A modified least action principle to handle dissipative phenomena
Application to the reconstruction of the early universe

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1. The least-action principle for a special class of dynamical systems with squared distance potentials
Outline

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2. Conservative vs gradient flow solutions and modified action principle
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2. Conservative vs gradient flow solutions and modified action principle
3. Application to the reconstruction of the early universe (EUR problem)
THE CLASSICAL PRINCIPLE OF LEAST ACTION

Let $H$ be a Euclidean space. Consider the dynamical system

\[
\frac{d^2X}{dt^2} = \nabla Q[X]
\]

where $Q$ is given and $\nabla$ denotes the gradient operator in $H$. 
THE CLASSICAL PRINCIPLE OF LEAST ACTION

Let $H$ be a Euclidean space. Consider the dynamical system

$$\frac{d^2X}{dt^2} = \nabla Q[X]$$

where $Q$ is given and $\nabla$ denotes the gradient operator in $H$. This equation can be derived from the "least action principle" which amounts to finding critical points of the "action"

$$\int_{t_0}^{t_1} 1 \frac{dX}{dt} \frac{dX}{dt} + Q[X] \ dt$$

among all curves $t \in [t_0, t_1] \rightarrow X(t)$ with fixed values at $t_0$ and $t_1$. 
Let $H$ be a Euclidean space and $S$ a bounded closed subset. We consider

$$Q[X] = \inf_{s \in S} \frac{||X - s||^2}{2} = \frac{||X||^2}{2} - R[X]$$

and the corresponding dynamical system

$$\frac{d^2X}{dt^2} = \nabla Q[X] = X - \nabla R[X]$$

Notice that $R$ is Lipschitz, convex, but not smooth unless $S$ is convex.
A SPECIAL CLASS OF DYNAMICAL SYSTEMS

Let $H$ be a Euclidean space and $S$ a bounded closed subset. We consider

$$Q[X] = \inf_{s \in S} \frac{||X - s||^2}{2} = \frac{||X||^2}{2} - R[X]$$

$$R[X] = \sup_{s \in S} ((X, s)) - \frac{1}{2} ||s||^2$$

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Notice that $R$ is Lipschitz, convex, but not smooth unless $S$ is convex.
AN EXAMPLE: GRAVITATION OF N PARALLEL "PANCAKES"

Let $H = \mathbb{R}^N$ the configuration space of $N$ points moving along the real line. We define

$$S = \{ (a_{\sigma_1}, \ldots, a_{\sigma_N}), \quad \sigma \in \text{Perm}_N \}$$

where $a_j = j/N - 1/2$, are $N$ uniformly spaced grid points and $\text{Perm}_N$ is the set of all permutations of the $N$ first integers.
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where $a_j = j/N - 1/2$, are $N$ uniformly spaced grid points and $\text{Perm}_N$ is the set of all permutations of the $N$ first integers. We find

$$\frac{d^2 X_i}{dt^2} = X_i - \frac{1}{2N} \sum_{j \neq i} \text{sgn}(X_i - X_j)$$

This describes the gravitational interaction of $N$ parallel planes ("pancakes") with a repulsive background.
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This describes the gravitational interaction of $N$ parallel planes ("pancakes") with a repulsive background. NB: a multi-d version can be defined and corresponds to a "Monge-Ampère" nonlinear correction to the classical Newton gravitation.
CONSERVATIVE SOLUTIONS

Since R is Lipschitz and convex, its gradient has bounded variation. Thus, according to the theory of Bouchut (and Ambrosio) for second-order ODEs with coefficients of bounded variation, the dynamical system

$$\frac{d^2X}{dt^2} = \nabla Q[X] = X - \nabla R[X]$$

admits a flow of global $C^1$ solutions, uniquely defined for "almost every initial condition" (in a sense made precise by Ambrosio)

$$X(t = 0), \quad \frac{dX(t = 0)}{dt}$$
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$$X(t = 0), \quad \frac{dX(t = 0)}{dt}$$

These solutions are "conservative" and time-reversible. For systems of particles, this corresponds to elastic, non-dissipative collisions.
REWRITING THE ACTION FOR "GOOD" CURVES

Unless S is convex, there is a non-empty, while small (in both the Baire category sense and the Lebesgue measure sense), "bad" set N on which Q is not differentiable, corresponding to points with multiple closest points on the subset S.

