INTERPOLATION IN THE NEVANLINNA AND SMIRNOV CLASSES AND HARMONIC MAJORANTS

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ABSTRACT. We consider a free interpolation problem in Nevanlinna and Smirnov classes and find a characterization of the corresponding interpolating sequences in terms of the existence of harmonic majorants of certain functions. We also consider the related problem of characterizing positive functions in the disk having a harmonic majorant. An answer is given in terms of a dual relation which involves positive measures in the disk with bounded Poisson balayage. We deduce necessary and sufficient geometric conditions, both expressed in terms of certain maximal functions.

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Date: May 3, 2004.

1991 *Mathematics Subject Classification.* 30E05, 32A35.

Key words and phrases. free interpolation, Smirnov and Nevanlinna class, harmonic majorants, Poisson balayage.

All authors supported by the PICS program no 1019 of Generalitat de Catalunya and CNRS. First and third author also supported by European Commission Research Training Network HPRN-CT-2000-00116. Second author also supported by the DGICYT grant BFM2002-04072-C02-01 and the CIRIT grant 2001-SGR00172. Third author also supported by DGICYT grant BFM2002-00571 and the CIRIT grant 2001-SGR00431.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. **Interpolating sequences for the Nevanlinna Class.** Let Λ be a discrete sequence of points in the unit disk $\mathbb{D}$. For a space of holomorphic functions X, the interpolation problem consists in describing the trace of X on Λ, i.e. the set of restrictions $X|\Lambda$, regarded as a sequence space. One approach is to fix a target space l and look for conditions so that $X|\Lambda = l$. An alternative approach, known as free interpolation, is to require that $X|\Lambda$ be *ideal*, i.e. stable under multiplication by ℓ^∞. See [Nik02, Section C.3.1 (Volume 2)], in particular, Theorem C.3.1.4, for functional analytic motivations. This approach is natural for those spaces that are stable under multiplication by H^∞, the space of bounded holomorphic functions on $\mathbb{D}$. For Hardy and Bergman spaces both definitions turn out to be equivalent, with the usual choice of l as an ℓ^p space with an appropriate weight (see [ShHSh], [Se93]).

The situation changes for the non-Banach classes we have in mind, namely the *Nevanlinna class*

$$N = \Big\{ f \in \operatorname{Hol}(\mathbb{D}) : \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta < \infty \Big\}$$

and the related *Smirnov class*

$$N^+ = \Big\{ f \in N : \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(e^{i\theta})|\, d\theta \Big\}.$$

We briefly discuss the known results. Naftalevič [Na56] described the sequences Λ for which the trace $N|\Lambda$ coincides with the sequence space $l_{\text{Na}} = \{(a_\lambda)_\lambda : \sup_\lambda (1-|\lambda|)\log^+|a_\lambda| < \infty\}$ (we state the precise result after Proposition 1.12). The choice of l_{Na} is motivated by the fact that $\sup_z (1-|z|)\log^+|f(z)| < \infty$ for $f \in N$, and this growth is attained. Unfortunately, the growth condition imposed in l_{Na} forces the sequences to be confined in a finite union of Stolz angles. Consequently a big class of Carleson sequences (i.e. sequences such that $H^\infty|\Lambda = \ell^\infty$), namely those containing a subsequence tending tangentially to the boundary, cannot be interpolating in the sense of Naftalevič. This does not seem natural, for H^∞ is in the multiplier space of N. In a sense, the target space l_{Na} is "too big". Further comments on Naftalevič's result can be found in [HaMa01] and below, after Proposition 1.12.

For the Smirnov class, Yanagihara [Ya74] proved that in order that $N^+|\Lambda$ contains the space $l_{\text{Ya}} = \{(a_\lambda)_\lambda : \sum_\lambda (1-|\lambda|)\log^+|a_\lambda| < \infty\}$, it is sufficient that Λ is a Carleson sequence. However there are Carleson sequences such that $N^+|\Lambda$ does not embed into l_{Ya} [Ya74, Theorem 3] : the target space l_{Ya} is "too small".

We now turn to the definition of free interpolation.

Definition. A sequence space l is called *ideal* if $\ell^\infty l \subset l$, i.e. whenever $(a_n)_n \in l$ and $(\omega_n)_n \in \ell^\infty$, then also $(\omega_n a_n)_n \in l$.

Definition. Let X be a space of holomorphic functions in $\mathbb{D}$. A sequence $\Lambda \subset \mathbb{D}$ is called *free interpolating for* X if $X|\Lambda$ is ideal. We denote $\Lambda \in \operatorname{Int} X$.

Remark 1.1. For any function algebra X containing the constants, $X|\Lambda$ is ideal if and only if

$$\ell^\infty \subset X|\Lambda.$$

The inclusion is obviously necessary. In order to see that it is sufficient notice that, by assumption, for any $(\omega_\lambda)_\lambda \in \ell^\infty$ there exists $g \in X$ such that $g(\lambda) = \omega_\lambda$. Thus, if $(f(\lambda))_\lambda \in X|\Lambda$, the sequence of values $(\omega_\lambda f(\lambda))_\lambda$ can be interpolated by $fg \in X$.

It is then clear that $\operatorname{Int} N^+ \subset \operatorname{Int} N$.

Free interpolation for these classes entails the existence of nonzero functions vanishing on all Λ except a given λ_0. Hence the Blaschke condition $\sum_{\lambda\in\Lambda}(1-|\lambda|) < \infty$ is necessary and will be assumed throughout this paper.

Given the Blaschke product $B_\Lambda = \prod_{\lambda\in\Lambda} b_\lambda$ with zero-sequence Λ, denote $B_\lambda = B_{\Lambda\setminus\{\lambda\}} = B_\Lambda/b_\lambda$. Here $b_\lambda = (|\lambda|/\lambda)(\lambda - z)(1-\overline{\lambda}z)^{-1}$. Define then

$$\varphi_\Lambda(z) := \begin{cases} \log|B_\lambda(\lambda)|^{-1} & \text{if } z = \lambda \in \Lambda \\ 0 & \text{if } z \notin \Lambda \end{cases}$$

Definition. We say that a Borel measurable function φ defined on the unit disk admits a *positive harmonic majorant* if and only if there exists a positive harmonic function h on the unit disk such that $h(z) \geq \varphi(z)$ for any $z \in \mathbb{D}$.

Let $\operatorname{Har}(\mathbb{D})$ denote the space of harmonic functions in $\mathbb{D}$ and $\operatorname{Har}_+(\mathbb{D})$ the subspace of its positive functions. Consider also the Poisson kernel in $\mathbb{D}$:

$$P(z,\zeta) = P_z(\zeta) = \operatorname{Re}\left(\frac{\zeta+z}{\zeta-z}\right) = \frac{1-|z|^2}{|\zeta-z|^2}.$$

Our characterization of interpolating sequences for the Nevanlinna class is as follows. Note that the existence of a harmonic majorant occurs at two junctures: first, to decide which sequences of points are free interpolating, second, to identify the trace space that arises for those sequences which are indeed free interpolating.

Theorem 1.2. *Let Λ be a sequence in $\mathbb{D}$. The following statements are equivalent:*

(a) *Λ is a free interpolating sequence for the Nevanlinna class N.*
(b) *The trace space is given by:*

$$N|\Lambda = l_N := \{(a_\lambda)_\lambda :\ \exists\, h \in \operatorname{Har}_+(\mathbb{D}) \text{ such that } h(\lambda) \geq \log^+|a_\lambda|,\ \lambda \in \Lambda\}.$$

(c) *φ_Λ admits a harmonic majorant.*
(d) *There exists $C > 0$ such that for any sequence of nonnegative numbers $\{c_\lambda\}$,*

$$\sum_{\lambda\in\Lambda} c_\lambda \varphi_\Lambda(\lambda) = \sum_{\lambda\in\Lambda} c_\lambda \log|B_\lambda(\lambda)|^{-1} \leq C \sup_{\zeta\in\partial\mathbb{D}} \sum_{\lambda\in\Lambda} c_\lambda P_\lambda(\zeta).$$

We recall that any positive harmonic function on the unit disk is the Poisson integral of a positive measure on the unit circle,

$$h(z) = P[\mu](z) = \int_{\partial\mathbb{D}} P_z(\zeta)\, d\mu(\zeta).$$

We will say that a harmonic function is *quasi-bounded* if and only if it admits an absolutely continuous boundary measure (for the reasons for this terminology, see [He69, pp. 6–7]). The analogous result for the Smirnov class will, as can be expected, involve quasi-bounded harmonic functions.

Let $d\sigma$ denotes the normalized Lebesgue measure in $\partial\mathbb{D}$. Also, for a nonnegative function φ on the unit disk, let $M\varphi$ denote the associated non-tangential maximal function (see (1.1) below).

Theorem 1.3. *Let Λ be a sequence in $\mathbb{D}$. The following statements are equivalent:*

(a) Λ *is a free interpolating sequence for the Smirnov class N^+.*
(b) *The trace space is given by*

$$N^+|\Lambda = l_{N^+} := \{(a_\lambda)_\lambda : \exists\, h \in \mathrm{Har}_+(\mathbb{D}) \text{ quasi-bounded } : h(\lambda) \geq \log^+ |a_\lambda|,\ \lambda \in \Lambda\}.$$

(c) φ_Λ *admits a quasi-bounded harmonic majorant.*
(d) $\displaystyle\lim_{n\to\infty} \sup_{\{c_\lambda\}\in\mathcal{B}(\Lambda)} \sum_{\lambda:\varphi_\Lambda(\lambda)\geq n} c_\lambda \varphi_\Lambda(\lambda) = 0$*, where $\mathcal{B}(\Lambda)$ denotes the set of nonegative sequences* $\{c_\lambda\}$ *such that* $\sup_{\zeta\in\partial\mathbb{D}} \sum_{\lambda\in\Lambda} c_\lambda P_\lambda(\zeta) \leq 1$.
(e) *(i)* $\sup_{t>0} t\sigma(\{\zeta \in \partial\mathbb{D} : M\varphi_\Lambda(\zeta) \geq t\}) < \infty$*, and*

(ii) $\lim_{n\to\infty} \sum_{\lambda\in\Lambda} c_\lambda^{(n)} \varphi_\Lambda(\lambda) = 0$ *for any sequence of sequences of nonnegative numbers* $\{c_\lambda^{(n)}\} \in \mathcal{B}(\Lambda)$ *such that* $\lim_{n\to\infty} \sum_{\lambda\in\Lambda} c_\lambda^{(n)} P_\lambda(\zeta) = 0$ *almost everywhere on $\partial\mathbb{D}$.*

The classical Carleson condition characterizing interpolating sequences for bounded analytic functions in the unit disk is $\sup_{\mathbb{D}} \varphi_\Lambda < \infty$, hence statements (c) in both results above can be viewed as Carleson-type conditions.

In view of Theorems 1.2 and 1.3, it seems natural to ask whether the measure μ such that $\varphi_\Lambda \leq P[\mu]$ can be obtained from Λ in a canonical way. We do not have an answer to this question, but with Propositions 1.12 and 1.13 it is easy to construct examples that discard natural candidates, such as the (weighted) sum of Dirac masses $\mu = \sum_\lambda (1-|\lambda|)\delta_{\lambda/|\lambda|}$, or Poisson balayage measures $d\nu = \sum_\lambda (1-|\lambda|) P_\lambda(\zeta)\, d\sigma(\zeta)$ (see definition below).

1.2. **Positive harmonic majorants.** The conditions in Theorems 1.2 and 1.3 (d) arise in the solution of a problem of independent interest:

Problem. *Which functions $\varphi : \mathbb{D} \longrightarrow \mathbb{R}_+$ admit a (quasi-bounded) harmonic majorant?*

Answers to this problem lead to rather precise theorems about the permissible decrease of the modulus of bounded holomorphic functions, e.g. Corollary 1.5 below. See [Hay], [LySe97]; [EiEs] also provides a survey of such results. The existence of harmonic majorants is relevant as well to the study of zero-sequences for Bergman and related spaces of holomorphic functions [Lu96].

An answer to the problem of positive harmonic majorants can be given in dual terms (see [BNT] for another characterization). The *Poisson balayage* (or swept-out function) of a finite positive measure μ in the closed unit disk is defined as

$$B(\mu)(\zeta) = \int_{\mathbb{D}} P_z(\zeta) d\mu(z) \qquad \zeta \in \partial\mathbb{D}.$$

We will be interested in the class of measures having bounded balayage. Recall that Carleson measures are those finite positive measures whose balayage has bounded mean oscillation (see [Gar81, Theorem VI.1.6, p. 229]); this is also an easy consequence of the H^1-BMO duality (see [Gar81, Theorem VI.4.4, p. 245]). Hence positive measures with bounded balayage form a subclass of the usual Carleson measures. It is easy to see (cf. Section 6) that positive measures with bounded balayage are precisely those which operate against positive harmonic functions, that is, those measures μ for which there exists a constant $C = C(\mu)$ such that

$$\int_{\mathbb{D}} h(z)d\mu(z) \leq Ch(0)$$

for any positive harmonic function in the unit disk $\mathbb{D}$.

