PHRAGMÉN-LINDELÖF-TYPE PROBLEMS FOR $A^{-\alpha}$.

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ABSTRACT. We want to formalize problems of the following kind about holomorphic functions on the unit disk: given a size condition (that the function f belongs to a certain growth space), and a restriction on the growth of f on a subset S of the disk, what restriction do we get on the global growth of f? More explicitly, given $0 \le \gamma \le \beta \le \alpha \le \infty$, we want to know when the norm inequality $\sup_{z\in\mathbb{D}}(1-|z|)^\beta|f(z)| \le M \sup_{s\in S}(1-|s|)^\gamma|f(s)|$ holds for any $f \in A^{-\alpha}$. We study the various implications between those properties for different values of the parameters, and give a necessary condition in an interesting special case.

1. DEFINITIONS AND STATEMENT OF RESULTS

Boris Korenblum often asks whether someone can find a characterization of the (discrete) subsets of the unit which have the Phragmén-Lindelöf property for the space $A^{-\infty}$, that is to say, such that any function in $A^{-\infty}$ that is bounded on the set must necessarily be bounded everywhere. We cannot answer this question, but we show that a certain generalization of the notion puts it into a common framework with a number of interesting problems in the area. The aim of this note is to take stock of the situation to date, and take a first step towards some further results. In particular, we give a (weak) necessary condition for the Phragmén-Lindelöf property.

1.1. **Definitions, implications.**

Given $0 \le \alpha < \infty$ and $S \subset \mathbb{D}$, let

$$A^{-\alpha}(S) = \{f \in H(\mathbb{D}) : \|f\|_{\alpha,S} := \sup_{z\in S}(1-|z|)^\alpha|f(z)| < \infty\} ,$$

where $H(\mathbb{D})$ stands for the set of all holomorphic functions in $\mathbb{D}$. In the case where $S = \mathbb{D}$, we write $\|f\|_{\alpha,\mathbb{D}} = \|f\|_\alpha$, and the spaces $A^{-\alpha}(\mathbb{D}) = A^{-\alpha}$ are the familiar growth spaces. Let also $A^{-\alpha}_+ := \bigcap_{\beta>\alpha} A^{-\beta}$ and $A^{-\infty} = \bigcup_{\alpha>0} A^{-\alpha}$.

We want to formalize problems of the following kind: given a size condition (that the function f belongs to a certain growth space), and a restriction on the growth of f on a subset S, what restriction do we get on the global growth of f?

Definition. Let $0 \le \gamma \le \beta \le \alpha \le \infty$, with $\beta < \infty$. A set $S \subset \mathbb{D}$ is *of type* (α, β, γ), denoted $S \in \mathcal{T}(\alpha, \beta, \gamma)$, if and only if there exists a constant $M > 0$ such that for all $f \in A^{-\alpha}$, $\|f\|_\beta \le M\|f\|_{\gamma,S}$.

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Even though Banach space methods do not apply immediately, this turns out, where it makes sense, to be the same as the seemingly weaker inclusion property.

Lemma 1. *Let $0 \leq \gamma \leq \beta < \alpha \leq \infty$, with $\beta < \infty$. A set $S \subset \mathbb{D}$ is of type (α, β, γ) if and only if*
$$A^{-\alpha} \cap A^{-\gamma}(S) \subset A^{-\beta}.$$

The proof is given in Section 2.

Special cases of the definition above coincide with previously studied properties:

- When $\alpha = \beta = \gamma > 0$, we have the $A^{-\alpha}$*-sampling* sets studied by Seip [Se93b]. The case $\alpha = \beta = \gamma = 0$ corresponds to the $H^\infty(\mathbb{D})$*-dominating sets* described earlier by Brown, Shields and Zeller [BrShZe60].
- Horowitz, Korenblum, and Pinchuk [HoKoPi97] studied $A^{-\infty}$ *-sampling sets*, which in our terminology would be $\cap_{\gamma>0} \cap_{\varepsilon>0} \mathcal{T}(\infty, \gamma + \varepsilon, \gamma)$, or equivalently, sets such that $A^{-\infty} \cap A^{-\gamma}(S) \subset A_+^{-\gamma}$ for any $\gamma > 0$.
- *Weakly sufficient sets* are those for which the topology of $A^{-\infty}$ is determined by the restrictions of $f \in A^{-\infty}$ on the set, see [KhTh98], [Sch74]. This corresponds to the sets S such that for any $\gamma > 0$, there exists $\beta = \beta(\gamma)$ with $S \in \mathcal{T}(\infty, \beta, \gamma)$.
- Bonet and Domański [BoDo03] studied the sets of type (β, β, γ), which they termed (γ, β) *-sampling sets*, and of type (∞, β, γ), which they termed (γ, β)*-dominating sets*.

We introduce another piece of terminology.

Definition. Let $0 \leq \beta < \alpha$. A set $S \subset \mathbb{D}$ is (α, β)*-Phragmén-Lindelöf* if and only if $S \in \mathcal{T}(\alpha, \beta, \beta)$.

