) Q^{\frac{1}{2}}.$$

The result follows from $K \geq \frac{20}{\alpha}$.

Proof of (ii). This is a straightforward computation which is left to the reader.

Proof of (iii). To prove (2.7), we hit (2.6) with iP_b and compute using the critical relation $(P, \Lambda P)_2 = 0$:

$$\frac{1}{2}\frac{d}{ds}\|P_b\|_2^2 = (i\partial_s P_b, iP_b)_2 = (\Psi_K e^{-i\frac{b|y|^2}{4}}, iP_b)$$

and (2.7) follows from (2.4). For (2.8), we have from scaling:

$$E(P_{b,\lambda,\gamma}) = \frac{1}{\lambda^2}\left(\frac{1}{2}\int_{\mathbb{R}^d} |\nabla P_b|^2 - \int_{\mathbb{R}^d} F(P_b) - \lambda^\alpha \int_{\mathbb{R}^d} G(P_b)\right) =: \frac{1}{\lambda^2}\tilde{E}(\lambda, P_b)$$

Therefore,

(2.15)
$$\frac{d}{ds}E(P_{b,\lambda,\gamma}) = \frac{1}{\lambda^2}\left(-2\frac{\lambda_s}{\lambda}\tilde{E}(\lambda, P_b) + \left\langle \tilde{E}'(\lambda, P_b), \partial_s P_b\right\rangle - \alpha\lambda^\alpha \frac{\lambda_s}{\lambda}\int_{\mathbb{R}^d} G(P_b)\right).$$

Using the equation (2.6) of P_b, we compute:

$$\left\langle \tilde{E}'(\lambda, P_b), \partial_s P_b\right\rangle = \frac{\lambda_s}{\lambda}\left\langle \tilde{E}'(\lambda, P_b), \Lambda P_b\right\rangle - \left(\frac{\lambda_s}{\lambda} + b\right)\left\langle \tilde{E}'(\lambda, P_b), \Lambda P_b\right\rangle \\ + (b_s + b^2 - \theta)\left\langle i\tilde{E}'(\lambda, P_b), \frac{|y|^2}{4}P_b\right\rangle + \left\langle i\tilde{E}'(\lambda, P_b), \Psi_K e^{-i\frac{b|y|^2}{4}}\right\rangle. \tag{2.16}$$

We now integrate by parts to estimate

$$\left\langle \tilde{E}'(\lambda, P_b), \Lambda P_b\right\rangle = \int_{\mathbb{R}^d} |\nabla P_b|^2 - 2\int_{\mathbb{R}^d} F(P_b) - \frac{d(p-1)}{2}\int_{\mathbb{R}^d} G(P_b) \\ = 2\tilde{E}(\lambda, P_b) + \alpha\lambda^\alpha \int_{\mathbb{R}^d} G(P_b), \tag{2.17}$$

where we have used $\alpha = 2 - \frac{d(p-1)}{2}$, from which:

$$\frac{d}{ds}E(P_{b,\lambda,\gamma}) \\ = \frac{1}{\lambda^2}\left[-2\frac{\lambda_s}{\lambda}\tilde{E}(\lambda, P_b) - \alpha\lambda^\alpha\frac{\lambda_s}{\lambda}\int_{\mathbb{R}^d} G(P_b) + \frac{\lambda_s}{\lambda}\left[2\tilde{E}(\lambda, P_b) + \alpha\lambda^\alpha\int_{\mathbb{R}^d} G(P_b)\right]\right] \\ + \frac{1}{\lambda^2}O\left(\left|\frac{\lambda_s}{\lambda} + b\right| + |b_s + b^2 - \theta| + (b^2 + \lambda^\alpha)^{K+2}\right).$$

The estimate (2.8) on the time-derivative of the energy then follows from (2.15), (2.16), (2.17), and (2.4).
Next,

$$\begin{aligned}\lambda^2 E(P_{b,\lambda,\gamma}) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla P_b|^2 - \int_{\mathbb{R}^d} F(P_b) - \lambda^\alpha \int_{\mathbb{R}^d} G(P_b)\\ &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla P|^2 + \frac{b^2}{8}\int_{\mathbb{R}^d} |y|^2|P|^2 - \int_{\mathbb{R}^d} F(P) - \lambda^\alpha \int_{\mathbb{R}^d} G(P).\end{aligned}$$

Thus, replacing $P = Q + \lambda^\alpha Z$,

$$\begin{aligned}\lambda^2 E(P_{b,\lambda,\gamma}) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla Q|^2 - \int_{\mathbb{R}^d} F(Q) + \frac{b^2}{8}\int_{\mathbb{R}^d} |y|^2 Q^2 - \lambda^\alpha \int_{\mathbb{R}^d} G(Q)\\ &+ \lambda^\alpha \int_{\mathbb{R}^d} (-\Delta Q - f(Q))\mathrm{Re}(Z) - \lambda^{2\alpha}\int_{\mathbb{R}^d} g(Q)\mathrm{Re}(Z) + \frac{b^2}{4}\lambda^\alpha \int_{\mathbb{R}^d} |y|^2 Q\mathrm{Re}(Z)\\ &+ \frac{\lambda^{2\alpha}}{2}\int_{\mathbb{R}^d} |\nabla Z|^2 + \frac{b^2\lambda^{2\alpha}}{8}\int_{\mathbb{R}^d} |y|^2|Z|^2 - \int_{\mathbb{R}^d} \{F(Q+\lambda^\alpha Z) - F(Q) - \lambda^\alpha f(Q)\mathrm{Re}(Z)\}\\ &- \lambda^\alpha \int_{\mathbb{R}^d} \{G(Q+\lambda^\alpha Z) - G(Q) - \lambda^\alpha g(Q)\mathrm{Re}(Z)\}.\end{aligned}$$

On the one hand, we recall that from Pohozaev identity,

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla Q|^2 - \int_{\mathbb{R}^d} F(Q) = 0,$$

and from the definition (2.13) of $\beta_{0,0}$,

$$\int_{\mathbb{R}^d} G(Q) = \frac{\beta}{2d(p-1)}\int_{\mathbb{R}^d} |y|^2 Q^2 = \frac{\beta}{4(2-\alpha)}\int_{\mathbb{R}^d} |y|^2 Q^2$$

and moreover

$$\Delta Q + f(Q) = Q.$$

On the other hand, we observe, since $\int_{\mathbb{R}^d} P^+_{0,0} Q = 0$,

$$\lambda^\alpha \int_{\mathbb{R}^d} ZQ = \lambda^\alpha \sum_{(j,k)\in\Sigma_K, j+k\geq 1} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{I}}_{j,k},$$

for some $\eta^{\mathrm{I}}_{j,k} \in \mathbb{R}$;

$$\lambda^{2\alpha}\int_{\mathbb{R}^d} Zg(Q) = \lambda^\alpha \sum_{(j,k)\in\Sigma_K, k\geq 1} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{II}}_{j,k},$$

for some $\eta^{\mathrm{II}}_{j,k} \in \mathbb{R}$;

$$\lambda^\alpha b^2 \int_{\mathbb{R}^d} |y|^2 Q\mathrm{Re}(Z) = \lambda^\alpha \sum_{(j,k)\in\Sigma_K, j\geq 1} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{III}}_{j,k},$$

for some $\eta^{\mathrm{III}}_{j,k} \in \mathbb{R}$;

$$\lambda^{2\alpha}\int_{\mathbb{R}^d}|\nabla Z|^2 + \frac{b^2\lambda^{2\alpha}}{8}\int_{\mathbb{R}^d}|y|^2Z^2 = \lambda^\alpha \sum_{(j,k)\in\Sigma_K, j\geq 1, k\geq 0} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{IV}}_{j,k},$$

for some $\eta^{\mathrm{IV}}_{j,k} \in \mathbb{R}$. Moreover, by Taylor expansion as before, for some $\eta^{\mathrm{V}}_{j,k}, \eta^{\mathrm{IV}}_{j,k} \in \mathbb{R}$

$$\left|\int_{\mathbb{R}^d}\left(F(Q+\lambda^\alpha Z) - F(Q) - \lambda^\alpha f(Q)\mathrm{Re}(Z) - \lambda^\alpha \sum_{(j,k)\in\Sigma_K, k\geq 1} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{V}}_{j,k}\right)\right| \lesssim \lambda^{(K+2)\alpha},$$

$$\left|\lambda^\alpha\int_{\mathbb{R}^d}\left(G(Q+\lambda^\alpha Z) - G(Q) - \lambda^\alpha g(Q)\mathrm{Re}(Z) - \lambda^\alpha \sum_{(j,k)\in\Sigma_K, k\geq 2} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{VI}}_{j,k}\right)\right| \lesssim \lambda^{(K+2)\alpha}.$$

Gathering these computations, we obtain (2.9). □

2.2. Approximate blow up law

For simplicity of notation, we set

$$\beta = \beta_{0,0} = \frac{2d(p-1)}{p+1}\frac{\|Q\|^{p+1}_{p+1}}{\|yQ\|_2^2}.$$

First, we find a relevant solution to the following approximate system

$$b_s + b^2 - \beta\lambda^\alpha = 0, \quad b + \frac{\lambda_s}{\lambda} = 0. \tag{2.18}$$

Indeed, for $|b| + \lambda \ll 1$, $\beta\lambda^\alpha$ is the main term in θ, and the only term in θ that will modify at the main order the blow up rate.

Lemma 2.2. *Let*

$$\lambda_{\mathrm{app}}(s) = \left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{2}{\alpha}} s^{-\frac{2}{\alpha}}, \quad b_{\mathrm{app}}(s) = \frac{2}{\alpha s}. \tag{2.19}$$

Then $(\lambda_{\mathrm{app}}(s), b_{\mathrm{app}}(s))$ *solves* (2.18) *for* $s > 0$.

