EQUIVALENCE OF SUMMATORY CONDITIONS ALONG SEQUENCES FOR BOUNDED HOLOMORPHIC FUNCTIONS

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ABSTRACT. A sequence of points z_k in the unit disk is said to be thin for a given positive function ρ, if there is a nontrivial bounded holomorphic function such that the infinite series $\sum_k \rho(1 - |z_k|)|f(z_k)|$ converges. All sequences will be assumed hyperbolically separated. We give necessary and sufficient conditions for the problem of thinness of a sequence to be non-trivial (one way or the other), and for two different positive functions ρ_1, ρ_2 to give rise to the same thin sequences. Along the way, some concrete conditions (necessary or sufficient) for a sequence to be thin are obtained.

This paper is dedicated to the memory of Matts Essén.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathcal{F}$ be a certain class of holomorphic functions in a domain $G \subset \mathbb{C}$. N. Nikolski [Ni] and S. Khavinson [Kh] established the existence of positive numbers $\{t_k\}$ (satisfying certain additional conditions) and of a sequence of points $\{z_k\} \subset G$ such that if $\sum_k t_k |f(z_k)| < \infty$, then $f \equiv 0$ for every $f \in \mathcal{F}$. Theorems of such kind arise in various applications: operator theory [Ni], approximation theory [Kh], [E3], sampling, etc. They are also of independent interest.

We shall consider the case when G is the unit disk $\mathbb{D} = \{|z| < 1\}$, $|z_k| \to 1$ as $k \to \infty$, and $\mathcal{F}$ is the class $H^\infty(\mathbb{D})$ of bounded holomorphic functions in $\mathbb{D}$. Note that since functions in the Nevanlinna class (also called "of bounded characteristic") can be written as the quotient of two bounded functions, all our results also apply automatically to that class. In [CTW], [E1], [E2], [EE] sufficient conditions on $\{z_k\}$ were obtained in the case $t_k \equiv 1$. In [NPT], [EE] the problem of description of sequences $\{z_k\}$ was studied for $t_k = 1 - |z_k|$ (see [EE] for a more detailed survey).

In order to avoid summing repeatedly values of the function f at nearby points, we impose a uniform condition of discreteness on our sequence, namely that it be separated in the pseudohyperbolic distance d_G.

Definition 1.1. *We shall say that $\{z_k\}$ is* separated, *if*

$$(1.1) \qquad \inf_{n \neq k} d_G(z_j, z_k) := \inf_{n \neq k} \left| \frac{z_n - z_k}{1 - \overline{z}_n z_k} \right| = \delta > 0.$$

A natural special case of the problem is that of radial coefficients: $t_k = \rho(1 - |z_k|)$, where $\rho(t)$, $t \in (0, 1)$, is a positive function. It will be convenient to write the function $\rho(t)$ in the form $\rho(t) = \rho_\theta(t) = t e^{\theta(t)}$.

Definition 1.2. *A sequence $\{z_k\}$ is said to be θ-thin, if there exists a function $f \in H^\infty$ such that $f \not\equiv 0$ and*

$$(1.2) \qquad \sum_k \rho_\theta(1 - |z_k|) |f(z_k)| = \sum_k (1 - |z_k|) e^{\theta(1 - |z_k|)} |f(z_k)| < \infty.$$

A non-θ-thin sequence is said to be θ-thick. The class of θ-thin separated sequences is denoted by Λ_θ.

Obviously, $\Lambda_{\theta_1} \subset \Lambda_{\theta_2}$ for $\theta_1 > \theta_2$.

