PLURICOMPLEX GREEN AND LEMPERT FUNCTIONS FOR EQUALLY WEIGHTED POLES

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1. Introduction

The pluricomplex Green function is an important tool of several variable complex analysis; in particular it provides a fundamental solution for the complex Monge-Ampère equation and information about the complex geometry of domains [10] (see [7] for an exposition of pluricomplex potential theory). For $n \geq 2$, the complex Monge-Ampère equation is nonlinear, so studying the several-pole analogue of the Green function (introduced in [8]) is no easy task, see [4], and [1] or [3] for some of the few cases where it can be explicitly computed.

Let Ω be a domain in $\mathbb{C}^n$, and poles and weights denoted by

$$S = \{(a_1, \nu_1); ...; (a_N, \nu_N)\} \subset \Omega \times \mathbb{R}_+,$$

where $\mathbb{R}_+ = [0, +\infty)$. The pluricomplex Green function is defined by

$$G_S(z) := \sup\{u(z) : u \in PSH_-(\Omega), u(x) \leq \nu_j \log \|x - a_j\| + C_j \text{ when } x \to a_j, j = 1, ..., N\}.$$

Note that if $N = 1$ we might as well take $\nu_1 = 1$, G_S is the pluricomplex Green function with one pole.

We also recall the definition of Coman's Lempert function [4]:

$$(1.1)\quad \ell_S(z) := \inf\{\sum_{j=1}^{N} \nu_j \log |\zeta_j| : \varphi(0) = z, \varphi(\zeta_j) = a_j, j = 1, ..., N \text{ for some } \varphi \in \mathcal{O}(\mathbb{D}, \Omega)\},$$

where $\mathbb{D}$ is the unit disc in $\mathbb{C}$.

It is easy to see that $\ell_S(z) \geq G_S(z)$ for all $z \in \Omega$.

A remarkable theorem of Lempert [10] says that equality holds in the case where Ω is convex and $N = 1$. Later Coman [4] proved with considerable effort that this assertion also holds when Ω is the unit ball, $N = 2$, and the weights are equal. At the same time he conjectured that the equality might hold for any number of points and any convex domain in $\mathbb{C}^n$. Recently, Carlehed and Wiegerinck [2], [3] proved that Coman's conjecture fails for the bidisc, with two poles lying on a coordinate axis and distinct weights. The main goal of this paper is to prove that Coman's conjecture does not even hold in the case when all weights are equal.

Weights on the Green function are analogous to multiplicities for zeros. Since the work of Carlehed and Wiegerinck [2], [3] uses weights greater than 1, we focus on the behavior of Coman's Lempert function with many poles when some group of poles tend to the same pole. Eventually, a counterexample is obtained Section 5 of this paper with the domain equal to the bidisc, and four poles at $(a, 0), (b, 0), (b, \varepsilon), (a, \varepsilon)$, with ε small enough (Theorem 5.1). As is [3], one can deduce from this that the Coman conjecture fails for strictly convex smoothly bounded domains which are close enough to the bidisc.

Along the way, we need to introduce more general notions of Lempert functions, coming from generalizations of the Green function. The reason is as follows : when we consider the pluricomplex Green function as a fundamental solution for the complex

Unfortunately, this was not fully successful, since our best candidate (see Definition 3.6) is not in general the limit of the Lempert functions for the natural systems of points which tend to the given "multiple poles" (see [14, Theorem 6.3]). However, we gather enough information to prove that in the four-point cases mentioned above, equality does not hold between the Lempert and Green functions.

Along the way to our counterexample, we give partial answers. There is equality between Lelong and Rashkovskii's Green function and our first generalization of Coman's Lempert function in the case of one pole, in the polydisc, with a simple enough singularity (Lemma 2.6; some hypothesis about integer multiplicities is of course necessary). We also prove equality between Lempert and Green functions in the case of the bidisc in $\mathbb{C}^2$, when all poles are on the first coordinate disc and all multiplicities equal to one ; also, our first generalization of the Lempert function provides a natural limit when poles collide along the first coordinate disc, producing "horizontal" non-isotropic singularities, and this is still equal to the appropriate generalized Green function (this is made precise in Theorem 4.1).

The organization of the present paper is as follows : in Section 2, we give notations and definitions, introduce our generalization of the Lempert function and give Lemma 2.6 as a first motivation of this particular definition. In Section 3, we generalize to this new Lempert functions some of the results of [17]; the proofs we give are restricted to the particular cases which do occur in the examples below. Section 4 includes Theorem 4.1 and provides a few negative examples in the bidisc, the latter motivating Definition 3.6, which amends our first generalization of Coman's Lempert function. Finally, the counterexample is proven in Section 5.

A longer version of this paper, with the proofs of some additional facts about our generalization of the Lempert function, is available as a preprint [14] and forms part of the second-named author's Ph. D. dissertation (Ha Noi, Viet Nam, november 2002).

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2. Definitions

Definition 2.1. [9]
We will say that $\Psi \in PSH_-(\mathbb{D}^n)$ is an indicator *(centered at 0) if and only if*

$$\Psi(z_1, \ldots, z_n) = g(\log|z_1|, \ldots, \log|z_n|),$$

where g is a convex continuous nonpositive valued function defined on $(\mathbb{R}_-)^n$, increasing with respect to each single variable, and positively homogeneous of degree 1: $g(\lambda x_1, \ldots, \lambda x_n) = \lambda g(x_1, \ldots, x_n)$, for any $\lambda > 0$.

This can be introduced in a less ad hoc way, see [9].

By [9], if Ψ is an indicator, it is a multiple of a fundamental solution to the complex Monge-Ampère equation, that is, there exists $\tau \geq 0$ such that

$$(dd^c\Psi(\cdot - a))^n = \tau\delta(a),$$

where $\delta(a)$ stands for the unit mass at the point $a \in \mathbb{C}^n$.

Let us fix the system $S := \{(a_j, \Psi_j)\}, 1 \leq j \leq N$, where $a_j \in \Omega, 1 \leq j \leq N$ and Ψ_j are indicators.

Definition 2.2. *The generalized Green function* [9] *is given by*

$$G_S(z) := \sup\{u(z) : u \in PSH_-(\Omega), u(x) \leq \Psi_j(x - a_j) + C_j, 1 \leq j \leq N\},$$

where the inequalities are required only for x belonging to a neighborhood of each a_j.

Remark 2.3. *If Ω is a hyperconvex domain in $\mathbb{C}^n$, then Lelong and Rashkovskii* [9] *also showed that the Green function is the unique solution of the following Dirichlet problem (for short we write G instead of G_S)*

(a) $G \in PSH_-(\Omega) \cap C(\overline{\Omega})$;

(b) $G(z) \to 0$ as $z \to \partial\Omega$;

(c) $\Psi_j(z) = \lim_{R\to\infty} R^{-1}G[a_j + (\exp(u_k + i\theta_k + R\log|z_k|))_{1\leq k\leq n}], 1 \leq j \leq N$, where the limit exists almost everywhere for $x = (u_k + i\theta_k)_{1\leq k\leq n}$ and doesn't depend on x;

(d) $(dd^cG)^n = \sum_{j=1}^N \tau_j\delta(a_j)$.