If X does not belong to this bad set and has a unique closest point \( \pi[X] \), then

\[
\nabla Q[X] = X - \pi[X],
Q[X] = \left\| X - \pi[X] \right\|^2 = \left\| \nabla Q[X] \right\|^2
\]

We call "good curve" a curve \( t \rightarrow X(t) \in H \) that stay valued outside of the bad set for almost every time. For such a curve, the action can be rewritten

\[
\frac{1}{2} \int_0^T \left\| \frac{dX}{dt} \right\|^2 dt + \left\| \nabla Q[X] \right\|^2 dt
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$$\frac{1}{2} \int_{t_0}^{t_1} \| \frac{dX}{dt} \|^2 + \|\nabla Q[X]\|^2 \ dt$$
REARRANING SQUARES

For any "good" curve (i.e. which almost never hits the bad set of points with multiple closest points) the action has been written

\[
\frac{1}{2} \int_{t_0}^{t_1} \left\| \frac{dX}{dt} \right\|^2 + \left\| \nabla Q[X] \right\|^2 \, dt
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\]

which can be rearranged as a perfect square up to a boundary term that does not play any role in the least action principle

\[
\frac{1}{2} \int_{t_0}^{t_1} \left\| \frac{dX}{dt} \right\|^2 - \left\| \nabla Q[X] \right\|^2 \, dt + Q[X(t_1)] - Q[X(t_0)]
\]
Due to the special structure of the action, we find as particular least action solutions any solution to the first-order "GRADIENT FLOW EQUATION"

\[
\frac{dX}{dt} = \nabla Q[X(t)] = X - \nabla R[X(t)] , \quad R[X] = \sup_{s \in S}\{(X, s) - \frac{1}{2}||s||^2\}
\]

(just like "instantons" in YANG-MILLS theory)
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\[ \frac{dX}{dt} = \nabla Q[X(t)] = X - \nabla R[X(t)] , \quad R[X] = \sup_{s \in S} \left\{ \left( (X, s) \right) - \frac{1}{2} \|s\|^2 \right\} \]

(just like "instantons" in YANG-MILLS theory)
However, this is correct only when \( t \to X(t) \in H \) is a "good" curve (i.e. almost never hits the bad set)
Since $R$ is Lipschitz and convex, we may use the classical theory of maximal monotone operators (going back to the 70’s, see Brezis book) to solve the initial value problem for the gradient-flow equation.

For each initial condition, there is a unique global solution $X \in C^0([t_0, +\infty[)$ in the sense

$$dX(t+\delta t) = X(t) - d0R[X(t)], \quad \forall t \geq t_0$$

where, for each $X$, $d0R[X]$ is the “relaxed” gradient of $R$ defined as the unique vector $w \in H$ with lowest norm $||w||$ such that

$$R[Z] \geq R[X] + ((w, Z - X)), \quad \forall Z \in H$$

which coincides with the usual gradient outside of the “bad set”.
GLOBAL SOLUTIONS OF THE GRADIENT FLOW

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\frac{dX(t + 0)}{dt} = X(t) - d^0 R[X(t)] , \quad \forall t \geq t_0
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DISSIPATIVE PROPERTIES OF GRADIENT FLOW SOLUTIONS

The gradient flow solutions are in general neither conservative solutions (in the sense of Bouchut-Ambrosio) to the original dynamical system nor action minimizers, strictly speaking. They are "bad" curves and hit the "bad" set for substantial amounts of time. However, they have interesting dissipative features. Indeed, the velocity may jump with instantaneous loss of kinetic energy. For systems of particles, for instance, this corresponds to sticky collisions instead of elastic collisions. These are interesting properties that we may like (or not, depending on the applications we have in mind) to extend to the original dynamical system.
A PROPOSAL FOR A MODIFIED ACTION

The conservative solutions, that are only defined for almost every initial condition, manage to hit the bad set only for a negligible amount of time, while the gradient flow solutions enjoy very much staying in it as soon as they touch it!!!
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The conservative solutions, that are only defined for almost every initial condition, manage to hit the bad set only for a negligible amount of time, while the gradient flow solutions enjoy very much staying in it as soon as they touch it!!! Our proposal is to pick up the nice dissipative property of the gradient flow solutions and to lift them to the full dynamical system. For that purpose, we introduce the modified action

\[
\int_{t_0}^{t_1} \left| \frac{dX}{dt} - X(t) + d_0 R[X(t)] \right|^2 dt
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\[ \int_{t_0}^{t_1} \left\| \frac{dX}{dt} - X(t) + d^0 R[X(t)] \right\|^2 dt \]