Our main results (Theorem 1.4 and Proposition 1.6) determine which pairs of functions θ_1, θ_2 give rise to the same class of θ-thin separated sequences. Those results exhibit a certain hierarchy among separated sequences in the disk. In particular, the classes Λ_θ corresponding to $\rho(t) = t^\alpha$, $0 \leq \alpha < 1$, are all the same, but the case $\rho(t) = t$ is distinct. Before the beginning of the spectrum (for sufficiently big ρ, in particular all such ρ must verify $\limsup_{t \to 0} \rho(t) = \infty$) all separated non Blaschke sequences are thick (Theorem 1.3). After the other end of the spectrum (for "small" ρ, in particular $\liminf_{t \to 0} \rho(t)/t = 0$ for all such ρ) all separated sequences (even maximal separated sequences, or "nets") are thin (Proposition 1.5).

Our first statement is about the set of all possible positive functions θ.

Theorem 1.3. *Suppose that $\theta(t)$ is a positive continuous nonincreasing function. There exists a non Blaschke separated θ-thin sequence $\{z_k\}$ if and only if*

$$(1.3) \qquad \int_0 \frac{dt}{t\theta(t)} = \infty.$$

Obviously, Theorem 1.3 implies the existence of θ-thin sequences for all negative functions $\theta(t)$.

Theorem 1.4. *a) If θ_1, θ_2 are positive functions and $\theta_1(t) \approx \theta_2(t)$, then $\Lambda_{\theta_1} = \Lambda_{\theta_2}$.*

b) If $\theta_1(t)/\theta_2(t) \to \infty$ as $t \to 0$ and $\theta_2(t)$ is a positive continuous nonincreasing function satisfying (1.3), then $\Lambda_{\theta_1} \neq \Lambda_{\theta_2}$, i. e. there exists a sequence $\{z_k\}$ such that $\{z_k\} \in \Lambda_{\theta_2}$, but $\{z_k\} \notin \Lambda_{\theta_1}$.

That (1.3) is a sufficient condition for the existence of elements in Λ_θ will follow from part (b) of this result, applied to $\theta_2 = \theta$. The necessity part in Theorem 1.3 will be proved in Section 2, Theorem 1.4 will be proved in Section 3.

Consider now functions θ bounded above (including negative ones).

Proposition 1.5. *Assume that $\rho_\theta(t)$ is a continuous nondecreasing function such that $\rho_\theta(t) \leq Ct$ with $C > 0$. There exists a separated θ-thick sequence $\{z_k\}$ if and only if*

$$(1.4) \qquad \int_0 \frac{\rho_\theta(t)\, dt}{t^2} = \infty, \qquad \rho_\theta(t) = t e^{\theta(t)}.$$

Proposition 1.5 implies the existence of θ-thick sequences for all nonnegative functions $\theta(t)$.

For simplicity of notation, we write $\rho_1 = \rho_{\theta_1}$, $\rho_2 = \rho_{\theta_2}$.

Proposition 1.6. *Assume that $\rho_1(t)$ satisfies the conditions of Proposition 1.5, (1.4) holds and $\rho_1(t)/\rho_2(t) \to \infty$ as $t \to 0$. Then $\Lambda_{\theta_1} \neq \Lambda_{\theta_2}$. Moreover, there exists a sequence $\{z_k\}$ such that $\{z_k\} \notin \Lambda_{\theta_1}$, but (1.2) with $\rho_\theta = \rho_2$ holds for every $f \in H^\infty$.*

Notice that the sufficient part of Proposition 1.5 (the existence of a θ-thick sequence when (1.4) is satisfied) follows from Proposition 1.6, taking $\theta_1 = \theta$ and an arbitrary θ_2.

In Section 2 we establish some preliminary results. Some of them are of independent interest. For example, Proposition 2.3 gives conditions on $\{z_k\}$ which imply that $\{z_k\}$ is θ-thick, as well as a sufficient condition of existence of θ-thin sequences. We develop these results in Section 5.

2. Some sufficient and some necessary conditions

First, we would like to show that our problem, which is a priori given in terms of infinite sums, reduces in many cases to a problem about the rate of decrease of $|f(z)|$ as a function of $(1 - |z|)$, when z is restricted to the sequence $\{z_k\}$. Such problems have already been investigated in [PT] and numerous other works, see the references in [EE].