We now introduce a new generalization of the Lempert function.

Definition 2.4.

$$L_S(z) := \inf\{\sum_{j=1}^N \tau_j \log|\zeta_j| : \exists\varphi \in \mathcal{O}(\mathbb{D}, \Omega), \varphi(0) = z,$$
$$\exists U_j \text{ a neighborhood of } \zeta_j : \Psi_j(\varphi(\zeta) - a_j) \leq \tau_j \log|\zeta - \zeta_j| + C_j, \forall\zeta \in U_j, 1 \leq j \leq N\}.$$

Note that for the non-trivial case where $\tau_j \neq 0$, the conditions imposed on the map φ force $\varphi(\zeta_j) = a_j$. In the basic case where $\Psi_j(z) = \log|z|$ for each j, we will have $\tau_j = 1$ for each j, and we simply find the usual Lempert function ℓ_S with simple poles ($\nu_j = 1$ for each j), since analytic maps are locally Lipschitz. But for $N > 1$, $\Psi_j(z) = \nu_j \log|z|$ with some $\nu_j > 1$ (and $\tau_j = \nu_j^n$), this is not a priori the same as the ℓ_S given in the Introduction (although we do not know of any example to exhibit this phenomenon, and do not know of any general inequality between the two functions).

Lemma 2.5. $G_S(z) \leq L_S(z)$, *for any $z \in \Omega$.*

Proof. If $\varphi : \mathbb{D} \to \Omega$ is an analytic disc in Ω, with $\varphi(0) = z, \varphi(\zeta_j) = a_j, 1 \leq j \leq N$ and $\Psi_j \circ \varphi(\zeta) \leq \tau_j \log|\zeta - \zeta_j| + C_j, 1 \leq j \leq N$, then $G_S \circ \varphi$ is a subharmonic function on $\mathbb{D}$, $G_S \circ \varphi$ is negative and

$$G_S \circ \varphi(\zeta) \leq C_j + \Psi_j \circ \varphi(\zeta) \leq C'_j + \tau_j \log|\zeta - \zeta_j|,\ 1 \leq j \leq N.$$

Thus $G_S \circ \varphi$ is a member in the defining family for the Green function on $\mathbb{D}$ with poles ζ_j and weights τ_j, and hence,

$$G_S \circ \varphi(\zeta) \leq \sum_{j=1}^N \tau_j \log \frac{|\zeta_j - \zeta|}{|1 - \zeta\bar{\zeta_j}|}.$$

It implies that

$$G_S(z) = G_S \circ \varphi(0) \leq \sum_{j=1}^{N} \tau_j \log |\zeta_j|.$$

Thus $G_S(z) \leq L_S(z)$, for all $z \in \Omega$. □

Recall (see e.g. [6], [12]) that the involutive Möbius map of $\mathbb{D}$ which exchanges $\xi \in \mathbb{D}$ and 0 is given by the following formula:

$$\phi_\xi(\zeta) := \frac{\xi - \zeta}{1 - \bar{\xi}\zeta}. \tag{2.1}$$

Therefore it is no loss of generality, in the case of a single pole a, to reduce ourselves to $a = 0$.

Lemma 2.6. *Let Ω be the polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$. If S has only one pole at $(0,0)$, and the indicator Ψ is of the following simple kind*

$$\Psi(z) = \max_{1 \leq j \leq n} c_j \log |z_j|,$$

where the numbers c_j are positive integers, then $L_S(z) = G_S(z) = \Psi(z)$, for any $z \in \mathbb{D}^n$.

Proof. By verifying the Dirichlet problem given by Lelong and Rashkovskii [9], we have

$$G_S(z) = \max_{1 \leq j \leq n} c_j \log |z_j|.$$

We may assume that $\max_{1 \leq j \leq n} c_j \log |z_j| = c_{j_0} \log |z_{j_0}|$ for some $1 \leq j_0 \leq n$. With this assumption we have $G_S(z) = c_{j_0} \log |z_{j_0}|$. To prove the Lemma, it suffices to show that there exists a mapping $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D}^n)$ and $\zeta_0 \in \mathbb{D}$ such that

(1) $\varphi(0) = z$,

(2) $\varphi(\zeta_0) = 0$,

(3) $\Psi \circ \varphi(\zeta) \leq m \log |\zeta - \zeta_0| + C, \forall \zeta \in \mathbb{D}$, where $m := \prod_{j=1}^{n} c_j =$ is the total mass of $(dd^c\Psi)^n$,

(4) $m \log |\zeta_0| = c_{j_0} \log |z_{j_0}|$.

The condition (3) can be rewritten as follows

(3') $\varphi_j^{(k)}(\zeta_0) = 0, 1 \leq k \leq m_j - 1, 1 \leq j \leq n$, where $m_j := m/c_j$.

We fulfill condition (4) by picking $\zeta_0 \in \mathbb{D}$ such that

$$|\zeta_0|^{m_{j_0}} = |z_{j_0}|$$

and put

$$\varphi_j(\zeta) := \Big[\phi_{\zeta_0}(\zeta)\Big]^{m_j} h_j\Big(\phi_{\zeta_0}(\zeta)\Big), \forall \zeta \in \mathbb{D}, 1 \leq j \leq n$$

where $h_j : \mathbb{D} \to \overline{\mathbb{D}}$ is such that $h_j(\zeta_0) = \dfrac{z_j}{\zeta_0^{m_j}}, 1 \leq j \leq n.$

Then the function $\varphi = (\varphi_1, \cdots, \varphi_n)$ and ζ_0 satisfy all properties (1), (2), (3') and (4). □

3. Existence of extremal discs

We now extend to this new Lempert function some known properties of its usual counterpart. The following generalizes [17, Theorem 2.4, p. 1054], or in the case of the unit ball [13, Proposition 3, p. 338] (see also [16, Papers V and VI]).

Proposition 3.1. *Let Ω be a convex domain and $S := \{(a_j, \Psi_j) : 1 \le j \le N\}$ and $S' := \{(a_j, \Psi_j) : 1 \le j \le N-1\}$ where $a_j \in \Omega$ and Ψ_j are indicators centered at a_j. Then*

$$L_S(z) \le L_{S'}(z), \text{ for all } z \in \Omega.$$

The proof of this proposition can be found in [14, Section 4]. Since that full proof is elementary and rather tedious, the one given below restricts itself to the special case where $n = 2$ and the indicators are of the type used in Lemma 2.6, with $1 \le c_j \le 2$.