Define

$$\mathcal{B} := \{\mu \text{ positive Borel measures on } \mathbb{D} \text{ such that } \sup_{\zeta\in\partial\mathbb{D}} B(\mu)(\zeta) \leq 1\}.$$

Theorem 1.4. *Let φ be a nonnegative Borel function on the unit disk $\mathbb{D}$. The following statements are equivalent:*

(a) *There exists a (positive) harmonic function h such that $\varphi(z) \leq h(z)$ for all $z \in \mathbb{D}$.*
(b) *There exists a constant $C = C(\varphi)$ such that*

$$\sup_{\mu\in\mathcal{B}} \int_{\mathbb{D}} \varphi(z)\, d\mu(z) \leq C.$$

The necessity of condition (b) is obvious (e. g. $C = h(0)$), while the sufficiency follows from a convenient version of a classical result in Convex Analysis, known as Minkowski-Farkas Lemma. The characterization of interpolating sequences in the Nevanlinna class in dual terms given by condition (d) in Theorem 1.2 follows from this result.

This can be applied to study the decrease of a non-zero bounded analytic function in the disk along a given non-Blaschke sequence.

Corollary 1.5. *Let Λ be a separated non-Blaschke sequence and $(\varepsilon_\lambda)_{\lambda\in\Lambda}$ a sequence of positive values. Then there exists a non-zero function $f \in H^\infty(\mathbb{D})$ with $|f(\lambda)| < \varepsilon_\lambda$, $\lambda \in \Lambda$, if and only if Λ is the union of a Blaschke sequence and a sequence Γ for which there exists a universal constant $C = C(\Gamma)$ such that*

$$\sum_{\gamma\in\Gamma} c_\gamma \log \varepsilon_\gamma^{-1} \leq C \sup_{\zeta\in\partial\mathbb{D}} \sum_{\gamma\in\Gamma} c_\gamma P_\gamma(\zeta)$$

for any sequence of nonnegative numbers $(c_\gamma)_{\gamma\in\Gamma}$.

In a similar way, Theorem 1.3 (d), (e) are obtained as an application of the following analogue of Theorem 1.4 for quasi-bounded harmonic functions (i.e. for the Smirnov class).

Theorem 1.6. *Let φ be a nonnegative Borel function on the unit disk $\mathbb{D}$. The following statements are equivalent:*

(a) *There exists a (positive) quasi-bounded harmonic function h such that $\varphi(z) \leq h(z)$ for all $z \in \mathbb{D}$.*
(b) *There is a convex increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \psi(t)/t = +\infty$ such that $\psi \circ \varphi$ admits a harmonic majorant on $\mathbb{D}$;*
(c) $\lim_{n\to\infty} \sup_{\mu\in\mathcal{B}} \int_{\{\varphi\geq n\}} \varphi\, d\mu = 0.$

(d) *(i)* $\sup_{t>0} t\sigma(\{\zeta \in \partial\mathbb{D} : M\varphi(\zeta) \geq t\}) < \infty$, *and*

(ii) $\lim_{n\to\infty} \int_{\mathbb{D}} \varphi d\mu_n = 0$ *for any sequence* $\{\mu_n\} \subset \mathcal{B}$ *such that* $\lim_{n\to\infty} B(\mu_n)(\zeta) = 0$ *almost everywhere on* $\partial\mathbb{D}$,

Condition (b) is inspired by a characterization of quasi-bounded harmonic functions given in Armitage and Gardiner's book [ArGa, Theorem 1.3.9, p. 10].

For the problem of harmonic majorants it is desirable to obtain criteria which, although only necessary or sufficient, are more geometric and easier to check than the duality conditions of Theorems 1.4 and 1.6.

Recall that the Stolz angle with vertex $\zeta \in \partial\mathbb{D}$ and aperture α is defined by

$$\Gamma_\alpha(\zeta) := \{z \in \mathbb{D} : |z - \zeta| \leq \alpha(1 - |z|^2)\}.$$

In our considerations the angle α is of no importance, so we will write $\Gamma(\zeta)$ for the generic Stolz angle with aperture α. Given a function f from $\mathbb{D}$ to $\mathbb{R}_+$, the non-tangential maximal function is defined as

$$Mf(\zeta) := \sup_{\Gamma(\zeta)} f. \tag{1.1}$$

Recall that σ denotes the normalized Lebesgue measure on $\partial\mathbb{D}$. Consider the weak-L^1 space

$$L^1_w(\partial\mathbb{D}) = \{f \text{ measurable } : \sup_{t>0} t\sigma(\{\zeta : |f(\zeta)| > t\} < \infty\})$$

and let

$$L^1_{w,0}(\partial\mathbb{D}) = \{f \text{ measurable } : \lim_{t\to\infty} t\sigma(\{\zeta : |f(\zeta)| > t\}) = 0\}.$$

It is well-known that the non-tangential maximal function of the Poisson transform of a positive finite measure belongs to L^1_w (see [Gar81, Theorem 5.1, p. 28]). A more careful analysis shows that if μ is absolutely continuous, then its Poisson transform is in $L^1_{w,0}$. This and some easy estimates imply the following result.

Proposition 1.7. (a) *If* φ *admits a harmonic majorant, then* $M\varphi \in L^1_w(\partial\mathbb{D})$.

(b) *If* φ *admits a positive quasi-bounded harmonic majorant, then* $M\varphi \in L^1_{w,0}(\partial\mathbb{D})$.

(c) *If* $M\varphi \in L^1(\partial\mathbb{D})$, *then the function* φ *admits* $P[M\varphi] := P[M\varphi d\sigma]$ *as a quasi-bounded harmonic majorant.*

As far as necessary conditions are concerned, there is a way to improve the previous result by using the Hardy-Littlewood maximal function. Given $f \geq 0$, this is defined as

$$f^*(x) := \sup \frac{1}{\sigma(I)} \int_I f,$$

where the supremum is taken over all arcs I containing x.

For $\varphi \geq 0$ define

$$\varphi^H(\zeta) := \sup_{z\in\mathbb{D}} \varphi(z)\chi^*_{I_z}(\zeta) = \sup_{z\in\mathbb{D}} \varphi(z) \sup_{I:\zeta\in I} \frac{\sigma(I \cap I_z)}{\sigma(I)},$$

where χ_E is the characteristic function of a set E and I_z is the "Privalov shadow" interval

$$I_z := \{\zeta \in \partial\mathbb{D} : z \in \Gamma(\zeta)\}. \tag{1.2}$$

Proposition 1.8. (a) *If φ admits a harmonic majorant, then $\varphi^H \in L^1_w(\partial\mathbb{D})$.*
(b) *If φ admits a quasi-bounded harmonic majorant, then $\varphi^H \in L^1_{w,0}(\partial\mathbb{D})$.*

We will give some examples in Proposition 7.4 that show that this is indeed stronger than the necessary condition given in the first part of Proposition 1.7, but still falls short of giving a sufficient condition for the existence of a harmonic majorant.

1.3. **Geometric criteria for interpolation.** We would like to obtain some geometric implications of the analytic conditions given in Theorems 1.2 and 1.3. To begin with, we would like to state the maybe surprising result that separated Blaschke sequences (with respect to the hyperbolic distance) are interpolating for the Smirnov class (and hence the Nevanlinna class). Recall that a sequence Λ is called *separated* if $\delta(\Lambda) := \inf_{\lambda\neq\lambda'} \rho(\lambda, \lambda') > 0$, where

$$\rho(z, w) := |b_z(w)| = \left|\frac{z - w}{1 - z\overline{w}}\right|,$$

is the pseudo-hyperbolic distance.

For such sequences, the values $\log |B_\lambda(\lambda)|^{-1}$ can always be majorized by the values at $\lambda \in \Lambda$ of the Poisson integral of an integrable function (see Proposition 4.1), thus the following corollary is immediate from Theorem 1.3.

Corollary 1.9. *Let Λ be a separated Blaschke sequence. Then $\Lambda \in \operatorname{Int} N^+$ (hence $\Lambda \in \operatorname{Int} N$).*

More precise conditions can be deduced from Propositions 1.7, 1.8 and (c) in Theorems 1.2 and 1.3.

Corollary 1.10. *Let Λ be a sequence in $\mathbb{D}$.*

(a) *If $\Lambda \in \operatorname{Int} N$ then $\varphi^H_\Lambda \in L^1_w(\partial\mathbb{D})$. If $\Lambda \in \operatorname{Int} N^+$ then $\varphi^H_\Lambda \in L^1_{w,0}(\partial\mathbb{D})$.*
(b) *If $M\varphi_\Lambda \in L^1(\partial\mathbb{D})$ then $\Lambda \in \operatorname{Int} N^+$ (and hence $\Lambda \in \operatorname{Int} N$).*

Notice that the necessary conditions obtained by replacing φ^H_Λ by $M\varphi_\Lambda$ in (a) also hold. This in an immediate consequence of the estimate $\varphi^H_\Lambda \geq M\varphi_\Lambda$.

This result implies the following Carleson-type conditions.

Corollary 1.11. (a) *If $\Lambda \in \operatorname{Int} N^+$, then*

$$\lim_{|\lambda|\to 1} (1 - |\lambda|) \log |B_\lambda(\lambda)|^{-1} = 0. \tag{1.3}$$

(b) *If $\Lambda \in \operatorname{Int} N$, then*

$$\sup_{\lambda\in\Lambda} (1 - |\lambda|) \log |B_\lambda(\lambda)|^{-1} < \infty. \tag{1.4}$$

(c) *If Λ is Blaschke and*

$$\sum_{\lambda\in\Lambda} (1 - |\lambda|) \log |B_\lambda(\lambda)|^{-1} < \infty, \tag{1.5}$$

then $\Lambda \in \operatorname{Int} N^+$ (and so $\Lambda \in \operatorname{Int} N$ as well).

Condition (1.3) already appeared in [Ya74, Theorem 1] as a necessary condition for the sequence space l_{Ya} (as defined in the beginning of Section 1.1) to be included in the trace of N^+. Condition (1.4) is discussed in Proposition 1.12 and the corollary thereafter.

In some situations the conditions above are indeed a characterization of interpolating sequences. For instance, the weak L^1-condition characterizes interpolating sequences lying on a radius, while for sequences approaching the unit circle very tangentially the characterization is given by the strong L^1-condition. This is collected in the next results.

Proposition 1.12. *Assume that $\Lambda \subset \mathbb{D}$ lies in a finite union of Stolz angles.*

(a) $\Lambda \in \operatorname{Int} N^+$ *if and only if* (1.3) *holds.*
(b) $\Lambda \in \operatorname{Int} N$ *if and only if* (1.4) *holds.*

It should be mentioned that (b) can also be derived from Naftalevič's result [Na56, Theorem 3]. On the other hand, his full characterization of the sequences such that $N|\Lambda = l_{\text{Na}}$ can also be deduced from Theorem 1.2.

Corollary (Naftalevič, 1956)**.** *$N|\Lambda = l_{Na}$ if and only if Λ is contained in a finite union of Stolz angles and* (1.4) *holds.*

Let us consider the other geometric extreme, sequences which in particular only approach the circle in a tangential fashion. Write

$$\mu_\Lambda := \sum_\lambda (1-|\lambda|)\delta_\lambda, \tag{1.6}$$

where δ_λ stands for the Dirac measure at λ.

Proposition 1.13. *If μ_Λ has bounded balayage, then $\Lambda \in \operatorname{Int} N$ if and only if $\Lambda \in \operatorname{Int} N^+$, and if and only if* (1.5) *holds.*

Note that the condition that μ_Λ has bounded balayage implies in particular that Λ approaches the circle tangentially. In Section 8, we will see more concrete conditions of geometric separation which are sufficient to imply that μ_Λ has bounded balayage (Proposition 8.2).

When μ_Λ has bounded balayage, the trace space will embed into Yanagihara's target space. More precisely, the following result holds.

Proposition 1.14. *The following are equivalent:*

(a) $N|\Lambda \subset l_{Ya}$,
(b) $N^+|\Lambda \subset l_{Ya}$,
(c) *μ_Λ has bounded balayage, i.e.* $\sup_{\zeta\in\partial\mathbb{D}} \sum_\lambda (1-|\lambda|)P_\lambda(\zeta) < \infty$.

Yanagihara considered the sequences such that $N^+|\Lambda \supset l_{\text{Ya}}$. These are automatically in $\operatorname{Int} N^+$, since for any Blachke sequence $l_{\text{Ya}} \supset \ell^\infty$. Conversely, Lemma 8.1 (see Section 8) implies that $l_{\text{Ya}} \subset l_{N^+}$, thus if $\Lambda \in \operatorname{Int} N^+$, then by Theorem 1.3(b) $N^+|\Lambda \supset l_{\text{Ya}}$. Therefore Theorem 1.3 characterizes in particular the sequences studied by Yanagihara.

Altogether, free interpolation for the Nevanlinna and Smirnov classes can be described in terms of the intermediate target spaces l_N and l_{N^+}. Notice first that always $N^+|\Lambda \subset l_{N^+}$ and

$N|\Lambda \subset l_N$ (this is proved at the beginning of Section 5). So, $\Lambda \in \operatorname{Int} N^+$ if and only if $N^+|\Lambda \supset l_{N^+}$, and $\Lambda \in \operatorname{Int} N$ if and only if $N|\Lambda \supset l_N$. Observe also that $l_{\text{Ya}} \subset l_{N^+} \subset l_N \subset l_{\text{Na}}$.