Proposition 2.1. *(a) Suppose that $\{z_k\} \in \Lambda_\theta$, $\theta(t) \geq 0$. Then there exists a non identically vanishing function $f_1 \in H^\infty(\mathbb{D})$ such that*

$$\sum_k (1 - |z_k|)e^{\theta(1-|z_k|)}|f_1(z_k)| < \infty \text{ and } |f_1(z_k)| \leq e^{-\theta(1-|z_k|)},$$

for all $k \in \mathbb{Z}_+$.

(b) Conversely, suppose that

$$\liminf_{t \to 0} \frac{\theta(t)}{\log \log \frac{1}{t}} =: L > 0,$$

that $\{z_k\}$ is separated, non-Blaschke, and that there exists a function $f \in H^\infty(\mathbb{D})$ such that

$$|f(z_k)| \leq e^{-\theta(1-|z_k|)},$$

for all $k \in \mathbb{Z}_+$. Then $\{z_k\} \in \Lambda_\theta$.

Proof. (a) There exists a non identically vanishing function $f \in H^\infty(\mathbb{D})$ such that $\sum_k (1 - |z_k|)e^{\theta(1-|z_k|)}|f(z_k)| < \infty$. Define

$$\{z_j'\} := \{z_k : |f(z_k)| \leq e^{-\theta(1-|z_k|)}\}.$$

Then

$$\sum_{k:z_k \notin \{z_j'\}} (1 - |z_k|) \leq \sum_{k:z_k \notin \{z_j'\}} (1 - |z_k|)e^{\theta(1-|z_k|)}|f(z_k)| < \infty.$$

So $\{z_k\} \setminus \{z_j'\}$ is a Blaschke sequence. Let B denote the Blaschke product with zeroes on $\{z_k\} \setminus \{z_j'\}$, then $f_1 := fB$ clearly satisfies the desired conclusion.

(b) Let

$$(2.1) \qquad Y_n := \{z : 1 - 2^{-n} \le |z| < 1 - 2^{-n-1}\}.$$

Since $\{z_k\}$ is separated, $\sum_{k:z_k \in Y_n}(1 - |z_k|) \le C_\delta$. Let $m \in \mathbb{Z}_+$ to be chosen later, $f_1 := f^m$. Then, because of the above and of the hypothesis, for any $\varepsilon > 0$,

$$\sum_k (1 - |z_k|)e^{\theta(1-|z_k|)}|f(z_k)^m| \le \sum_n \sum_{k:z_k \in Y_n}(1 - |z_k|)e^{-(m-1)\theta(1-|z_k|)}$$

$$\le C_\varepsilon + C_\delta \sum_n e^{-(m-1)(L-\varepsilon)\log n} = C_\varepsilon + C_\delta \sum_n \frac{1}{n^{(m-1)(L-\varepsilon)}} < \infty$$

for m large enough (for $L = \infty$, replace $L - \varepsilon$ by any positive number). $\qquad \square$

Proof of the necessity part in Theorem 1.3.

The existence of a non-trivial $f_1 \in H^\infty(\mathbb{D})$ such that $|f_1(z)| \le e^{-\theta(1-|z|)}$ over the sequence $\{z_k\}$, which is non-Blaschke and separated, implies that $g(t) := e^{-\theta(1-t)}$ is *not* an *essential minorant* in the language of [LS]. The main result of that paper is that g is an essential minorant if and only if

$$\int^1 \frac{dt}{(1-t)\log(\frac{1}{g(t)})} < \infty,$$

which is exactly the negation of (1.3). Therefore, if (1.3) does not hold, there are no sequences in Λ_θ. $\qquad \square$

For a given sequence $\{z_i\}$ let N_m be a number of points z_i in Y_m, where Y_m is as in (2.1).

