Maxwell’s equations and the Lorentz force

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An introduction to magnetohydrodynamics

The motion equations

The dynamics of a plasma results from the combination of fluid effects (pression, viscosity...), and electromagnetic effects (Lorentz force).

\[ \partial_t u + (u \cdot \nabla u) + \nabla p - \mu \Delta u - (\rho E + j \wedge B) = 0 \]

\( \text{Navier-Stokes dynamics} \quad \text{Lorentz force} \)

where

- \((u, p)\) are the bulk velocity and pressure of the plasma: for incompressible fluids, \(p\) is the Lagrange multiplier associated with the incompressibility condition \(\nabla \cdot u = 0\).
- \((\rho, j)\) are the charge and current densities
  \[ \partial_t \rho + \nabla \cdot j = 0. \]
- \((E, B)\) are the electric and magnetic fields
Equations for the electromagnetic field

If the electromagnetic field is self-induced, it satisfies Maxwell’s equations, namely

- the Faraday equation for the magnetic induction
  \[ \partial_t B - \text{rot} E = 0 \]

- the Ampère equation with Maxwell’s correction
  \[ \varepsilon_0 \mu_0 \partial_t E - \text{rot} B = -\mu_0 j \]

where \( \varepsilon_0 \) and \( \mu_0 \) are the permittivity and permeability of vacuum.

Note that the Ampère equation, together with the conservation of charge, implies the Gauss equation

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} . \]
Which closure relation?

For a plasma consisting of two species of charged particles (electrons and ions), we expect to get different closure relation depending on the ratio of masses, ratio of charges, Debye length...

Zero mass approximation
In the case when electrons move in a fixed (homogeneous) background of (heavy) ions, the current is proportional to the bulk velocity of electrons

\[ j = q u , \]

where \( q \) is the elementary charge of electrons.

Bi-fluid system
In the case when both species have comparable masses, it is natural to define the velocity of the fluid and the current by

\[ u = \frac{m_+ u_+ + m_- u_-}{m_+ + m_-} , \quad j = q_+ u_+ + q_- u_- , \]

which implies in particular that both quantities are independent variables.
Ohm’s law in incompressible regime

Ohm’s law is the most common closure relation used by physicists

\[ j = \sigma (E + u \wedge B) \]

where \( \sigma \) is the conductivity of the plasma.

- Our goal was to justify such a law starting from a microscopic model, i.e. to get the scaling conditions under such an approximation is valid.

- In incompressible regimes, we will actually obtain some modified Ohm’s laws

\[ j = \sigma P(E + u \wedge B) \]

when the density of charges \( \rho = 0 \) (where \( P \) is the Leray projection), and

\[ j - \rho u = \sigma (E + u \wedge B) - \frac{1}{2} \nabla \rho \]

when quasineutrality is not achieved.
A program for deriving Ohm’s law

▶ The microscopic point of view

At the microscopic level, the dynamical system consists of a huge number $N$ of charged particles, governed by Newton’s dynamics

$$\frac{dX_i}{dt} = V_i, \quad m_i \frac{dV_i}{dt} = q_i (E + V_i \wedge B)$$

coupled with Maxwell’s equations where $j = \sum_{i=1}^{N} q_i V_i$.

Assuming that the plasma is rarefied, we get - for large $N$- a statistical description of the plasma. The distribution function of each species is then governed by a kinetic equation of the type

$$\partial_t f_i + v \cdot \nabla_x f_i + \frac{q_i}{m_i} (E + v \wedge B) \cdot \nabla_v f_i = Q(f_i, f_i) + \sum_{i \neq j} Q(f_i, f_j)$$

where $Q$ is typically the Boltzmann or Landau collision operator, and $(E, B)$ is the electromagnetic field given by Maxwell’s equations with $j = \sum_{i} q_i \int f_i v dv$. 
Physical parameters and scalings

In the absence of electromagnetic fields, the dynamics is characterized by two nondimensional parameters

- the Knudsen number $Kn$ measuring the relaxation towards local thermodynamic equilibrium, i.e. the accuracy of the hydrodynamic approximation;
- the Mach number $Ma$ measuring the compressibility of the fluid.