Proof of Proposition 3.1

This proof adapts the ideas of [17], [13, Proposition 3], and [15, Theorem 2.7].

Given any $\delta > 0$, there exists a holomorphic map φ from the disc to Ω and points $\zeta_j^0 \in \mathbb{D}$, $1 \le j \le N-1$, such that $\varphi(z) = 0$,

$$L_{S'}(z) \le \sum_{j=1}^{N-1} \tau_j \log|\zeta_j^0| \le L_{S'}(z) + \delta,$$

and $\Psi_j \circ \varphi(\zeta) \le \tau_j \log|\zeta - \zeta_j^0| + C_j$, $1 \le j \le N-1$. Let $r < 1$, to be specified later. We set $\varphi^r(\zeta) := \varphi(r\zeta)$. If $r > \max|\zeta_j^0|$, $1 \le j \le N-1$, we have $\frac{\zeta_j^0}{r} \in \mathbb{D}$ and

$$\varphi^r(\frac{\zeta_j^0}{r}) = a_j, 1 \le j \le N-1,$$

and more generally

$$\Psi_j \circ \varphi^r(\zeta) \le \tau_j \log|r(\zeta - \frac{\zeta_j^0}{r})| + C_j \le \tau_j \log|(\zeta - \frac{\zeta_j^0}{r})| + C_j, 1 \le j \le N-1. \tag{3.1}$$

We will introduce a correcting term to ensure that the same property hold for $j = N$, without destroying it for $j \le N-1$.

Let K denote the convex hull of $\varphi^r(\overline{\mathbb{D}}) \cup \{a_N\}$. Since $\varphi^r(\overline{\mathbb{D}}) \cup \{(a,0)\} \subset\subset \Omega$, we can find an $\varepsilon > 0$ such that the distance between K and $\partial\Omega$ is at least εM_1 where $M_1 := \sup_{r\overline{\mathbb{D}}} |a_N - \varphi|$.

Lemma 3.2. *Given any $m \in \mathbb{N}^*$, there exists h a holomorphic function on $\mathbb{D}$ and some $\zeta^* \in \mathbb{D}$ satisfying*

- $h(\mathbb{D}) \subset U_\varepsilon := \cup_{x \in [0,1]} D(x, \varepsilon)$,
- $h(0) = 0$,
- $h(\frac{\zeta_j^0}{r}) = h'(\frac{\zeta_j^0}{r}) = 0$, $1 \le j \le N-1$,
- $h(\zeta^*) = 1$, *and* $h'(\zeta^*) = 0$.

Accepting this lemma temporarily, define

$$\tilde{\varphi}(\zeta) = \varphi^r(\zeta) + h(\zeta)(a_N - \varphi^r(\zeta)).$$

The definition of ε and the first condition above show that $\tilde{\varphi}(\mathbb{D}) \subset \Omega$. Clearly, $\tilde{\varphi}(0) = z$.

We have $h(\zeta) = O((\zeta - \frac{\zeta_j^0}{r})^2)$ for $1 \le j \le N-1$, so that the conditions (3.1), which reduce under our restrictive hypotheses to the vanishing of the derivatives of certain coordinate functions of φ^r, still hold for $\tilde{\varphi}$.

Finally, one also checks that $\tilde{\varphi}(\zeta) = a_N + (h(\zeta)-1)(a_N - \varphi^r(\zeta)) = a_N + O((\zeta-\zeta^*)^2)$, which will imply $\Psi_N \circ \tilde{\varphi}(\zeta) \le \tau_N \log|\zeta - \zeta^*| + C_N$. For the mapping $\tilde{\varphi}$, the logarithmic sum of the preimages yields

$$\sum_{j=1}^{N-1} \log\left|\frac{\zeta_j^0}{r}\right| + \log|\zeta^*| \le \sum_{j=1}^{N-1} \log|\zeta_j^0| + (N-1)\log\frac{1}{r} \le L_{S'}(z) + \delta + (N-1)\log\frac{1}{r}.$$

Since this construction can be carried out for any r arbitrarily close to 1, we have $L_S(z) \le L_{S'}(z)$. □

Proof of Lemma 3.2. Let ρ be a Riemann map from $\mathbb{D}$ to U_ε so that $\rho(0) = 0$. We look for h under the form $h = \rho \circ h_1$, where h_1 is a holomorphic map from $\mathbb{D}$ to itself such that

- $h_1(0) = 0$,
- $h_1(\frac{\zeta_j^0}{r}) = h_1'(\frac{\zeta_j^0}{r}) = 0$, $1 \le j \le N-1$, and
- there exists $\zeta^* \in \mathbb{D}$ such that $h_1(\zeta^*) = \rho^{-1}(1)$ and $h_1'(\zeta^*) = 0$.

Let B_0 be the finite Blaschke product with a single zero at the origin and double zeroes at the points $\frac{\zeta_j^0}{r}$, $1 \le j \le N-1$, and look for h_1 under the form $h_1 = B_0 g$, where g is holomorphic and bounded by 1 in modulus on the unit disc. For ease of notation, write $\gamma := \rho^{-1}(1) \in \mathbb{D}$.

The function h_1 will fulfill the above conditions if and only if

$$g(\zeta^*) = \frac{\gamma}{B_0(\zeta^*)}, \quad g'(\zeta^*) = -g(\zeta^*)\frac{B_0'(\zeta^*)}{B_0(\zeta^*)} = -\gamma\frac{B_0'(\zeta^*)}{B_0^2(\zeta^*)}.$$

By the Schwarz-Pick lemma, such a function can be found if and only if $|g'(\zeta^*)| \le (1-|g(\zeta^*)|^2)/(1-|\zeta^*|^2)$, i.e.

$$(1-|\zeta^*|^2)\frac{|B_0'(\zeta^*)|}{|B_0^2(\zeta^*)|} \le \frac{1-|\gamma|^2|B_0(\zeta^*)|^{-2}}{|\gamma|}.$$

Since $|\gamma|$ is fixed and $\lim_{\zeta\to 1}|B_0(\zeta)| = 1$, $\lim_{\zeta\to 1}|B_0'(\zeta)| = |B_0'(1)| < \infty$, this is achieved for ζ^* close enough to 1. □

We will use the shorthand $S' \subset S$ to mean that the sets of poles are included as noted, and that the indicators remain the same for all points of the smaller set, as in Proposition 3.1.

Proposition 3.3. *Let Ω be a bounded taut domain, and $S = \{(a_j, \Psi_j)\}_{j=1,..,N}, N \ge 2$. If $L_S(z)$ is not attained by any analytic disc, then*

$$L_S(z) \ge \min_{S' \subsetneq S} L_{S'}(z).$$

In particular, if Ω is convex and bounded, the conclusion becomes

$$L_S(z) = \min_{S' \subsetneq S} L_{S'}(z).$$

Proof. The proof of this Proposition is adapted from that of [17, Theorem 2.2, p. 1053].