Lemma 2.2. *Let $\sum_m N_m 2^{-m} = \infty$. Then for every $f \in H^\infty$, $f \not\equiv 0$, there exists a set $\mathcal{J}$ of indices with the following properties: $\sum_{m \in \mathcal{J}} N_m 2^{-m} < \infty$, and*

$$|f(z_k)| > \exp\{-C(f,\delta)2^m/N_m\} \quad \text{for at least } N_m/2 \text{ points } z_k \in Y_m, \ m \notin \mathcal{J},$$

where $C(f,\delta)$ is a positive constant and δ is the constant from (1.1).

Lemma 2.2 was in fact proved in [EE] (see proof of Proposition 7.9 (a)), although it was not formulated in this explicit form. In Section 4 we give another, essentially self-contained proof (without resorting to Govorov and Grishin as in [EE]).

Proposition 2.3. *Let $\theta(t)$ be a positive nonincreasing function and $\rho_\theta(t) = te^{\theta(t)}$.*
  a) Suppose that $\{z_k\}$ is a separated sequence and

$$(2.2) \qquad \sum_{m \notin \mathcal{J}} N_m \rho_\theta(2^{-m})\exp\{-\gamma 2^m/N_m\} = \infty \quad \text{for every } \gamma > 0$$

and for every set $\mathcal{J}$ of indices with $\sum_{m \in \mathcal{J}} N_m 2^{-m} < \infty$. Then $\{z_k\} \not\subseteq \Lambda_\theta$.
  b) Suppose that a set $\mathcal{L}$ of indices and a sequence $\{N_m\}$ of nonnegative integers are such that $N_m \uparrow \infty$, $N_m 2^{-m} \downarrow 0$ as $m \in \mathcal{L}$, $m \to \infty$, and

$$(2.3) \qquad \sum_{m \in \mathcal{L}} N_m \rho_\theta(2^{-m})\exp\{-\gamma 2^m/N_m\} < \infty \quad \text{for some } \gamma > 0.$$

Then there exists a separated θ-thin sequence $\{z_k\}$, for which N_m are the given numbers.

Examples

a) If $N_m = p_m 2^m / (\theta(2^{-m}) + \log m)$, where $p_m \to \infty$, then $\{z_k\} \not\subset \Lambda_\theta$.

b) If $N_m = p 2^m / (\theta(2^{-m}) + \log m)$ with a positive constant p, then there exists a separated sequence $\{z_k\} \in \Lambda_\theta$, for which N_m are the given numbers.

Proposition 2.3 generalizes and refines Proposition 7.9 in [EE]. Its proof will be given in Section 4.

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.4 (a). It will be enough to show that if $\theta_1 \geq c\theta_2$, for some $c > 0$, then $\Lambda_{\theta_1} \subset \Lambda_{\theta_2}$. Without loss of generality, assume that $c < 1$.

Suppose that $\{z_k\} \in \Lambda_{\theta_1}$. By Proposition 2.1, we can pick a non identically vanishing function $f_1 \in H^\infty(\mathbb{D})$ such that

$$|f_1(z_k)| \leq e^{-\theta_1(1-|z_k|)}, \quad \forall k \in \mathbb{Z}_+.$$

Pick an integer $m \geq c^{-1}$. Then

$$|f_1(z_k)^m| \leq e^{-c^{-1}\theta_1(1-|z_k|)} e^{\theta_1(1-|z_k|)} |f_1(z_k)|,$$

so

$$\sum_k (1-|z_k|) e^{\theta_2(1-|z_k|)} |f_1(z_k)^m| \leq \sum_k (1-|z_k|) e^{\theta_2(1-|z_k|)-c^{-1}\theta_1(1-|z_k|)} e^{\theta_1(1-|z_k|)} |f_1(z_k)|$$

$$\leq \sum_k (1-|z_k|) e^{\theta_1(1-|z_k|)} |f_1(z_k)| < \infty,$$

so f_1^m is a non identically vanishing function in $H^\infty(\mathbb{D})$ which proves that $\{z_k\} \in \Lambda_{\theta_2}$. $\square$

Proof of Theorem 1.4 (b). Assume that (1.3) holds with $\theta = \theta_2$. First we construct a sequence $\{z_k\} \in \Lambda_{\theta_2}$.