which favor "bad" curves that stay on the bad set for a substantial amount of time. Let us recall that \( d^0 R[X] \) is the "relaxed" gradient of

\[ R[X] = \sup\{((X, s)) - \frac{1}{2}\|s\|^2, \ s \in S\} \]
St George-cross test- 1: conservative solutions
St George-cross test- 2:solutions with dissipation
St George-cross test- 3: conservative solutions - zoom out
St George-cross test- 4:solutions with dissipation-zoom out
St George-cross test- 5:solutions with dissipation-rough time step
APPLICATION TO THE RECONSTRUCTION OF THE EARLY UNIVERSE

Following Peebles 1989, Frisch and coauthors (Nature 417) 2002, we want to reconstruct the history of the Universe from the knowledge of the present mass distribution in the universe. We consider an expanding universe with self-gravitating matter.
THE EARLY UNIVERSE GRAVITATIONAL MODEL

Each particle of matter, labelled by $a$, follows a trajectory $t \to X(t, a)$ driven by

$$
\frac{2t}{3} \frac{d^2X}{dt^2} + \frac{dX}{dt} + \nabla \varphi(t, X(t, a)) = 0
$$

where $\rho$ and $\varphi$ respectively denote the density field and the gravitational potential.

This is a (crude) semi-Newtonian approximation to the Einstein equations where all terms in red come from general relativity (Einstein-de Sitter "big bang" universe).

Notice that at early times $t \downarrow 0$, "friction" takes over "inertia" (by combining Einstein and Newton, we go back to Aristoteles...)
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\rho(t, x) = \int_a \delta(x - X(t, a)) = 1 + t \nabla^2 \varphi(t, x)
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INITIAL CONSTRAINTS

Observe the degeneracy of the model at time $t = 0$

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\frac{2t}{3} \frac{d^2 X}{dt^2} + \frac{dX}{dt} + \nabla \varphi(t, X(t, a)) = 0, \quad 1 + t \nabla^2 \varphi = \rho = \int \delta(x - X(t, a)) \, da
\]

Necessarily, at $t = 0$, the matter has unit density and, at each point, the velocity is slaved by the gravitational potential, in sharp contrast with classical Newtonian and Vlasovian descriptions.
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$$\rho_0(x) = 1, \quad X_0(a) = a, \quad \frac{dX_0}{dt}(a) = -\nabla \varphi_0(a)$$
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Granularity is not possible at $t = 0$ and must come up AT LATER TIMES through some mass concentration mechanism.
RECONSTRUCTING THE EARLY UNIVERSE?

Notice that the ONLY free initial values at \( t = 0 \) are the initial density fluctuations

\[
\rho'_0(x) = \lim_{t \downarrow 0} \frac{\rho(t, x) - 1}{t} = \nabla^2 \varphi_0(x)
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This makes plausible the EUR problem, which amounts to, following Peebles 1989, Frisch and coauthors (Nature 417) 2002, reconstructing the history of the Universe from the only observation of the HIGHLY CONCENTRATED (with essentially no Lebesgue component) density field \( \rho(T, x) \) at present time \( T \).
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Figure 7: Vortex dynamics in a driving fluid.
AN ALTERNATING MODEL:
MONGE-AMPERE GRAVITATION

Let us substitute the Monge-Ampère equation for the Poisson equation

\[ \rho(t, x) = \det(I + tD^2 \phi(t, x)) \quad \text{vs} \quad \rho(t, x) = 1 + t \nabla^2 \phi(t, x) \]

This Monge-Ampère equation i) is exact for pancakes, ii) is asymptotically correct both at early times and for weak fields iii) might be as accurate as the Poisson equation as an approximation of the Einstein equations.