Take a sequence of analytic discs φ^k, where

$$\varphi^k(0) = z \text{ and } \Psi_j \circ \varphi^k(\zeta) \le \tau_j \log|\zeta - \zeta_j^k| + C_j^k,\ \forall \zeta \in \mathbb{D},\ k \ge 1,\ 1 \le j \le N$$

such that $\sum_{j=1}^N \tau_j \log|\zeta_j^k|$ converges to $L_S(z)$, as k tends to 0.

By passing to a subsequence, using that Ω is taut, we may assume that φ^k converges locally uniformly to some $\varphi \in \mathcal{O}(\mathbb{D}, \Omega)$. Also (if necessary, by passing to a subsequence again), we may assume that $\zeta_j^k \to \zeta_j \in \overline{\mathbb{D}}$, for each $1 \le j \le N$, as $k \to \infty$.

We need to see that for each $\zeta_j \in \mathbb{D}$,

$$\Psi_j \circ \varphi(\zeta) \le \tau_j \log|\zeta - \zeta_j| + C_j, \text{ for } \zeta \text{ in a neighborhood of } \zeta_j. \tag{3.2}$$

Recall (from [9]) that Ψ being an indicator (centered at 0) means that

$$\Psi(z_1, \dots, z_n) = g(\log|z_1|, \dots, \log|z_n|),$$

where g is a convex continuous nonpositive valued function defined on $(\mathbb{R}_-)^n$, increasing with respect to each single variable, and positively homogeneous of degree 1: $g(\lambda x_1, \dots, \lambda x_n) = \lambda g(x_1, \dots, x_n)$, for any $\lambda > 0$.

We study the situation for a fixed pole a_j. We must have for each $k \ge 0$,

$$\varphi^k(\zeta_j^k + h) = (\varphi_l^k(\zeta_j^k + h), 1 \le l \le n) = (\alpha_{k,l} h^{m_{k,l}} + O(|h|^{m_{k,l}+1}), 1 \le l \le n).$$

From the above expression,

$$\Psi_j(\varphi^k(\zeta_j^k + h)) = g\left(-m_{k,l} + \frac{\log|\alpha_{k,l}| + O(h)}{|\log|h||}\right) \log|h|,$$

so the conditions on φ^k imply that

$$g(-m^k) \le \tau_j, \text{ where } m^k := (m_{k,1}, \dots, m_{k,n}). \tag{3.3}$$

Passing to a subsequence if needed, we may assume that $m^k \to m =: (m_1, \dots, m_n) \in (\mathbb{N} \cup \{\infty\})^n$. The uniform convergence on compacta of the sequence φ^k implies that of all derivatives, and that in the limit $\varphi_l^{(q)}(\zeta_j) = 0$ for $q \le m_l - 1$. This, together with (3.3), proves (3.2).

If no $\zeta_j \in \partial\mathbb{D}$, φ is an analytic disc attaining the infimum in the definition of $L_S(z)$. That is excluded by our hypothesis. Otherwise, assume after renumbering the coordinates that $\zeta_j \in \mathbb{D}, 1 \le j \le M$ and $\zeta_j \in \partial\mathbb{D}$ for $M+1 \le j \le N$. (Note that not every ζ_j can be in $\partial\mathbb{D}$, as this would imply that $L_S(z) = 0$.) Then φ is a member in the defining family for $L_{S'}$, where $S' := \{(a_j, \Psi_j)\}_{j=1,..,M}$, and thus $L_S(z) \ge L_{S'}(z)$. □

Corollary 3.4. *Let Ω be a bounded taut domain in $\mathbb{C}^n$, and let S be as above. Then for every $z \in \Omega$ there exists an analytic disc φ, such that $\varphi(0) = z$, passing through a (non empty) $S_0 \subset S$ such that φ attains the infimum in the definition of $L_{S_0}(z)$, and $L_{S_0}(z) = \min_{\emptyset \ne S' \subset S} L_{S'}(z)$.*

Proof. If S is a singleton, a normal family argument close to the one used in the previous proof will show that the corollary is true for this case.

Otherwise, by the previous proposition, either there is an analytic attaining the infimum, or $\min_{\emptyset \ne S' \subset S} L_{S'}(z) = L_{S_0}(z)$ for some proper subset $S_0 \subset S$, and $L_{S_0}(z)$ is attained by an analytic disc passing though z and the points in S_0 (otherwise one could pass to a still smaller subset). □

As the consequence of Corollary 3.4 and Proposition 3.1 we have the following.

Theorem 3.5. *Let Ω be a bounded convex domain, then the infimum in the definition of the function L_S is attained by an extremal disc that passes through a (non-empty) subset $S' \subset S$ (possibly the whole system S).*

However, it would be natural to consider as well the more general case of the relationship between the Lempert functions of two systems $S := \{(a_j, \Psi_j) : 1 \leq j \leq N\}$ and $S' := \{(a_j, \Psi'_j) : 1 \leq j \leq N\}$, where $\Psi_j \leq \Psi'_j$, for any $1 \leq j \leq N$ ($S' \subset S$ corresponds to the case where the Ψ'_j have $\tau_j = 0$ for a_j outside the pole set of S'). Unfortunately, our generalized Lempert function is not in general monotone when we compare two such generalized pole sets, see a counter-example below (Proposition 4.2). We therefore introduce a corrected Lempert function $\tilde{L}$.

Definition 3.6. *Let $S := \{(a_j, \Psi_j) : 1 \leq j \leq N\}$ and $S_1 := \{(a_j, \Psi^1_j) : 1 \leq j \leq N\}$ where $a_j \in \Omega$ and Ψ_j, Ψ^1_j are indicators. We define*

$$\tilde{L}_S(z) := \inf\{L_{S^1}(z) : \Psi^1_j \geq \Psi_j + C_j, 1 \leq j \leq N\}.$$

Lemma 3.7. $G_S(z) \leq \tilde{L}_S(z) \leq L_S(z)$.

Proof. The fact that $\tilde{L}_S(z) \leq L_S(z)$ follows from the definition. For any S_1 as in the definition, $L_{S^1}(z) \geq G_{S^1}(z) \geq G_S(z)$, as follows from Lemma 2.5 and the definition of the pluricomplex Green function. □

In the situation related to the example in Proposition 4.2 where two fixed poles a_1, a_2 lie on a coordinate axis, a_3 lies on a line orthogonal to this axis at a_1, and a_3 tends to a_1, then the limit of the ordinary Lempert functions is given by an $\tilde{L}_S$, and not by the corresponding L_S (the limit of the corresponding Green functions is not known in this case). A precise statement and a proof can be found in [14, Theorem 5.5]. However, there are other examples where $\tilde{L}$ also fails to be the limit of the Lempert functions for single poles [14, Theorem 6.3].