We note that (1.3) is equivalent to the following condition:

$$(3.1) \qquad \sum_{m=1}^\infty [\theta_2(2^{-m})]^{-1} = \infty.$$

Without loss of generality we may assume that $\theta_2(t) \uparrow \infty$ as $t \to 0$. We show that there exist a set $\mathcal{L}$ of indices and a nondecreasing sequence $\{N_m\}$ of positive integers with the following properties:

$$(3.2) \qquad \sum_{m \in \mathcal{L}} [\theta_2(2^{-m})]^{-1} = \infty, \quad \sum_{m \in \mathcal{L}} [\theta_2(2^{-m})]^{-2} < \infty,$$

$$(3.3) \qquad 2^{m-1}/\theta_2(2^{-m}) < N_m \leq 2^m/\theta_2(2^{-m}), \quad N_m 2^{-m} \downarrow 0 \text{ as } m \in \mathcal{L}, \ m \to \infty.$$

Denote $\varepsilon_m = [\theta_2(2^{-m})]^{-1}$. Then $\varepsilon_m \downarrow 0$ and $\sum_m \varepsilon_m = \infty$. Choose integers m_n, M_n such that $m_n < M_n \leq m_{n+1}$ and

$$\frac{1}{2n} < \sum_{m=m_n+1}^{M_n} \varepsilon_m < \frac{1}{n},$$

and set $\mathcal{L}_1 = \bigcup_n \{m_n + 1, \ldots, M_n\}$. Then (3.2) holds with $\mathcal{L} = \mathcal{L}_1$. Let

$$\mathcal{J}_m = \{j : j > m, \ 2^j \varepsilon_j \leq 2^m \varepsilon_m\}, \quad \mathcal{L} = \mathcal{L}_1 \setminus \bigcup_{m \in \mathcal{L}_1} \mathcal{J}_m.$$

Since

$$\sum_{j \in \mathcal{J}_m} \varepsilon_j \leq 2^m \varepsilon_m \sum_{j=m+1}^{\infty} 2^{-j} = \varepsilon_m,$$

the relations (3.2) with $\mathcal{L} = \mathcal{L}_1$ imply the same relations with $\mathcal{L}$. For $m, k \in \mathcal{L}$, $m < k$ we have $2^{m-k} \varepsilon_m < \varepsilon_k < \varepsilon_m$. Fix $m_1 \in \mathcal{L}$ and set $N_{m_1} = [2^{m_1} \varepsilon_{m_1}]$, where $[\cdot]$ denotes the integral part of a number. Suppose that N_m is already defined, and $k = \min\{j : j \in \mathcal{L}, \ j > m\}$. Then we set

$$N_k = \min\{2^{k-m} N_m, \ [2^k \varepsilon_k]\}.$$

Since $2^m \varepsilon_m < 2^k \varepsilon_k$, we have $N_k \geq N_m$.

Now we prove (3.3). If $N_k = [2^k \varepsilon_k]$, then the inequalities in (3.3) are evident. If $N_k = 2^{k-m} N_m$, then by induction we have

$$2^{k-1} \varepsilon_k < 2^{k-1} \varepsilon_m = 2^{k-m} 2^{m-1} \varepsilon_m < 2^{k-m} N_m = N_k.$$

Moreover, the inequality $N_k \leq N_m 2^{k-m}$ implies that $N_k 2^{-k} \leq N_m 2^{-m}$. Thus, (3.3) is established.