The paper is organized as follows. The next section is devoted to collecting some basic results on functions in the Nevanlinna class. In Section 3 we prove the sufficiency for interpolation of the conditions (c) of Theorems 1.3 and 1.2. We essentially use a result by Garnett allowing interpolation by H^∞ functions on sequences which are denser than Carleson sequences, under some decrease assumptions on the interpolated values. In Section 4 we study the necessity of these conditions. We first observe that in the product $B_\lambda(\lambda)$ appearing in Theorem 1.2, only the factors $b_\lambda(\lambda')$ with λ' close to λ are relevant. Then we split the sequence into four pieces, thereby reducing the interpolation problem, in a way, to that on separated sequences. The trace space characterization will be discussed in Section 5. In Section 6 we consider measures with bounded balayage, show that they operate against positive harmonic functions and prove Theorems 1.4 and 1.6. In Section 7, we prove Proposition 1.8, and provide examples to show that the sufficient condition is not necessary, and the necessary condition not sufficient. Section 8 is devoted to the proofs of Corollary 1.11, Propositions 1.12, 1.14, and 1.13, as well as the deduction of Naftalevič's result from Theorem 1.2. Also, we give examples of measures with bounded balayage. In the final section, we exploit the reasoning of Section 3 to construct non-Carleson interpolating sequences for "big" Hardy-Orlicz classes.

Acknowledgements. The authors wish to express special thanks to Jean-Baptiste Hiriart-Urruty for introducing them to Farkas' Lemma, to Stephen Gardiner for pointing out an efficient characterization of quasi-bounded harmonic functions, and to Alexander Borichev for Lemma 6.6, and his discussions with us about harmonic majorants.

2. PRELIMINARIES

We next recall some standard facts about the structure of the Nevanlinna and Smirnov classes (general references are e.g. [Gar81], [Nik02] or [RosRov]).

A function f is called *outer* if it can be written in the form

$$f(z) = C \exp\left\{ \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} \log v(\zeta) d\sigma(\zeta) \right\},$$

where $|C| = 1$, $v > 0$ a.e. on $\partial\mathbb{D}$ and $\log v \in L^1(\partial\mathbb{D})$. Such a function is the quotient $f = f_1/f_2$ of two bounded outer functions $f_1, f_2 \in H^\infty$ with $\|f_i\|_\infty \le 1$, $i = 1, 2$. In particular, the weight v is given by the boundary values of $|f_1/f_2|$. Setting $w = \log v$, we have

$$\log |f(z)| = P[w](z) = \int_{\partial\mathbb{D}} P_z(\zeta) w(\zeta) d\sigma(\zeta).$$

This formula allows us to freely switch between assertions about outer functions f and the associated measures $wd\sigma$.

Another important family in this context are *inner* functions: $I \in H^\infty$ such that $|I| = 1$ almost everywhere on $\partial\mathbb{D}$. Any inner function I can be factorized into a Blaschke product B_Λ carrying the zeros $\Lambda = \{\lambda_n\}_n$ of I, and a singular inner function S defined by

$$S(z) = \exp\left\{ -\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right\},$$

for some positive Borel measure μ singular with respect to Lebesgue measure.

According to the Riesz-Smirnov factorization, any function $f \in N^+$ is represented as

$$f = \alpha \frac{BSf_1}{f_2},$$

where f_1, f_2 are outer with $\|f_1\|_\infty, \|f_2\|_\infty \le 1$, S is singular inner, B is a Blaschke product and $|\alpha| = 1$. Similarly, functions $f \in N$ are represented as

$$f = \alpha \frac{BS_1 f_1}{S_2 f_2},$$

with f_i outer, $\|f_i\|_\infty \le 1$, S_i singular inner, B is a Blaschke product and $|\alpha| = 1$.

In view of the Riesz-Smirnov factorization described above, the essential difference between Nevanlinna and Smirnov functions is the extra singular factor appearing in the denominator in the Nevanlinna case. This is reflected in the corresponding result for free interpolation in N by the fact that φ_Λ is bounded by a harmonic function, not necessarily quasi-bounded.

3. From harmonic majorants to interpolation

For a given Blaschke sequence $\Lambda \subset \mathbb{D}$ set $\delta_\lambda = |B_\lambda(\lambda)|$. The key result to the proof of the sufficient condition is the following theorem by Garnett [Gar77], that we cite for our purpose in a slightly weaker form (see also [Nik02] as a general source, in particular C.3.3.3(g) (Volume 2) for more results of this kind).

Theorem. *Let $\psi : [0, \infty) \longrightarrow [0, \infty)$ be a decreasing function such that $\int_0^\infty \psi(t)\, dt < \infty$. If a sequence $a = (a_\lambda)_\lambda$ satisfies*

$$|a_\lambda| \le \delta_\lambda \psi(\log \frac{e}{\delta_\lambda}), \quad \lambda \in \Lambda,$$

then there exists a function $f \in H^\infty$ such that $f|\Lambda = a$.

Observe that according to our former notation we have $\log(e/\delta_\lambda) = 1 + \varphi_\Lambda(\lambda)$.

As we have already noted in Remark 1.1, in order to have free interpolation in the Nevanlinna and Smirnov classes, it is sufficient that $\ell^\infty \subset N|\Lambda$ and $\ell^\infty \subset N^+|\Lambda$ respectively. Our aim will be to accommodate the decrease given in Garnett's result by an appropriate function in N or N^+. This is the crucial step in the proof given hereafter of the sufficiency of conditions (c) in both Theorems 1.3 and 1.2.

Proof of sufficiency of 1.3 (c) and 1.2 (c). The proof will be presented for the more difficult case of the Nevanlinna class. So, assume that $h \in \mathrm{Har}_+(\mathbb{D})$ majorizes φ_Λ. Then h is the Poisson integral of a positive measure μ on the circle and the function

$$g(z) = \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \tag{3.1}$$

has positive real part in the disk. By Smirnov's theorem, g is an outer function in some H^p, $p < 1$, and therefore in N^+ (see [Nik02], in particular A.4.2.3 (Volume 1)). Also $\exp(g)$ is in the Nevanlinna class. By assumption we have $\log(1/\delta_\lambda) \le \mathrm{Re}\; g(\lambda)$, $\lambda \in \Lambda$.

Take now $\psi(t) = (1+t)^{-2}$, which obviously satisfies the hypothesis of Garnett's theorem, and set $H = (2+g)^2$, which is still outer in N^+. We have the estimate

$$|H(\lambda)| = |2 + g(\lambda)|^2 \geq (2 + \text{Re } g(\lambda))^2 \geq (1 + \log \frac{e}{\delta_\lambda})^2 = \frac{1}{\psi(\log(e/\delta_\lambda))},$$

hence the sequence $(\gamma_\lambda)_\lambda$ defined by

$$\gamma_\lambda = \frac{1}{H(\lambda)\psi(\log(e/\delta_\lambda))}, \quad \lambda \in \Lambda,$$

is bounded by 1.

In order to interpolate an arbitrary $\omega = (\omega_\lambda)_\lambda \in \ell^\infty$ by a function in N, split

$$\omega_\lambda = \Big(\frac{\omega_\lambda \gamma_\lambda \exp(-g(\lambda))}{\delta_\lambda} \delta_\lambda \psi(\log \frac{e}{\delta_\lambda})\Big) \cdot \frac{H(\lambda)}{\exp(-g(\lambda))}.$$

Since by hypothesis $(\omega_\lambda \gamma_\lambda \exp(-g(\lambda))/\delta_\lambda)_\lambda$ is bounded, we can apply Garnett's result to interpolate the sequence

$$a_\lambda = \frac{\omega_\lambda \gamma_\lambda \exp(-g(\lambda))}{\delta_\lambda} \delta_\lambda \psi(\log \frac{e}{\delta_\lambda}), \quad \lambda \in \Lambda,$$

by a function $f \in H^\infty$. Now $F = fH\exp(g)$ is a function in N with $F|\Lambda = \omega$.

The proof for the Smirnov case is obtained by observing that if the measure μ is absolutely continuous, then $\exp(g)$ is in the Smirnov class and so is the interpolating function F. ■

4. From interpolation to harmonic majorants

We first show that in order to construct the appropriate function estimating $\log|B_\lambda(\lambda)|^{-1}$ we only need to consider the factors of B_λ given by points $\lambda' \in \Lambda$ which are close to λ. This is in accordance with the results for some related spaces of functions [HaMa01, Theorem 1], and it obviously implies Corollary 1.9.

Proposition 4.1. *Let Λ be a Blaschke sequence. For any $\delta \in (0,1)$, there exists a quasi-bounded positive harmonic function $h = P[w]$, $w \in L^1(\partial\mathbb{D})$, such that*

$$-\log \prod_{\lambda:\rho(\lambda,z)\geq\delta} |b_\lambda(z)| \leq h(z), \quad z \in \mathbb{D},$$

and therefore an outer function $G \in N^+$, where $G = \exp(-g)$ and g is given by (3.1) *with $d\mu = w d\sigma$, such that*

$$\prod_{\lambda:\rho(\lambda,z)\geq\delta} |b_\lambda(z)| \geq |G(z)|, \quad z \in \mathbb{D}.$$

Proof. We shall use the intervals I_z introduced in (1.2). In [NPT, p. 124, lines 3 to 17], it is proved that the function w given by

$$w(\zeta) = c_0 \sum_{\lambda\in\Lambda} \chi_{I_\lambda}(\zeta),$$

where c_0 is an appropriate positive constant, is suitable. At this juncture, the separation hypothesis made in [NPT, Lemma 4] is no longer used. ■

Proof of the necessity of 1.3 (c) and 1.2 (c). We will use a dyadic partition of the disk: for any n in $\mathbb{N}$, let

$$I_{n,k} := \{e^{i\theta} : \theta \in [2\pi k 2^{-n}, 2\pi(k+1)2^{-n})\},\ 0 \le k < 2^n. \tag{4.1}$$

and the associated Whitney partition in "dyadic squares":

$$Q_{n,k} := \{re^{i\theta} : e^{i\theta} \in I_{n,k}, 1 - 2^{-n} \le r < 1 - 2^{-n-1}\}. \tag{4.2}$$

Observe that the hyperbolic diameter of each Whitney square $Q_{n,k}$ is bounded between two absolute constants.

We split the sequence into four pieces: $\Lambda = \bigcup_{i=1}^4 \Lambda_i$ such that each piece Λ_i lies in a union of dyadic squares that are uniformly separated from each other. More precisely, set

$$\Lambda_1 = \Lambda \cap Q^{(1)},$$

where the family $Q^{(1)}$ is given by $\{Q_{2n,2k}\}_{n,k}$ (for the remaining three sequences we respectively choose $\{Q_{2n,2k+1}\}_{n,k}$, $\{Q_{2n+1,2k}\}_{n,k}$ and $\{Q_{2n+1,2k+1}\}_{n,k}$). In order to avoid technical difficulties we count only those Q containing points of Λ (in case $\Lambda \cap Q$ is empty there is nothing to prove). In what follows we will argue on one sequence, say Λ_1. The arguments are the same for the other sequences.

Our first observation is that, by construction, for $Q, L \in Q^{(1)}$, $Q \ne L$,

$$\rho(Q, L) := \inf_{z \in Q, w \in L} \rho(z, w) \ge \delta > 0,$$

for some fixed δ. In what follows, the letters j, k... will stand for indices in $\mathbb{N}^2$ of the form $(n, l), 0 \le l < 2^n$. The closed rectangles $\overline{Q_j}$ are compact in $\mathbb{D}$ so that $\Lambda_1 \cap Q_j$ can only contain a finite number of points (they contain at least one point, by assumption). Therefore

$$0 < m_j := \min_{\lambda \in \Lambda_1 \cap Q_j} |B_\lambda(\lambda)|$$

(note that we consider the entire Blaschke product B_λ associated with $\Lambda \setminus \{\lambda\}$). Take $\lambda_j^1 \in Q_j$ such that $m_j = |B_{\lambda_j^1}(\lambda_j^1)|$.

Assume now that $\Lambda \in \operatorname{Int} N$. Since $\ell^\infty \subset N|\Lambda$, there exists a function $f_1 \in N$ such that

$$f_1(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda \in \{\lambda_j^1\}_j \\ 0 & \text{if} \quad \lambda \in \Lambda \setminus \{\lambda_j^1\}_j. \end{cases}$$

By the Riesz-Smirnov factorization we have

$$f_1 = B_{\Lambda \setminus \{\lambda_j^1\}_j} \frac{h_1}{h_2 T_2}, \tag{4.3}$$

where T_2 is singular inner, h_1 is some function in H^∞ and h_2 is outer in H^∞. Again, we can assume $\|h_i\|_\infty \le 1$, $i = 1, 2$. Hence

$$1 = |f_1(\lambda_k^1)| \le |B_{\Lambda \setminus \{\lambda_j^1\}_j}(\lambda_k^1)| \cdot \frac{1}{|h_2(\lambda_k^1) T_2(\lambda_k^1)|},$$

and

$$|B_{\Lambda \setminus \{\lambda_j^1\}_j}(\lambda_k^1)| \ge |h_2(\lambda_k^1) T_2(\lambda_k^1)|, \quad k \in \mathbb{N}.$$

Since h_2T_2 does not vanish and is bounded above by 1, the function $\log|h_2T_2|$ is a negative harmonic function. By Harnack's inequality, there exists an absolute constant $c \geq 1$ such that

$$\frac{1}{c}|\log|h_2(\lambda_k^1)T_2(\lambda_k^1)|| \leq |\log|h_2(z)T_2(z)|| \leq c|\log|h_2(\lambda_k^1)T_2(\lambda_k^1)||, \quad z \in Q_k,$$

hence

$$|h_2(\lambda_k^1)T_2(\lambda_k^1)|^c \leq |h_2(z)T_2(z)| \leq |h_2(\lambda_k^1)T_2(\lambda_k^1)|^{1/c}, \quad z \in Q_k.$$

This yields

$$|(h_2T_2)^c(\lambda')| \leq |(h_2T_2)(\lambda_k^1)| \leq |B_{\Lambda\setminus\{\lambda_j^1\}_j}(\lambda_k^1)| \tag{4.4}$$

for every $\lambda' \in \Lambda_1 \cap Q_k$.