4. Examples in the bidisc

First, we would like to give one case where the Green function with several poles and indicator singularities is equal to its generalized Lempert counterpart. This is analogous in spirit to the result of Carlehed and Wiegerinck about the Green function with several poles in the bidisc [1], [3] (but easier).

Theorem 4.1. *Let $\Psi_m(z) = \max\{m \log|z_1|; \log|z_2|\}$, for any $m \in \mathbb{N}^*$.*
Let $a_1, a_2, \ldots, a_N \in \mathbb{D}$, and

$$S := \{((a_1, 0); \Psi_{m_1}); \ldots, ((a_N, 0); \Psi_{m_N})\}.$$

Then for any $z \in \mathbb{D}^2$,

$$L_S(z) = G_S(z) = \max\{\sum_{j=1}^{N} m_j \log\left|\phi_{a_j}(z_1)\right|; \log|z_2|\}.$$

As a consequence, if $a^{(k)}_{j,i} \in \mathbb{D}$, $1 \leq j \leq N$, $1 \leq i \leq m_j$, are distinct points which verify

$$\lim_{k\to\infty} a^{(k)}_{j,i} = a_j, \quad , 1 \leq i \leq m_j,$$

and $S^{(k)}$ *the pole system made up of all the* $a_{j,i}^{(k)}$ *with equal weight* 1, *then* $\lim_{k\to\infty} L_{S^{(k)}}(z) = L_S(z)$ *and* $\lim_{k\to\infty} G_{S^{(k)}}(z) = G_S(z)$, *for any* $z \in \mathbb{D}^2$.

Proof. First of all, the Green function has the formula given above. To prove this assertion it suffices to show that the function defined by the right hand side verifies the Dirichlet problem in Remark 2.3. Indeed the conditions (a), (b) and (c) are trivially fulfilled. The last condition follows from the following theorem of Zeriahi [18], [19].

Theorem. *For* $i = 1, 2$, *let* Ω_i *be an open set in* $\mathbb{C}^{n_i}$, *and* u_i *a locally bounded plurisubharmonic function in* Ω_i, *such that* $(dd^c u_i)^{n_i} = 0$ *in* Ω_i. *Define* $v(z_1, z_2) = \max\{u_1(z_1), u_2(z_2)\}, n = n_1 + n_2$. *Then* $(dd^c v)^n = 0$ *in* $\Omega_1 \times \Omega_2$.

By our definition,

$$L_S(z) = \inf \left\{ \sum_{j=1}^{N} m_j \log |\zeta_j| : \exists \varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D}^2), \right.$$

$$\left. \varphi(0) = z, \varphi_1(\zeta_j) = a_j, \varphi_2^{(k)}(\zeta_j) = 0,\ 0 \le k \le m_j - 1, 1 \le j \le N \right\}.$$

If $z_1 \in \{a_1, \dots, a_n\}$, say $z_1 = a_1$, then picking $\zeta_1^{m_1} = z_2$ and $\varphi(\zeta) = (a_1, \zeta^{m_1})$, we see by Proposition 3.1 that

$$\log |z_2| = m_1 \log |\zeta_1| \ge L_{((a_1,0),\Psi_{m_1})}(z) \ge L_S(z) \ge G_S(z) = \log |z_2|,$$

so there is equality throughout.

If $z_1 \notin \{a_1, \dots, a_n\}$, we may reduce ourselves to $z = (0, \gamma)$ and $|a_1| \ge |a_2| \ge ... \ge |a_N| > 0$. Then

$$G_S(z) = \max\{\log |a_1^{m_1} \cdot a_2^{m_2} \cdots a_N^{m_N}|; \log |\gamma|\}.$$

We will use induction on N. When $N = 1$ the equality follows from Lemma 2.6. Suppose that $N > 1$ and the theorem is proved for $N - 1$. We consider three cases.

Case 1. $|\gamma| \le |a_1^{m_1} \cdot a_2^{m_2} \cdots a_N^{m_N}|$.

Then $G_S(z) = \log |a_1^{m_1} \cdot a_2^{m_2} \cdots a_N^{m_N}|$. The map

$$\zeta \mapsto \left(\zeta, \frac{\gamma}{a_1^{m_1} \cdot a_2^{m_2} \cdots a_N^{m_N}} \prod_{j=1}^{N} \left(\frac{a_j - \zeta}{1 - \overline{a_j}\zeta} \right)^{m_j} \right)$$

verifies all the requirements with $\zeta_j = a_j$. This implies that $G(z) = L(z)$.

Case 2. $|\gamma| \ge |a_2^{m_2} \cdots a_N^{m_N}|$.

Then $G(z) = \log |\gamma|$. Moreover, $G(z)$ is also equal to the Green function $G_1(z)$ for the system with $N - 1$ poles

$$S_1 := \{((a_2, 0); \Psi_{m_2}); \dots, ((a_N, 0); \Psi_{m_N})\}.$$

By induction, $G_1 = L_1$, where L_1 is the generalized Lempert function with respect to S_1. On the other hand, we always have $L_S(z) \le L_1(z)$ by Proposition 3.1. Hence $G_S(z) = L_S(z)$.

Case 3. $|a_1^{m_1} \cdot a_2^{m_2} \cdots a_N^{m_N}| < |\gamma| < |a_2^{m_2} \cdots a_N^{m_N}|$.

We now show that the $G_S(z) = \log |\gamma|$ is also equal to the new Lempert function, and the infimum in the definition of the new Lempert function is attained by an extremal disc φ passing through all poles $(a_1, 0); (a_2, 0); ...; (a_N, 0)$ and z.

Set $M := \sum_{j=1}^{N} m_j$ and define $r \in (0,1)$ by

$$r = \sqrt[M]{\frac{|a_1^{m_1} \cdot a_2^{m_2} \cdots a_N^{m_N}|}{|\gamma|}}.$$

We have, for any $1 \le j \le N$,

$$|a_j|^M < |a_j|^{m_1} \le |a_1|^{m_1} < r^M$$

by the hypothesis on γ. So $a_j/r \in \mathbb{D}$. We introduce the map $\varphi : \mathbb{D} \mapsto \mathbb{D}^2$ given by

$$\varphi(\zeta) = \left(r\zeta, e^{i\theta} \prod_{j=1}^{N} \left(\frac{\zeta_j - \zeta}{1 - \overline{\zeta_j}\zeta} \right)^{m_j} \right),$$

where $\zeta_j = \dfrac{a_j}{r}$, $1 \le j \le N$, and θ is chosen such that

$$e^{i\theta} \left(\frac{a_1}{r}\right)^{m_1} \left(\frac{a_2}{r}\right)^{m_2} \cdots \left(\frac{a_N}{r}\right)^{m_N} = \gamma.$$

It is easy to verify that φ verifies the conditions in the definition of L_S and that $|\zeta_1^{m_1} \cdot \zeta_2^{m_2} \cdots \zeta_N^{m_N}| = |\gamma|$. Hence, φ is an extremal disc for the new Lempert function, and $G_S(z) = L_S(z)$ in this case. □

We will now give some negative results, mainly that the generalized Green function can be different from the generalized Lempert function as given in Definition 2.4.