Let $l_m = N_m 2^{-m}$. By (3.3) we have $\theta_2(2^{-m}) \leq 1/l_m < 2\theta_2(2^{-m})$, and

$$\begin{aligned}
N_m \rho_2(2^{-m}) \exp\{-\gamma 2^m / N_m\} &= l_m \exp\{\theta_2(2^{-m}) - \gamma/l_m\} \\
&\leq l_m \exp\{-1/l_m\} < l_m^2,
\end{aligned}$$

if γ is sufficiently big. By (3.2), $\sum_{m \in \mathcal{L}} l_m^2 < \infty$. Thus, the condition (2.3) is satisfied, and the existence of a separated θ_2-thin sequence $\{z_k\}$ follows from Proposition 2.3 b).

Now we show that $\{z_k\} \notin \Lambda_{\theta_1}$. Let $p_m = \theta_1(2^{-m})/\theta_2(2^{-m})$. By (3.3), $\theta_2(2^{-m}) \leq 2^m/N_m < 2\theta_2(2^{-m})$. For $\rho_1(t) = te^{\theta_1(t)}$ and for a fixed constant γ we have

$$\begin{aligned}
N_m \rho_1(2^{-m}) \exp\{-\gamma 2^m / N_m\} &= l_m \exp\{\theta_1(2^{-m}) - \gamma 2^m / N_m\} \\
&\geq l_m \exp\{\theta_2(2^{-m})(p_m - 2\gamma)\} > l_m,
\end{aligned}$$

for sufficiently big $m \in \mathcal{L}$. By (3.2) and (3.3) $\sum_{m \in \mathcal{L}} l_m = \infty$. We see that (2.2) holds, and by Proposition 2.3 a) $\{z_k\} \notin \Lambda_{\theta_1}$. $\qquad\square$

Proof of the necessity part of Proposition 1.5.

Suppose that (1.4) fails. By taking a Riemann sum, we will show that any separated sequence is thin, even better : that we can take as function f the constant 1. The separation condition (1.1) implies that every Whitney square contains at most $M = M(\delta)$ points z_k. Hence, $N_m = O(2^m)$ for every separated sequence $\{z_k\}$. Then

$$\sum_k \rho(1 - |z_k|) \leq \sum_m N_m \rho(2^{-m}) \leq C \sum_m 2^m \rho(2^{-m});$$

the convergence of the last series is equivalent to the negation of (1.4). $\qquad\square$

To prove Proposition 1.6, we need the following elementary fact about numerical series.

Lemma 3.1. *If $\sum_n 2^n \rho_1(2^{-n}) = \infty$, $\rho_1(t) \leq Ct$ and $\rho_1(t)/\rho_2(t) \to \infty$ as $t \to 0$, we can choose a subset of the integers $\mathcal{A} \subset \mathbb{Z}_+$ such that*

$$(3.4) \qquad \sum_{n \in \mathcal{A}} 2^n \rho_2(2^{-n}) < \infty,$$

and

$$(3.5) \qquad \sum_{n \in \mathcal{A}} 2^n \rho_1(2^{-n}) = \infty.$$

Proof. We can recursively define a strictly increasing sequence of integers $\{n_j\}$ such that

1. $\rho_2(2^{-n}) \leq 2^{-j}\rho_1(2^{-n})$, $n \geq n_j$;
2. $\sum_{n_j \leq n < n_{j+1}} 2^n \rho_1(2^{-n}) \geq 1$.

Now we pick

$$n'_j := \min\{m \geq n_j : \sum_{n_j \leq n \leq m} 2^n \rho_1(2^{-n}) \geq 1\},$$

and set

$$\mathcal{A}_j := \{n \in \mathbb{Z}_+ : n_j \leq n \leq n'_j\}, \quad \mathcal{A} := \cup_j \mathcal{A}_j.$$