Korenblum's Phragmén-Lindelöf sets correspond to our $(\infty, 0)$ case. An easy adaptation of [Sch74, Theorem 3.10] to our situation shows that any weakly sufficient set is $(\infty, 0)$-Phragmén-Lindelöf. Also, it is easy to see (taking powers) that any set of type $(\infty, \beta, 0)$ is really $(\infty, 0)$-Phragmén-Lindelöf.

We would like to clarify the relationships between those various classes of sets. First notice that it is trivial that $\mathcal{T}(\alpha, \beta, \gamma) \subset \mathcal{T}(\alpha', \beta', \gamma')$ when $\alpha \geq \alpha'$, $\beta \leq \beta'$, and $\gamma \geq \gamma'$, and readily seen by taking powers that $\mathcal{T}(\alpha, \beta, \gamma) \subset \mathcal{T}(\alpha/m, \beta/m, \gamma/m)$ whenever $m \in \mathbb{N}$.

Theorem 2. (a) *If $\alpha \geq \alpha'$, $\beta \leq \beta' + \min(\alpha - \alpha', \gamma - \gamma')$, and $\gamma \geq \gamma'$, then*
$$\mathcal{T}(\alpha, \beta, \gamma) \subset \mathcal{T}(\alpha', \beta', \gamma').$$

(b) *Conversely, if $\alpha < \alpha'$; or if $\gamma < \gamma'$ and $\frac{\gamma}{\beta} < \frac{\gamma'}{\beta'}$, then $\mathcal{T}(\alpha, \beta, \gamma) \not\subset \mathcal{T}(\alpha', \beta', \gamma')$.*

(c) *In the special situation of Phragmén-Lindelöf sets, we can give a more complete result: $\mathcal{T}(\alpha, \beta, \beta) \subset \mathcal{T}(\alpha', \beta', \beta')$ if and only if $\alpha \geq \alpha'$ and $\beta \geq \beta'$, and the inclusion is proper when $\alpha > \alpha'$ or $\beta > \beta'$.*

The proof is given in Section 2. Notice that a number of cases remain open.

Corollary 3. (a) *If S is $A^{-\alpha}$-sampling, then S is (α, β)-Phragmén-Lindelöf for all $\beta \leq \alpha$.*

(b) *If S is (α, β)-Phragmén-Lindelöf for some $\beta \leq \alpha$, then S is $A^{-\beta}$-sampling.*

1.2. **A necessary condition.** We now would like to give some conditions for Phragmén-Lindelöf sets in terms of densities.

For an arc $I \subset \mathbb{T}$, let $|I| = \sigma(I)$ stand for its normalized arc-length. We also use a normalized distance on $\mathbb{T}$:

$$d(e^{i\theta}, e^{i\psi}) = \frac{|\theta - \psi|}{\pi}, \quad d(\zeta, F) := \inf_{\xi \in F} d(\zeta, \xi) \quad \text{when } F \subset \mathbb{T}.$$

For a closed proper subset $F \subset \mathbb{T}$ such that $\mathbb{T} \setminus F = \bigcup_j I_j$, where the I_j are disjoint open arcs, the *Beurling-Carleson characteristic* of F is

$$\hat{\kappa}(F) := \sum_j |I_j|(1 + \log \frac{1}{|I_j|}) = \int_{\mathbb{T}} \log \frac{1}{d(\zeta, F)} d\sigma(\zeta).$$

The *density* of a set $S \subset \mathbb{D}$ is defined as

$$D(S) := \limsup_{\hat{\kappa}(F) \to \infty} \frac{\sum_{s \in S \cap \Gamma(F)} \log \frac{1}{|s|}}{\hat{\kappa}(F)},$$

where $\Gamma(F)$ denotes the Korenblum star over F (see precise definition below).

S is a zero-set for $A_+^{-\alpha}$ if and only if $D(S) \leq \alpha$ ([Se95], [HeKoZh00, Theorem 4.15, p. 112]).

The Seip densities can be seen as versions of this, made uniform over automorphisms of the disk. More precisely, for $\frac{1}{2} < r < 1$, let

$$D(S, r) := \frac{\sum_{s \in S \cap (\mathbb{D} \setminus \bar{D}(0, \frac{1}{2}))} \log \frac{1}{|s|}}{\log \frac{1}{1-r}}.$$

The *uniform upper and lower densities* are then defined respectively as

$$\begin{aligned} D_u^+(S) &:= \limsup_{r \to 1} \sup_{\varphi \in \mathrm{Aut}(\mathbb{D})} D(\varphi(S), r) \\ D_u^-(S) &:= \liminf_{r \to 1} \inf_{\varphi \in \mathrm{Aut}(\mathbb{D})} D(\varphi(S), r). \end{aligned}$$

A set S is called *hyperbolically separated* if and only if there exists $\delta > 0$ such that for any $s \neq s' \in S$, $|\varphi_s(s')| \geq \delta$, where φ_s is an automorphism of $\mathbb{D}$ sending s to 0.