We shall need some notation, to be used in this section and the next one.

For $z \in \mathbb{D}^2$, we will use the following indicators:

$$(4.1)\quad \Psi_0(z) := \max(\log|z_1|, \log|z_2|),$$
$$\Psi_H(z) := \max(2\log|z_1|, \log|z_2|), \quad \Psi_V(z) := \max(\log|z_1|, 2\log|z_2|).$$

Here H stands for "horizontal" and V for "vertical", for the obvious reasons : for $a \in \mathbb{D}^2$,
$\Psi_j(\varphi(\zeta) - a) \le \tau_j \log|\zeta - \zeta_0| + C$ translates to ($\tau_0 = 1$, $\tau_H = \tau_V = 2$):

$$\begin{aligned} \varphi(\zeta_0) = a, &\quad \text{when} \quad j = 0, \\ \varphi(\zeta_0) = a, \varphi_2'(\zeta_0) = 0 &\quad \text{when} \quad j = H, \\ \varphi(\zeta_0) = a, \varphi_1'(\zeta_0) = 0 &\quad \text{when} \quad j = V. \end{aligned}$$

For a, $b \in \mathbb{D}$, let

$$\begin{aligned} S_{a0} &:= \{((a,0), \Psi_0)\} = \{(a,0)\} \\ S_{a0b0} &:= \{((a,0), \Psi_0); ((b,0), \Psi_0)\} = \{(a,0); (b,0)\} \\ S_{aV} &:= \{((a,0), \Psi_V)\} \\ S_{bV} &:= \{((a,0), \Psi_V)\} \\ S_{a0bV} &:= \{((a,0), \Psi_0); ((b,0), \Psi_V)\} \\ S_{aVbV} &:= \{((a,0), \Psi_V); ((b,0), \Psi_V)\}. \end{aligned}$$

We will denote with the corresponding subscripts the pertinent Green and Lempert functions, e.g. G_{a0bV}, L_{a0bV}, $\tilde{L}_{a0bV}$, etc. A special case of Theorem 4.1 is that $L_{aHb0} = G_{aHb0}$ for any a and b in the disc, for instance.

We start by giving an example of a situation where $\tilde{L}_S(z) < L_S(z)$, with $S = S_{a0bV}$.

Proposition 4.2. *For $z_1 \in \mathbb{D}$, $L_{a0bV}(z_1,0) > L_{a0b0}(z_1,0)$, and therefore $L_{a0bV}(z_1,0) > \tilde{L}_{a0bV}(z_1,0) \geq G_{a0bV}(z_1,0)$.*

Proof. From the above, $L_{a0b0}(z_1,0) = G_{a0b0}(z_1,0) = \log|\phi_a(z_1)| + \log|\phi_b(z_1)|$, where ϕ_a and ϕ_b are as in (2.1). We have

$$L_{a0bV}(z_1,0) = \inf\{\log|\zeta_1| + 2\log|\zeta_2| : \exists\varphi \in \mathcal{O}(\mathbb{D},\mathbb{D}^2), \varphi(0) = (z_1,0), \varphi(\zeta_1) = (a,0), \varphi(\zeta_2) = (b,0) \text{ and } \varphi_1'(\zeta_2) = 0\},$$

$$L_{a0}(z_1,0) = \inf\{\log|\zeta_1| : \exists\varphi \in \mathcal{O}(\mathbb{D},\mathbb{D}^2), \varphi(0) = (z_1,0) \text{ and } \varphi(\zeta_1) = (a,0)\},$$

$$L_{bV}(z_1,0) = \inf\{2\log|\zeta_2| : \exists\varphi \in \mathcal{O}(\mathbb{D},\mathbb{D}^2), \varphi(0) = (z_1,0), \varphi(\zeta_2) = (b,0) \text{ and } \varphi_1'(\zeta_2) = 0\}.$$

So $L_{a0bV}(z_1,0) \geq L_{a0}(z_1,0) + L_{bV}(z_1,0)$, since each of the infima on the right hand side is taken over a family of maps φ which is wider than the one used in the definition of L_{a0bV}.

By Lemma 2.6, $L_{a0}(z_1,0) = \log|\phi_a(z_1)|$, $L_{bV}(z_1,0) = \log|\phi_b(z_1)|$.

Now suppose that $L_{a0bV}(z_1,0) \leq L_{a0b0}(z_1,0)$. This means

$$L_{a0bV}(z_1,0) \leq G_{a0b0}(z_1,0) = L_{a0}(z_1,0) + L_{bV}(z_1,0),$$

so there is equality throughout. Since $L_{a0bV}(z_1,0) < \min(L_{a0}(z_1,0), L_{bV}(z_1,0))$, Proposition 3.3 shows that the infimum in the definition of L_{a0bV} is attained by a map φ. It follows from the Schwarz Lemma applied to a and z_1 that its first coordinate φ_1 is a Möbius map of the disc. But we also had to have $\varphi_1'(\zeta_2) = 0$. This is a contradiction. □

The following example is similar, and will be useful in the final construction.

Proposition 4.3. *If $a \neq b \in \mathbb{D}$ and $|\gamma|^2 < |ab|$, then*

$$G_{aVbV}(0,\gamma) < L_{aVbV}(0,\gamma).$$

Proof. First of all we can rewrite the generalized Lempert function as follows

$$L_{aVbV}(z) = \inf\{2\log|\zeta_1| + 2\log|\zeta_2| : \exists\varphi \in \mathcal{O}(\mathbb{D},\mathbb{D}^2), \varphi(0) = z, \varphi(\zeta_1) = (a,0), \varphi(\zeta_2) = (b,0) \text{ and } \varphi_1'(\zeta_1) = 0, \varphi_1'(\zeta_2) = 0\}.$$

As in the proof of Proposition 4.2, by Lemma 2.6 we have

$$L_{aV}(z) = G_{aV}(z) = \max\{\log|\phi_a(z_1)|; 2\log|z_2|\}, \quad \forall z \in \mathbb{D}^2,$$

and similarly for $L_{bV}(z) = G_{bV}(z)$.

By using the Dirichlet problem given by Lelong and Rashkovskii [9], we can verify that

$$G_{aVbV}(z) = \max\{\log|\phi_a(z_1)| + \log|\phi_b(z_1)|; 2\log|z_2|\}.$$

Since $|\gamma|^2 < |ab|$, $G_{aVbV}(0,\gamma) = \log|a| + \log|b|$.