For perfect gases, the inverse viscosity is related to these two parameters by the Von Karman relation $Re = Ma/Kn$. For the sake of simplicity, we will assume here that $Kn = Ma = \varepsilon << 1$.

Up to change of units, the electric and magnetic effects depend essentially on three physical parameters

- the speed of light $(\varepsilon_0\mu_0)^{-1/2}$;
- the strength of the magnetic induction;
- the strength of the electrostatic repulsion.

These two last parameters depend on the species, via the elementary mass and charge.
A range of limiting models

Define $\rho$ as the limit of $(\rho_+ - \rho_-)$ and $j$ as the limit of $\varepsilon^{-\beta}(u_+ - u_-)$.

- From the mass equations
  \[
  \partial_t (\rho_+ - \rho_-) + \frac{1}{\varepsilon} \text{div}(u_+ - u_-) = 0
  \]
  we deduce the conservation of charge:
  \[
  \text{div} j = 0 \text{ if } \beta < 1, \quad \partial_t \rho + \text{div} j = 0 \text{ if } \beta = 1.
  \]

- At leading orders, the momentum equations express the balance between the pressure, the Lorentz force and the friction
  \[
  \frac{1}{\varepsilon} \nabla(\rho_+ - \rho_-) - \frac{1}{\varepsilon^{\alpha}} \left(2E + (u_+ + u_-) \wedge B\right) = 2 \frac{1}{\sigma \varepsilon^{2\beta}} (u_- - u_+ - \varepsilon \rho_- u_- + \varepsilon \rho_+ u_+)
  \]
  We get some Ohm’s law if $\alpha = \beta$. More precisely,
  \[
  \nabla \rho = 0 \text{ and } j = \sigma P(E + u \wedge B) \text{ if } \alpha = \beta < 1, \quad j = \sigma (E + u \wedge B) + \rho u - \frac{1}{2} \nabla \rho \text{ if } \alpha = \beta = 1.
  \]
Strategy of proof

As usual for hydrodynamic limits, the best we can do is to recognize the structure of the limiting system in the scaled Maxwell-Boltzmann system.

• In incompressible viscous regime, in the absence of electromagnetic field, we take advantage of the analogy between the entropy inequality for the scaled Boltzmann equation and the Leray energy inequality, and prove some global weak convergence to the Navier-Stokes equations.

The main specific point is the fact that the spatial regularity is obtained by combining regularity with respect to $v$ (due to the relaxation), and some hypoellipticity property (the transport being balanced by the relaxation).

• When adding the electromagnetic effects, the same strategy can be applied as long as constraint equations remain linear, and the Lorentz force depends linearly on $E$ and $B$.

For instance, in the zero mass limit, the program should be fulfilled up to technical difficulties. Deriving Ohm’s law is much more difficult.
Some mathematical challenges

The kinetic equations and their fast relaxation limit?

- The Vlasov-Maxwell-Boltzmann system is not known to have global (even renormalized) solutions. We have to prove the existence of very weak solutions (in the sense of Alexandre and Villani).
- For these solutions, local conservation laws are not satisfied. We have to recover them in the fast relaxation limit.
- In the case of many species, the collision operator forces some coupling between equilibrium states. Understanding relaxation requires then a refined study of the linearized collision operator.

Oscillations generated by the macroscopic perturbation?

- When the singular perturbation is linear, it generates a group of high frequency oscillations. The point is to check that these waves have no constructive interferences (compensated compactness).
- When the singular perturbation is nonlinear, there is no systematic method to study deviations from the equilibrium.

Stability of the Navier-Stokes-Maxwell system?
The Navier-Stokes-Maxwell equations with Ohm’s law

For the sake of simplicity, we will focus on the simplified model

\[
\begin{align*}
\partial_t u + (u \cdot \nabla u) + \nabla p - \mu \Delta u - j \wedge B &= 0 \\
\mu &
\end{align*}
\]

\[
 j = \sigma \textbf{P}(E + u \wedge B), \quad \rho = \text{div} E = 0 .
\]

A natural framework to study these equations (coming from physics!) should be the energy space, i.e. the functional space defined by the (formal) energy conservation. We indeed expect solutions in this space to be global.