Let us now exploit Proposition 4.1. By construction, the sequence $\{\lambda_j^1\}_j \subset \Lambda_1$ is separated. Therefore, there exists an outer function G_1 in the Smirnov class such that

$$|B_{\{\lambda_j^1\}_j\setminus\{\lambda_k^1\}}(\lambda_k^1)| \geq |G_1(\lambda_k^1)|, \quad k \in \mathbb{N}.$$

Again, G_1 is a quotient of two bounded outer functions and we can suppose that G_1 is outer in H^∞ with $\|G_1\|_\infty \leq 1$. Also, we can use Harnack's inequality as above to get

$$|G_1(\lambda_k^1)| \geq |G_1^c(\lambda')|$$

for every $\lambda' \in \Lambda_1 \cup Q_k$. This together with (4.4) and our definition of λ_k^1 give

$$\begin{aligned} |B_{\Lambda\setminus\{\lambda'\}}(\lambda')| &\geq |B_{\Lambda\setminus\{\lambda_k^1\}}(\lambda_k^1)| = |B_{\Lambda\setminus\{\lambda_j^1\}_j}(\lambda_k^1)| \cdot |B_{\{\lambda_j^1\}_j\setminus\{\lambda_k^1\}}(\lambda_k^1)| \\ &\geq |(h_2T_2)^c(\lambda')| \cdot |G_1^c(\lambda')| \end{aligned}$$

for every $\lambda' \in Q_k$ and $Q_k \in Q^{(1)}$. Set $g_1 = (h_2G_1)^c$ and $S_1 = T_2^c$; by construction, g_1 is outer with $\|g_1\|_\infty \leq 1$ and S_1 is singular inner.

Construct in a similar way functions g_i, S_i for the sequences Λ_i, $i = 2, 3, 4$, and define the products $g = \prod_{i=1}^4 g_i$ and $S = \prod_{i=1}^4 S_i$. Of course g is outer in H^∞, and S is singular inner. So, whenever $\lambda' \in \Lambda$, there exists $k \in \{1, 2, 3, 4\}$ such that $\lambda' \in \Lambda_k$, and hence

$$|B_\lambda(\lambda)| \geq |g_k(\lambda)S_k(\lambda)| \geq |g(\lambda)S(\lambda)|. \tag{4.5}$$

Therefore, the positive harmonic function $h = -\log|gS|$ satisfies $h(\lambda) \geq -\log|B_\lambda(\lambda)|$. The proof for N^+ goes along the same lines, except that singular inner factors do not occur in (4.3), and so will not appear in (4.5) either. ■

5. The trace spaces

In this short section we prove the trace space characterization of free interpolation given in Theorems 1.2 and 1.3.

In order to see that (b) in each theorem implies free interpolation it suffices to observe that $\ell^\infty \subset l_{N^+} \subset l_N$ and use Remark 1.1.

For the proof of the converse, we will only consider the situation in the Nevanlinna class, since the case of the Smirnov class is again obtained by removing the singular part of the measure and the singular inner factors.

Assume that $(a_\lambda)_\lambda \in N|\Lambda$ and that $f \in N$ is such that $f(\lambda) = a_\lambda$, $\lambda \in \Lambda$. Since f can be written as $f = f_1/(S_2 f_2)$, where $f_1 \in H^\infty$, $\|f_1\|_\infty \leq 1$, S_2 is singular inner with associated singular measure μ_S, and $f_2 \in H^\infty$ is an outer function with $\|f_2\|_\infty \leq 1$, we can define the positive finite measure $\mu = \log(1/|f_2|)\, d\sigma + d\mu_S$ which obviously satisfies $P[\mu](\lambda) \geq \log^+ |a_\lambda|$, $\lambda \in \Lambda$.

Conversely, suppose that $(a_\lambda)_\lambda$ is such that there is a positive finite measure μ with $P[\mu](\lambda) \geq \log^+ |a_\lambda|$. The Radon-Nikodym decomposition of μ is given by $d\mu = w\, d\sigma + d\mu_S$, where $w \in L^1(\partial\mathbb{D})$ is positive and μ_S is a positive finite singular measure. Let S be the singular inner function associated with μ_S, and let f be the function defined by

$$f(z) = \exp\left(\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} w(\zeta)\, d\sigma(\zeta)\right), \quad z \in \mathbb{D}.$$

By definition, f is outer in N^+ and $F = f/S \in N$. Clearly, $\log^+ |a_\lambda| \leq \log|F(\lambda)|$, thus $|a_\lambda| \leq |F(\lambda)|$. Since $N|\Lambda$ is ideal by assumption, there exists $f_0 \in N$ interpolating $(a_\lambda)_\lambda$. ∎

6. Harmonic majorants and measures with bounded balayage

Let us start by proving that positive measures with bounded balayage are precisely those which operate against positive harmonic functions. Recall that $\mathcal{B}(\mu)(\zeta) = \int_{\mathbb{D}} P_z(\zeta) d\mu(\zeta)$ and

$$\mathcal{B} := \{\mu \text{ positive Borel measures on } \mathbb{D} \text{ such that } \sup_{\zeta\in\partial\mathbb{D}} B(\mu)(\zeta) \leq 1\}.$$

Proposition 6.1. *Let μ be a positive Borel measure on the disk. Then $\int_{\mathbb{D}} h d\mu$ is finite for any positive harmonic function h on the disk if and only if there exists some $c > 0$ such that μ has balayage uniformly bounded by c. Furthermore, the relevant constants are related:*

$$\sup_{\zeta\in\partial\mathbb{D}} B(\mu)(\zeta) = \sup\left\{\int_{\mathbb{D}} h d\mu : h \in \mathrm{Har}_+(\mathbb{D}), h(0) = 1\right\},$$

and for any positive harmonic function h,

$$h(0) = \max_{\mu\in\mathcal{B}} \int_{\mathbb{D}} h d\mu.$$

Proof. Let $h = P[\nu]$, where $\nu \geq 0$ is a measure on $\partial\mathbb{D}$. If μ has balayage bounded by c,

$$\int_{\mathbb{D}} h(z) d\mu(z) = \int_{\partial\mathbb{D}} \int_{\mathbb{D}} P_z(\zeta) d\mu(z) d\nu(\zeta) \leq c\nu(\partial\mathbb{D}) = ch(0).$$

Conversely, since $z \mapsto P_z(\zeta)$ is a harmonic function for any fixed ζ, $\int_{\mathbb{D}} P_z(\zeta)\, d\mu(z)$ is pointwise defined. Pick a sequence ζ_n such that

$$\lim_{n\to\infty} \int_{\mathbb{D}} P_z(\zeta_n)\, d\mu(z) = \sup_{\zeta\in\partial\mathbb{D}} \int_{\mathbb{D}} P_z(\zeta)\, d\mu(z),$$

where the supremum on the right hand side might a priori be infinite. Since the set $E := \{h \in \mathrm{Har}_+(\mathbb{D}) : h(0) = 1\}$ is uniformly bounded on compact sets in $\mathbb{D}$, a normal family argument shows that $\sup\{\int_{\mathbb{D}} h d\mu : h \in E\} < \infty$. Observe that the mapping $z \mapsto P_z(\zeta_n)$ is in E for every ζ_n, $n \in \mathbb{N}$. Hence $\sup_n \int_{\mathbb{D}} P_z(\zeta_n) d\mu(z) < \infty$.

This proves that μ has bounded balayage, and the equalities between constants that we had announced. ∎

The next result is a refined version of Theorem 1.4 stated in the introduction.

Theorem 6.2. *Let φ be a nonnegative Borel function on the unit disk. Then there exists a harmonic function h such that $h(z) \geq \varphi(z)$ for any $z \in \mathbb{D}$ if and only if*

$$M_\varphi := \sup_{\mu \in \mathcal{B}} \int_{\mathbb{D}} \varphi \, d\mu < \infty. \tag{6.1}$$

Furthermore,

$$M_\varphi = \inf \left\{ h(0) : h \in \operatorname{Har}(\mathbb{D}), h \geq \varphi \right\}.$$

That (6.1) is necessary is clear from the above considerations. In order to prove that it is sufficient, we will reduce ourselves to a discrete version of it. We will use the dyadic squares introduced in (4.2). As in the previous section, choose a point $z_{n,k}$ in each square, say

$$z(Q_{n,k}) = z_{n,k} := (1 - 2^{-n}) \exp(2\pi k 2^{-n}).$$

Observe that by Harnack's inequality, there exists a universal constant K such that : if z, z' lie in the same Whitney square $Q_{n,k}$ (as defined in (4.2)), then $K^{-1} P_{z'}(\zeta) \leq P_z(\zeta) \leq K P_{z'}(\zeta)$, for any $\zeta \in \partial\mathbb{D}$.

Lemma 6.3. *The function φ satisfies condition (6.1) if and only if there exists a constant M'_φ such that for any sequence of nonnegative coefficients $\{c_{n,k}\}$ such that*

$$\sup_{\zeta \in \partial\mathbb{D}} \sum_{n,k} c_{n,k} P_{z_{n,k}}(\zeta) \leq 1, \tag{6.2}$$

then

$$\sum_{n,k} c_{n,k} \sup_{Q_{n,k}} \varphi \leq M'_\varphi. \tag{6.3}$$

Furthermore, $C^{-1} M_\varphi \leq M'_\varphi \leq C M_\varphi$, where $C > 1$ is an absolute constant.

Proof of Lemma 6.3. Pick $z^*_{n,k} \in Q_{n,k}$ such that $\varphi(z^*_{n,k}) \geq (\sup_{Q_{n,k}} \varphi)/2$ and define the measure $\mu := \sum_{n,k} c_{n,k} \delta_{z^*_{n,k}}$. Then, if $\{c_{n,k}\}$ satisfies (6.2),

$$B(\mu)(\zeta) = \int_{\mathbb{D}} P_z(\zeta) \, d\mu(z) = \sum_{n,k} c_{n,k} P_{z^*_{n,k}}(\zeta) \leq K \sum_{n,k} c_{n,k} P_{z_{n,k}}(\zeta) \leq K.$$

So if φ satisfies (6.1),

$$\sum_{n,k} c_{n,k} \sup_{Q_{n,k}} \varphi \leq 2 \sum_{n,k} c_{n,k} \varphi(z^*_{n,k}) = 2 \int_{\mathbb{D}} \varphi \, d\mu \leq 2K M_\varphi.$$

The converse direction is easier, and left to the reader (it also follows from the proof of the theorem, below). ∎

We now need a classical result in convex analysis. Recall that the convex hull of a subset $A \subset \mathbb{R}^d$ is defined as

$$\operatorname{Conv}(A) := \Big\{ \sum_{i=1}^{N} \alpha_i a_i : a_i \in A, \alpha_i \geq 0, \sum_i \alpha_i = 1 \Big\}.$$

If we write $\mathbb{R}_+A := \{\lambda x : \lambda \geq 0, x \in A\}$, then the conical convex hull of A is defined as

$$\operatorname{Cone}(A) := \operatorname{Conv}(\mathbb{R}_+A) = \Big\{\sum_{i=1}^{N} \alpha_i a_i : a_i \in A, \alpha_i \geq 0\Big\}.$$

When A is a finite set, the conical convex hull is equal to its closure: $\overline{\operatorname{Cone}}(A) = \operatorname{Cone}(A)$ (for this and other facts, see [HULL]). The key fact for us will be the generalized form of the Minkowski-Farkas Lemma (see [HULL, Chapter III, Theorem 4.3.4]) that we cite here only for finite A. Let $\langle \cdot, \cdot \rangle$ stand for the standard Euclidean scalar product in $\mathbb{R}^d$.

Theorem 6.4. *Let $(a_j, b_j) \in \mathbb{R}^d \times \mathbb{R}$, $1 \leq j \leq N$, be such that $X := \{x \in \mathbb{R}^d : \langle a_j, x\rangle \leq b_j\} \neq \emptyset$. Denote $A := \{(a_j, b_j), 1 \leq j \leq N\} \subset \mathbb{R}^d \times \mathbb{R}$. Then the following properties are equivalent for $(v, r) \in \mathbb{R}^d \times \mathbb{R}$:*

(a) *For any $x \in X$, $\langle v, x\rangle \leq r$.*
(b) $(v, r) \in \operatorname{Cone}(A)$.

We will use the following special case. For a vector $v \in \mathbb{R}^d$, the coordinates are denoted by v^i, $1 \leq i \leq d$. Also, $\mathbb{R}^d_+$ denotes the set of points of $\mathbb{R}^d$ with nonnegative coordinates.