From Lemma 2.5 we already know $G_{aVbV}(z) \leq L_{aVbV}(z)$, for any $z \in \mathbb{D}^2$. Suppose equality holds at $z_0 := (0,\gamma)$. Then, by using Lemma 2.6 and the definition of L_{aVbV}

we have

$$G_{aVbV}(z_0) = \log|a| + \log|b| = \\ G_{aV}(z_0) + G_{bV}(z_0) = L_{aV}(z_0) + L_{bV}(z_0) \leq L_{aVbV}(z_0) = G_{aVbV}(z_0).$$

Hence equality would hold throughout. Now, by Proposition 3.3, the infimum in the definition of L_{aVbV} is attained by an extremal disc φ that passes through both $(a,0)$ and $(b,0)$. It follows that φ must be extremal for L_{aV} and L_{bV}. We will prove that this is impossible.

First of all we characterize all extremal discs for L_{aV}. Let $\varphi = (\varphi_1, \varphi_2)$ be such a disc. By the definition there exists $\zeta_1 \in \mathbb{D}$ such that $\varphi(0) = (0,\gamma)$, $\varphi(\zeta_1) = (a,0)$, $\varphi_1'(\zeta_1) = 0$, $|\zeta_1|^2 = |a|$.

Setting $g := \phi_a \circ \varphi_1 \circ \phi_{\zeta_1}$, we have

$$g(0) = 0, g'(0) = 0, g(\zeta_1) = a, |\zeta_1|^2 = |a|.$$

The Schwarz Lemma now gives $g(\zeta) = e^{i\theta}\zeta^2$, where $\theta \in \mathbb{R}$. It implies that

$$\varphi_1(\zeta) = \phi_a\left(e^{i\theta}\left(\phi_{\zeta_1}(\zeta)\right)^2\right), \quad \forall \zeta \in \mathbb{D}.$$

If the function φ is an extremal disc for L_{bV}, then there is $\zeta_2 \in \mathbb{D}$ such that

$$\varphi_1(0) = 0, \varphi_1(\zeta_2) = b, \varphi_1'(\zeta_2) = 0, |\zeta_2|^2 = |b|.$$

Clearly $\zeta_1 \neq \zeta_2$ since $a \neq b$. Since φ_1 only has one critical point, the condition $\varphi_1'(\zeta_2) = 0$ is not verified, so we have a contradiction. □

Proposition 4.4. *If $a \neq b \in \mathbb{D}$, $|\gamma| < |a|$, and $|\gamma|^2 < |ab|$, then*

$$G_{aVbV}(0,\gamma) < L_{a0bV}(0,\gamma).$$

Proof. The arguments are similar to those in the proof of the above proposition, so we only indicate the differences. As in the proof of Proposition 4.2, $L_{a0bV}(z) \geq L_{a0}(z) + L_{bV}(z) = G_{a0}(z) + G_{bV}(z)$ by Lemma 2.6 ; because of the value of $|\gamma|$, this is equal to $G_{aVbV(z)}$. So if the conclusion was not true, equality would have to hold throughout, but the extremal disc φ in the definition of $L_{a0}(0,\gamma)$ would have to have a Möbius map for its first coordinate φ_1, and since this has no critical point, it could not be extremal for $L_{bV}(0,\gamma)$. □

5. The main counterexample

Theorem 5.1. *Coman's question admits a negative answer in the bidisc for equal weights. More precisely, consider, for $\varepsilon \in \mathbb{C}$,*

$$S^\varepsilon := \{(a,0); (b,0); (b,\varepsilon); (a,\varepsilon)\} \text{ with } b = -a,$$

where the weights are all equal to 1. Denote by G^ε and L^ε the corresponding Green and generalized Lempert functions. Let $z = (0,\gamma)$ with $|a|^{3/2} < |\gamma| < |a|$. Then, $\liminf_{\varepsilon\to 0} L^\varepsilon(z) > G_{aVbV}(z)$ and therefore, for $|\varepsilon|$ small enough,

$$G^\varepsilon(z) < L^\varepsilon(z).$$

Proof. Using the result of Edigarian about the product property of the Green function, [5], we have

$$G^{\varepsilon}(0,\gamma) = \max\left\{ \log|a| + \log|b|; \log|\gamma| + \log\left|\frac{\varepsilon-\gamma}{1-\bar{\varepsilon}\gamma}\right|\right\}.$$

Thus

$$G_{aVbV}(0,\gamma) = \lim_{\varepsilon\to 0} G^{\varepsilon}(0,\gamma) = \log|a| + \log|b| = \log|a|^2.$$

By Propositions 4.3 and 4.4, and since $L_{a0b0}(z) = \log|\gamma| > \log|a|^2 = G_{aVbV}(z)$, with $z = (0,\gamma)$, we have

$$\lim_{\varepsilon\to 0} G^{\varepsilon} = G_{aVbV}(z) < \tilde{L}_{aVbV}(z) = \min\{L_{a0b0}(z), L_{aVb0}(z), L_{a0bV}(z), L_{aVbV}(z)\}. \tag{5.1}$$

We consider $I := \liminf_{\varepsilon\to 0} L^{\varepsilon}(z)$. We want to prove that $I > G_{aVbV}(z)$. In many cases, this will follow from $I \geq \tilde{L}_{aVbV}(z)$.

Recall that L^{ε} is a Lempert function with simple poles, and thus the usual definition (the ℓ_S in the introduction) coincides here with our generalization given in Definition 2.4. For each ε, pick an analytic disc $\varphi^{\varepsilon} \in \mathcal{O}(\mathbb{D},\mathbb{D}^2)$ such that

$$\varphi^{\varepsilon}(0) = z, \varphi^{\varepsilon}(\zeta_1^{\varepsilon}) = (a,0), \varphi^{\varepsilon}(\zeta_2^{\varepsilon}) = (b,0), \varphi^{\varepsilon}(\zeta_3^{\varepsilon}) = (b,\varepsilon), \varphi^{\varepsilon}(\zeta_4^{\varepsilon}) = (a,\varepsilon)$$

and such that $\sum_{j=1}^{4}\log|\zeta_j^{\varepsilon}|$ converges to I as $\varepsilon\to 0$.

By passing to a subsequence, we may assume that φ^{ε} converges locally uniformly to some $\varphi \in \mathcal{O}(\mathbb{D},\mathbb{D}^2)$. Also (if necessary, by passing to a subsequence again), we may assume that $\zeta_j^{\varepsilon} \to \zeta_j \in \overline{\mathbb{D}}$, for each j, as $\varepsilon \to 0$.

Denote $K = \{k \in \{1,2,3,4\} : \zeta_k \in \mathbb{D}\}$. It is easy to see that $\mathbb{D}\cap\{\zeta_1,\zeta_4\}\cap\{\zeta_2,\zeta_3\} = \emptyset$.