Proof. We compute:

$$\left(\frac{b^2}{\lambda^2}\right)_s = 2\frac{b}{\lambda}\frac{b_s + b^2}{\lambda} = -2\beta\frac{\lambda_s}{\lambda}\lambda^{\alpha-2},$$

and so

$$\frac{b^2}{\lambda^2} - \frac{2\beta}{2-\alpha}\lambda^{\alpha-2} = c_0. \tag{2.20}$$

Taking the constant $c_0 = 0$, and using $b = -\frac{\lambda_s}{\lambda} > 0$, we find

$$\frac{\lambda_s}{\lambda^{1+\frac{\alpha}{2}}} = \sqrt{\frac{2\beta}{2-\alpha}}.$$

Therefore,

$$\lambda(s) = \left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{2}{\alpha}} s^{-\frac{2}{\alpha}}, \quad b(s) = -\frac{\lambda_s}{\lambda}(s) = \frac{2}{\alpha}\frac{1}{s}$$

is solution of (2.18). □

Remark 2.3. We now express this solution in the time variable t_{app} related to λ_{app}. Let

$$dt_{\text{app}} = \lambda_{\text{app}}^2 ds = \left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{4}{\alpha}} s^{-\frac{4}{\alpha}} ds.$$

Therefore (with the convention that $t_{\text{app}} \to 0^-$ as $s \to +\infty$)

$$t_{\text{app}} = -C_s s^{-\frac{4-\alpha}{\alpha}} \quad \text{where} \quad C_s = \frac{\alpha}{4-\alpha}\left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{4}{\alpha}}. \tag{2.21}$$

As a consequence, we obtain for $t_{\text{app}} < 0$,

$$\lambda_{\text{app}}(t_{\text{app}}) = C_\lambda |t_{\text{app}}|^{\frac{2}{4-\alpha}} \quad \text{where} \quad C_\lambda = \left(\frac{4-\alpha}{\alpha} C_s^{-\frac{\alpha}{4-\alpha}}\right)^{\frac{1}{2}}, \tag{2.22}$$

$$b_{\text{app}}(t_{\text{app}}) = C_b |t_{\text{app}}|^{\frac{\alpha}{4-\alpha}}, \quad \text{where} \quad C_b = \frac{2}{\alpha} C_s^{-\frac{\alpha}{4-\alpha}}. \tag{2.23}$$

Now, we choose suitable initial conditions b_1 and λ_1 for $b(s)$ and $\lambda(s)$ at some large time s_1, first to adjust the value of the energy of $P_{b,\lambda,\gamma}$ (up to the small error term in (2.9)) and second to be able to close the perturbed dynamical system of (λ, b) at the end of the proof (see proof of Lemma 6.1 below). Let $E_0 \in \mathbb{R}$ and

$$C_0 = \frac{8E_0}{\int_{\mathbb{R}^d} |y|^2 Q^2}.$$

Fix $0 < \lambda_0 \ll 1$ such that $\frac{2\beta}{2-\alpha} + C_0 \lambda_0^{2-\alpha} > 0$. For $\lambda \in (0, \lambda_0]$, let

$$\mathcal{F}(\lambda) = \int_\lambda^{\lambda_0} \frac{d\mu}{\mu^{\frac{\alpha}{2}+1}\sqrt{\frac{2\beta}{2-\alpha} + C_0 \mu^{2-\alpha}}}. \tag{2.24}$$

Note that the function $\mathcal{F}$ is related to the resolution of the system (2.20) for $c_0 = C_0$, see proof of Lemma 6.1.

Lemma 2.4. *Let $s_1 \gg 1$. There exist b_1 and λ_1 such that*

$$\left|\frac{\lambda_1^{\frac{\alpha}{2}}}{\lambda_{\mathrm{app}}^{\frac{\alpha}{2}}(s_1)} - 1\right| + \left|\frac{b_1}{b_{\mathrm{app}}(s_1)} - 1\right| \lesssim s_1^{-\frac{1}{2}} + s_1^{2-\frac{4}{\alpha}}, \tag{2.25}$$

$$\mathcal{F}(\lambda_1) = s_1, \quad \mathcal{E}(b_1, \lambda_1) = C_0. \tag{2.26}$$

Proof. First, we choose λ_1. Note that $\mathcal{F}$ is a decreasing function of λ satisfying $\mathcal{F}(\lambda_0) = 0$ and $\lim_{\lambda \downarrow 0} \mathcal{F}(\lambda) = +\infty$. Thus, there exists a unique $\lambda_1 \in (0, \lambda_0)$ such that $\mathcal{F}(\lambda_1) = s_1$.

For $\lambda \in (0, \lambda_0]$,

$$\begin{aligned}\left|\mathcal{F}(\lambda) - \frac{2}{\alpha\sqrt{\frac{2\beta}{2-\alpha}}\lambda^{\frac{\alpha}{2}}}\right| &\lesssim 1 + \left|\int_\lambda^{\lambda_0} \frac{d\mu}{\mu^{\frac{\alpha}{2}+1}}\left[\frac{1}{\sqrt{\frac{2\beta}{2-\alpha} + C_0\mu^{2-\alpha}}} - \frac{1}{\sqrt{\frac{2\beta}{2-\alpha}}}\right]\right| \\ &\lesssim 1 + \int_\lambda^{\lambda_0} \frac{d\mu}{\mu^{1+\frac{\alpha}{2}-(2-\alpha)}}.\end{aligned}$$

Thus,

$$\left|\mathcal{F}(\lambda) - \frac{2}{\alpha\sqrt{\frac{2\beta}{2-\alpha}}\lambda^{\frac{\alpha}{2}}}\right| \lesssim \begin{cases} 1 & \text{for } \alpha \in (0, \frac{4}{3}), \\ |\log \lambda| & \text{for } \alpha = \frac{4}{3}, \\ \lambda^{2-\frac{3\alpha}{2}} & \text{for } \alpha \in (\frac{4}{3}, 2). \end{cases}$$

To simplify, we will use the non sharp but sufficient estimate

$$\left|\mathcal{F}(\lambda) - \frac{2}{\alpha\sqrt{\frac{2\beta}{2-\alpha}}\lambda^{\frac{\alpha}{2}}}\right| \lesssim \lambda^{-\frac{\alpha}{4}} + \lambda^{2-\frac{3\alpha}{2}}. \tag{2.27}$$

Applied to λ_1, it gives

$$\left|s_1 - \frac{2}{\alpha\sqrt{\frac{2\beta}{2-\alpha}}\lambda_1^{\frac{\alpha}{2}}}\right| \lesssim \lambda_1^{-\frac{\alpha}{4}} + \lambda_1^{2-\frac{3\alpha}{2}} \quad \text{and thus} \quad \left|\frac{\lambda_1^{\frac{\alpha}{2}}}{\lambda_{\mathrm{app}}^{\frac{\alpha}{2}}(s_1)} - 1\right| \lesssim s_1^{-\frac{1}{2}} + s_1^{2-\frac{4}{\alpha}}.$$

Second, we choose b_1. From the definition of $\mathcal{E}$, we have

$$\begin{aligned} h(b) &:= \lambda_1^2 \mathcal{E}(b, \lambda_1) \\ &= b^2 - \left(\frac{2}{\alpha s_1}\right)^2 - \frac{2\beta}{2-\alpha}\left(\lambda_1^\alpha - \lambda_{\mathrm{app}}^\alpha(s_1)\right) + \lambda_1^\alpha \sum_{(j,k)\in\Sigma_K,\ j+k\geq 1} b^{2j}\lambda_1^{-k\alpha}\eta_{j,k} \\ &= b^2 - \left(\frac{2}{\alpha s_1}\right)^2 + O(s_1^{-\frac{5}{2}}) + O(s_1^{-\frac{4}{\alpha}}). \end{aligned}$$

Observe that

$$|h(b_{\mathrm{app}}(s_1))| \lesssim s_1^{-\frac{4}{\alpha}}, \quad |h'(b_{\mathrm{app}}(s_1))| \geq 2b_{\mathrm{app}}(s_1) + O(s_1^{-3}) \geq s_1^{-1}.$$

Since $\lambda_1^2 \approx s_1^{-\frac{4}{\alpha}}$, it follows that there exists a unique b_1 such that

$$|b_1 - b_{\mathrm{app}}(s_1)| \lesssim s_1^{-\frac{3}{2}} + s_1^{1-\frac{4}{\alpha}}, \quad h(b_1) = C_0\lambda_1^2,$$

and so $\mathcal{E}(b_1, \lambda_1) = C_0$. □

3. Existence proof assuming uniform estimates

This section is devoted to the proof of Theorem 1.4 by a compactness argument, assuming uniform estimates on specific solutions of (1.6). These estimates are given in Proposition 3.2.

3.1. Uniform estimates in rescaled time variable

The rescaled time depending on a suitable modulation of the solution $u(t)$, we first recall without proof the following standard result (see e.g. [24]).

Lemma 3.1 (Modulation). *Let $u(t) \in \mathcal{C}(I, H^1(\mathbb{R}^d))$ for some interval I, be such that*

$$\sup_{t\in I} \inf_{\lambda_0>0,\gamma_0} \left\| \lambda_0^{\frac{d}{2}} u(t, \lambda_0 y) e^{i\gamma_0} - Q(y) \right\|_{H^1} \le \delta, \tag{3.1}$$

for $\delta > 0$ small enough. Then, there exist $\mathcal{C}^1$ functions $\lambda \in (0, +\infty)$, $b \in \mathbb{R}$, $\gamma \in \mathbb{R}$ on I such that u admits a unique decomposition of the form

$$u(t, x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} \left(P_{b(t)} + \varepsilon(t, y)\right) e^{i\gamma(t)}, \qquad y = \frac{x}{\lambda(t)} \tag{3.2}$$

On I, ε satisfies the following orthogonality conditions

$$(\varepsilon, i\Lambda P_b)_2 = \left(\varepsilon, |y|^2 P_b\right)_2 = (\varepsilon, i\rho_b)_2 = 0, \tag{3.3}$$

where $\rho_b(t, y) = \rho(y) e^{-i\frac{b(t)|y|^2}{4}}$.

Recall that P_b was defined in (2.5).

Let $E_0 \in \mathbb{R}$. Given $t_1 < 0$ close to 0, following Remark 2.3, we define the initial rescaled time s_1 as

$$s_1 := \left|C_s^{-1} t_1\right|^{-\frac{\alpha}{4-\alpha}}.$$

Let λ_1 and b_1 be given by Lemma 2.4 for this value of s_1. Let $u(t)$ be the solution of (1.6) for $t \le t_1$, with data

$$u(t_1, x) = \frac{1}{\lambda_1^{\frac{d}{2}}} P_{b_1}\left(\frac{x}{\lambda_1}\right). \tag{3.4}$$

As long as the solution $u(t)$ satisfies (3.1), we consider its decomposition $(\lambda, b, \gamma, \varepsilon)$ from Lemma 3.1 and we define the rescaled time s by

$$s = s_1 - \int_t^{t_1} \frac{1}{\lambda^2(\tau)} d\tau. \tag{3.5}$$

The heart of the proof of Theorem 1.4 is the following result, giving uniform backwards estimates on the decomposition of $u(s)$ on $[s_0, s_1]$ for some s_0 independent of s_1.

Proposition 3.2 (Uniform estimates in rescaled time). *There exists $s_0 > 0$ independent of s_1 such that the solution u of (1.6) defined by (3.4) exists and satisfies (3.1) on $[s_0, s_1]$. Moreover, its decomposition*

$$u(s,x) = \frac{1}{\lambda^{\frac{d}{2}}(s)} (P_b + \varepsilon)(s,y)\, e^{i\gamma(s)}, \qquad y = \frac{x}{\lambda(s)},$$

satisfies the following uniform estimates on $[s_0, s_1]$,

$$\|\varepsilon(s)\|_{H^1} \lesssim s^{-(K+1)}, \qquad \left|\frac{\lambda^{\frac{\alpha}{2}}(s)}{\lambda_{\text{app}}^{\frac{\alpha}{2}}(s)} - 1\right| + \left|\frac{b(s)}{b_{\text{app}}(s)} - 1\right| \lesssim s^{-\frac{1}{2}} + s^{2-\frac{4}{\alpha}}. \tag{3.6}$$

In addition,

$$|E(P_{b,\lambda,\gamma}(s)) - E_0| = O(s^{-6}).$$

Let us insist again that the key point in Proposition 3.2 is that s_0 and the constants in the estimates are independent of $s_1 \to +\infty$.