According to a well-known geometric characterization given by Seip, S is $A^{-\alpha}$-sampling if and only if it contains a hyperbolically separated subset $S' \subset S$ with $D_u^-(S') > \alpha$ ([Se93b], [HeKoZh00, Theorem 5.18, p. 153]).

A set of density less than α cannot be of type (α, β, γ) (see Lemma 5 below). From Corollary 3 we see also that an (α, β)-Phragmén-Lindelöf set must be of uniform lower density greater than β, and since this notion is invariant under automorphisms, *every* point of the boundary must be a non-tangential limit point of the set.

On the other hand, it follows from examples given in [HoKoPi97] and [KhTh98] that there exist sets of type (α, β, γ) with lower density equal to 0 as soon as $\beta > \gamma$. But it does follow from Theorem 2 that all of the sets we are studying are dominating sets for H^∞, and therefore that almost every point of the unit circle is a non-tangential limit point of the set.

We now define a more quantitative notion of non-tangential density, related to the above densities (see Lemma 8).

Given $z \in \mathbb{D}$, let

$$I_z^\gamma = \Big\{ \zeta \in \mathbb{T} : d(\zeta, z/|z|) < \gamma(1-|z|) \Big\}.$$

Also, for $\zeta \in \mathbb{T}$, $\gamma \in (0,1/2]$, $t \in (0,1)$ and $A \subset \mathbb{T}$ define:

$$\begin{aligned} \Gamma_\gamma(\zeta) &= \{z \in \mathbb{D} : \zeta \in I_z^\gamma\} \qquad & {\Gamma_\gamma}^t(\zeta) = \Gamma_\gamma(\zeta) \cap \bar{D}(1-t) \\ \Gamma_\gamma(A) &= \bigcup_{\zeta \in A} \Gamma_\gamma(\zeta) & \Gamma_\gamma^t(A) = \Gamma_\gamma(A) \cap \bar{D}(1-t)\ , \end{aligned}$$

where $D(t) = \{z \in \mathbb{D} : |z| < t\}$. Notice that $\Gamma_\gamma(\zeta)$ is contained in the half disk $\{z \in \mathbb{D} : |\arg z - \arg \zeta| < \pi/2\}$.

Given a sequence S in $\mathbb{D}$, consider the counting function

$$n_\gamma(\zeta, t; S) = \#({\Gamma_\gamma}^t(\zeta) \cap S)\ .$$

Definition. Let S be a sequence in $\mathbb{D}$ and let $A \subset \mathbb{T}$. The *boundary density of S in A* is

$$BD_S^\gamma(A) := \lim_{t \searrow 0} \sup_{\substack{\zeta \in A \\ 0 < u \le t}} \frac{n_\gamma(\zeta, u; S)}{\gamma |\log u|}$$

Theorem 4. *Let $S \in \mathcal{T}(\infty, 0, 0)$. Then, for all γ and all $I \subset \mathbb{T}$ interval*

$$BD_S^\gamma(I) = \infty\ .$$

The proof is given in Section 3. This theorem is quite far from being optimal. It only shows that any subset of $\mathbb{T}$ in which S has finite boundary density must be of empty interior. It would be natural to expect that the boundary density is infinite outside of a small exceptional set, in fact smaller than merely negligible for the 1-dimensional Lebesgue measure.

Indeed, as follows from the proofs in [Hr78, Section 9], or the more explicit version in [BeCo86] using [Da77] (see also [BeCo88] and [Ko02]), the correct measure should be the Hausdorff measure H_L associated to the function $L(x) := x|\log x|$. More precisely, any $f \in A^{-\infty}$ such that

$$\limsup_{r \to 1} |f(r\zeta)| < \infty, \quad \text{for all } \zeta \in \mathbb{T} \setminus E, \quad \text{with } H_L(E) = 0,$$

must belong to $H^\infty(\mathbb{D})$. And conversely, given a closed set $E \subset \mathbb{T}$ with $H_L(E) > 0$, one can find a positive measure μ supported on $E' \subset E$ such that $f(z) = \exp\{\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\mu(\zeta)\}$ belongs to $A^{-\alpha} \setminus H^\infty(\mathbb{D})$ (in fact α can be chosen arbitrarily small) and is bounded on the set $\Big\{z \in \mathbb{D} : 1-|z|^2 \le (d(z,E))^2\Big\}$, which is as dense as one could possibly want near the points of $\mathbb{T} \setminus E$.

2. INEQUALITIES AND INCLUSIONS

In this section, we prove Lemma 1 and Theorem 2. The existence of regular uniformly distributed sequences will be useful.

Proposition A. *For any $t > 0$ there exists a hyperbolically separated sequences S_t such that*

$$D_u^+(S_t) = D_u^-(S_t) = t \, .$$

For such sequences also $D(S_t) = t$ (see [Se93b, Proposition 3.1], [HeKoZh00, Section 5.4]).

We first need an auxiliary result which is of some interest in itself.