▶ The energy inequality

The kinetic energy is obtained by integrating the motion equation:

\[
\begin{align*}
\frac{1}{2} \|u(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \\
= \frac{1}{2} \|u_0\|_{L^2}^2 + \int_0^t \int j \wedge B \cdot u \, dx \, ds
\end{align*}
\]
The **electromagnetic energy** is given by Maxwell’s equations

\[
\|E(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \\
= \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 + 2 \int_0^t \int (E \cdot (\text{rot } B - j) - B \cdot \text{rot } E) \, dx \, ds
\]

By Ohm’s law,

\[
\int j \wedge B \cdot u \, dx = - \int j \cdot (\sigma^{-1} j - E) \, dx.
\]

Combining these formal identities leads to the **global conservation** of

\[
\mathcal{E}(t) = \|u(t)\|_{L^2}^2 + \|E(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + \int_0^t (2\mu \|\nabla u(s)\|_{L^2}^2 + 2\sigma^{-1} \|j\|_{L^2}^2) \, ds
\]
Lack of compactness for $E$ and $B$

The energy estimate shows that all terms in the motion equation and in Ohm’s law make sense. But it does not guarantee weak stability of the Lorentz force $j \wedge B$.

Furthermore, we do not expect to gain regularity (or even compactness) on the electromagnetic field $(E, B)$. Maxwell’s equations are indeed the archetype of hyperbolic equations, meaning that singularities are propagated.

These oscillations may be created

- either by boundary or initial data
- or by the nonlinear source terms

They are localized on the corresponding light cones. A natural question is to know whether or not they can be an obstacle to the stability of the Lorentz force $j \wedge B$. 
About the Cauchy problem

Global strong solutions in 2D (N. Masmoudi)
Assume that \((u_0, E_0, B_0) \in L^2(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2\) for \(s \in ]0, 1[\).
Then there exists a unique global strong solution. This comes from

- the conservation of energy,
- some logarithmic loss estimate to bound the \(L^\infty\) norm of \(u\) in terms of its \(H^1\) norm,
- some propagation of regularity for the electromagnetic field.

Small solutions in 3D (S. Ibrahim & S. Keraani)
Assume that \((u_0, E_0, B_0) \in B^{1/2}_{2,1}(\mathbb{R}^3) \times (H^{1/2}(\mathbb{R}^3))^2\) is sufficiently small.
Then there exists a unique global strong solution. The proof uses

- refined a priori estimates obtained by para-differential calculus,
- some fixed point argument.

This does not imply the local existence of strong solutions for large data.
Stability of smooth solutions and dissipative solutions

Strong-weak uniqueness

Around smooth solutions, we get stability by Dafermos’ uniqueness principle: define the modulated energy

\[
\delta \mathcal{E}(t) = \|(u - \bar{u})(t)\|^2_{L^2} + \|(E - \bar{E})(t)\|^2_{L^2} + \|(B - \bar{B})(t)\|^2_{L^2}
\]
\[
+ \int_0^t (2\mu\|\nabla (u - \bar{u})(s)\|^2_{L^2} + 2\sigma^{-1}\|(j - \bar{j})(s)\|^2_{L^2}) \, ds
\]

A direct computation based on the energy inequality and the evolution equations leads to the modulated energy inequality

\[
\delta \mathcal{E}(t) \leq \delta \mathcal{E}(0) - \int_0^t \int \nabla \bar{u} : (u - \bar{u}) \otimes^2 dxds
\]
\[
- \int_0^t \int ((j - \bar{j}) \wedge (B - \bar{B}) \cdot \bar{u} + (u - \bar{u}) \wedge (B - \bar{B}) \cdot \bar{j}) \, dxds
\]
By Cauchy-Schwarz inequality and Sobolev embeddings, we can estimate the flux terms

\[
\delta \mathcal{E}(t) \leq \delta \mathcal{E}(0) + \int_0^t \left( \frac{2}{\mu} \| \bar{u} \|_{L^\infty}^2 \| u - \bar{u} \|_{L^2}^2 + \frac{\mu}{2} \| \nabla (u - \bar{u}) \|_{L^2}^2 \right) \, ds \\
+ \int_0^t \left( \frac{1}{2\sigma} \| j - \bar{j} \|_{L^2}^2 + 2\sigma \| B - \bar{B} \|_{L^2}^2 \| \bar{u} \|_{L^\infty}^2 \right) \, ds \\
+ \int_0^t \left( \frac{\mu}{\kappa} \| u - \bar{u} \|_{L^6}^2 + \frac{\kappa}{\mu} \| B - \bar{B} \|_{L^2}^2 \| \bar{j} \|_{L^3}^2 \right) \, ds
\]