3.2. Proof of Theorem 1.4 assuming Proposition 3.2

First, we convert the estimates of Proposition 3.2 in the original time variable t. We claim:

Lemma 3.3 (Estimates in the t variable). *There exists $t_0 < 0$ such that under the assumptions of Proposition 3.2, for all $t \in [t_0, t_1]$,*

$$b(t) = C_b |t|^{\frac{\alpha}{4-\alpha}} (1 + o_{t\uparrow 0}(1)), \quad \lambda(t) = C_\lambda |t|^{\frac{2}{4-\alpha}} (1 + o_{t\uparrow 0}(1)) \tag{3.7}$$

$$\|\varepsilon(t)\|_{H^1} \lesssim |t|^{\frac{(K+1)\alpha}{4-\alpha}} \tag{3.8}$$

$$|E(P_{b,\lambda,\gamma}(t)) - E_0| = o_{t\uparrow 0}(1) \tag{3.9}$$

Proof of Lemma 3.3. Using (3.6), (3.5), for all large $s < s_1$,

$$t_1 - t(s) = \int_s^{s_1} \lambda^2(\sigma) d\sigma = \int_s^{s_1} \lambda_{\text{app}}^2(\sigma) \left[1 + O(\sigma^{-\frac{1}{2}}) + O(\sigma^{2-\frac{4}{\alpha}})\right] d\sigma.$$

Recall that t_{app} given by (2.21) corresponds to the normalization

$$t_{\text{app}}(s) = -\int_s^{+\infty} \lambda_{\text{app}}^2(\sigma), \quad t_{\text{app}}(s_1) = t_1,$$

from which we obtain

$$t(s) = t_{\rm app}(s)(1+o(1)) = -C_s s^{-\frac{4-\alpha}{\alpha}}\left[1+o(1)\right].$$

The estimates of Lemma 3.3 now follow directly follow from (2.19) and Proposition 3.2 (see the definition of C_λ and C_b in (2.22) and (2.23)). □

Now, we finish the proof of Theorem 1.4 assuming Proposition 3.2.

Proof of Theorem 1.4. Let $(t_n) \subset (t_0, 0)$ be an increasing sequence of time such that $t_n \to 0$ as $n \to +\infty$. For each n, let u_n be the solution of (1.6) on $[t_0, t_n]$ with final data at t_n

$$u_n(t_n, x) = \frac{1}{\lambda^{\frac{d}{2}}(t_n)} P_{b(t_n)}\left(\frac{x}{\lambda(t_n)}\right), \tag{3.10}$$

where $\lambda(t_n) = \lambda_1$ and $b(t_n) = b_1$ are given by Lemma 2.4 for $s_1 = |C_s^{-1} t_n|^{-\frac{\alpha}{4-\alpha}}$, so that $u_n(t)$ satisfies the conclusions of Proposition 3.2 and of Lemma 3.3 on the interval $[t_0, t_n]$. The minimal mass blow up solution for (1.6) is now obtained as the limit of a subsequence of (u_n). In a first step, we prove that a subsequence of $(u_n(t_0))$ converges to a suitable initial data. Indeed, from Lemma 3.3, we infer that $(u_n(t_0))$ is bounded in $H^1(\mathbb{R}^d)$. Hence there exists a subsequence of $(u_n(t_0))$ (still denoted by $(u_n(t_0))$ and $u_\infty(t_0) \in H^1(\mathbb{R}^d)$ such that

$$u_n(t_0) \rightharpoonup u_\infty(t_0) \quad \text{weakly in } H^1(\mathbb{R}^d) \text{ as } n \to +\infty.$$

Now, we obtain strong convergence in H^s (for some $0 < s < 1$) by direct arguments. Let $\chi : [0, +\infty) \to [0, 1]$ be a smooth cut-off function such that $\chi \equiv 0$ on $[0, 1]$ and $\chi \equiv 1$ on $[2, +\infty)$. For $R > 0$, define $\chi_R : \mathbb{R}^d \to [0, 1]$ by $\chi_R(x) = \chi(|x|/R)$. Take any $\delta > 0$. By the expression of $u_n(t_n)$ in (3.10), we can choose R large enough (independent of n) so that

$$\int_{\mathbb{R}^d} |u_n(t_n)|^2 \chi_R dx \le \delta. \tag{3.11}$$

It follows from elementary computations that

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u_n|^2 \chi_R dx = 2\,\mathrm{Im}\int_{\mathbb{R}^d} \nabla\chi_R \cdot \nabla u_n\, \bar{u}_n dx.$$

Hence from the geometrical decomposition

$$u_n(t, x) = \frac{1}{\lambda_n^{\frac{d}{2}}(t)}\left(P_{b_n(t)} + \varepsilon_n)(t, y)\right) e^{i\gamma_n(t)}, \qquad y = \frac{x}{\lambda_n(t)},$$

and the smallness (3.7)-(3.8) of ε_n and λ_n we infer

$$\left|\frac{d}{dt}\int_{\mathbb{R}^d} |u_n(t)|^2 \chi_R dx\right| \le \frac{C}{\lambda_n(t) R}\left(e^{-\frac{R}{2\lambda_n(t)}} + \|\varepsilon_n(t)\|_{H^1}^2\right) \le \frac{C}{R}|t|^{\left(-\frac{2}{\alpha}+K+1\right)\frac{\alpha}{4-\alpha}}.$$

Integrating between t_0 and t_n, we obtain

$$\int_{\mathbb{R}^d} |u_n(t_0)|^2 \chi_R dx \leq \frac{C}{R}|t_0|^{\left(-\frac{2}{\alpha}+K+1\right)\frac{\alpha}{4-\alpha}+1} + \int_{\mathbb{R}^d} |u_n(t_n)|^2 \chi_R dx.$$

Combined with (3.11), for a possibly larger R, this implies

$$\int_{\mathbb{R}^d} |u_n(t_0)|^2 \chi_R dx \leq 2\delta.$$

We conclude from the local compactness of Sobolev embeddings that for $0 \leq s < 1$:

$$u_n(t_0) \to u_\infty(t_0) \quad \text{strongly in } H^s(\mathbb{R}^d), \quad \text{as } n \to +\infty.$$

Let $u_\infty(t)$ be the solution of (1.6) with $u_\infty(t_0)$ as initial data at $t = t_0$. From [6, 7] there exists $0 < s_0 < 1$ such that the Cauchy problem for (1.6) is locally well-posed in $H^{s_0}(\mathbb{R}^d)$. This implies that u_∞ exists on $[t_0, 0)$ and for any $t \in [t_0, 0)$,

$$u_n(t) \to u_\infty(t) \quad \text{strongly in } H^{s_0}(\mathbb{R}^d), \;\; \text{weakly in } H^1(\mathbb{R}^d), \;\; \text{as } n \to +\infty.$$

Moreover, since $\lim_{n\to\infty} \int_{\mathbb{R}^d} u_n^2(t_n) = \int_{\mathbb{R}^d} Q^2$, we have $\int_{\mathbb{R}^d} u_\infty^2 = \int_{\mathbb{R}^d} Q^2$. By weak convergence in $H^1(\mathbb{R}^d)$ and the estimates from Lemma 3.3 applied to u_n, $u_\infty(t)$ satisfies (3.1), and denoting $(\varepsilon_\infty, \lambda_\infty, b_\infty, \gamma_\infty)$ its decomposition, we have by standard arguments (see e.g. [24]), for any $t \in [t_0, 0)$, and as $n \to \infty$,

$$\lambda_n(t) \to \lambda_\infty(t), \quad b_n(t) \to b_\infty(t), \quad \gamma_n(t) \to \gamma_\infty(t), \quad \varepsilon_n(t) \rightharpoonup \varepsilon_\infty(t) \quad H^1\text{-weak}.$$

The uniform estimates on u_n from Lemma 3.3 give, on $[t_0, 0)$,

$$b_\infty(t) = C_b|t|^{\frac{\alpha}{4-\alpha}}\left(1 + o_{t\uparrow 0}(1)\right), \quad \lambda_\infty(t) = C_\lambda|t|^{\frac{2}{4-\alpha}}\left(1 + o_{t\uparrow 0}(1)\right), \tag{3.12}$$

$$\|\varepsilon_\infty(t)\|_{H^1} \lesssim |t|^{\frac{(K+1)\alpha}{4-\alpha}}, \tag{3.13}$$

and therefore we have

$$\begin{aligned} \frac{b_\infty(t)}{\lambda_\infty^2(t)} &= \frac{C_b}{C_\lambda^2}|t|^{\frac{\alpha}{4-\alpha}-\frac{4}{4-\alpha}}\left(1 + o_{t\uparrow 0}(1)\right) \\ &= \frac{2}{4-\alpha}\frac{1}{|t|}\left(1 + o_{t\uparrow 0}(1)\right) = \frac{\sigma}{|t|}\left(1 + o_{t\uparrow 0}(1)\right), \end{aligned} \tag{3.14}$$

This justifies the form (1.8) and the blow up rate (1.7). Finally, we prove that $E(u_\infty) = E_0$. Let $t_0 < t < 0$. We have by (3.9) and (2.9),

$$\mathcal{E}(b_n(t), \lambda_n(t)) - \frac{8E_0}{\int_{\mathbb{R}^d} |y|^2 Q^2} = o_{t\uparrow 0}(1)$$

where the $o_{t\uparrow 0}(1)$ is independent of n, and thus

$$\mathcal{E}(b_\infty(t), \lambda_\infty(t)) - \frac{8E_0}{\int_{\mathbb{R}^d} |y|^2 Q^2} = o_{t\uparrow 0}(1)$$

Using (2.9), we deduce

$$E(P_{b_\infty,\lambda_\infty,\gamma_\infty}(t)) - E_0 = o_{t\uparrow 0}(1)$$

and thus, by (3.12)-(3.13),

$$E(u_\infty(t)) - E_0 = o_{t\uparrow 0}(1).$$

Thus, by conservation of energy and passing to the limit $t \uparrow 0$, we obtain

$$E(u_\infty(t)) = E_0.$$

This concludes the proof. □

3.3. Bootstrap estimates

The rest of the paper is devoted to the proof of Proposition 3.2. We use a bootstrap argument involving the following estimates:

$$\|\varepsilon(s)\|_{H^1} < s^{-K}, \quad \left|\frac{\lambda^{\frac{\alpha}{2}}(s)}{\lambda_{\mathrm{app}}^{\frac{\alpha}{2}}(s)} - 1\right| + \left|\frac{b(s)}{b_{\mathrm{app}}(s)} - 1\right| < s^{-\delta(\alpha)} \tag{3.15}$$

for some small enough universal constant $\delta(\alpha) > 0$. The following value is suitable in this paper

$$\delta(\alpha) = \min\left(\frac{1}{4}, \frac{2}{\alpha} - 1\right) > 0. \tag{3.16}$$

For $s_0 > 0$ to be chosen large enough (independently of s_1), we define

$$s_* = \inf\{\tau \in [s_0, s_1]; (3.15) \text{ holds on } [\tau, s_1]\}. \tag{3.17}$$

Observe from (2.25) that

$$\left|\frac{\lambda_1^{\frac{\alpha}{2}}}{\lambda_{\mathrm{app}}^{\frac{\alpha}{2}}(s_1)} - 1\right| + \left|\frac{b_1}{b_{\mathrm{app}}(s_1)} - 1\right| \lesssim s_1^{-\frac{1}{2}} + s_1^{2-\frac{4}{\alpha}} \ll s_1^{-\delta(\alpha)},$$

for s_1 large, and hence by the definition (3.4) of $u(s_1)$, s_* is well-defined and $s_* < s_1$. In §5, §6 and §7, we prove that (3.6) holds on $[s_*, s_1]$. By a standard continuity argument, provided that s_0 is large enough, we obtain $s_* = s_0$ which implies Proposition 3.2. The main lines of the proof are as follows: first, we derive modulation equations from the construction of P_b, second we control the remaining error using a mixed Energy/Morawetz functional first derived in [31].