This leads to the MONGE-AMPERE GRAVITATIONAL MODEL

\[ 2t^3 d^2 X dt^2 + dX dt + \nabla \phi(t, X(t)) = 0 \]

\[ \rho(t, x) = \int \delta(x - X(t, a)) da = \det(I + tD^2 \phi(t, x)) \]

\[ Y \]
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\[ \frac{2t}{3} \frac{d^2X}{dt^2} + \frac{dX}{dt} + \nabla\varphi(t, X(t)) = 0 \]

\[ \rho(t, x) = \int \delta(x - X(t, a))da = \det(1 + t\nabla^2\varphi(t, x)) \]
A LEAST ACTION PRINCIPLE FOR MAG

The configuration space is the Hilbert space $H$ of all $L^2$ maps. Using "optimal transport" tools, we can derive the MAG model from the following MAG action

$$\int_{t_0}^{t_1} \frac{2}{3} t^{3/2} \left\| \frac{dX}{dt} \right\|^2 + t^{-1/2} Q[X(t)] \, dt, \quad Q[X] = \inf_{s \in S} \frac{\|X - s\|^2}{2}$$

for curves $t \to X(t) \in H$ where $S \subset H$ is the subset of all volume-preserving maps.
SPECIAL STRUCTURE OF THE MAG ACTION

In the MAG action, the potential part satisfies:

\[
Q[X] = \inf_{s \in S} \frac{||X - s||^2}{2} = \frac{1}{2} ||\nabla Q[X]||^2
\]

and we can rewrite the MAG action:

\[
\frac{1}{2} \int_{t_0}^{t_1} \left( \frac{4}{3} t^{3/2} \left| \frac{dX}{dt} \right|^2 + t^{-1/2} \left| \nabla Q[X(t)] \right|^2 \right) dt
\]
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By reorganizing the squares and integrating by part in time, we find

\[ \frac{8}{3} \int_{t_0}^{t_1} t^{-1/2} \left| t \frac{dX}{dt} - \nabla Q[X(t)] \right|^2 dt + \text{time boundary term} \]
Due to the special structure of the MAG action, we find as particular least action solutions any solution to the GRADIENT FLOW EQUATION

\[ t \frac{dX}{dt} = \nabla Q[X(t)] , \quad Q[X] = \inf \{ ||X - s||^2 ; s \in S \} \]

(just like "instantons" in YANG-MILLS theory)
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It turns out that these gradient flow solutions exactly correspond to the well-known Zeldovich approximate solutions introduced for the semi-newtonian model in the 1970’.
THE MODIFIED MAG ACTION TAKING CONCENTRATIONS INTO ACCOUNT

It turns out that the "relaxed" gradient favors mass concentrations!
Thus, we take concentration into account by introducing a modified MAG action, by substituting the "relaxed" gradient for the regular gradient

\[
\int_{t_0}^{t_1} t^{-1/2} \left| t \frac{dX}{dt} - X(t) + d^0 R[X(t)] \right|^2 \, dt
\]

where we recall that

\[
R[X] = \sup \left\{ \left( (X, s) \right) - \frac{1}{2} \| s \|^2, \ s \in S \right\}
\]
In the next slides, we show samples of 1D simulations, directly based on the minimization of the fully space and time discrete version of the action.
In the next slides, we show samples of 1D simulations, directly based on the minimization of the fully space and time discrete version of the action. As a matter of fact, the discrete scheme does not even rely on the computation of "relaxed" gradients. The calculation entirely relies on many ($\sim 10^5$) iterations of an elementary sorting algorithm.
EUR-case 1: (Zeldovich) solution of the gradient flow equation

horizontal : 51 grid points in x  /vertical : 60 grid points in t
EUR-case 1: reconstructed trajectories

horizontal: 51 grid points in $x$  
vertical: 60 grid points in $t$
EUR-case 1: IVP with reconstructed velocities

horizontal : 51 grid points in x  /vertical : 60 grid points in t
EUR-case 2: reconstructed trajectories

**horizontal**: 51 grid points in $x$  
**vertical**: 60 grid points in $t$
EUR-case 2: IVP with reconstructed velocities

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Among the minimizers of the modified action, we recover, by construction, all solutions of the gradient-flow equation in the sense of maximal monotone operator theory, which do take into account concentration phenomena.
DISCUSSION

Among the minimizers of the modified action, we recover, by construction, all solutions of the gradient-flow equation in the sense of maximal monotone operator theory, which do take into account concentration phenomena. This leads, in our opinion, to a much better handling of the EUR problem, with a drawback: the substitution of the Monge-Ampère gravitation for the Newton gravitation, which is, of course, questionable from the physical viewpoint, except in 1D.
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Abridged bibliography: preprint 2010 available on YB website and HAL preprint server

The EUR problem and the pressure-less Euler-Poisson model

Zeldovich approximation and burglaryence

Conservative and dissipative solutions
H. Brezis, North-Holland Mathematics Studies (1973)

Optimal transportation