Lemma 5. *Let $\alpha > \beta \geq \gamma$. Any set S such that $A^{-\alpha} \cap A^{-\gamma}(S) \subset A^{-\beta}$ is a uniqueness set for $A^{-\alpha}$.*

Proof. Suppose that $f \in A^{-\alpha} \setminus \{0\}$ and $f|_S = 0$. Then $f \in A^{-\alpha} \cap A^{-\gamma}(S) \subset A^{-\beta}$.

Let $\delta := D(S) \leq \beta$ by the necessary condition on zero-sets. Since $\alpha > \beta$, we can pick $t \in (\beta - \delta, \alpha - \delta)$. Choose then S_t, disjoint from S, as in Proposition A. Then $D(S \cup S_t) = \delta + t \in (\beta, \alpha)$, and we can find $g \in A_+^{-(\delta+t)} \subset A^{-\alpha}$, g not identically zero, such that $g|_{S \cup S_t} = 0$. Since $D(S \cup S_t) > \beta$, we have $g \notin A^{-\beta}$, thus contradicting the inclusion. ■

The proof of Lemma 1 now proceeds verbatim as the proof of [KhTh98, Proposition 3, p. 439], where the roles of α, β, and γ are played respectively by $q + 2, q$, and p. What was denoted by $A^{-p}(S)$ in [KhTh98] would now be denoted by $A^{-p}(S) \cap A^{-\infty}$, and one should now choose $p_m^{\alpha-\beta} \geq 3^{2m}$ in [KhTh98, p. 440, line 6] to take into account our slightly more general range of exponents.

Proof of Theorem 2. (a) By the remarks before the statement of the theorem, there is no loss of generality to assume that $\beta' \leq \beta$. This proof is then an adaptation of that of [BoDo03, Theorem 1.4].

(c) Since $\mathcal{T}(\alpha, \beta', \beta') \subset \mathcal{T}(\alpha', \beta', \beta')$ trivially, it will be enough to prove $\mathcal{T}(\alpha, \beta, \beta) \subset \mathcal{T}(\alpha, \beta', \beta')$ when $\beta > \beta'$.

Let $f \in A^{-\alpha} \cap A^{-\beta'}(S)$ and define $f_n(z) = z^n f(z)$. For $z \in S$:

$$(1 - |z|)^\beta |f_n(z)| \leq |z|^n (1 - |z|)^{\beta - \beta'} \|f\|_{\beta', S} \, ,$$

and taking the supremum of the right hand side over z,

$$\|f_n\|_{\beta, S} \leq \frac{t^t n^n}{(t+n)^{t+n}} \|f\|_{\beta', S} \ \text{ where } t := \beta - \beta'.$$

By Lemma 1, $\|g\|_\beta \leq M \|g\|_{\beta, S}$, for any $g \in A^{-\alpha}$. Thus, for z with $|z| = r_{n,t} := n/(n+t)$ we find:

$$\begin{aligned} \frac{t^t n^n}{(t+n)^{t+n}} (1 - |z|)^{\beta'} |f(z)| = (1 - |z|)^\beta |f_n(z)| &\leq \|f_n\|_\beta \leq M \|f_n\|_{\beta, S} \\ &\leq M \frac{t^t n^n}{(t+n)^{t+n}} \|f\|_{\beta', S} \, , \end{aligned}$$

i.e. $(1 - |z|)^{\beta'} |f(z)| \leq M \|f\|_{\beta', S}$.

For arbitrary z there exists n such that $|z| \in [r_{n,t}, r_{n+1,t})$, and the maximum principle yields:

$$(1-|z|)^{\beta'}|f(z)| \leq \frac{(1-r_{n,t})^{\beta'}}{(1-r_{n+1,t})^{\beta'}} M\|f\|_{\beta',S} \ .$$

This implies the existence of $C>0$ such that $\|f\|_{\beta'} \leq CM\|f\|_{\beta',S}$, because $\lim\limits_{n\to\infty} \frac{1-r_{n,t}}{1-r_{n+1,t}} = 1$.

Given the previous results, to prove the converse and that the inclusions are proper when the inequalities are strict, it will be enough to see that:

(i) For every $\alpha > \alpha'$ there exists $S \in \mathcal{T}(\alpha', \alpha', \alpha') \setminus \mathcal{T}(\alpha, 0, 0)$.
(ii) If $\beta > \beta'$ there exists $S \in \mathcal{T}(\infty, \beta', \beta') \setminus \mathcal{T}(\beta, \beta, \beta)$.

Proof of (i). This is a special case of the first statement in (b), see proof below.

Proof of (ii). Let $t \in (\beta', \beta)$ and let S_t be as in Proposition A. Take also a sequence of radii $\{r_n\} \nearrow 1$ with $\lim\limits_{n\to\infty} \frac{1-r_{n+1}}{1-r_n} = 0$, and a sequence $\Lambda \subset \cup_{n\in\mathbb{N}}\{z : |z| = r_n\}$, as in [HoKoPi97, Example 2.5], such that

$$A^{-\infty} \cap A^{-\alpha}(\Lambda) \subset A^{\alpha}_{+} \ .$$

Notice that $D_u^-(S_0) = 0$ for all separated subsequence $S_0 \subset \Lambda$, hence Λ is not sampling for any $A^{-\alpha}$, $\alpha > 0$.