4. Modulation equations

In this section, we work with the solution $u(t)$ of Proposition 3.2 on the time interval $[s_*, s_1]$ (see (3.15)-(3.17)). We justify that the dynamical system satisfied by the modulation parameters λ, b is at the main order given by (2.18). Define

$$\mathrm{Mod}(s) = \begin{pmatrix} b + \frac{\lambda_s}{\lambda} \\ b_s + b^2 - \theta \\ 1 - \gamma_s \end{pmatrix}.$$

Lemma 4.1 (Modulation, additional orthogonality, error estimate). *For all* $s \in [s_*, s_1]$,

$$|\,\mathrm{Mod}(s)| \lesssim s^{-(K+2)}, \tag{4.1}$$

$$|\,(\varepsilon(s), Q)_2\,| \lesssim s^{-(K+1)}, \tag{4.2}$$

$$\sup_{y\in\mathbb{R}^d} \left(e^{\frac{|y|}{2}} (|\psi_K| + |\nabla\psi_K|) \right) \lesssim s^{-(K+4)}. \tag{4.3}$$

Proof of Lemma 4.1. The proofs of the two estimates are combined. Since $\varepsilon(s_1) \equiv 0$, we may define

$$s_{**} = \inf\{s \in [s_*, s_1];\ |\,(\varepsilon(\tau), P_b)_2\,| < \tau^{-(K+2)} \text{ holds on } [s, s_1]\}.$$

We work on the interval $[s_{**}, s_1]$.

Since P_b verifies equation (2.6), we obtain the following equation for ε:

$$\begin{aligned} &i\varepsilon_s + \Delta\varepsilon - \varepsilon + ib\Lambda\varepsilon + (f(P_b+\varepsilon) - f(P_b)) + \lambda^\alpha(g(P_b+\varepsilon) - g(P_b)) \\ &\quad - i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda(P_b+\varepsilon) + (1-\gamma_s)(P_b+\varepsilon) + (b_s + b^2 - \theta)\frac{|y|^2}{4}P_b \\ &\qquad = -\Psi e^{-i\frac{b|y|^2}{4}}. \end{aligned} \tag{4.4}$$

where $\Psi := \Psi_K$. Recall that equation (4.4) combined with the orthogonality conditions chosen on ε – see (3.3) – contains the equations of the modulation parameters. Technically, one differentiates in time the orthogonality conditions for ε, then uses the equation (4.4) on ε and the estimate (2.4) on the error term Ψ. Here, as in [31], the orthogonality conditions are chosen to obtain quadratic control in ε. Since it is a standard argument (see e.g. [25, 28, 31]), we only sketch relevant computations.

Consider for example the orthogonality condition $(\varepsilon, i\Lambda P_b)_2 = 0$. Differentiating in s, we obtain $\langle\varepsilon_s, i\Lambda P_b\rangle + \langle\varepsilon, i\partial_s(\Lambda P_b)\rangle = 0$. Since

$$\frac{d}{ds}(\Lambda P_b) = \left((\Lambda P)_s - i\frac{b_s}{4}|y|^2\Lambda P\right)e^{-i\frac{b}{4}|y|^2},$$

and

$$\begin{aligned} (\Lambda P)_s = \lambda^\alpha\Bigg(&\alpha\frac{\lambda_s}{\lambda}\Bigg(Z + \sum_{(j,k)\in\Sigma_K} kb^{2j}\lambda^{k\alpha-1}(P^+_{j,k} + bP^-_{j,k})\Bigg) \\ &+ b_s\Bigg(\sum_{(j,k)\in\Sigma_K} 2jb^{2j-1}\lambda^{k\alpha}P^+_{j,k} + \sum_{(j,k)\in\Sigma_K}(2j+1)b^{2j}\lambda^{k\alpha}P^-_{j,k}\Bigg)\Bigg), \end{aligned}$$

proceeding as in the proof of Proposition 2.1, and using the properties of the functions $P^{\pm}_{j,k}$, we note that

$$\sup_{y\in\mathbb{R}}\left(e^{\frac{y}{2}}\left|\frac{d}{ds}(\Lambda P_b)(y)\right|\right)\lesssim|\operatorname{Mod}(s)|+b^2(s)+\lambda^\alpha(s).$$

Thus, by (3.15),

$$|\left(\varepsilon,i\partial_s(\Lambda P_b)\right)_2|\lesssim\|\varepsilon(s)\|_2\left(|\operatorname{Mod}(s)|+b^2(s)+\lambda^\alpha(s)\right)\lesssim s^{-2}|\operatorname{Mod}(s)|+s^{-(K+2)}.$$

Next, we write $\langle\varepsilon_s,i\Lambda P_b\rangle=-\langle i\varepsilon_s,\Lambda P_b\rangle$ and we use the equation of ε. We start by the contribution of the first line of (4.4). Remark that by (3.15),

$$\begin{aligned}f(P_b+\varepsilon)-f(P_b)&=e^{-ib\frac{|y|^2}{4}}\left(f\left(P+e^{ib\frac{|y|^2}{4}}\varepsilon\right)-f(P)\right)\\&=e^{-ib\frac{|y|^2}{4}}df(P)\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right)+O(|\varepsilon|^2)\\&=e^{-ib\frac{|y|^2}{4}}df(P)\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right)+O(s^{-2}|\varepsilon|),\end{aligned}\tag{4.5}$$

$$\lambda^\alpha\left(g(P_b+\varepsilon)-g(P_b)\right)=O(\lambda^\alpha|\varepsilon|)=O(s^{-2}|\varepsilon|),\tag{4.6}$$

$$\Delta\varepsilon+ib\Lambda\varepsilon=e^{-ib\frac{|y|^2}{4}}\Delta\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right)+b^2\frac{|y|^2}{4}\varepsilon,\tag{4.7}$$

and

$$\Lambda P_b=e^{-ib\frac{|y|^2}{4}}\left(\Lambda P-ib\frac{|y|^2}{2}P\right).$$

Therefore, using (3.15) and $P=Q+O_{H^1}(s^{-2})$ (see the definition of P in (2.3)), we have

$$\begin{aligned}&\langle-\Delta\varepsilon+\varepsilon-ib\Lambda\varepsilon-(f(P_b+\varepsilon)-f(P_b))+\lambda^\alpha\left(g(P_b+\varepsilon)-g(P_b)\right),\Lambda P_b\rangle\\&=\left\langle-\Delta\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right)+e^{ib\frac{|y|^2}{4}}\varepsilon-pQ^{p-1}\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right),\Lambda Q-ib\frac{|y|^2}{2}Q\right\rangle+O(s^{-2}\|\varepsilon\|_2)\\&=\left\langle L_+\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right),\Lambda Q\right\rangle-\frac{b}{2}\left\langle L_-\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right),i|y|^2Q\right\rangle+O(s^{-2}\|\varepsilon\|_2)\\&=\left\langle e^{ib\frac{|y|^2}{4}}\varepsilon,L_+(\Lambda Q)\right\rangle-\frac{b}{2}\left\langle e^{ib\frac{|y|^2}{4}}\varepsilon,iL_-(|y|^2Q)\right\rangle+O(s^{-2}\|\varepsilon\|_2)\\&=-2\left(\varepsilon,e^{-ib\frac{|y|^2}{4}}Q\right)_2+2b\left(\varepsilon,ie^{-ib\frac{|y|^2}{4}}\Lambda Q\right)_2+O(s^{-2}\|\varepsilon\|_2)\\&=-2\left(\varepsilon,P_b\right)_2+2b\left(\varepsilon,i\Lambda P_b\right)_2+O(s^{-2}\|\varepsilon\|_2)=O(s^{-(K+2)}).\end{aligned}$$

Note that we have used algebraic relations from (1.9), then (3.15), $(\varepsilon,i\Lambda P_b)_2=0$ and the definition of s_{**}.

The part corresponding to the second line of (4.4) gives

$$
\begin{aligned}
(MOD, \Lambda P_b)_2 &= -(b_s + b^2 - \theta)\|yP_b\|_2^2 + O(|\operatorname{Mod}(s)|\|\varepsilon\|_2) \\
&= -(b_s + b^2 - \theta)(\|yQ\|_2^2 + O\left(s^{-2}\right)) + O(s^{-2}|\operatorname{Mod}(s)|),
\end{aligned}
$$

where

$$
MOD = -i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda(P_b + \varepsilon) + (1 - \gamma_s)(P_b + \varepsilon) + (b_s + b^2 - \theta)\frac{|y|^2}{4}P_b.
$$

Finally, from the estimate (2.4) on Ψ, we have

$$
\left|\left(\Psi, \Lambda P - ib\frac{|y|^2}{2}P\right)_2\right| \lesssim s^{-2}|\operatorname{Mod}(s)| + s^{-2(K+2)}.
$$

Combining the previous estimates, we find

$$
|b_s + b^2 - \theta| \lesssim s^{-2}|\operatorname{Mod}(s)| + s^{-(K+2)}.
$$

Using the other orthogonality conditions in (3.3) in a similar way, together with (1.9), we find

$$
|\operatorname{Mod}(s)| \lesssim s^{-2}|\operatorname{Mod}(s)| + s^{-(K+2)}.
$$

We deduce that for all $s \in [s_{**}, s_1]$,

$$
|\operatorname{Mod}(s)| \lesssim s^{-(K+2)}. \tag{4.8}
$$

By conservation of the L^2 norm and (3.4), we have

$$
\|u(s)\|_2^2 = \|u(s_1)\|_2^2 = \|P_b(s_1)\|_2^2.
$$

Thus, by (3.2),

$$
\begin{aligned}
(\varepsilon(s), P_b)_2 &= \frac{1}{2}\left(\|u(s)\|_2^2 - \|P_b(s)\|_2^2 - \|\varepsilon(s)\|_2^2\right) \\
&= -\frac{1}{2}\|\varepsilon(s)\|_2^2 + \frac{1}{2}\left(\|P_b(s_1)\|_2^2 - \|P_b(s)\|_2^2\right).
\end{aligned}
$$

Moreover, by (2.7), (3.15) and (4.8),

$$
\frac{d}{ds}\int_{\mathbb{R}^d} |P_b|^2 \lesssim s^{-(K+4)}.
$$

Integrating and combining the previous estimates with (3.15), we obtain, for all $s \in [s_{**}, s_1]$,

$$
|(\varepsilon(s), P_b)_2| \lesssim s^{-(K+3)}. \tag{4.9}
$$

Therefore, $s_{**} = s_*$ and the estimates (4.8) and (4.9) are proved on $[s_*, s_1]$. Combining the definitions of P (2.3) and of P_b (2.5), the expressions of λ_{app} and b_{app} (2.19), and the smallness from (3.15), we get

$$
|P_b - Q| \lesssim Q^{\frac{1}{2}}s^{-1}.
$$

Together with (4.9) this gives (4.2).

