

**Interaction energy of domain walls of logarithmically decaying tails in  
a nonlocal variational model**

RADU IGNAT (JOINT WORK WITH ROGER MOSER)

**The model.** For  $\alpha \in (0, \pi)$ , consider maps  $m = (m_1, m_2) : (-1, 1) \rightarrow \mathbb{S}^1$  with

$$(1) \quad m_1(-1) = m_1(1) = \cos \alpha.$$

For  $\varepsilon > 0$ , consider the energy

$$E_\varepsilon(m) = \varepsilon \int_{-1}^1 |m'|^2 dx_1 + \int_{\mathbb{R}_+^2} |\nabla u|^2 dx,$$

where  $u : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  is determined (up to a constant) by the boundary value problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \mathbb{R}_+^2, \\ \frac{\partial u}{\partial x_2} &= -m'_1 && \text{on } \mathbb{R} \times \{0\}, \end{aligned}$$

where  $m_1$  is extended by  $\cos \alpha$  outside of  $(-1, 1)$  and  $x = (x_1, x_2)$ . The energy  $E_\varepsilon$  can be written as a strictly convex functional in  $m_1$ :

$$E_\varepsilon(m) = \varepsilon \int_{-1}^1 \frac{(m'_1)^2}{1 - m_1^2} dx_1 + \|m_1\|_{H^{1/2}(\mathbb{R})}^2.$$

This represents a simplified version of the free energy of a magnetisation vector field  $m$  in a thin film of a ferromagnetic material (for more details on the model, see e.g. [3]) and  $u$  is called the stray field potential.

**Néel walls.** We are interested in transition layers corresponding to rotations between  $(\cos \alpha, \pm \sin \alpha)$  and  $(\cos \alpha, \mp \sin \alpha)$  on the unit circle  $\mathbb{S}^1$ . Such a transition is called Néel wall and is typically a two-length scale object (a core and two logarithmically decaying tails) with an energy  $E_\varepsilon$  of order  $\pi\gamma_\pm^2/|\log \varepsilon|$  as  $\varepsilon \rightarrow 0$  (see [5]). Here,  $\gamma_\pm = \pm 1 - \cos \alpha$  stands for the height of the transition in  $m_1$  when  $m_1$  passes through  $\pm 1$ .

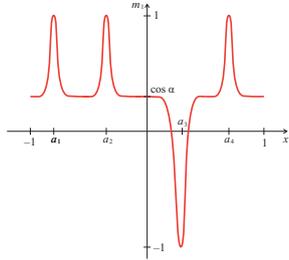


FIGURE 1. Several Néel walls of positions  $a_n$ ,  $1 \leq n \leq 4$ .

We are particularly interested in the interaction of several transitions (see Figure 1). For fixed  $-1 < a_1 < \dots < a_N < 1$  and  $d_n \in \{\pm 1\}$ ,  $n = 1, \dots, N$ , set

$$M(a, d) = \left\{ m : (-1, 1) \rightarrow \mathbb{S}^1 \text{ with (1) and } m_1(a_n) = d_n \text{ for } 1 \leq n \leq N \right\}.$$

Note that minimizers of  $E_\varepsilon$  over  $M(a, d)$  exist and have a unique component  $m_1$  that is smooth away from the positions  $a_n$ ,  $1 \leq n \leq N$ .

**Main result.** We estimate the minimal energy  $E_\varepsilon$  required for a profile in  $M(a, d)$ .

**Theorem 1** (Ignat-Moser [4]). *As  $\varepsilon \rightarrow 0$ , we have*

$$\inf_{M(a, d)} E_\varepsilon = \pi \sum_{n=1}^N \frac{\gamma_n^2}{\log \frac{1}{\delta}} + \frac{W(a, d)}{(\log \frac{1}{\delta})^2} + o\left(\frac{1}{(\log \frac{1}{\delta})^2}\right)$$

where  $\delta = \varepsilon |\log \varepsilon|$ ,  $\gamma_n = d_n - \cos \alpha$  and

$$W(a, d) = \sum_{n=1}^N (e(d_n) - \pi \gamma_n^2 \log(2 - 2a_n^2)) - \pi \sum_{n=1}^N \sum_{k \neq n} \gamma_k \gamma_n \log \left( \frac{1 + \sqrt{1 - \rho(a_k, a_n)}}{\rho(a_k, a_n)} \right)$$

where  $e(\pm 1) > 0$  and  $\rho(a_k, a_n) = \frac{|a_k - a_n|}{1 - a_k a_n}$ .

In analogy to the theory of Ginzburg-Landau vortices (see [1]), we call  $W(a, d)$  the renormalised energy for the  $N$  walls placed at  $a = (a_1, \dots, a_N)$  with signs  $d = (d_1, \dots, d_N)$ . As the theorem shows,  $W(a, d)$  represents the next-to-leading order term in the expansion of  $\inf_{M(a, d)} E_\varepsilon$  in  $1/|\log \delta|$ . This is an improvement of the result in [2] giving only the first leading order term of  $E_\varepsilon$ .