Corollary 6.5. *Given $a_j \in \mathbb{R}^d$, $1 \leq j \leq N$, let $X_+ := \{x \in \mathbb{R}^d_+ : \langle a_j, x\rangle \leq 1\}$, and suppose that $X_+ \neq \emptyset$. Then the following properties are equivalent for $v \in \mathbb{R}^d_+$:*

(a) *For any $x \in X_+$, $\langle v, x\rangle \leq 1$.*
(b) *There exist $\alpha_j \geq 0, 1 \leq j \leq N$ such that $\sum_{j=1}^N \alpha_j = 1$ and for any $i = 1, \ldots, d$,*

$$v^i \leq \sum_{j=1}^{N} \alpha_j a_j^i.$$

Proof. Let $\{e_i\}_{1\leq i\leq d}$ be the canonical basis of $\mathbb{R}^d$ and consider

$$A := \{(a_j, 1), 1 \leq j \leq N\} \cup \{(-e_i, 0), 1 \leq i \leq d\}.$$

Then X_+ corresponds to the X in Theorem 6.4, from what we see that (a) implies that there exist $\alpha_j \geq 0, \beta_i \geq 0$, $1 \leq j \leq N$, $1 \leq i \leq d$, such that

$$(v, 1) = \sum_{j=1}^{N} \alpha_j (a_j, 1) - \sum_{i=1}^{d} \beta_i (e_i, 0).$$

When applied to each coordinate, this yields $1 = \sum_{j=1}^N \alpha_j$ and

$$v^i = \sum_{j=1}^{N} \alpha_j a_j^i - \beta_i \leq \sum_{j=1}^{N} \alpha_j a_j^i.$$

The converse implication is immediate. ∎

Proof of Theorem 6.2. Suppose that φ satisfies (6.1). For each nonnegative integer m, we define

$$a_j := \Big(P_{z_{n,k}}(\exp(ij \cdot 2^{-m} 2\pi))\Big)_{\substack{0\leq n\leq m \\ 0\leq k\leq 2^n-1}} \quad \text{for } 0 \leq j \leq 2^m - 1,$$

$d := \sum_{n=0}^{m} 2^n$ and

$$X_+ := \Big\{ \{c_{n,k}\}_{\substack{0\le n\le m\\ 0\le k\le 2^n-1}} \in \mathbb{R}_+^d : \sum_{\substack{0\le n\le m\\ 0\le k\le 2^n-1}} c_{n,k} P_{z_{n,k}}(\exp(ij\cdot 2^{-m}2\pi)) \le 1, \text{ for } 1\le j\le 2^m-1 \Big\}.$$

Obviously, X_+ is not empty: for instance $c_{0,0}=1$ and $c_{n,k}=0$ for $n\ge 1$ gives a point in X_+. We claim that any $\{c_{n,k}\}\in X_+$ will satisfy (6.2) up to a constant. Indeed, for any $\theta\in[0,2\pi)$, there is an index $j<2^m$ so that $j\cdot 2^{-m}2\pi \le \theta < (j+1)\cdot 2^{-m}2\pi$, therefore by Harnack's inequality, for any z such that $|z|\le 1-2^{-m}$,

$$P_z(e^{i\theta}) = P_{z\exp(i(j\cdot 2^{-m}2\pi-\theta))}(\exp(ij\cdot 2^{-m}2\pi)) \le K P_z(\exp(ij\cdot 2^{-m}2\pi).$$

Therefore $\{K^{-1}c_{n,k}\}$ satisfies (6.2), and by Lemma 6.3 and the hypothesis, φ satisfies (6.3) with constant KM'_φ. Corollary 6.5 then implies the existence of positive coefficients $(\alpha_j^m)_{j=0}^{2^m-1}$ with sum equal to KM'_φ, such that

$$\sup_{Q_{n,k}} \varphi \le \sum_{j=0}^{2^m-1} \alpha_j^m P_{z_{n,k}}(\exp(ij\cdot 2^{-m}2\pi)) = \int_{\partial\mathbb{D}} P_{z_{n,k}} d\nu^m,$$

where ν^m is the discrete measure on the circle given by the following combination of Dirac masses:

$$\nu^m = \sum_{j=0}^{2^m-1} \alpha_j^m \delta_{\exp(ij\cdot 2^{-m}2\pi)}.$$

Since the mass of ν^m is uniformly bounded by KM'_φ, we can take a weak* limit ν of this sequence of measures, so that for any (n,k),

$$\sup_{Q_{n,k}} \varphi \le \int_{\partial\mathbb{D}} P_{z_{n,k}} d\nu = h(z_{n,k}),$$

where $h := P[\nu]$. Harnack's inequality now implies that there is an absolute constant C_1 such that $C_1 h(z)\ge\varphi(z)$ for any $z\in\mathbb{D}$. This proves the theorem, with the inequality

$$\inf\{h(0) : h\in \mathrm{Har}(\mathbb{D}), h\ge\varphi\} \le C_1 K M'_\varphi \le C C_1 K M_\varphi.$$

The constants C, K and C_1 only depend on the discretization we have chosen. Picking a discretization with smaller "squares", we may make all three constants as close to 1 as we wish. ∎

Now we can prove Corollary 1.5.

Proof of Corollary 1.5. Given a non-Blaschke sequence Λ, arguing as in [NPT] one can show that there exists a function $f\in H^\infty(\mathbb{D})$ in the unit disk with $|f(\lambda)|<\varepsilon_\lambda$ for any $\lambda\in\Lambda$ if and only if Λ is the union of a Blaschke sequence and a sequence Γ for which there exists a positive harmonic function h in the unit disk with $h(\gamma)>-\log\varepsilon_\gamma$ for all $\gamma\in\Gamma$. Then the result follows from Theorem 1.4. ∎

We finish this section with the proof of Theorem 1.6.

Proof of Theorem 1.6. (a) $\Rightarrow$ (d). Part (i) holds whenever φ admits a harmonic majorant, be it quasi-bounded or not (see Proposition 1.7), while (ii) follows from the dominated convergence theorem.

(d) $\Rightarrow$ (c). We proceed by contradiction. Suppose that there exist $\delta > 0$ and a sequence of measures $\mu_n \in \mathcal{B}_n$ such that

$$\int_{\{\varphi \geq n\}} \varphi \, d\mu_n \geq \delta. \tag{6.4}$$

Let $\tilde{\mu}_n = \chi_{\{\varphi \geq n\}} \mu_n$. Then

$$\int_{\partial\mathbb{D}} B(\tilde{\mu}_n)(\zeta) d\sigma(\zeta) = \tilde{\mu}_n(\mathbb{D}) = \mu_n\left(\{\varphi \geq n\}\right).$$

Since $\mu_n \in \mathcal{B}$, their Carleson norms are uniformly bounded by some $C_0 > 0$. We apply the direct part of [Gar81, Lemma I.5.5, p. 32] to φ; the lemma is stated for harmonic functions, but harmonicity plays no role in the proof of the direct part. We obtain

$$\tilde{\mu}_n(\mathbb{D}) = \mu_n\left(\{\varphi \geq n\}\right) \leq c_1 C_0 \sigma(\{M\varphi \geq n\}) \leq c_1 C_0 C_\varphi / n,$$

by (d) (i). Since the sequence $(B(\tilde{\mu}_n))_n$ tends to 0 in $L^1(\partial\mathbb{D})$, some subsequence must tend to 0 almost everywhere, and applying (d) (ii) to that subsequence, we find a contradiction with (6.4).

(c) $\Rightarrow$ (b). We define a function ψ on $\mathbb{R}_+$ by

$$\psi(t) = \psi_n(t) = a_n t + b_n \text{ for } t \in [n, n+1],$$

where (a_n) is an increasing sequence of positive numbers tending to infinity, to be determined later, and (b_n) is given recursively by $b_0 = 0$ and $\psi_n(n+1) = \psi_{n+1}(n+1)$. Observe that each ψ_n is defined on the whole real line (they give supporting hyperplanes for the polygonal convex graph of ψ). We shall also use $u^+ = \max(u, 0)$ for $u \in \mathbb{R}$.

We prove that $\psi \circ \varphi$ admits a harmonic majorant using Theorem 1.4. Let $\varphi_n = \varphi \chi_{\{\varphi \geq n\}}$ and $\varepsilon_n := \sup_{\mu \in \mathcal{B}} \left(\int_{\mathbb{D}} \varphi_n \, d\mu \right)$. If $\mu \in \mathcal{B}$, then

$$\begin{aligned}
\int_{\mathbb{D}} \psi \circ \varphi(z) \, d\mu(z) &= \sum_{n \geq 0} \int_{\{n \leq \varphi < n+1\}} \psi_n \circ \varphi_n(z) d\mu(z) \\
&= \sum_{n \geq 0} \int_{\mathbb{D}} [\psi_n^+ \circ \varphi_n(z) - \psi_n^+ \circ \varphi_{n+1}(z)] d\mu(z) \\
&= \int_{\mathbb{D}} \psi_0 \circ \varphi(z) \, d\mu(z) + \sum_{n \geq 1} \int_{\mathbb{D}} [\psi_n^+ \circ \varphi_n(z) - \psi_{n-1}^+ \circ \varphi_n(z)] d\mu(z) \\
&\leq a_0 \int_{\mathbb{D}} \varphi(z) \, d\mu(z) + \sum_{n \geq 1} \int_{\mathbb{D}} (a_n - a_{n-1})(\varphi_n(z) - n)^+ d\mu(z) \\
&\leq a_0 \varepsilon_0 + \sum_{n \geq 1} (a_n - a_{n-1}) \varepsilon_n.
\end{aligned}$$

Since $\lim_n \varepsilon_n = 0$, we can choose an increasing sequence (a_n) such that $\lim_n a_n = \infty$, but $\sum_{n \geq 1} (a_n - a_{n-1}) \varepsilon_n < \infty$, and we are done.

(b) $\Rightarrow$ (a). First notice that ψ can be replaced by a function $\tilde{\psi} \leq \psi$ with the same properties as ψ and the additional explicit description:

$$\tilde{\psi}(t) = \tilde{\psi}_n(t) = a_n t + b_n \geq a_n' t \quad \text{for} \quad t \in [\gamma_n, \gamma_{n+1}],$$

where $a_n \geq a'_n > 0$ for $n \geq 1$, $\gamma_0 = 0$ and $(\gamma_n)_{n\geq 1}$ is an increasing sequence of positive numbers tending to infinity fast enough so that $\sum_{n\geq 1} 1/a'_n < \infty$.

Define $\varphi_n := \varphi\chi_{\{\gamma_n \leq \varphi < \gamma_{n+1}\}}$; thus $\tilde{\psi} \circ \varphi = \sum_n \tilde{\psi}_n \circ \varphi_n$.

The following Lemma is due to Alexander Borichev.

Lemma 6.6. *There exists an absolute constant $C > 0$ such that whenever $\varphi \geq 0$ is bounded and $\varphi \leq h$ for some $h \in \mathrm{Har}_+(\mathbb{D})$, then there exists $\tilde{h} \in \mathrm{Har}_+(\mathbb{D})$ quasi-bounded such that $\varphi \leq \tilde{h}$ and*

$$\int_{\partial\mathbb{D}} \tilde{h}(\zeta)d\sigma(\zeta) = \tilde{h}(0) \leq Ch(0).$$

In order to prove (a) let h_0 be a harmonic majorant of $\psi \circ \varphi$. Each φ_n is then bounded and majorized by h_0/a'_n, hence by applying the previous lemma we find $\tilde{h}_n$ quasi-bounded such that $\varphi_n \leq \tilde{h}_n$ and $\tilde{h}_n(0) \leq Ch_0(0)/a'_n$. The series $\tilde{h} := \sum \tilde{h}_n$ converges in $L^1(\partial\mathbb{D})$, since $\tilde{h}_n \geq 0$ for all n and

$$\tilde{h}(0) = \sum_n \tilde{h}_n(0) \leq Ch_0(0) \sum_n 1/a'_n < \infty,$$

and defines therefore a quasi-bounded harmonic majorant of φ. ■

Proof of Lemma 6.6. Set $M := \max(\|\varphi\|_\infty, 2h(0))$. Let μ denote the boundary measure of h, i.e. the measure such that $h = P[\mu]$. We use the standard dyadic decomposition of the circle given in (4.1).

Let $E_0 = \emptyset$. For any $n \geq 1$, let E_n be the union of the dyadic intervals $I_{n,k} \subset \partial\mathbb{D} \setminus \bigcup_{l<n} E_l$ such that

$$\mu(I_{n,k}) > M\sigma(I_{n,k}).$$

Note that E_n cannot contain two contiguous intervals such that $I_{n,k} \cup I_{n,k+1} = I_{n-1,k'}$, because then $I_{n-1,k'} \subset \bigcup_{l<n} E_l$, a contradiction. Therefore, if $I_{n,k} \subset E_n$, then

$$\mu(I_{n,k}) \leq \mu(I_{n-1,k'}) \leq M\sigma(I_{n-1,k'}) = 2M\sigma(I_{n,k}) < 2\mu(I_{n,k}).$$

For any interval I, let $\tilde{I}$ be the interval of same center and triple length, and let $\tilde{E} := \cup\tilde{I}$, where the union is taken over all the dyadic intervals included in $E := \cup_n E_n$. We write

$$d\tilde{\mu} := C_2 M\chi_{\tilde{E}}\, d\sigma(\zeta) + \chi_{\partial\mathbb{D}\setminus E}\, d\mu = d\tilde{\mu}_1 + d\tilde{\mu}_2,$$

where $C_2 > 0$ is to be chosen. This measure is absolutely continuous with respect to arc length.