Finally, note that (4.3) is a direct consequence of (2.4), (3.15) and (4.1). □

5. The mixed energy Morawetz monotonicity formula

In this section, following [31], we introduce a mixed Energy/Morawetz functional to control the remaining part of the solution in $H^1(\mathbb{R}^d)$. First, define the energy of ε

$$H(s,\varepsilon) := \frac{1}{2}\|\nabla\varepsilon\|_2^2 + \frac{1}{2}\|\varepsilon\|_2^2 - \int_{\mathbb{R}^d}(F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon)dy$$
$$- \lambda^\alpha \int_{\mathbb{R}^d}(G(P_b+\varepsilon) - G(P_b) - dG(P_b)\varepsilon)dy.$$

Note that as in [31], the time derivative of the linearized energy H for ε cannot be controlled alone, and one has to add a virial type functional such as $\frac{b}{2}\mathrm{Im}\int_{\mathbb{R}^d}\nabla\left(\frac{|y|^2}{2}\right)\nabla\varepsilon\bar{\varepsilon}dy$. In practice, due to the lack of control on $\|y\varepsilon\|_2$, we replace $\frac{1}{2}|y|^2$ by a function whose gradient is bounded, which we introduce now.
Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth even and convex function, nondecreasing on $\mathbb{R}^+$, such that

$$\phi(r) = \begin{cases} \frac{1}{2}r^2 & \text{for } r < 1, \\ 3r + e^{-r} & \text{for } r > 2, \end{cases}$$

and set $\phi(x) = \phi(|x|)$. Let $A \gg 1$ to be fixed. Define ϕ_A by $\phi_A(y) = A^2\phi\left(\frac{y}{A}\right)$ and

$$J(\varepsilon) = \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^d}\nabla\phi_A\cdot\nabla\varepsilon\bar{\varepsilon}dy.$$

Finally, set

$$S(s,\varepsilon) = \frac{1}{\lambda^4(s)}(H(s,\varepsilon) + b(s)J(\varepsilon(s))).$$

The relevance of the functional S lies on the following two properties.

Proposition 5.1 (Coercivity of S). *For any $s \in [s_*, s_1]$,*

$$S(s,\varepsilon(s)) \gtrsim \frac{1}{\lambda^4(s)}\left(\|\varepsilon(s)\|_{H^1}^2 + O(s^{-2(K+1)})\right).$$

Proposition 5.2. *For any $s \in [s_*, s_1]$,*

$$\frac{d}{ds}\left[S(s,\varepsilon(s))\right] \gtrsim \frac{b}{\lambda^4(s)}\left(\|\varepsilon(s)\|_{H^1}^2 + O(s^{-2(K+1)})\right).$$

The rest of this section is organized as follows. We first prove Proposition 5.1 in §5.1. In §5.2 we compute the time derivative of H and in §5.3, the time derivative of J. We finish the proof of Proposition 5.2 in §5.4.

5.1. Coercivity of S

We prove Proposition 5.1. We first claim a coercivity property for H, consequence of the properties of L_+ and L_- (see (1.12)) and of the orthogonality conditions of ε (see (3.3)).

Lemma 5.3 (Coercivity of H). *For all $s \in [s_*, s_1]$,*

$$H(s,\varepsilon) \gtrsim \|\varepsilon\|_{H^1}^2 + O(s^{-2(K+1)}).$$

Proof. From the orthogonality conditions (3.3), (4.2), and estimates (3.15), the following holds:

$$\begin{aligned}
\left(\varepsilon, |y|^2 Q\right)_2 &= \left(\varepsilon, |y|^2 P_b\right)_2 + O(|b|\|\varepsilon\|_2) + O(\lambda^\alpha \|\varepsilon\|_2) = O(s^{-1}\|\varepsilon\|_{H^1}),\\
(\varepsilon, i\rho)_2 &= (\varepsilon, i\rho_b)_2 + O(|b|\|\varepsilon\|_2) = O(s^{-1}\|\varepsilon\|_{H^1}),\\
(\varepsilon, Q)_2 &= O(s^{-(K+1)}).
\end{aligned}$$

From (3.15), we have

$$\lambda^\alpha \int_{\mathbb{R}^d} (G(P_b+\varepsilon) - G(P_b) - dG(P_b)\varepsilon)dx = O(s^{-2}\|\varepsilon\|_{H^1}^2).$$

Next, (denoting $\varepsilon = \varepsilon_1 + i\varepsilon_2$),

$$\left| F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon - \left(1+\frac{4}{d}\right) Q^{\frac{4}{d}}\varepsilon_1^2 - \frac{1}{2} Q^{\frac{4}{d}}\varepsilon_2^2 \right| \lesssim e^{-\frac{1}{2}|y|}|\varepsilon|^3 + |\varepsilon|^{2+\frac{4}{d}} + |\varepsilon|^2(|b| + \lambda^\alpha).$$

Thus, from (3.15),

$$\left| \int_{\mathbb{R}^d} F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon - \left(1+\frac{4}{d}\right) Q^{\frac{4}{d}}\varepsilon_1^2 - \frac{1}{2} Q^{\frac{4}{d}}\varepsilon_2^2 \right| \lesssim O(s^{-1}\|\varepsilon\|_{H^1}^2),$$

and

$$\left| H(s,\varepsilon) - \frac{1}{2}\langle L_+\varepsilon_1, \varepsilon_1\rangle - \frac{1}{2}\langle L_-\varepsilon_2, \varepsilon_2\rangle \right| \lesssim O(s^{-1}\|\varepsilon\|_{H^1}^2).$$

Combining these estimates with the coercivity properties of L_+, L_- (see (1.12)), we obtain the result. □

Since

$$|bJ(\varepsilon)| \le |b|\|\nabla\phi_A\|_\infty \|\varepsilon\|_{H^1}^2 \lesssim O(s^{-1}\|\varepsilon\|_{H^1}^2)$$

(from (3.15)), Lemma 5.3 implies Proposition 5.1.

For future reference, we also claim a similar localized coercivity property (see similar statement in [17] and [31, Eq. (3.72)]). Define

$$H_A(s,\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^d} \nabla\varepsilon^T \nabla^2\phi_A \nabla\bar{\varepsilon} dy + \frac{1}{2}\|\varepsilon\|_2^2 - \int_{\mathbb{R}^d} (F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon)dy - \lambda^\alpha \int_{\mathbb{R}^d} (G(P_b+\varepsilon) - G(P_b) - dG(P_b)\varepsilon)dy.$$

Lemma 5.4. *There exists $A_0 > 1$ such that for any $A > A_0$, for all $s \in [s_*, s_1]$,*

$$H_A(s,\varepsilon) \gtrsim \|\varepsilon\|_2^2 + O(s^{-2(K+1)}).$$

For now on, we consider $A > A_0$.

5.2. Time variation of the energy of ε

Lemma 5.5. *For all $s \in [s_*, s_1]$, we have*

$$\frac{d}{ds}[H(s,\varepsilon(s))] = \frac{\lambda_s}{\lambda}\left(\|\nabla\varepsilon\|_2^2 - \langle f(P_b+\varepsilon) - f(P_b), \Lambda\varepsilon\rangle\right) + O(s^{-(2K+3)}) + O(s^{-2}\|\varepsilon\|_{H^1}^2).$$

Proof of Lemma 5.5. The time derivative for H separates into two parts:

$$\frac{d}{ds}[H(s,\varepsilon(s))] = D_s H(s,\varepsilon) + \langle D_\varepsilon H(s,\varepsilon), \varepsilon_s\rangle,$$

where D_s (respectively, D_ε) denotes differentiation of the functional with respect to s (respectively, ε). In particular,

$$\begin{aligned} D_s H(s,\varepsilon) = &-\int_{\mathbb{R}^d} (P_b)_s\left(f(P_b+\varepsilon) - f(P_b) - df(P_b)\varepsilon\right) \\ &- \lambda^\alpha \int_{\mathbb{R}^d} (P_b)_s\left(g(P_b+\varepsilon) - g(P_b) - dg(P_b)\varepsilon\right) \\ &- \alpha\frac{\lambda_s}{\lambda}\lambda^\alpha \int_{\mathbb{R}^d} \left(G(P_b+\varepsilon) - G(P_b) - dG(P_b)\varepsilon\right). \end{aligned}$$

Note that

$$e^{i\frac{b|y|^2}{4}}(P_b)_s = P_s - ib_s\frac{|y|^2}{4}P = P_s - i\left(b_s + b^2 - \beta\lambda^\alpha\right)\frac{|y|^2}{4}P + i\left(b^2 - \beta\lambda^\alpha\right)\frac{|y|^2}{4}P.$$

By (2.11), (3.15) and Lemma 4.1, we obtain

$$|(P_b)_s| \lesssim s^{-2}e^{-\frac{|y|}{2}} \quad \text{and} \quad \left|\frac{\lambda_s}{\lambda}\right|\lambda^\alpha \lesssim s^{-3}.$$

Thus,

$$|D_s H(s,\varepsilon)| \lesssim s^{-2}\|\varepsilon\|_{H^1}^2.$$

Now, we compute $\langle D_\varepsilon H(s,\varepsilon), \varepsilon_s\rangle$. Note that (4.4) rewrites

$$i\varepsilon_s - D_\varepsilon H(s,\varepsilon) + \mathrm{Mod_{op}}(s)P_b - i\frac{\lambda_s}{\lambda}\Lambda\varepsilon + (1-\gamma_s)\varepsilon + e^{-ib\frac{|y|^2}{4}}\Psi = 0, \tag{5.1}$$

where

$$\mathrm{Mod_{op}}(s)P_b := -i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda P_b + (1-\gamma_s)P_b + (b_s + b^2 - \theta)\frac{|y|^2}{4}P_b.$$

Using (5.1), since $\langle iD_\varepsilon H(s,\varepsilon), D_\varepsilon H(s,\varepsilon)\rangle = 0$, we have

$$\begin{aligned}\langle D_\varepsilon H(s,\varepsilon), \varepsilon_s\rangle &= \langle iD_\varepsilon H(s,\varepsilon), i\varepsilon_s\rangle \\ &= -\langle iD_\varepsilon H(s,\varepsilon), \mathrm{Mod}_{\mathrm{op}}(s)P_b\rangle + \frac{\lambda_s}{\lambda}\langle iD_\varepsilon H(s,\varepsilon), i\Lambda\varepsilon\rangle \\ &\quad -(1-\gamma_s)\langle iD_\varepsilon H(s,\varepsilon), \varepsilon\rangle - \left\langle iD_\varepsilon H(s,\varepsilon), e^{-ib\frac{|y|^2}{4}}\Psi\right\rangle. \end{aligned} \tag{5.2}$$