Define $S = S_t \cup \Lambda$. It is clear by Seip's characterization that S is not $A^{-\beta}$-sampling, since for every separated subsequence $S_0 \subset S$:

$$D_u^-(S_0) \leq D_u^+(S_0 \cap S_t) + D_u^-(S_0 \cap \Lambda) = D_u^+(S_0 \cap S_t) \leq D_u^+(S_t) = t < \beta \ .$$

To see that $S \in \mathcal{T}(\infty, \beta', \beta')$, let $f \in A^{-\beta'}(S)$. For any $\delta \in (\beta', t)$:

$$f \in A^{-\infty} \cap A^{-\beta'}(S) \subset A^{-\beta'}_{+} \subset A^{-\delta} \ .$$

S_t is $A^{-\delta}$-sampling, because it is hyperbolically separated and $D_u^-(S_t) = t > \delta$. Thus there exists $M = M(\delta) > 0$ such that

$$\|f\|_{\delta} \leq M(\delta)\|f\|_{\delta, S_t} \leq M(\delta)\|f\|_{A^{-\delta}(S)} \ .$$

Since the sampling constant $M(\delta)$ depends only on $D_u^-(S_t)$ and the separation of S_t, we can take the same constant M_0 for all δ close to β':

$$\|f\|_{\delta} \leq M(\delta)\|f\|_{\delta,S} \leq M_0\|f\|_{\delta,S} \qquad \text{for } \ \delta \searrow \beta' \ .$$

Letting δ tend to β' finishes the proof.

(b) Suppose that $\alpha' > \alpha$. Let $t \in (\alpha, \alpha')$ and let S_t be as in Proposition A. Thus S_t is $A^{-\alpha}$-sampling. Also, by [Se95], S_t is a zero set for $A^{-\alpha'}$, therefore by Lemma 5 it cannot be of type $\mathcal{T}(\alpha', \beta', \gamma')$.

Suppose now that $\gamma' > \gamma$, and $\gamma' > \gamma\frac{\beta'}{\beta}$. First we deal with the case when $\gamma' = \beta'$: by the proof of (c), above, we know that $\mathcal{T}(\alpha, \gamma, \gamma) \not\subset \mathcal{T}(\alpha, \gamma', \gamma')$, but trivially $\mathcal{T}(\alpha, \gamma, \gamma) \subset \mathcal{T}(\alpha, \beta, \gamma)$.

Assume then that $\beta' > \gamma'$. Set $M := \beta'/\gamma' \in (1, \beta/\gamma)$. By [BoDo03, Example 4.1] (itself a refinement of [KhTh98, Proposition 5]), there exists a countable union of circles centered at the origin E, such that for any $\delta, \delta' > 0$ and $t > \delta$

$$A^{-t} \cap A^{-\delta}(E) \subset A^{-\delta'} \text{ if and only if } \delta' > M\delta.$$

In particular, $E \in \mathcal{T}(\alpha, \beta, \gamma)$. But since $\beta' = M\gamma'$, we have $A^{-\alpha'} \cap A^{-\gamma'}(E) \not\subset A^{-\beta'}$.

Note that E could be replaced by a discrete set (provided the points in each circle are close enough; we omit the details). ■

3. Density at the boundary

We may want to consider the true cones of vertex $\zeta \in \mathbb{T}$ and aperture $\pi\gamma$, $0 < \gamma < 1/\pi$, defined as the rotation $\tilde{\Gamma}_\gamma(\zeta) := \zeta \cdot \tilde{\Gamma}_\gamma(1)$ of

$$\tilde{\Gamma}_\gamma(1) = \{z \in \mathbb{D} : \text{Re } z > 0\,; |\text{Im } z| < \pi\gamma(1 - \text{Re } z)\} \cup \{z \in \mathbb{D} : \text{Re } z \le 0\,; |z| < \pi\gamma\}\,.$$

By construction there exists $r_0 = r_0(\gamma)$ such that

$$\Gamma_\gamma(\zeta) \subset \tilde{\Gamma}_\gamma(\zeta) \qquad \text{and} \qquad \tilde{\Gamma}_{\gamma/2}(\zeta) \cap (\mathbb{D} \setminus D(r_0)) \subset \Gamma_\gamma(\zeta)\,. \tag{1}$$

Theorem 4 could also be stated using the density $B\tilde{D}^\gamma_S(I)$ defined as $BD^\gamma_S(I)$ but replacing the counting function $n_\gamma(\zeta, t; S)$ by $\tilde{n}_\gamma(\zeta, t; S) = \#(\tilde{\Gamma}^t_\gamma(\zeta) \cap S)$. It is immediate from (1) that for all γ and I:

$$B\tilde{D}^{\gamma/2}_S(I) \le BD^\gamma_S(I) \le B\tilde{D}^\gamma_S(I) \tag{2}$$

We first state some basic properties of the density.