The function we are looking for is $\tilde{h} := P[\tilde{\mu}]$. Indeed, let $z \in \mathbb{D}$ and suppose that there exist a dyadic interval $I \subset E$, maximal among the dyadic intervals contained in E, such that

$$\int_{\tilde{I}} P_z(\zeta)d\sigma(\zeta) \geq \frac{1}{C_2}. \tag{6.5}$$

Then clearly $\tilde{h}(z) \geq M \geq \varphi(z)$. We claim that if z is such that (6.5) does not hold for any maximal dyadic interval $I \subset E$, then $\tilde{h}(z) \geq h(z)$, which will finish the proof.

Under that assumption, since the level sets of the Poisson integral in (6.5) are arcs of circles connecting the extremities of $\tilde{I}$, where they make a fixed angle with $\partial\mathbb{D}$ depending on C_2, we must have $|z - \zeta| \geq c_3\sigma(I)$ for any $\zeta \in I$ and any maximal dyadic subinterval I of E, so that all the values $P_z(\zeta)$ for $\zeta \in I$ are comparable, say to the value at its center ζ_I. Therefore for any such I,

$$\int_I P_z(\zeta)d\mu(\zeta) \leq c_4 P_z(\zeta_I)\int_I d\mu(\zeta) \leq 2c_4 P_z(\zeta_I)\int_I M d\sigma(\zeta) \leq 2c_4^2 \int_I M P_z(\zeta)d\sigma(\zeta)$$
$$= \frac{2c_4^2}{C_2}\int_I P_z(\zeta)d\tilde{\mu}(\zeta).$$

Since c_3 is an increasing function of C_2, and therefore $c_4 > 1$ a decreasing function of C_2, we may choose a value of $C_2 > 1$ large enough so that $C_2 \geq 2c_4^2$, and therefore, since E is the union of its maximal dyadic subintervals,

$$\int_{\tilde{E}} P_z(\zeta)d\tilde{\mu}(\zeta) \geq \int_E P_z(\zeta)d\tilde{\mu}(\zeta) \geq \int_E P_z(\zeta)d\mu(\zeta).$$

By construction, $\int_{\partial\mathbb{D}\setminus E} P_z(\zeta)d\tilde{\mu}(\zeta) = \int_{\partial\mathbb{D}\setminus E} P_z(\zeta)d\mu(\zeta)$, and we are done. ∎

7. WEAKER CONDITIONS FOR THE EXISTENCE OF HARMONIC MAJORANTS

In this section we state first a sufficient condition implied by a result of Borichev on a similar problem. On the other hand, we also prove the necessary condition of Proposition 1.8 and show that it is not sufficient.

Theorem 7.1. [BNT] *Given a collection of nonnegative data $\{\varphi_{n,k}\} \subset \mathbb{R}_+$, there exists a finite positive measure ν on $\partial\mathbb{D}$ such that*

$$\frac{\nu(I_{n,k})}{\sigma(I_{n,k})} \geq \varphi_{n,k}$$

if and only if

$$S =: \sup\Big\{ \sum_{(n,k)\in A} \varphi_{n,k}\sigma(I_{n,k}) : \{I_{n,k}\}_{(n,k)\in A} \text{ is a disjoint family}\Big\} < \infty. \tag{7.1}$$

This is an analogue of the discretized version of Theorem 1.2(d), (as in Lemma 6.3) obtained by considering only measures of type $\mu_A := \sum_{(n,k)\in A}\sigma(I_{n,k})\delta_{z_{n,k}}$, and by replacing the Poisson kernel P_z by the "square" kernels

$$K_z(e^{i\theta}) := K_{I_z}(e^{i\theta}) := \frac{1}{\sigma(I_z)}\chi_{I_z}(e^{i\theta}).$$

Here I_z denote the intervals defined in (1.2) and χ_E stands again for the characteristic function of E.

The similarity of Theorem 1.2 with this result leads us to an:

Open Question. Is condition (d) in Theorem 1.2 still sufficient if we restrict it to $\{c_\lambda\}$ such that for any $\lambda \in \Lambda$, $c_\lambda = 0$ or $(1-|\lambda|)$?

Theorem 7.1 together with the estimate $K_z \lesssim P_z$ provide a sufficient (but not necessary) condition for domination by true harmonic functions, which is clearly less restrictive than requiring that $M\varphi \in L^1(\partial\mathbb{D})$, but easier to check in concrete examples than the characterizing condition of Theorem 1.4.

Corollary 7.2. *Any positive function φ such that $\varphi_{n,k} := \sup_{Q_{n,k}} \varphi$ satisfies* (7.1) *admits a harmonic majorant. On the other hand, the positive harmonic function $z \mapsto P_z(1)$ does not satisfy* (7.1) *for certain choices of A.*

Proof. It is well known and easy to see that there exists a constant c such that $P_z \geq cK_{I_{n,k}}$ for any $z \in Q_{n,k}$ (the constant c depends on the aperture α of the Stolz angle). Therefore, for any $z \in Q_{n,k}$

$$P[\nu](z) \geq c \int_{\partial\mathbb{D}} K_{I_{n,k}}(\zeta) d\nu(\zeta) \geq c \frac{\nu(I_{n,k})}{\sigma(I_{n,k})} \geq c\varphi_{n,k} = c \sup_{Q_{n,k}} \varphi,$$

which proves that $P[(1/c)\nu]$ is the harmonic majorant we are looking for.

To see that the condition is not necessary, consider any $A \subset \{(n,1) : n \in \mathbb{N}\}$. Then the intervals $I_{n,1}$ are all disjoint; however $P_{z_{n,1}}(1) \simeq 2^n \simeq \sigma(I_{n,1})^{-1}$, so that condition (7.1) will fail (the sum is comparable to $\#A$). ■

In the same way as in Corollary 1.11, Corollary 7.2 and Proposition 1.8 imply the following result. For $Q = Q_{n,k}$, write $I(Q) = I_{n,k}$ (the radial projection of the square to an arc of the circle).

Corollary 7.3. *Assume that Λ is contained in a union A of Whitney squares Q of center $z(Q)$ and that*

$$\sup\Big\{ \sum_{Q \in A'} (1 - |z(Q)|) \sup_{\lambda \in \Lambda \cap Q} \log |B_\lambda(\lambda)|^{-1} \Big\} < \infty,$$

where the supremum is taken over all $A' \subset A$ such that $\{I(Q), Q \in A'\}$ is a disjoint family, then Λ is interpolating for the Nevanlinna class.

We move next to the proof of the necessary condition in terms of the Hardy-Littlewood maximal function.

Proof of Proposition 1.8. (a) The problem can be localized, so we may work on the upper half plane, $\mathbb{C}_+ := \{x + iy : y > 0\}$, with $I_{x+iy} := (x - y, x + y)$, restricting ourselves to positive harmonic functions which are Poisson integrals of positive measures with finite mass. Here the Poisson kernel is given by

$$P_{x+iy}(s) = \frac{1}{\pi} \frac{y}{(x-s)^2 + y^2}.$$

For convenience we shall write here $|E|$ for the Lebesgue measure of a measurable set $E \subset \mathbb{R}$. Also, we only need to look at boundary points in a fixed bounded interval, say $-1 \leq x \leq 1$.

For any $t > 0$, let $E_t := \{s \in [-1,1] : \varphi^H(s) > t\}$. For any $s \in E_t$, there exists $z = z(s)$ and $J = J(s) \supset I_z$ such that

$$\varphi(z) \int_J \chi_{I_z} > t|J|, \quad \text{i.e.} \quad \varphi(z)|I_z| > t|J|. \tag{7.2}$$

By Vitali's covering lemma, there exist an absolute constant $c_1 \in (0,1)$ and a disjoint family of intervals $J_j := J(s_j)$, $1 \leq j \leq N$, such that $\sum_j |J_j| \geq c_1 |E_t|$.

Write $z_j := z(s_j) =: x_j + iy_j$. Note that since the point z_j is contained in the "tent" over I_{z_j} (therefore in the tent over J_j) the points z_j are separated in the hyperbolic metric.

Now define new points z'_j in the following way : let $y'_j := |J_j|/2 = y_j|J_j|/|I_{z_j}| \geq y_j$ and $z'_j := x_j + iy'_j$. Note that $|J_j \cap I_{z'_j}| \geq |J_j|/2$.

We claim that $h(z'_j) \geq t$, where h is a harmonic majorant of φ. Indeed, writing $h = P[\mu]$,

$$h(z'_j) = \frac{1}{\pi y'_j} \int_{\mathbb{R}} \frac{1}{1 + \left(\frac{t-x_j}{y'_j}\right)^2} \, d\mu(t) \geq \frac{1}{\pi y'_j} \int_{\mathbb{R}} \frac{1}{1 + \left(\frac{t-x_j}{y_j}\right)^2} \, d\mu(t) = \frac{y_j}{y'_j} h(z_j),$$

and, by (7.2), $h(z_j) \geq \varphi(z_j) > t|J_j|/|I_{z_j}| = ty'_j/y_j$.

Therefore, since $Mh \in L^1_w(\mathbb{R})$,

$$\frac{c_1}{2}|E_t| \leq \frac{1}{2}\sum_j |J_j| \leq \sum_j |J_j \cap I_{z'_j}| \leq |\{Mh > t\}| \leq \frac{C_h}{t}.$$

(b) Similarly. ∎

We now give two examples showing that the necessary condition of Proposition 1.8 is strictly stronger than that of Proposition 1.7 but still not sufficient.

Proposition 7.4. (a) *There are functions φ such that $\varphi^H \in L^1_w(\partial\mathbb{D})$, but that do not admit a harmonic majorant.*

(b) *There are functions φ such that $M\varphi \in L^1_w(\partial\mathbb{D})$, but $\varphi^H \notin L^1_w(\partial\mathbb{D})$.*

Proof. The proof rests on the following family of examples. Note that it is easy to turn those examples into examples of sequences which are (or are not) interpolating for the Nevanlinna class.

Again we will work on $\mathbb{C}_+$. Our functions φ will vanish everywhere on the upper half plane, except on the sequence $\lambda_k := x_k + iy_k$, where $x_k = k^{-\alpha}$ and $y_k = k^{-\beta}$. To ensure that $y_k \leq (x_{k+1}-x_k)^2$ we take $\beta \geq 2(\alpha+1)$. With this choice, it can be deduced from Proposition 8.2 (or the remark before Lemma 8.4), that a necessary and sufficient condition for the existence of a harmonic majorant is that $M\varphi \in L^1$, that is,

$$\sum_k \varphi(\lambda_k) y_k < \infty. \tag{7.3}$$

We note that

$$\chi^*_{I_{\lambda_k}}(x) = \frac{2}{1 + \max(1, \frac{|x-x_k|}{y_k})}.$$

Henceforth we only study data $\{\varphi_k\} := \{\varphi(\lambda_k)\}$ which are increasing sequences of positive numbers tending to infinity. We also assume that $\{(\varphi_k y_k + \varphi_{k+1}y_{k+1})/(x_k - x_{k+1})\}_k$ forms an increasing sequence. Let $k_0(t) := \min\{k : t < \varphi_k\}$. The necessary condition arising from the fact that $M\varphi \in L^1_w(\mathbb{R})$ reads

$$\sum_{k \geq k_0(t)} y_k \simeq k_0^{1-\beta}(t) \leq \frac{C}{t}, \qquad \forall t > 0. \tag{7.4}$$

This condition will be assumed for both examples.

Now, for $k \geq k_0(t)$, define $J_k := \{x : \varphi_k \chi_k^*(x) > t\} \simeq (x_k - y_k\varphi_k/t, x_k + y_k\varphi_k/t)$, and let $k_1(t) := \min\{k : J_k \cap J_{k+1} \neq \emptyset\}$. Then,

$$\bigcup_{k \geq k_1(t)} J_k = (0, x_{k_1(t)} + y_{k_1(t)} \frac{\varphi_{k_1(t)}}{t}) = (0, k_1^{-\alpha}(t) + k_1^{-\beta}(t)\frac{\varphi_{k_1(t)}}{t})$$

and

$$\left|\{x : \varphi^H(x) > t\}\right| \simeq k_1^{-\alpha}(t) + k_1^{-\beta}(t)\frac{\varphi_{k_1(t)}}{t} + \frac{2}{t}\sum_{k=k_0(t)}^{k_1(t)-1} \frac{\varphi_k}{k^\beta} \simeq k_1^{-\alpha}(t) + \frac{2}{t}\sum_{k=k_0(t)}^{k_1(t)} \frac{\varphi_k}{k^\beta}. \tag{7.5}$$

In order to prove (a), choose $\varphi_k := \varepsilon_k k^{\beta-1}$. Since $t \simeq \varepsilon_{k_0(t)} k_0(t)^{\beta-1}$, condition (7.4) becomes that $(\varepsilon_k)_k$ remains bounded above, while the necessary and sufficient condition (see (7.3)) is

$$\sum_k \frac{\varepsilon_k}{k} < \infty.$$

With $\varepsilon_k := (\log k)^{-1}$, this condition fails, so that φ admits no harmonic majorant.