From (4.5), (4.6) and (4.7) in the proof of Lemma 4.1

$$\begin{aligned}&D_\varepsilon H(s,\varepsilon) = -\Delta\varepsilon + \varepsilon - (f(P_b+\varepsilon) - f(P_b)) - \lambda^\alpha(g(P_b+\varepsilon) - g(P_b)) \\ &= e^{-ib\frac{|y|^2}{4}}\left(L_+\mathrm{Re}\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right) + iL_-\mathrm{Im}\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right)\right) + ib\Lambda\varepsilon + b^2\frac{|y|^2}{4}\varepsilon + O(s^{-2}|\varepsilon|).\end{aligned}$$

Therefore, using the orthogonality conditions (3.3), (4.2), (4.9) and estimates (3.15), we have (see also proof of Lemma 4.1),

$$\langle D_\varepsilon H(s,\varepsilon), \Lambda P_b\rangle = -2\,(\varepsilon, P_b)_2 + b\,(\varepsilon, i\Lambda P_b)_2 + O(s^{-2}\|\varepsilon\|_2) = O(s^{-(K+1)}).$$

Thus, from Lemma 4.1,

$$\left|\frac{\lambda_s}{\lambda} + b\right| |\langle D_\varepsilon H(s,\varepsilon), \Lambda P_b\rangle| \lesssim O(s^{-(2K+3)}).$$

Using similar arguments we get

$$\langle D_\varepsilon H(s,\varepsilon), iP_b\rangle = -4\,(\varepsilon, \Lambda P_b)_2 + O(s^{-1}\|\varepsilon\|_2) = O(s^{-1}\|\varepsilon\|_2) = O(s^{-(K+1)})$$

and

$$\left\langle D_\varepsilon H(s,\varepsilon), i\frac{|y|^2}{4}P_b\right\rangle = (\varepsilon, \rho_b)_2 + O(s^{-1}\|\varepsilon\|_2) = O(s^{-1}\|\varepsilon\|_2) = O(s^{-(K+1)}).$$

Using Lemma 4.1, we obtain in conclusion for this term

$$\langle iD_\varepsilon H(s,\varepsilon), \mathrm{Mod}_{\mathrm{op}}(s)P_b\rangle = O(s^{-(2K+3)}).$$

Next, we have

$$\begin{aligned}&\langle iD_\varepsilon H(s,\varepsilon), i\Lambda\varepsilon\rangle = \langle D_\varepsilon H(s,\varepsilon), \Lambda\varepsilon\rangle = \\ &\qquad \langle -\Delta\varepsilon + \varepsilon - (f(P_b+\varepsilon) - f(P_b)) - \lambda^\alpha(g(P_b+\varepsilon) - g(P_b)), \Lambda\varepsilon\rangle.\end{aligned}$$

Note that (by direct computations)

$$\langle -\Delta\varepsilon, \Lambda\varepsilon\rangle = \|\nabla\varepsilon\|_2^2, \quad \langle \varepsilon, \Lambda\varepsilon\rangle = 0,$$

and by (3.15),

$$|\langle \lambda^\alpha(g(P_b+\varepsilon) - g(P_b)), \Lambda\varepsilon\rangle| \lesssim O(s^{-2}\|\varepsilon\|_{H^1}^2).$$

Thus,

$$\frac{\lambda_s}{\lambda}\langle iD_\varepsilon H(s,\varepsilon), i\Lambda\varepsilon\rangle = \frac{\lambda_s}{\lambda}\Big(\|\nabla\varepsilon\|_2^2 - \langle f(P_b+\varepsilon)-f(P_b), \Lambda\varepsilon\rangle\Big) + O(s^{-3}\|\varepsilon\|_{H^1}^2).$$

For the first term of (5.2), we claim

$$\begin{aligned}
&|(1-\gamma_s)\langle iD_\varepsilon H(s,\varepsilon),\varepsilon\rangle| \\
&\quad = \Big|(1-\gamma_s)\langle (f(P_b+\varepsilon)-f(P_b)) + \lambda^\alpha(g(P_b+\varepsilon)-g(P_b)),\varepsilon\rangle\Big| \\
&\quad \lesssim |\operatorname{Mod}(s)|\Big(\|\varepsilon\|_2^2 + \|\varepsilon\|_{H^1}^{2+\frac{4}{d}}\Big) = O(s^{-4}\|\varepsilon\|_{H^1}^2).
\end{aligned}$$

Finally, the second term of (5.2) is estimated by (4.3) and (3.15)

$$|\langle iD_\varepsilon H(s,\varepsilon),\Psi\rangle| \le O(s^{-(K+4)}\|\varepsilon\|_{H^1}) \le O(s^{-(2K+3)}) + O(s^{-5}\|\varepsilon\|_{H^1}^2).$$

Gathering these estimates, we have proved the lemma. □

5.3. The time derivative of the Morawetz part

Lemma 5.6. *For all* $s \in [s_*, s_1]$,

$$\begin{aligned}
\frac{d}{ds}[J(\varepsilon(s))] &= \int_{\mathbb{R}^d} \nabla\varepsilon^T\nabla^2\phi_A\nabla\bar\varepsilon dy - \frac14\int_{\mathbb{R}^d}|\varepsilon|^2\Delta^2\phi_A dy \\
&- \Big\langle f(P_b+\varepsilon)-f(P_b), \frac12\Delta\phi_A\varepsilon + \nabla\phi_A\nabla\varepsilon\Big\rangle + O(s^{-(2K+2)}) + O(s^{-2}\|\varepsilon\|_{H^1}^2).
\end{aligned}$$

Proof. From the definition of $J(\varepsilon)$, we have

$$\frac{d}{ds}[J(\varepsilon(s))] = \operatorname{Re}\int_{\mathbb{R}^d} i\varepsilon_s\left(\frac12\Delta\phi_A\bar\varepsilon + \nabla\phi_A\nabla\bar\varepsilon\right)dy.$$

We replace $i\varepsilon_s$ using (4.4). First, from standard computations

$$\begin{gathered}
\operatorname{Re}\int_{\mathbb{R}^d} -\Delta\varepsilon\left(\frac12\Delta\phi_A\bar\varepsilon + \nabla\phi_A\nabla\bar\varepsilon\right)dy = \int_{\mathbb{R}^d}\nabla\varepsilon^T\nabla^2\phi_A\nabla\bar\varepsilon dy - \frac14\int_{\mathbb{R}^d}|\varepsilon|^2\Delta^2\phi_A dy, \\
\operatorname{Re}\int_{\mathbb{R}^d}\varepsilon\left(\frac12\Delta\phi_A\bar\varepsilon + \nabla\phi_A\nabla\bar\varepsilon\right)dy = 0, \\
\frac{\lambda_s}{\lambda}\operatorname{Re}\int_{\mathbb{R}^d} i\Lambda\varepsilon\left(\frac12\Delta\phi_A\bar\varepsilon + \nabla\phi_A\nabla\bar\varepsilon\right)dy = 0.
\end{gathered}$$

Next,

$$\begin{aligned}
&\lambda^\alpha \operatorname{Re}\int_{\mathbb{R}^d}(g(P_b+\varepsilon)-g(P_b))\left(\frac12\Delta\phi_A\bar\varepsilon + \nabla\phi_A\nabla\bar\varepsilon\right)dy \\
&\qquad = O(\lambda^\alpha\|\varepsilon\|_{H^1}^2) = O(s^{-2}\|\varepsilon\|_{H^1}^2).
\end{aligned}$$

The term corresponding to the second line of (4.4) is estimated as follows.

$$\left|\left\langle -i(b+\frac{\lambda_s}{\lambda})\Lambda(P_b+\varepsilon)+(1-\gamma_s)(P_b+\varepsilon)-(b_s+b^2-\theta)\frac{|y|^2}{4}P_b, \frac{1}{2}\Delta\phi_A\varepsilon+\nabla\phi_A\nabla\varepsilon\right\rangle\right| \lesssim |\operatorname{Mod}(s)|\|\varepsilon\|_{H^1} \lesssim O(s^{-(2K+2)}).$$

Finally, by (4.3),

$$\left|\left\langle \Psi e^{-i\frac{b|y|^2}{4}}, \frac{1}{2}\Delta\phi_A\bar{\varepsilon}+\nabla\phi_A\nabla\bar{\varepsilon}\right\rangle\right| \leq O(s^{-(K+4)}\|\varepsilon\|_{H^1}) \leq O(s^{-(2K+4)}).$$

The result follows. □

5.4. The Lyapunov property

Proof of Proposition 5.2. By definition of S, we have

$$\frac{d}{ds}[S(s,\varepsilon(s))] = \frac{1}{\lambda^4}\left(-4\frac{\lambda_s}{\lambda}\left(H(s,\varepsilon)+bJ(\varepsilon)\right)+\frac{d}{ds}[H(s,\varepsilon(s))]+b\frac{d}{ds}[J(\varepsilon(s))]+b_sJ(\varepsilon)\right)$$

First, we claim the following estimate

$$\frac{d}{ds}[H(s,\varepsilon(s))]+b\frac{d}{ds}[J(\varepsilon(s))] = b\int_{\mathbb{R}^d}\nabla\varepsilon^T\nabla^2\phi_A\nabla\bar{\varepsilon}dy - b\|\nabla\varepsilon\|_2^2+\frac{b}{A}O(\|\varepsilon\|_{H^1}^2)+O(s^{-(2K+3)}). \tag{5.3}$$

Proof of (5.3). It is essential to see from Lemmas 5.5 and 5.6 that the main nonlinear terms are cancelling. Indeed, by integration by parts, we have

$$\begin{aligned}
&-\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\Lambda\bar{\varepsilon}dy \\
&= -\frac{d}{2}\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\bar{\varepsilon}dy - \operatorname{Re}\int_{\mathbb{R}^d}y\nabla(F(P_b+\varepsilon)-F(P_b)-dF(P_b)\varepsilon)dy \\
&\qquad + \operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b)-df(P_b)\varepsilon)y\nabla\bar{P}_bdy \\
&= -\frac{d}{2}\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\bar{\varepsilon}dy + d\operatorname{Re}\int_{\mathbb{R}^d}(F(P_b+\varepsilon)-F(P_b)-dF(P_b)\varepsilon)dy \\
&\qquad + \operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b)-df(P_b)\varepsilon)y\nabla\bar{P}_bdy,
\end{aligned}$$

For the localized part, this gives

$$-\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\left(\frac{1}{2}\Delta\phi_A\bar{\varepsilon}+\nabla\phi_A\nabla\bar{\varepsilon}\right)dy=$$
$$-\frac{1}{2}\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\Delta\phi_A\bar{\varepsilon}dy+\operatorname{Re}\int_{\mathbb{R}^d}\Delta\phi_A(F(P_b+\varepsilon)-F(P_b)-dF(P_b)\varepsilon)dy$$
$$+\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b)-df(P_b)\varepsilon)\nabla\phi_A\nabla\bar{P}_bdy.$$

Writing these two terms as above, it becomes clear that when y or $\nabla\phi_A$ appear, they are multiplied by ∇P_b, which is exponentially decaying in space (see Proposition 2.1). Therefore, such terms are controlled by expressions involving only $\|\varepsilon\|_{H^1}$.