We now briefly discuss how the above expression comes about. Suppose that for a given  $a \in A_N$ , we study minimisers  $m$  of  $E_\varepsilon$  in  $M(a, d)$ . When  $\varepsilon$  is small, we expect to have a typical Néel wall profile near each of the points  $a_1, \dots, a_N$  with the prescribed signs  $d_1, \dots, d_N$ , and the full transition layer  $m$  is essentially a superposition of all of these. We can think of a Néel wall as consisting of two parts: a small core around  $a_n$  and two logarithmically decaying tails. In our situation, the walls are confined in the relatively short interval  $(-1, 1)$  and each tail will interact with the other walls and with the boundary as well. We can then account for the full energy  $\inf_{M(a, d)} E_\varepsilon$  (at leading and next-to-leading order) as follows.

**Core energy.** The core of each wall requires a certain amount of energy, namely  $\frac{e(\pm 1)}{(\log \frac{1}{\delta})^2}$  for a positive and a negative wall, respectively. The constants  $e(\pm 1)$  represent the rescaled energy of the core profile as  $\varepsilon \rightarrow 0$ . This is the only term where we have a contribution from the Dirichlet integral of  $m$  and it appears only at next-to-leading order in the full energy. All the remaining terms below come from the stray field energy alone.

**Tail energy.** The two tails of the wall at  $a_n$  give rise to the energy  $\frac{\pi \gamma_n^2}{\log \frac{1}{\delta}}$ . This is the leading order term of the full energy.

**Tail-boundary interaction.** Moving a wall relative to the boundary points  $\pm 1$  will deform the tail profile, resulting in a change of the energy. This phenomenon gives rise to the energy  $\frac{\pi\gamma_n^2 \log(2-2a_n^2)}{(\log \frac{1}{\delta})^2}$  for the wall at  $a_n$ . (The sign here is not a mistake; it is the opposite of the sign of the corresponding expression in Theorem 1.) This means that the tails are attracted by the boundary, in the sense that the energy decreases if  $a_n$  approaches  $\pm 1$ .

**Tail-tail interaction.** There is an energy contribution coming from reinforcement or cancellation between the stray fields generated by different walls. For the walls at  $a_k$  and  $a_n$  with  $k \neq n$ , this amounts to

$$\frac{\pi\gamma_k\gamma_n}{(\log \frac{1}{\delta})^2} \log \left( \frac{1 + \sqrt{1 - \varrho(a_k, a_n)^2}}{\varrho(a_k, a_n)} \right).$$

(Again we have the opposite sign relative to the above theorem.) A conclusion is that the tails of two walls attract each other if they have opposite signs and repel each other if they have the same sign.

**Tail-core interaction.** Since the profile of a Néel wall decays only logarithmically, it will change the turning angle of the neighbouring walls slightly. This has an effect on the energy as well (at the next-to-leading order). Indeed, the tail of the wall at  $a_k$  and the core of the wall at  $a_n$  with  $k \neq n$  lead to a contribution of

$$-\frac{2\pi\gamma_k\gamma_n}{(\log \frac{1}{\delta})^2} \log \left( \frac{1 + \sqrt{1 - \varrho(a_k, a_n)^2}}{\varrho(a_k, a_n)} \right).$$

We also have an interaction between the two tails of a wall and its own core: if  $k = n$ , then we obtain the energy  $-\frac{2\pi\gamma_n^2 \log(2-2a_n^2)}{(\log \frac{1}{\delta})^2}$ . This is twice the size of the terms from the tail-boundary interaction and tail-tail interaction, but with the opposite signs, resulting in a net repulsion between walls of opposite signs and a net attraction between walls of the same sign. Furthermore, we have a net repulsion of the walls by the boundary.

Notwithstanding the term ‘energy’ used in this description, strictly speaking, these are energy differences and therefore some of them may be negative. All except one of these contributions occur similarly in the theory of Ginzburg-Landau vortices. The core-tail interaction, on the other hand, is new and more delicate to handle.

## REFERENCES

- [1] F. Béthuel, H. Brezis, F. Hélein, *Ginzburg-Landau vortices*, Birkhäuser Boston Inc., 1994.
- [2] A. DeSimone, R.V. Kohn, S. Müller, F. Otto, *Repulsive interaction of Néel walls, and the internal length scale of the cross-tie wall*, Multiscale Model. Simul. **1** (2003), 57–104.
- [3] A. DeSimone, R.V. Kohn, S. Müller, F. Otto, *Recent analytical developments in micromagnetics*, The Science of Hysteresis, Vol. 2, 269–381, Elsevier Academic Press 2005.
- [4] R. Ignat, R. Moser, *Interaction energy of domain walls in a nonlocal Ginzburg-Landau type model from micromagnetics*, Arch. Ration. Mech. Anal. **221** (2016), 419–485.
- [5] C. Melcher, *The logarithmic tail of Néel walls*, Arch. Ration. Mech. Anal. **168** (2003), 83–113.