However, $k_0(t) \simeq (t \log t)^{1/(\beta-1)}$. Since $x_k - x_{k+1} \simeq k^{-\alpha-1}$, then $1/k_1(t)^{\alpha+1} \simeq \varepsilon_{k_1(t)}/(tk_1(t))$, thus $k_1(t) \simeq (t/\varepsilon_{k_1(t)})^{1/\alpha}$, and $k_1(t) \simeq (t \log t)^{1/\alpha}$.

Therefore equation (7.5) becomes

$$\begin{aligned}\left|\{x : \varphi^H(x) > t\}\right| &\simeq \frac{1}{t \log t} + \frac{2}{t}\sum_{k=k_0(t)}^{k_1(t)} \frac{1}{k \log k} \simeq \frac{1}{t \log t} + \frac{2}{t}\log\left(\frac{\log k_1(t)}{\log k_0(t)}\right) \\ &\leq \frac{1}{t \log t} + \frac{C}{t} \leq \frac{C'}{t},\end{aligned}$$

and this choice of φ does satisfy the necessary condition given in Proposition 1.8.

To prove the second statement in the Lemma, choose $\varepsilon_k := 1$. With similar but easier calculations one sees that $k_0(t) \simeq t^{1/(\beta-1)}$ and $k_1(t) \simeq t^{1/\alpha}$. Therefore (7.5) becomes

$$\left|\{x : \varphi^H(x) > t\}\right| \simeq \frac{1}{t} + \sum_{k=k_0(t)}^{k_1(t)-1} \frac{1}{k} \simeq \frac{1}{t} + \frac{2}{t}\log\left(\frac{k_1(t)}{k_0(t)}\right) \simeq \frac{\log t}{t},$$

so the weak L^1 condition fails for φ^H, even though φ satisfies the necessary condition in Proposition 1.7. ■

8. Proofs of the geometric conditions

Proof of Corollary 1.11. Since

$$I_\lambda \subset \left\{\zeta \in \partial\mathbb{D} : M\varphi_\Lambda(\zeta) \geq \log|B_\lambda(\lambda)|^{-1}\right\}, \quad \lambda \in \Lambda,$$

to prove (a) and (b) it suffices to apply condition (a) of Corollary 1.10. Statement (c) follows from the next Lemma applied to φ_Λ.

Lemma 8.1. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{R}_+$ satisfy $\sum_{\lambda\in\mathbb{D}}(1-|\lambda|)\varphi(\lambda) < \infty$. Then φ admits a quasi-bounded harmonic majorant.*

Proof. Let $u = \sum_\lambda \varphi(\lambda)\chi_{I_\lambda}$. By assumption $u \in L^1(\partial\mathbb{D})$ and obviously $M\varphi \le u$, hence the result follows from Corollary 1.10(b) and Theorem 1.3. ■

Parts (b) and (c) also follow directly from Theorem 1.2(d), by a simple argument based on the ℓ^1, ℓ^∞ duality. ■

Proof of Proposition 1.12. It is enough to consider the case where Λ is contained in only one Stolz angle. Indeed, if $\Lambda = \bigcup_{i=1}^n \Lambda_i$ with $\Lambda_l \subset \Gamma_{\zeta_l}$, $l = 1, \dots, n$, and $\zeta_i \neq \zeta_j$, then

$$\lim_{z\to\zeta_i, z\in\Gamma_{\zeta_i}} |B_{\Lambda_j}(z)| = 1, \quad j \neq i,$$

so that $\log|B_\lambda(\lambda)|^{-1}$ behaves asymptotically like $\log|B_{\Lambda_i\setminus\{\lambda\}}(\lambda)|^{-1}$ in Γ_{ζ_i} (here $\lambda \in \Lambda_i$). Also, we can assume that the sequence is radial (this means that we replace the initial sequence by one which is in a uniform pseudo-hyperbolic neighborhood of the initial one; by Harnack's inequality such a perturbation does not change substantially the behavior of positive harmonic functions).

According to Corollary 1.11 it is enough to prove the sufficiency of the conditions. Let us first show that condition (1.3) implies interpolation in N^+. In order to construct a function $w \in L^1(\partial\mathbb{D})$ meeting the requirement of Theorem 1.3(c) assume that $\Lambda = \{\lambda_n\}_n \subset [0,1)$ is arranged in increasing order and set $\tilde\varepsilon_n = (1-|\lambda_n|)\log|B_{\lambda_n}(\lambda_n)|^{-1}$. Clearly there exists a decreasing sequence $(\varepsilon_n)_n$ with $\tilde\varepsilon_n \le \varepsilon_n$, $n \in \mathbb{N}$, and $\lim_n \varepsilon_n = 0$. Now, if $I_n = I_{\lambda_n}$, set $J_n = I_n \setminus I_{n+1}$, $\beta_n = \varepsilon_n - \varepsilon_{n+1}$, and set

$$w(\zeta) = \sum_n \frac{\beta_n}{\sigma(J_n)}\chi_{J_n}(\zeta), \quad \zeta \in \partial\mathbb{D}.$$

Then $w \in L^1(\partial\mathbb{D})$, and

$$\begin{aligned} P[w](\lambda_n) &\ge \int_{I_n} P(\lambda_n,\zeta)\sum_k \frac{\beta_k}{\sigma(J_n)}\chi_{J_k}(\zeta)\,d\sigma(\zeta) \gtrsim \sum_{k\ge n}\frac{\beta_k}{\sigma(J_n)}\frac{1}{(1-|\lambda_n|)}\int_{J_k} d\sigma(\zeta) \\ &= \frac{\sum_{k\ge n}\beta_k}{1-|\lambda_n|} = \frac{\varepsilon_n}{1-|\lambda_n|} \ge \frac{\tilde\varepsilon_n}{1-|\lambda_n|} = \log|B_{\lambda_n}(\lambda_n)|^{-1}. \end{aligned}$$

This and Theorem 1.3 prove the assertion.

The proof for the Nevanlinna class is even simpler. Set $d\mu_s = \delta_1$, the Dirac mass on $1 \in \partial\mathbb{D}$. From (1.4) we get

$$\log|B_{\lambda_n}(\lambda_n)|^{-1} \lesssim \frac{1}{1-|\lambda_n|} \lesssim P[\mu_s](\lambda_n),$$

and we finish by applying Theorem 1.2. ■

Proof of Proposition 1.13. By Corollary 1.11(c), we already know that (1.5) is a sufficient condition for Λ to be interpolating for N^+. Conversely, suppose that Λ is interpolating for N, that is, φ_Λ admits a harmonic majorant. Since μ_Λ has bounded balayage, then $\int_{\mathbb{D}} \varphi_\Lambda d\mu_\Lambda < \infty$, which is exactly (1.5). ■

Proof of Proposition 1.14. It is obvious that (a) implies (b). If we assume (c), μ_Λ will act against any positive harmonic function. Suppose $F \in N$. As seen in Section 5, there exists a positive harmonic function h so that $\log^+|F| \le h$. Thus, taking μ_λ as in (1.6),

$$\sum_{\lambda\in\Lambda}(1-|\lambda|)\log^+|F(\lambda)| = \int_{\mathbb{D}} \log^+|F(\lambda)|\, d\mu_\Lambda(\lambda) \le \int_{\mathbb{D}} h(\lambda)\, d\mu_\Lambda(\lambda) < \infty.$$

Finally, to prove that (b) implies (c), suppose that (c) doesn't hold, i.e. $g_\Lambda := \sum_\lambda (1-|\lambda|)P_\lambda$ is unbounded. Since g_Λ is lower semi-continuous, this implies that $g_\Lambda \notin L^\infty(\partial\mathbb{D})$. Since L^∞ is the dual of L^1, there exists $f \in L^1(\partial\mathbb{D})$ such that $\int_{\partial\mathbb{D}} f g_\Lambda = \infty$. Taking an outer function $F \in N^+$ with $\log|F| = P[f]$ we see that

$$\sum_{\lambda\in\Lambda}(1-|\lambda|)\log|F(\lambda)| = \sum_{\lambda\in\Lambda}(1-|\lambda|)\int_{\partial\mathbb{D}} P_\lambda f = \int_{\partial\mathbb{D}} f g_\Lambda = \infty,$$

so (b) doesn't hold. ■

Proof of Naftalevič's theorem. Assume that Λ is contained in a finite union of Stolz angles and (1.4) holds. By Proposition 1.12, $\Lambda \in \operatorname{Int} N$, hence the trace $N|\Lambda$ is given by the majorization condition of Theorem 1.2(b). Taking as majorizing function the Poisson integral of the sum of the Dirac masses at the vertices, we see that $N|\Lambda \supset l_{\mathrm{Na}}$.

Conversely, if $N|\Lambda = l_{\mathrm{Na}}$ then the trace is ideal, so Λ is free interpolating and by Corollary 1.11(b) (1.4) holds. According to Theorem 1.2(b) and the definition of l_{Na}, the function

$$\varphi(z) = \begin{cases} (1-|\lambda|)^{-1} & \text{if } z = \lambda \in \Lambda \\ 0 & \text{if } z \notin \Lambda \end{cases}$$

admits a harmonic majorant h. Let $h(z) = P[\mu](z)$ and consider the intervals

$$I_z^\alpha = \{\zeta \in \partial\mathbb{D} : z \in \Gamma_\alpha(\zeta)\}.$$

There exist constants α and C_α such that $\mu(I_z^\alpha) > C_\alpha$ for any z such that $h(z) \ge (1-|z|)^{-1}$.

If Λ is not contained in a finite union of Stolz angles, then there is an accumulation point $\zeta \in \partial\mathbb{D}$ of $\Lambda' \subset \Lambda$ such that $\Lambda' \not\subset \Gamma_\beta(\zeta)$ for any β. Pick $\beta > \alpha$; then for $\lambda' \in \Lambda'$, $I_{\lambda'}^\alpha \not\ni \zeta$ and we can construct an infinite subsequence $\Lambda'' \subset \Lambda'$ such that the Privalov shadows $\{I_{\lambda'}^\alpha\}_{\lambda'\in\Lambda''}$ are disjoint. This prevents h from being the Poisson integral of a finite positive measure. ■

We now give an example of a concrete separation condition implying that μ_Λ has bounded balayage.

Proposition 8.2. *Assume that $\Lambda \subset \mathbb{D}$ is contained in the union of a family A of Whitney squares such that*

$$|\operatorname{Arg}(z(Q)) - \operatorname{Arg}(z(L))| > g^{-1}(1-|z(Q)|)$$

for any $Q, L \in A$, $Q \ne L$, where $z(Q)$ is the center of Q and g is a positive function, with $g(x)/x$ decreasing and

$$\int_0 \frac{g(x)}{x^2} < \infty\,.$$

Then $\Lambda \in \operatorname{Int} N$ if and only if $\Lambda \in \operatorname{Int} N^+$, and if and only if

$$\sum_{Q\in A}(1-|z(Q)|)\sup_{\lambda\in\Lambda\cap Q}\log|B_\lambda(\lambda)|^{-1} < \infty$$

Note that this covers some cases where μ_Λ does not have bounded balayage, even though another measure associated with the sequence will (see the proof).

In order to prove Proposition 8.2 consider the "Carleson window" $Q(e^{i\theta}, r)$ centered at $e^{i\theta}$, of side r:

$$Q(e^{i\theta}, r) := \{z \in \mathbb{D} : 1 - |z| \le r, |\operatorname{Arg}(z) - \theta| \le r\}.$$

The next result is a Carleson-type condition which implies boundedness of the balayage.

Lemma 8.3. *Suppose that $\mu(Q(e^{i\theta}, r)) \le g(r)$, where g is a nondecreasing function on $[0, 2)$ with*

$$\int_0 \frac{g(x)}{x^2} dx < \infty.$$

Then μ is a measure with bounded balayage.

A discrete version of this condition is

$$\sum_n 2^n \sup_{\theta \in \mathbb{R}} \mu(Q(e^{i\theta}, 2^{-n})) < \infty,$$

as can be checked by writing a Riemann sum.

Proof. For any $t > 0$, let $\Omega_t(\theta) := \{z \in \mathbb{D} : P_z(e^{i\theta}) \ge t\}$. This is a disk, tangent to the unit circle at the point $e^{i\theta}$, of radius $1/(t + 1)$. Therefore $\Omega_t(\theta) \subset Q(e^{i\theta}, C/t)$ for $t \ge 1$, say, with $C > 0$ an absolute constant.

Using the distribution function $\mu(\Omega_t(\theta))$ and the fact that the measure μ is bounded, we get the following estimate for the balayée of μ:

$$\begin{aligned}
\int_{\mathbb{D}} P_z(e^{i\theta}) d\mu(z) &= \int_0^\infty \mu(\Omega_t(\theta)) dt \le C_1 + \int_1^\infty \mu(\Omega_t(\theta)) dt \le C_1 + \int_1^\infty \mu(Q(e^{i\theta}, C/t)) dt \\
&\le C_1 + \int_1^\infty g(C/t) dt \le C_1 + C \int_0^1 \frac{g(x)}{x^2} dx < \infty.
\end{aligned}$$

∎

We will now compare measures satisfying the condition in Lemma 8.3, measures with bounded balayage and Carleson measures. Each set is included in the next, and the examples will show that both inclusions are strict.