Lemma 6. *Given S and an interval $I \subset \mathbb{T}$, then*

$$BD^\gamma_{S\cap\Gamma_\gamma(I)}(\mathbb{T}) = BD^\gamma_S(I)\,.$$

Proof. Let $S^\gamma_I = S \cap \Gamma_\gamma(I)$. By definition $BD^\gamma_S(I) = BD^\gamma_{S^\gamma_I}(I)$, thus it suffices to prove that $BD^\gamma_{S^\gamma_I}(I) = BD^\gamma_{S^\gamma_I}(\mathbb{T})$. It is clear that $BD^\gamma_{S^\gamma_I}(I) \le BD^\gamma_{S^\gamma_I}(\mathbb{T})$. In order to prove the reverse inequality let $\zeta \notin I$ and take the closest end $\eta \in I$ to ζ. Then $n_\alpha(\zeta, t; S^\gamma_I) \le n_\alpha(\eta, t; S^\gamma_I)$, hence taking the supremum and the limit we obtain the desired inequality. ■

Lemma 7. *Let $S \subset \mathbb{D}$ and let $J = \{e^{i\theta} \in \mathbb{T} : \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$. Given γ and $I \subset \mathbb{T}$ there exists $\phi \in Aut(\mathbb{D})$ such that*

(a) $\phi(I) = J$;
(b) $BD^{\gamma/2}_{\phi(S)}(J) \le 4\, BD^\gamma_S(I)$.

Proof. It will be more convenient to use the cones $\tilde{\Gamma}_\gamma$.

Apply first a rotation so that the interval is transformed into a symmetric interval $I=(e^{-i\alpha},e^{i\alpha})$. Apply now the automorphism $\phi_a(z)=\frac{a+z}{1+az}$, where $a\in(-1,1)$ is chosen so that (a) is satisfied. Explicitly, $a=-\frac{2\cos\alpha}{|1-ie^{i\alpha}|^2}$.

Let $\zeta\in I$. Since ϕ_a is a Möbius transformation, the straight lines along the sides of the cone $\tilde\Gamma_\gamma(\zeta)$ are transformed into circles containing the points $\phi_a(\zeta)$, $\phi_a(\infty)=1/a$, and cutting the unit circle at $\phi_a(\zeta)$ with angle γ. Hence

$$\tilde\Gamma_{\gamma/2}(\phi_a(\zeta))\subset\phi_a\Big(\tilde\Gamma_\gamma(\zeta)\Big)\ .$$

The circle $|z|=\tau$ is transformed into a circle passing through $\phi_a(\tau)=\frac{a+\tau}{1+a\tau}$ and $\phi_a(-\tau)=\frac{a-\tau}{1-a\tau}$ and cutting the real line at these points with angle $\pi/2$. Thus, if $a\le 0$, that is, $|I|\le 1/2$, taking $t=1-\tau$ and $u=t\frac{1-a}{1+a-at}$ we have $\tilde\Gamma^u_{\gamma/2}(\phi_a(\zeta))\subset\phi_a\Big(\tilde\Gamma^t_\gamma(\zeta)\Big)$, and

$$\frac{1}{\gamma/2}\frac{\tilde n_{\gamma/2}(\phi_a(\zeta),u;\phi_a(S))}{|\log u|}\le\frac{2}{\gamma}\frac{\tilde n_\gamma(\zeta,t;S)}{|\log u|}=\frac{2}{\gamma}\ \frac{\tilde n_\gamma(\zeta,t;S)}{\log\frac1t}\ \frac{\log\frac1t}{\log\frac1t-\log\frac{1-a}{1+a-at}}\ .$$

Hence

$$\begin{aligned}B\tilde D^{\gamma/2}_{\phi_a(S)}(\phi_a(I)) &\le 2\ \lim_{T\searrow 0}\sup_{\substack{\zeta\in I\\ t\le T}}\frac{\tilde n_\gamma(\zeta,t;S)}{\gamma\log\frac1t}\ \frac{\log\frac1t}{\log\frac1t-\log\frac{1-a}{1+a-at}}\\ &\le 2\ B\tilde D^\gamma_S(I)\lim_{T\searrow 0}\frac{\log\frac1T}{\log\frac1T-\log\frac{1-a}{1+a-aT}}=2B\tilde D^\gamma_S(I)\ .\end{aligned}$$

The conclusion is then an immediate consequence of (2). In the case where $a>0$, an analogous computation with $u=t\frac{1+a}{1-a+at}$ will yield the same result. ∎

Lemma 8. *Let $S\subset\mathbb{D}$. Then $D(S)\le BD^\gamma_S(\mathbb{T})$ for all $\gamma>0$.*

Proof. Let $S=(s_k)_k$ and define the measure $\mu=\sum_k\delta_{s_k}$. Then, for $E\subset\mathbb{D}$:

$$\begin{aligned}\gamma\sum_{s_k\in E}1-|s_k| &= \int_E\int_{I^\gamma_z}d\sigma(\zeta)\ d\mu(z)=\int_{\mathbb{T}}\int_{E\cap\Gamma_\gamma(\zeta)}d\mu(z)\ d\sigma(\zeta)=\\ &= \int_{\mathbb{T}}\#(S\cap E\cap\Gamma_\gamma(\zeta))\ d\sigma(\zeta)\ .\end{aligned}$$