Therefore, combining Lemma 5.5 and Lemma 5.6, we have

$$\frac{d}{ds}[H(s,\varepsilon(s))]+b\frac{d}{ds}[J(\varepsilon(s))]=b\int_{\mathbb{R}^d}\nabla\varepsilon^T\nabla^2\phi_A\nabla\bar{\varepsilon}dy-b\|\nabla\varepsilon\|_2^2$$
$$+\left(b+\frac{\lambda_s}{\lambda}\right)\Bigg(\|\nabla\varepsilon\|_2^2-\frac{d}{2}\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\bar{\varepsilon}dy$$
$$+d\operatorname{Re}\int_{\mathbb{R}^d}(F(P_b+\varepsilon)-F(P_b)-dF(P_b)\varepsilon)+(f(P_b+\varepsilon)-f(P_b)-df(P_b)\varepsilon)y\nabla\bar{P}_bdy\Bigg)$$
$$+b\Bigg(-\frac{1}{2}\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))(\Delta\phi_A-d)\bar{\varepsilon}dy+$$
$$\operatorname{Re}\int_{\mathbb{R}^d}(F(P_b+\varepsilon)-F(P_b)-dF(P_b)\varepsilon)(\Delta\phi_A-d)dy$$
$$+\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b)-df(P_b)\varepsilon)(\nabla\phi_A-y)\nabla\bar{P}_bdy\Bigg)$$
$$-b\frac{1}{4}\int_{\mathbb{R}^d}|\varepsilon|^2\Delta^2\phi_Ady+O(s^{-(2K+3)})+O(s^{-2}\|\varepsilon\|_{H^1}^2).$$

By $\left|b+\frac{\lambda_s}{\lambda}\right|\lesssim O(s^{-4})$, we have

$$\Bigg|\left(b+\frac{\lambda_s}{\lambda}\right)\Bigg(\|\nabla\varepsilon\|_2^2-\frac{d}{2}\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\bar{\varepsilon}dy$$
$$+d\operatorname{Re}\int_{\mathbb{R}^d}(F(P_b+\varepsilon)-F(P_b)-dF(P_b)\varepsilon)dy$$
$$+\operatorname{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b)-df(P_b)\varepsilon)y\nabla\bar{P}_bdy\Bigg)\Bigg|\lesssim s^{-4}\|\varepsilon\|_{H^1}^2.$$

Next,

$$\begin{aligned}
&|b|\left|-\frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\Delta(\phi_A-d)\bar{\varepsilon}dy\right| \\
&\quad\lesssim \frac{1}{s}\int_{\mathbb{R}^d}\left||P|^{\frac{4}{d}}|\varepsilon|^2|\Delta\phi_A-d|+|\varepsilon|^{2+\frac{4}{d}}\right|dy \lesssim \frac{e^{-\frac{A}{2}}}{s}\|\varepsilon\|_2^2+O\left(s^{-1}\|\varepsilon\|_{H^1}^{2+\frac{4}{d}}\right),
\end{aligned}$$

and similarly for the terms

$$\begin{aligned}
b\mathrm{Re}\int_{\mathbb{R}^d}(F(P_b+\varepsilon)-F(P_b)-dF(P_b)\varepsilon)(\Delta\phi_A-d)dy,\\
b\mathrm{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b)-df(P_b)\varepsilon)(\nabla\phi_A-y)\nabla\bar{P}_b dy.
\end{aligned}$$

Next,

$$\left|-b\int_{\mathbb{R}^d}|\varepsilon|^2\Delta^2\phi_A dy\right|\lesssim\frac{b}{A^2}\|\varepsilon\|_2^2.$$

In conclusion for this term, we have obtained (5.3)

Using $-\frac{\lambda_s}{\lambda}=b+O(s^{-2})$ and the expressions of H and H_A, we have

$$\begin{aligned}
&-4\frac{\lambda_s}{\lambda}H(s,\varepsilon)+\frac{d}{ds}[H(s,\varepsilon(s))]+b\frac{d}{ds}[J(\varepsilon(s))]\\
&\gtrsim 4bH(s,\varepsilon)+b\int_{\mathbb{R}^d}\nabla\varepsilon^T\nabla^2\phi_A\nabla\bar{\varepsilon}dy-b\|\nabla\varepsilon\|_2^2\\
&+\frac{b}{A}O(\|\varepsilon\|_{H^1}^2)+O(s^{-2}\|\varepsilon\|_{H^1}^2)+O(s^{-(2K+3)})\\
&\gtrsim 2bH_A(s,\varepsilon)+2bH(s,\varepsilon)+\frac{b}{A}O(\|\varepsilon\|_{H^1}^2)+O(s^{-2}\|\varepsilon\|_{H^1}^2)+O(s^{-(2K+3)}).
\end{aligned}$$

Thus, by the coercivity properties Lemma 5.3 and Lemma 5.4, we obtain (for A large enough and s_0 large enough)

$$-4\frac{\lambda_s}{\lambda}H(s,\varepsilon)+\frac{d}{ds}[H(s,\varepsilon(s))]+b\frac{d}{ds}[J(\varepsilon(s))]\gtrsim b\|\varepsilon\|_{H^1}^2+O(s^{-(2K+3)}).$$

Since $|\frac{\lambda_s}{\lambda}|\sim|b|=O(s^{-1})$, $b_s=O(s^{-2})$ and $J(\varepsilon)=O(\|\varepsilon\|_{H^1}^2)$, we have

$$\left(\left|\frac{\lambda_s}{\lambda}\right|b+|b_s|\right)|J(\varepsilon)|\lesssim s^{-2}O(\|\varepsilon\|_{H^1}^2)$$

and thus

$$\frac{d}{ds}[S(s,\varepsilon(s))]\gtrsim\frac{b}{\lambda^4}\left(\|\varepsilon\|_{H^1}^2+O(s^{-(2K+2)})\right).$$

This finishes the proof. □

6. End of the proof of Proposition 3.2

In this section, we finish the proof of Proposition 3.2. Recall from §3.3 that our objective is to prove $s_* = s_0$ by improving estimates (3.15) into (3.6). Therefore, it is sufficient to prove the following lemma which closes the bounds (3.15) provided $\delta(\alpha) > 0$ has been chosen small enough (e.g. as in (3.16)).

Lemma 6.1 (Refined estimates). *For all $s \in [s_*, s_1]$,*

$$\|\varepsilon(s)\|_{H^1} \lesssim s^{-(K+1)}, \tag{6.1}$$

$$\left|\frac{\lambda^{\frac{\alpha}{2}}(s)}{\lambda_{\mathrm{app}}^{\frac{\alpha}{2}}(s)} - 1\right| + \left|\frac{b(s)}{b_{\mathrm{app}}(s)} - 1\right| \lesssim s^{-\frac{1}{2}} + s^{2-\frac{4}{\alpha}}. \tag{6.2}$$

Proof. First, we prove (6.1). From Proposition 5.1, and the expression of S, there exists a universal constant $\kappa > 1$ such that for any $s \in [s_*, s_1]$,

$$\frac{1}{\kappa}\frac{1}{\lambda^4}\left(\|\varepsilon\|_{H^1}^2 - \kappa^2 s^{-2(K+1)}\right) \le S(s,\varepsilon) \le \frac{\kappa}{\lambda^4}\|\varepsilon\|_{H^1}^2. \tag{6.3}$$

From Proposition 5.2, possibly taking a larger κ,

$$\frac{d}{ds}[S(s,\varepsilon(s))] \ge \frac{1}{\kappa}\frac{b}{\lambda^4}\left(\|\varepsilon\|_{H^1}^2 - \kappa^2 s^{-2(K+1)}\right). \tag{6.4}$$

Define

$$s_\dagger := \inf\{s \in [s_*, s_1], \quad \|\varepsilon(\tau)\|_{H^1} \le 2\kappa^2 \tau^{-(K+1)} \quad \text{for all } \tau \in [s, s_1]\}.$$

Since $\varepsilon(s_1) = 0$, by continuity $s_\dagger$ is well-defined and $s_\dagger < s_1$. For the sake of contradiction, assume that $s_\dagger > s_*$. In particular, $\|\varepsilon(s_\dagger)\|_{H^1} = 2\kappa^2 s_\dagger^{-(K+1)}$. Define

$$s_\ddagger := \sup\{s \in [s_\dagger, s_1], \quad \|\varepsilon(\tau)\|_{H^1} \ge \kappa \tau^{-(K+1)} \quad \text{for all } \tau \in [s_\dagger, s]\}.$$

In particular, $s_\dagger < s_\ddagger < s_1$ and $\|\varepsilon(s_\ddagger)\|_{H^1} = \kappa s_\ddagger^{-(K+1)}$, and from (6.4), S is nondecreasing on $[s_\dagger, s_\ddagger]$. From equations (6.3)-(6.4) and the estimates on λ (see (3.6)), we obtain

$$\begin{aligned}
\|\varepsilon(s_\dagger)\|_{H^1}^2 - \kappa^2 s_\dagger^{-2(K+1)} &\le \kappa\lambda^4(s_\dagger) S(s_\dagger, \varepsilon(s_\dagger)) \\
&\le \kappa\lambda^4(s_\dagger) S(s_\ddagger, \varepsilon(s_\ddagger)) \le \kappa^2 \frac{\lambda^4(s_\dagger)}{\lambda^4(s_\ddagger)}\|\varepsilon(s_\ddagger)\|_{H^1}^2 \le \kappa^4 \frac{\lambda^4(s_\dagger)}{\lambda^4(s_\ddagger)} s_\ddagger^{-2(K+1)} \\
&\le 2\kappa^4 \left(\frac{s_\ddagger}{s_\dagger}\right)^{\frac{8}{\alpha}} s_\ddagger^{-2(K+1)} \le 2\kappa^4 s_\dagger^{-2(K+1)},
\end{aligned}$$

since $K > 4/\alpha$. Therefore $\|\varepsilon(s_\dagger)\|_{H^1}^2 \le 3\kappa^4 s_\dagger^{-2(K+1)}$, which is a contradiction. Hence $s_\dagger = s_*$ and (6.1) is proved.