If $K = \emptyset$ then $I = 0$, and hence we have $I \geq \tilde{L}_{aVbV}(z) > G_{aVbV}(z)$, by (5.1). So now we only consider the cases where $K \neq \emptyset$.

If $\zeta_j \neq \zeta_k$, $\forall j \neq k \in K$, then $I = \sum_{k\in K}\log|\zeta_k|$, $\varphi_2 \in \mathcal{O}(\mathbb{D},\mathbb{D})$, $\varphi_2(0) = \gamma$ and $\varphi_2(\zeta_k) = 0, k \in K$. It implies that

$$\varphi_2(\zeta) = \prod_{k\in K}\left(\frac{\zeta_k - \zeta}{1-\zeta\overline{\zeta_k}}\right)h(\zeta),$$

where $h \in \mathcal{O}(\mathbb{D},\overline{\mathbb{D}})$ and $h(0) = \dfrac{\gamma}{\prod_{k\in K}\zeta_k}$. Thus we have

$$L_{a0b0}(z) = \log|\gamma| \leq \sum_{k\in K}\log|\zeta_k| = I,$$

and hence, $I \geq \tilde{L}_{aVbV}(z) > G_{aVbV}(z)$.

If $K = \{2,3\}$ and $\zeta_2 = \zeta_3$, then, since $\zeta_2^{\varepsilon} \to \zeta_2$, $\zeta_3^{\varepsilon} \to \zeta_2$ and $|\zeta_3^{\varepsilon} - \zeta_2^{\varepsilon}| \geq |\varepsilon|$,

$$\varphi_1'(\zeta_2) = \lim_{\varepsilon\to 0}\frac{0}{\zeta_3^{\varepsilon} - \zeta_2^{\varepsilon}} = 0.$$

Thus $I \geq L_{bV}(z) \geq L_{a0bV}(z)$ by Proposition 3.1. So that $I \geq \tilde{L}_{aVbV}(z) > G_{aVbV}(z)$.

Similarly, if $K = \{1,4\}$ and $\zeta_1 = \zeta_4$, then $\varphi_1'(\zeta_1) = 0$. Thus $I \geq L_{aV}(z) \geq L_{aVb0}(z) \geq \tilde{L}_{aVbV}(z) > G_{aVbV}(z)$.

If $K = \{1,2,3\}, \zeta_2 = \zeta_3$, then $\varphi_1'(\zeta_2) = 0$. Thus $I = \log|\zeta_1| + 2\log|\zeta_2| \geq L_{a0bV}(z) \geq \tilde{L}_{aVbV}(z) > G_{aVbV}(z)$. The same reasoning obtains if $K = \{4,2,3\}$, $\zeta_2 = \zeta_3$.

Similarly, if either $K = \{1, 2, 4\}, \zeta_1 = \zeta_4$ or $K = \{1, 3, 4\}, \zeta_1 = \zeta_4$, then $\varphi_1'(\zeta_1) = 0$. This implies that $I \geq L_{aVb0}(z) \geq \tilde{L}_{aVbV}(z) > G_{aVbV}(z)$.

If $K = \{1, 2, 3, 4\}$ and $\zeta_1 = \zeta_4, \zeta_2 = \zeta_3$, then $\varphi_1'(\zeta_1) = \varphi_1'(\zeta_2) = 0$. It implies that $I = 2\log|\zeta_1| + 2\log|\zeta_2| \geq L_{aVbV}(z) \geq \tilde{L}_{aVbV}(z) > G_{aVbV}(z)$.

Suppose now that $K = \{1, 2, 3, 4\}$ and $\zeta_1 \neq \zeta_4$, $\zeta_2 = \zeta_3$. This is the final and most delicate case (the proof of [14, Theorem 6.3] suggests that it does occur). Both previous types of argument now break down, because we only get

$$I < \min(\log|\zeta_1|, \log|\zeta_4|) + 2\log|\zeta_2| \geq L_{a0bV}(z);$$

or, from the fact that $\varphi_2(\zeta_1) = \varphi_2(\zeta_4) = \varphi_2(\zeta_2) = 0$ and $\varphi_2(0) = \gamma$,

$$I < \log|\zeta_1| + \log|\zeta_4| + \log|\zeta_2| \geq \log|\gamma| \geq L_{a0b0}(z).$$

By using a rotation in the first coordinate we can assume that $a > 0$. We will prove that $I > G_{aVbV}(z)$. If not, we would have

$$\log|\zeta_1| + \log|\zeta_4| + 2\log|\zeta_2| = I = G_{aVbV}(z) = 2\log a. \tag{5.2}$$

Then the function φ_1 has the following properties:

$$\varphi_1(0) = 0;\ \varphi_1(\zeta_1) = \varphi_1(\zeta_4) = a;\ \varphi_1(\zeta_2) = -a;\ \varphi_1'(\zeta_2) = 0. \tag{5.3}$$

Setting $f := \phi_{-a} \circ \varphi_1 \circ \phi_{\zeta_2}$, with ϕ_ξ defined as in (2.1), we have $f(0) = 0$, $f'(0) = 0$ and $f(\zeta_2) = -a$. The Schwarz Lemma shows that $|\zeta_2|^2 \geq a$, and hence

$$2\log|\zeta_2| \geq \log a \tag{5.4}$$

Setting $g := \phi_a \circ \varphi_1$, we have $g(\zeta_1) = g(\zeta_4) = 0$ and $g(0) = a$. Thus the function g must have the following form

$$g(\zeta) = \phi_{\zeta_1}(\zeta)\phi_{\zeta_4}(\zeta)h_1(\zeta), \forall\zeta \in \mathbb{D},\ \text{where } h_1 \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}}) \text{ and } h_1(0) = \frac{a}{\zeta_1\zeta_4},\ \text{hence}$$

$$\log|\zeta_1| + \log|\zeta_4| \geq \log a. \tag{5.5}$$

The assumption (5.2) implies that all the inequalities in (5.4) and (5.5) become equalities. Now, since $\varphi_2(0) = \gamma$ and $\varphi_2(\zeta_1) = \varphi_2(\zeta_2) = \varphi_2(\zeta_4) = 0$,

$$\varphi_2(\zeta) = \prod_{j=1, j\neq 3}^{4} \left(\frac{\zeta_j - \zeta}{1 - \bar{\zeta_j}\zeta}\right) h_2(\zeta), \qquad \text{where } h_2 \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}}) \text{ and } h_2(0) = \frac{\gamma}{\zeta_1\zeta_2\zeta_4}.$$

It implies that $|\gamma| \leq |\zeta_1\zeta_2\zeta_4| = a^{3/2}$. This contradicts the hypothesis $|\gamma| > a^{3/2}$, and the inequality $I > G_{aVbV}(z)$ is proved.

If $K = \{1, 2, 3, 4\}$ and $\zeta_1 = \zeta_4$, $\zeta_2 \neq \zeta_3$, the proof is similar. □

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