Take $F\subset\mathbb{T}$ finite or countable, and $E=\Gamma_\gamma(F)$. By construction, there exists c_γ such that $\Gamma_\gamma(F)\cap\Gamma_\gamma(\zeta)\subset\Gamma^{c_\gamma d(\zeta,F)}_\gamma(\zeta)$. Thus

$$\gamma\sum_{s_k\in\Gamma_\gamma(F)}(1-|s_k|)\le\int_{\mathbb{T}}\#(S\cap\Gamma_\gamma(F)\cap\Gamma_\gamma(\zeta))\ d\sigma(\zeta)\le\int_{\mathbb{T}}n_\gamma(\zeta,c_\gamma d(\zeta,F);S)\ d\sigma(\zeta)\ .$$

By hypothesis, given $\varepsilon>0$ there exists $t_\varepsilon>0$ such that

$$n_\gamma(\zeta,t;S)\le\gamma(BD^\gamma_S(\mathbb{T})+\varepsilon)|\log t|\qquad\text{for all } 0<t<t_\varepsilon \text{ and all } \zeta\in\mathbb{T}.$$

We separate the integral above into two terms. For ζ such that $c_\gamma d(\zeta, F) \geq t_\varepsilon$ we have $\Gamma_\gamma^{c_\gamma d(\zeta,F)}(\zeta) \subset D(1 - c_\gamma d(\zeta, F)) \subset D(1 - t_\varepsilon)$. Hence

$$\begin{aligned}\gamma \sum_{s_k\in\Gamma_\gamma(F)} (1 - |s_k|) &= \int_{\mathbb{T}} \#(S \cap D(1 - t_\varepsilon))\, d\sigma(\zeta) + \int_{\{\zeta\colon c_\gamma d(\zeta,F)\leq t_\varepsilon\}} n_\gamma(\zeta, c_\gamma d(\zeta, F); S)\, d\sigma(\zeta)\\ &\leq C_1(\varepsilon, S) + \gamma(BD_S^\gamma(\mathbb{T}) + \varepsilon) \int_{\mathbb{T}} \log\Big(\frac{1}{c_\gamma d(\zeta, F)}\Big)\, d\sigma(\zeta)\\ &\leq C_2(\varepsilon, S) + \gamma(BD_S^\gamma(\mathbb{T}) + \varepsilon)\, \hat{\kappa}(F)\end{aligned}$$

and, as desired,

$$\limsup_{\hat{\kappa}(F)\to\infty} \frac{\sum_{s_k\in\Gamma_\gamma(F)}(1 - |s_k|)}{\hat{\kappa}(F)} \leq BD_S^\gamma(\mathbb{T}) + \varepsilon \qquad \text{for all } \varepsilon > 0\ .$$

■

Proof of Theorem 4 . Assume there exists $I \subset \mathbb{T}$ with $BD_S^\gamma(I) < \infty$. We want to construct $H \in A^{-\infty} \setminus H^\infty$ such that $\sup_S |H| < \infty$, so that $S \notin \mathcal{T}(\infty, 0, 0)$.

By Lemma 7 we can assume that $I = J$. Take $\Omega = D(-1, \sqrt{2}) \cup \mathbb{D}$, so that the intersection angle between $\mathbb{T}$ and $\partial D(-1, \sqrt{2})$ at $\pm i$ is $\pi/4$. Consider then a conformal mapping $\psi : \Omega \longrightarrow \mathbb{D}$ with $\psi(J) = J$. Explicitly $\psi = \psi_3 \circ \psi_2 \circ \psi_1$, where $\psi_1(z) = \dfrac{i + z}{i - z}$, $\psi_2(z) = (iz)^{\frac{4}{5}}$ (the argument of iz running from 0 to $5\pi/4$), $\psi_3(z) = i\dfrac{z - i}{z + i}$, from which we see that $\psi(i) = i, \psi(-i) = -i, \psi(1) = 1$ and there exists a constant $\eta > 0$ such that

$$1 - |\psi(w)| \;\geq\; \eta(1 - |w|) \qquad \text{for all } w \in \mathbb{D}, \text{ and} \tag{3}$$

$$\psi(\mathbb{D}) \;\subset\; \Gamma_{\gamma'}(J) \cup D(r_0) \qquad \text{for some } r_0 \text{ and } \gamma' \leq \frac{\pi}{2} - \frac{\pi}{5}. \tag{4}$$

Similarly to (4) we have $\psi(\mathbb{D}) \subset \tilde{\Gamma}_{\gamma'}(J)$.