Gronwall’s lemma gives then the stability inequality

\[
\delta \mathcal{E}(t) \leq \delta \mathcal{E}(0) \exp \left( 2(\mu^{-1} + \sigma) \int_0^t \| \bar{u}(s) \|_{L^\infty}^2 \, ds + \kappa \mu^{-1} \int_0^t \| \bar{j}(s) \|_{L^3}^2 \, ds \right)
\]

which provides in particular uniqueness.
Notion of dissipative solution

By analogy with Lions dissipative solutions to the incompressible Euler equations, we define dissipative solutions to the Navier-Stokes-Maxwell equations \((u, B, E, j) \in (L^\infty(L^2_{\text{div}}))^2 \times L^\infty(L^2) \times L^2(L^2)\), weakly continuous in time, and such that for any smooth \((\bar{u}, \bar{E}, \bar{B}, \bar{j})\)

\[
\delta \mathcal{E}(t) \leq \delta \mathcal{E}(0) \exp(\gamma(t)) + \int_0^t \int (j - \bar{j}) \cdot (\sigma^{-1}\bar{j} - \bar{E} - \bar{u} \wedge \bar{B}) \exp(\gamma(s)) \, dx ds
\]

\[
- \int_0^t \int \begin{pmatrix} u - \bar{u} \\ E - \bar{E} \\ B - \bar{B} \end{pmatrix} \cdot \mathbf{A}(\bar{u}, \bar{E}, \bar{B}, \bar{j}) \exp(\gamma(s)) \, dx ds
\]

where the acceleration operator is defined by

\[
\mathbf{A}(\bar{u}, \bar{E}, \bar{B}, \bar{j}) = \begin{pmatrix} \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} - \mu \Delta \bar{u} - \bar{j} \wedge \bar{B} \\ \partial_t \bar{E} - \text{rot} \bar{B} + \bar{j} \\ \partial_t \bar{B} + \text{rot} \bar{E} \end{pmatrix}
\]

and the growth rate is given by

\[
\gamma(t) = C(\mu^{-1} + \sigma) \int_0^t \|\bar{u}(s)\|^2_{L^\infty} ds + C\mu^{-1} \int_0^t \|\bar{j}(s)\|^2_{L^3} ds
\]
Such solutions **always exist**. One way to get the existence is to take weak limits in some coupled Navier-Stokes-Maxwell system

\[
\begin{cases}
\partial_t u_+ + (u_+ \cdot \nabla u_+) + \nabla p_+ - \mu \Delta u_+ - \frac{1}{\varepsilon} (E + u_+ \wedge B) = \frac{u_- - u_+}{\varepsilon} \\
\partial_t u_- + (u_- \cdot \nabla u_-) + \nabla p_- - \mu \Delta u_- + \frac{1}{\varepsilon} (E + u_- \wedge B) = \frac{u_+ - u_-}{\varepsilon} \\
\partial_t E - \text{rot } B = -\frac{u_+ - u_-}{\varepsilon}, \\
\partial_t B + \text{rot } E = 0
\end{cases}
\]

Dissipative solutions **coincide with the unique smooth solution** with same initial data as long as the latter does exist. The required smoothness is given by the growth rate.

Dissipative solutions **are not known to be weak solutions** of the Navier-Stokes-Maxwell equations in conservative form.
Maxwell’s equations and the Lorentz force
Stability of smooth solutions and dissipative solutions
Convergence

▶ Convergences

• By definition, for well-prepared initial data (i.e. for data satisfying the nonlinear constraint), one has the convergence from the bi-fluid system to dissipative solutions of the Navier-Stokes-Maxwell system.

• For general initial data, we expect Ohm’s law not to be satisfied only in some initial layer. Indeed the deviation from equilibrium is governed by some relaxation process. We should therefore be able to also prove convergence (work in progress).

• Starting from the Vlasov-Maxwell-Boltzmann system, it should be possible to get similar convergence results using some improved relative entropy method, allowing to consider only approximate conservation laws (work in progress).