Example 1. Let $\alpha = \{\alpha_n\}$ be a sequence of nonnegative reals. Let μ_α be the measure concentrated on the circles centered at the origin of radius $1 - 2^{-n}$ given in dual terms by

$$\int_{\mathbb{D}} f(z) d\mu_\alpha(z) := \sum_{n \ge 1} \alpha_n \frac{1}{2\pi} \int_0^{2\pi} f((1 - 2^{-n}) e^{i\theta}) d\theta.$$

One can check that μ_α is a Carleson measure if and only if it has bounded balayage and this happens if and only if $\sum_n \alpha_n < \infty$. Also μ_α satisfies the condition in Lemma 8.3 if and only if $\sum_n \sum_{k \ge n} \alpha_k < \infty$.

Example 2. Let m be a nonnegative-valued function on the interval $[0, 1)$. Let μ_m be the measure concentrated on the ray from the origin to 1 given by

$$\int_{\mathbb{D}} f(z) d\mu_m(z) := \int_0^1 f(x) m(x) dx.$$

One can check that μ_m is a Carleson measure if and only if there exists a constant K such that

$$\int_{1-\delta}^{1} m(x)dx \le K\delta, \qquad \forall \delta > 0$$

and μ_m is a measure with bounded balayage if and only if it satisfies the condition in Lemma 8.3, which happens if and only if

$$\int^{1} \frac{m(x)}{1-x} dx < \infty.$$

In particular, if we take $\alpha_k = k^{-\gamma}$, with $1 < \gamma \le 2$, μ_m is a measure with bounded balayage but it does not satisfy the condition in Lemma 8.3; if we take $m(x) = 1$, μ_m is a Carleson measure, but it does not have bounded balayage.

In view of Proposition 1.13, among other things, it is interesting to understand for which separated sequences Λ the corresponding measure μ_Λ has bounded balayage. It is easy to see that this is the case when $|\lambda/|\lambda| - \lambda'/|\lambda'|| \ge (1-|\lambda|)^{1/2}$, $\lambda' \ne \lambda$, but more is true.

Lemma 8.4. *Suppose that g is a positive valued function such that $g(x)/x$ is increasing and*

$$\int_0 \frac{g(x)}{x^2} dx < \infty.$$

Let g^{-1} stand for the inverse function of g. Then, if we have a sequence $\Lambda \subset \mathbb{D}$ such that

$$\left| \frac{\lambda}{|\lambda|} - \frac{\lambda'}{|\lambda'|} \right| \ge g^{-1}(1-|\lambda|), \quad \forall \lambda' \ne \lambda,$$

the measure μ_Λ has bounded balayage.

Examples of such functions g are given by $x(\log \frac{1}{x})^{-1-\varepsilon}$, with $\varepsilon > 0$. In that case, $g^{-1}(t) \simeq t(\log \frac{1}{t})^{1+\varepsilon}$.

On the other hand, we can see that for the above lemma to hold, we must have $g^{-1}(t) >> t$. More precisely, take the sequence in the upper half-plane given by

$$\lambda_k := e^{-k} + ik^{-1/2}e^{-k}.$$

Then, Re $\lambda_k -$ Re $\lambda_{k+1} \simeq e^{-k}$, so the sequence $\{\lambda_k\}_k$ verifies the separation condition in Lemma 8.4 with $g^{-1}(x) \simeq x(\log \frac{1}{x})^{1/2}$, but

$$\sum_k (\text{Im } \lambda_k) P_{\lambda_k}(0) \simeq \sum_k \left(\frac{\text{Re } \lambda_k}{\text{Im } \lambda_k} \right)^2 = \sum_k \frac{1}{k} = \infty.$$

Proof of Lemma 8.4. Let $\theta \in [0, 2\pi)$. By hypothesis, there is at most one $\lambda \in \Lambda$ such that

$$\theta \in J_\lambda := \Big(\text{Arg}(\lambda) - \frac{1}{2} g^{-1}(1-|\lambda|), \text{Arg}(\lambda) + \frac{1}{2} g^{-1}(1-|\lambda|) \Big).$$

Let $\mu' := \sum_{\lambda' \ne \lambda} (1-|\lambda'|)\delta_{\lambda'}$. Then

$$\int_{\mathbb{D}} P_z(e^{i\theta}) d\mu_\Lambda(z) = (1-|\lambda|)P_\lambda(e^{i\theta}) + \int_{\mathbb{D}} P_z(e^{i\theta}) d\mu'(z) \le C + \int_{\mathbb{D}} P_z(e^{i\theta}) d\mu'(z).$$

By the proof of Lemma 8.3 for this specific value of θ, we see that it will be enough to check that for some absolute constants C_1, C_2, one has

$$\mu'(Q(e^{i\theta}, r)) \le C_1 g(C_2 r), \text{ for } 0 < r < 2.$$

Consider $\Sigma_r := \{\lambda' \neq \lambda : \lambda' \in Q(e^{i\theta}, r)\}$. For any $\lambda' \in \Sigma_r$ we have

$$\sigma(J_{\lambda'}) = g^{-1}(1 - |\lambda'|) \leq 2|\theta - \mathrm{Arg}(\lambda')| \leq 2r,$$

so the intervals $J_{\lambda'}$ are all contained in $[\theta - 3r, \theta + 3r]$. Since they are disjoint, $\sum_{\lambda' \in \Sigma_r} \sigma(J_{\lambda'}) \leq 6r$.

Using that $g(x)/x$ is increasing we have

$$\sup_{\lambda' \in \Sigma_r} \frac{1 - |\lambda'|}{\sigma(J_{\lambda'})} \leq \frac{g\Big(\sup\limits_{\lambda' \in \Sigma_r} \sigma(J_{\lambda'})\Big)}{\sup\limits_{\lambda' \in \Sigma_r} \sigma(J_{\lambda'})} \leq \frac{g(2r)}{2r}.$$

Finally,

$$\mu'(Q(e^{i\theta}, r)) = \sum_{\lambda' \in \Sigma_r} 1 - |\lambda'| \leq \sup_{\lambda' \in \Sigma_r} \frac{1 - |\lambda'|}{\sigma(J_{\lambda'})} \sum_{\lambda' \in \Sigma_r} \sigma(J_{\lambda'}) \leq \frac{g(2r)}{2r} 6r = 4g(2r).$$

■

Proof of Proposition 8.2. For each Whitney square Q in A, let $\lambda(Q)$ be the point in $\Lambda \cap Q$ such that

$$\log |B_{\lambda(Q)}(\lambda(Q))|^{-1} = \max\{\log |B_\lambda(\lambda)|^{-1} : \lambda \in \Lambda \cap Q\}.$$

Let Σ be the sequence formed by $\{\lambda(Q) : Q \in A\}$. By Lemma 8.4 the corresponding measure μ_Σ has bounded balayage. Therefore, there exists a positive harmonic function h with $h(\lambda(Q)) \geq \log |B_{\lambda(Q)}(\lambda(Q))|^{-1}$ if and only if

$$\sum (1 - |\lambda(Q)|) \log |B_{\lambda(Q)}(\lambda(Q))|^{-1} < \infty\,.$$

According to condition (c) in Theorem 1.2 one deduces that $\Lambda \in \mathrm{Int}\, N$ if and only if the last sum converges. Furthermore, when this is the case, the function h can always be taken quasi-bounded (see Lemma 8.1), so that interpolation can actually be performed in the Smirnov class. ■

9. HARDY-ORLICZ CLASSES

Let $\phi : \mathbb{R} \longrightarrow [0, \infty)$ be a convex, nondecreasing function satisfying

(i) $\lim_{t\to\infty} \phi(t)/t = \infty$

(ii) Δ_2-condition: $\phi(t+2) \leq M\phi(t) + K$, $t \geq t_0$ for some constants $M, K \geq 0$ and $t_0 \in \mathbb{R}$.

Such a function is called strongly convex (see [RosRov]), and one can associate with it the corresponding *Hardy-Orlicz class*

$$\mathcal{H}_\phi = \{f \in N^+ : \int_{\partial\mathbb{D}} \phi(\log |f(\zeta)|)\, d\sigma(\zeta) < \infty\},$$

where $f(\zeta)$ is the non-tangential boundary value of f at $\zeta \in \partial\mathbb{D}$, which exists almost everywhere. In [Har99], the following result was proved.

Theorem. *Let ϕ be a strongly convex function satisfying (i), (ii) and the V_2-condition:*

$$2\phi(t) \leq \phi(t + \alpha), \quad t \geq t_1$$

where $\alpha > 0$ is a suitable constant and $t_1 \in \mathbb{R}$. Then $\Lambda \subset \mathbb{D}$ is free interpolating for $\mathcal{H}_\phi$ if and only if Λ is a Carleson sequence, and in this case

$$\mathcal{H}_\phi|\Lambda = \{a = (a_\lambda)_\lambda : |a|_\varphi = \sum_{\lambda\in\Lambda}(1-|\lambda|)\varphi(\log|a_\lambda|) < \infty\}.$$

The conditions on ϕ imply that there exist $p, q \in (0,\infty)$ such that $H^p \subset \mathcal{H}_\phi \subset H^q$. In particular, the V_2-condition implies the inclusion $H^p \subset \mathcal{H}_\phi$ for some $p > 0$. This V_2-condition has a strong topological impact on the spaces. In fact, it guarantees that metric bounded sets are also bounded in the topology of the space (and so the usual functional analysis tools still apply in this situation; see [Har99] for more on this and for further references). It was not clear whether this was only a technical problem or if there existed a critical growth for ϕ (below exponential growth $\phi(t) = e^{pt}$ corresponding to H^p spaces) giving a breakpoint in the behavior of interpolating sequences for $\mathcal{H}_\phi$.

We can now affirm that this behavior in fact changes between exponential and polynomial growth. Let ϕ be a strongly convex function with associated Hardy-Orlicz space $\mathcal{H}_\phi$. Assume moreover that ϕ satisfies

$$\phi(a+b) \le c(\phi(a)+\phi(b)), \tag{9.1}$$

for some fixed constant $c \ge 1$ and for all $a, b \ge t_0$. The standard example in this setting is $\phi_p(t) = t^p$ for $p > 1$. We have the following result.

Theorem 9.1. *Let $\phi : \mathbb{R} \longrightarrow [0,\infty)$ be a strongly convex function such that* (9.1) *holds. If there exists a positive weight $w \in L^1(\partial\mathbb{D})$ such that $\phi\circ w \in L^1(\partial\mathbb{D})$ and $\varphi_\Lambda \le P[w]$, then $\Lambda \in \operatorname{Int}\mathcal{H}_\phi$.*

Proof. Note first that (9.1) implies that $\mathcal{H}_\phi$ is an algebra contained in N^+, hence it is sufficient to interpolate bounded sequences (see Remark 1.1). As in Section 3, we set

$$g(z) = \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} w(\zeta)d\sigma(\zeta).$$

The reasoning carried out in Section 3 leads to an interpolating function of the form $fH\exp(g)$, with $f \in H^\infty$, and $H = (2+g)^2$ outer in H^p for all $p < 1$ (note that the measure μ defining g here is absolutely continuous, in fact $\mu = w\,d\sigma$). Also, $H^p \subset \mathcal{H}_\phi$ for any $p > 0$ by our conditions on ϕ. By construction, $\int \phi(\log|\exp g|) = \int \phi\circ w < \infty$ so that $\exp(g) \in \mathcal{H}_\phi$. Since $\mathcal{H}_\phi$ is an algebra, we deduce that $fH\exp(g) \in \mathcal{H}_\phi$. ■

Example 9.2. We give an example of an interpolating sequence for $\mathcal{H}_\phi$ which is not Carleson, thus justifying our claim that there is a breakpoint between Hardy-Orlicz spaces verifying the V_2-condition and those that do not.

Consider the functions ϕ_p and let $\Lambda_0 = \{\lambda_n\}_n \subset \mathbb{D}$ be a Carleson sequence verifying $I_n \cap I_k = \emptyset$, $n \ne k$, where I_n are the arcs defined in (1.2). Since $\sum_n(1-|\lambda_n|) < \infty$, there exists a strictly increasing sequence of positive numbers $(\gamma_n)_n$ such that $\sum_n(1-|\lambda_n|)\gamma_n < \infty$ and $\lim_{n\to\infty}\gamma_n = \infty$. Setting

$$w = \sum_n \gamma_n^{1/p}\chi_{I_n},$$

we obtain $\int \phi_p \circ w = \sum_n(1-|\lambda_n|)\gamma_n < \infty$ and $w \in L^1(\partial\mathbb{D})$ since $p > 1$. Associate with Λ_0 a second Carleson sequence $\Lambda_1 = \{\lambda_n'\}_n$ such that the pseudo-hyperbolic distance between corresponding points satisfies $|b_{\lambda'_n}(\lambda_n)| = e^{-\gamma_n^{1/p}}$. Since $\gamma_n \to \infty$ the elements of the sequence $\Lambda = \Lambda_0 \cup \Lambda_1$ are arbitrarily close and Λ cannot be a Carleson sequence. By construction, $\log |B_\lambda(\lambda)|^{-1} \leq P[w](\lambda)$ (as before, we may possibly have to multiply u with some constant c to have that condition also in the points λ'_n, but this operation conserves the integrability condition), and therefore $\Lambda \in \operatorname{Int} \mathcal{H}_\phi$.

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