Lemma 9. *Let ψ be the conformal mapping above. Then*

$$BD_{\psi(S)}^{\gamma/2}(J) \leq 2\, BD_S^\gamma(J)\ .$$

Proof. By construction of ψ there exists r_0 such that for any $\zeta \in J$

$$\Gamma_{\gamma/2}^t(\psi(\zeta)) \subset \psi(\Gamma_\gamma^{\eta t}(\zeta)) \cup D(r_0)\ .$$

Thus $\frac{n_{\gamma/2}(\psi(\zeta),t;\psi(S))}{\frac{\gamma}{2}|\log t|} \leq 2\ \frac{n_\gamma(\zeta,\eta t;S)+C_{r_0}}{\gamma|\log t|}$, and taking the supremum and the limit we obtain the desired inequality. ■

Using this and Lemma 6

$$BD_{\psi(S)\cap\Gamma_\gamma(J)}^{\gamma/2}(\mathbb{T}) = BD_{\psi(S)}^{\gamma/2}(J) \leq 2\, BD_S^\gamma(J)\ .$$

Applying Lemma 8 we get $D\Big(\psi(S)\cap\Gamma_{\gamma/2}(J)\Big)\leq 2BD_S^\gamma(J)$, therefore there exists $f\in A_+^{-\alpha}\setminus\{0\}$, $\alpha=2BD_S^\gamma(J)$, such that $f\equiv 0$ on $\psi(S)\cap\Gamma_{\gamma/2}(J)$.

We want to multiply f by a function g with a suitable decrease on $\mathbb{D}\setminus\Gamma_{\gamma/2}(J)$. By (4), if $z\in\psi(S)$, there exists $\delta>0$ such that $1-|z|\geq\delta d(z/|z|,J)$.

Define, for a certain constant $c_1>0$ to be chosen later on, the function

$$g(z)=\exp\Big\{-c_1\int_{\mathbb{T}\setminus J}\frac{\zeta+z}{\zeta-z}\ \log\frac{1}{d(\zeta,J)}\ d\sigma(\zeta)\Big\}\ .$$

Then

$$-\log|g(z)|=c_1\int_{\mathbb{T}\setminus J}P_z(\zeta)\log\frac{1}{d(\zeta,J)}\ d\sigma(\zeta)\ ,$$

where $P_z(\zeta)$ stands for the Poisson kernel, and in particular $\|g\|_\infty\leq 1$.

For $z\notin\Gamma_{\gamma/2}(J)$ there exist $c_2,c_3>0$ such that

$$I_{c_2}(z)=:\{\zeta\in\mathbb{T}:d(z/|z|,\zeta)<c_2(1-|z|)\}\subset\mathbb{T}\setminus J$$

and $d(I_{c_2}(z),J)\geq c_3(1-|z|)$. Thus, for some $c_4=c_4(c_2,\gamma)$:

$$\begin{aligned}-\log|g(z)| &\geq c_1\int_{I_{c_2}(z)}P_z(\zeta)\ \log\frac{1}{d(\zeta,J)}\ d\sigma(\zeta)\\ &\geq c_1c_3\ \log\frac{1}{d(I_{c_2}(z),J)}\geq c_1c_4\ \log\frac{1}{(1-|z|)}\ .\end{aligned}$$

Hence $|g(z)|\leq(1-|z|)^{c_1c_4}$ for $z\notin\Gamma_{\gamma/2}(J)$.

For any $e^{i\theta}\in J$,

$$\lim_{z\to e^{i\theta},z\in\Gamma_{\gamma/2}(e^{i\theta})}|g(z)|=1.$$

By Privalov's theorem, we can choose $e^{i\theta}\in J$ such that

$$\limsup_{z\to e^{i\theta},z\in\Gamma_{\gamma/2}(e^{i\theta})}|f(z)|>0.$$

Thus for any $k>0$

$$\limsup_{z\to e^{i\theta},z\in\Gamma_{\gamma/2}(e^{i\theta})}\frac{|f(z)g(z)|}{(1-|z|)^k}=\infty\ . \tag{5}$$

Choose now $c_1>0$ such that $c_1c_4>2BD_S^\gamma(J)+k$. Define $F(z)=\frac{f(z)g(z)}{(1-e^{-i\theta}z)^k}$ and $H=F\circ\psi$, where θ is as before.

Let us check the required properties for H:

(i) $\sup_S|H|<\infty$. If w is such that $\psi(w)\in\psi(S)\cap\Gamma_{\gamma/2}(J)$ we have then $f(\psi(w))=0$, and obviously $H(w)=0$.

If w is such that $\psi(w) \in \psi(S) \cap (\mathbb{D} \setminus \Gamma_{\gamma/2}(J))$, and $z = \psi(w)$; then

$$|H(w)| = \frac{|f(z)g(z)|}{|1 - e^{-i\theta}z|^k} \leq \frac{|f(z)|(1-|z|)^{c_1c_4}}{(1-|z|)^k} \leq \|f\|_{A^{-c_1c_4+k}} < \infty ,$$

since $c_1c_4 - k > \alpha$.

(ii) $H \in A^{-\infty}$. It is clear that $F \in A_+^{-(\alpha+k)}$, since $f \in A_+^{-\alpha}$ and $g \in H^\infty$. The statement follows then from (3).

(iii) $H \notin H^\infty$. This follows from (5), since $|1 - e^{-i\theta}z| \simeq (1-|z|)$ on $\Gamma_{\gamma/2}(e^{i\theta})$. ■

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