Now, we prove (6.2). The main idea is to use a conservation law on (b, λ) which can be found from the differential system satisfied by (b, λ), but that we rather derive from energy properties of the blow up profile. Recall that $\lambda(s_1) = \lambda_1$ and $b(s_1) = b_1$ are chosen in Lemma 2.4 so that $\mathcal{F}(\lambda(s_1)) = s_1$ and $\mathcal{E}(b(s_1), \lambda(s_1)) = \frac{8E_0}{\int_{\mathbb{R}^d} |y|^2 Q^2}$. In particular, we deduce from (2.9) that $|E(P_{b_1,\lambda_1,\gamma_1}) - E_0| \lesssim s_1^{-6}$. Using (2.8) and (3.15), (4.1), for all $s \in [s_*, s_1]$,

$$\left|\frac{d}{ds} E(P_{b,\lambda,\gamma})\right| \lesssim s^{-(K+2)+\frac{4}{\alpha}}.$$

In particular, by integration, we find, for all $s \in [s_*, s_1]$, $|E(P_{b,\lambda,\gamma}(s)) - E_0| \lesssim s^{-6}$ (recall $K > 20/\alpha$) and using (2.9) at s,

$$\left|\mathcal{E}(b(s), \lambda(s)) - \frac{8E_0}{\int_{\mathbb{R}^d} |y|^2 Q^2}\right| \lesssim s^{-6}.$$

We obtain from the expression (2.10) of $\mathcal{E}$ with $C_0 = \frac{8E_0}{\int_{\mathbb{R}^d} |y|^2 Q^2}$:

$$\left|b^2 - \frac{2\beta}{2-\alpha}\lambda^\alpha - C_0\lambda^2\right| \lesssim \frac{\lambda^\alpha}{s^2}$$

where the error term $O(\frac{\lambda^\alpha}{s^2})$ comes from θ and cannot be improved. In this estimate, since $\lambda^2 \approx s^{-\frac{4}{\alpha}}$ and $\frac{\lambda^\alpha}{s^2} \approx s^{-4}$, whether or not $C_0\lambda^2$ is controled by the error term depends on the value of α. We address both cases at once in what follows. Since $b \approx \lambda^{\frac{\alpha}{2}}$,

$$\left|b - \sqrt{\frac{2\beta}{2-\alpha}\lambda^\alpha + C_0\lambda^2}\right| \lesssim \frac{\lambda^{\frac{\alpha}{2}}}{s^2}, \tag{6.5}$$

and with $\left|\frac{\lambda_s}{\lambda} + b\right| \lesssim s^{-(K+1)}$, we obtain (see (2.24) for the definition of $\mathcal{F}$)

$$\left|\frac{\lambda_s}{\lambda^{\frac{\alpha}{2}+1}\sqrt{\frac{2\beta}{2-\alpha} + C_0\lambda^{2-\alpha}}} - 1\right| = |\mathcal{F}'(s) - 1| \lesssim s^{-2}. \tag{6.6}$$

Integrating (6.6) on $[s, s_1]$, we obtain

$$|\mathcal{F}(\lambda(s_1)) - \mathcal{F}(\lambda(s)) - (s_1 - s)| \lesssim s^{-1}$$

and thus, by $\mathcal{F}(\lambda(s_1)) = s_1$ (this choice was done in Lemma 2.4), we obtain

$$\mathcal{F}(\lambda(s)) = s + O(s^{-1}).$$

Therefore, using (2.27) and the definition of $\lambda_{\text{app}}(s)$ in (2.19),

$$\left|\frac{\lambda_{\text{app}}^{\frac{\alpha}{2}}(s)}{\lambda^{\frac{\alpha}{2}}(s)} - 1\right| \lesssim s^{-\frac{1}{2}} + s^{2-\frac{4}{\alpha}}.$$

We reinject this estimate into (6.5) and use the definition of b_{app} to conclude:

$$b(s) = b_{\text{app}}(s) + O(s^{-\frac{3}{2}} + s^{-\frac{4-\alpha}{\alpha}}).$$

This finishes the proof. □

A. Proof of Lemma 1.1 and 1.2

Lemma 1.1. By contradiction, assume that there exists a blow up solution $u(t)$ of (NLS) with $\epsilon = -1$ and $\|u(t)\|_2 = \|Q\|_2$. Let a sequence $t_n \to T^* \in (0, +\infty]$ with $\|\nabla u(t_n)\|_2 \to +\infty$ and consider the renormalized sequence

$$v_n(x) = \lambda(t_n)^{\frac{d}{2}} u(t_n, \lambda(t_n)x), \quad \lambda(t_n) = \frac{\|\nabla Q\|_2}{\|\nabla u(t_n)\|_2}.$$

Then, by conservation of mass,

$$\|v_n\|_2 = \|Q\|_2$$

and conservation of energy and $\epsilon < 0$,

$$E_0 = E(u_n) \geq E_{\rm crit}(u_n) = \frac{E_{\rm crit}(v_n)}{\lambda^2(t_n)}.$$

Therefore, the sequence v_n satisfies:

$$\|v_n\|_2 = \|Q\|_2, \quad \|\nabla v_n\|_2 = \|\nabla Q\|_2, \quad \limsup_{n\to+\infty} E_{\rm crit}(v_n) \leq 0.$$

From standard concentration compactness argument, see [24, 32], there holds, up to a subsequence, for some $x_n \in \mathbb{R}^d, \gamma_n \in \mathbb{R}$,

$$v_n(. - x_n)e^{i\gamma_n} \underset{n\to+\infty}{\to} Q \quad \text{in} \quad H^1(\mathbb{R}^d).$$

In particular,

$$\|u(t_n)\|_{p+1} = \frac{\|v_n\|_{p+1}}{\lambda^{\frac{d(p-1)}{2(p+1)}}(t_n)} \to +\infty \quad \text{as} \quad n \to \infty,$$

which contradicts the a priori bound from the energy conservation law and (1.2):

$$E_0 = E(u) \geq E_{\rm crit}(u) + \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} \geq \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}.$$

This concludes the proof. □

Proof of Lemma 1.2. Let $\delta > 0$. We first recall the virial identity

$$\begin{aligned}\frac{d^2}{dt^2}\|xu(t)\|_2^2 = 8\left(\|\nabla u\|_2^2 - \frac{2}{2+\frac{4}{d}}\|u\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} + \frac{d(p-1)}{2(p+1)}\|u\|_{p+1}^{p+1}\right)\\ = 16E(u) - \frac{4(4-d(p-1))}{p+1}\|u\|_{p+1}^{p+1}.\end{aligned}$$

Define for $a, b \in \mathbb{R}$ the rescaled function $Q_{a,b}$ by $Q_{a,b}(x) = a^{\frac{d}{2}} Q(abx)$. We have

$$\|Q_{a,b}\|_2 = b^{-\frac{d}{2}} \|Q\|_2,$$
$$E(Q_{a,b}) = a^2 b^{-d} \left(b^2 \frac{1}{2} \|\nabla Q\|_2^2 - \frac{1}{2+\frac{4}{d}} \|Q\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} + a^{\frac{d(p-1)}{2}-2} \frac{1}{p+1} \|Q\|_{p+1}^{p+1} \right).$$

Recall that the critical energy vanishes at Q:

$$\frac{1}{2} \|\nabla Q\|_2^2 - \frac{1}{2+\frac{4}{d}} \|Q\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} = 0.$$

Take $b \in \mathbb{R}$ such that $\|Q_{a,b}\|_2 = \|Q\|_2 + \delta$. Then $b < 1$ and for a large enough we have

$$E(Q_{a,b}) < 0.$$

Choose now $u_0 = Q_{a,b}$. By conservation of energy and the virial identity, we have

$$\frac{d^2}{dt^2} \|xu(t)\|_2^2 < 16 E(Q_{a,b}) < 0,$$

which implies blow-up in finite time of u for positive and negative times and concludes the proof. □

B. Proof of Lemma 1.3

For the sake of simplicity, we give the proof only for $d \geq 2$. The case $d = 1$ would require an additionnal (standard) concentration compactness argument (see [32]). For $M < \|Q\|_2$, set

$$A_M = \{u \in H^1_{\text{rad}}(\mathbb{R}^d) \quad \text{with} \quad \|u\|_2 = M\}$$

and consider the minimization problem

$$I_M = \inf_{u \in A_M} E(u).$$

First, we claim

$$-\infty < I_M < 0. \tag{B.1}$$

Indeed, from (1.2) and

$$\int_{\mathbb{R}^d} |u|^{p+1} \leq C_{\text{GN}}(p) \|\nabla u\|_2^{\frac{d(p-1)}{2}} \|u\|_2^{p+1-\frac{d(p-1)}{2}}, \tag{B.2}$$

with $1 < p < 1 + \frac{4}{d}$, we note that $I_M > -\infty$ and that any minimizing sequence is bounded in $H^1(\mathbb{R}^d)$. Let $u \in A_M$ and $v_\lambda(x) = \lambda^{\frac{d}{2}} u(\lambda x)$, then $v_\lambda \in A_M$ and

$$E(v_\lambda) = \lambda^2 \left[E_{\text{crit}}(u) - \frac{1}{\lambda^{2-\frac{d(p-1)}{2}}} \frac{1}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \right].$$

In particular, for $0 < \lambda \ll 1$ and $u \not\equiv 0$, $E(v_\lambda) < 0$ and (B.1) follows.

Second, let $u_\lambda = \lambda^{\frac{2}{p-1}} u(\lambda x)$, so that

$$E(u_\lambda) = \lambda^{\frac{4}{p-1}+2-d}\left[\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}\right] - \frac{\lambda^{\frac{2}{p-1}(2+\frac{4}{d})-d}}{2+\frac{4}{d}}\int_{\mathbb{R}^d}|u|^{2+\frac{4}{d}}.$$

We observe that

$$\begin{aligned}\frac{d}{d\lambda}E(u_\lambda)_{|\lambda=1} &= \left(\frac{4}{p-1}+2-d\right)\left[\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}\right]\\ &\quad -\frac{\frac{2}{p-1}(2+\frac{4}{d})-d}{2+\frac{4}{d}}\int_{\mathbb{R}^d}|u|^{2+\frac{4}{d}}\\ &= \left(\frac{4}{p-1}+2-d\right)E(u) - \frac{\frac{4}{d}}{2+\frac{4}{d}}\left(\frac{2}{p-1}-\frac{d}{2}\right)\int_{\mathbb{R}^d}|u|^{2+\frac{4}{d}}.\end{aligned}$$

Together with $\|u_\lambda\|_2 = \lambda^{\frac{2}{p-1}-\frac{d}{2}}\|u\|_2$, which implies $\frac{d}{d\lambda}\|u_\lambda\|_{2|\lambda=1} > 0$, this proves that

$$I(M) \text{ is decreasing in } M. \tag{B.3}$$

To finish, let (u_n) be a minimizing sequence. Up to a subsequence and from the standard radial compactness of Sobolev embeddings (see [3])

$$u_n \rightharpoonup u \quad \text{in} \quad H^1(\mathbb{R}^d), \quad u_n \to u \quad \text{in} \quad L^q, \quad 2 < q \le 2 + \frac{4}{d}.$$

Hence

$$E(u) \le I_M \quad \text{and} \quad \|u\|_2 \le M.$$

From (B.3) and the definition of I_M, we deduce $\|u\|_2 = M$ and $E(u) = I_M$. From a standard Lagrange multiplier argument, u satisfies

$$\Delta u + |u|^{1+\frac{4}{d}}u + |u|^{p-1}u = \omega u$$

for a constant $\omega \in \mathbb{R}$. The sign $\omega > 0$ now follows from a standard Pohozaev type argument.

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