It is clear that $\sum_{\mathcal{A}} 2^n \rho_1(2^{-n}) \geq \sum_j 1 = \infty$. Because $2^n \rho_1(2^{-n}) \leq C$ for all n, $\sum_{\mathcal{A}_j} 2^n \rho_1(2^{-n}) \leq 1 + C$ for all j, therefore by (1)

$$\sum_{\mathcal{A}} 2^n \rho_2(2^{-n}) \leq \sum_j \sum_{\mathcal{A}_j} 2^n \rho_2(2^{-n}) \leq \sum_j 2^{-j}(1 + C) < \infty.$$

$\square$

Proof of Proposition 1.6. Consider the sequence $S = \{z_{m,j} : m \in \mathcal{A}, \ 0 \leq j < 2^m\}$, where

$$z_{m,j} := (1 - 2^{-m})\exp(2^{-m}2\pi ij), \quad 0 \leq j < 2^m,$$

and $\mathcal{A}$ is the set from Lemma 3.1. For any $f \in H^\infty$, by (3.4),

$$\sum_{z \in S} \rho_2(1 - |z|)|f(z)| \leq \|f\|_\infty \sum_{z \in S} \rho_2(1 - |z|) = \|f\|_\infty \sum_{n \in \mathcal{A}} 2^n \rho_2(2^{-n}) < \infty.$$

Now we will show that $S \not\subseteq \Lambda_{\theta_1}$. Let $f \in H^\infty$, $f \not\equiv 0$, be given, and let $\mathcal{J}$ be the set from Lemma 2.2. Since $N_m = 2^m$ as $m \in \mathcal{A}$, the set $\mathcal{J}$ is finite, and by (3.5) we have

$$\sum_k \rho_1(1 - |z_k|)|f(z_k)| \geq \frac{1}{2} \sum_{m \in \mathcal{A} \setminus \mathcal{J}} 2^m \rho_1(2^{-m})\exp\{-C(f, \delta)\} = \infty.$$

$\square$

4. Proof of Lemma 2.2 and Proposition 2.3.

Our alternative prove of Lemma 2.2 is based on the following Lemma, which is proved in [NPT, p. 124, lines 3 to 17] and is the main ingredient in [PT, Lemma 3].

Lemma A. *Let $f \in H^\infty(\mathbb{D})$ and $\delta' \in (0,1)$. Then there exists a function h, positive and harmonic on $\mathbb{D}$, such that for all z for which $d_G(z, f^{-1}(0)) \geq \delta'$,*

$$e^{-h(z)} \leq |f(z)|.$$

In particular, if $\{z_k\}$ is a separated sequence, there exists a Blaschke sequence $b \subset \{z_k\}$ such that for any $z_k \notin b$,

$$e^{-h(z_k)} \leq |f(z_k)|.$$

Proof of Lemma 2.2. By Lemma A, there exists a positive harmonic function h and a Blaschke sequence $b \subset \{z_k\}$ such that

$$\text{For } z_k \notin b, \; e^{-h(z_k)} < |f(z_k)|.$$

Let

$$\mathcal{J} := \{m : \#b \cap Y_m \geq N_m/4\}.$$

Since b is a Blaschke sequence, we have $\sum_{m \in \mathcal{J}} N_m 2^{-m} < \infty$.

Let $C(f,\delta) > 0$ be a constant to be chosen later. For a point $a \in Y_m$, let $I_a \subset \partial\mathbb{D}$ be the arc centered at $a/|a|$, of length 2^{-m}. By the separation condition, the I_{z_k} for $z_k \in Y_m$ form a covering of finite multiplicity of $\partial\mathbb{D}$, that is to say, there exists a positive constant $C_1(\delta)$ such that

$$\sum_{z_k \in Y_m} \chi_{I_{z_k}} \leq C_1(\delta).$$

By Harnack's inequality, for θ such that $e^{i\theta} z_k/|z_k| \in I_{z_k}$, we have

$$h(e^{i\theta}(1 - 2^{-m})z_k/|z_k|) \geq C_2 h(z_k).$$

For $m \in \mathcal{J}$, let

$$N'_m := \#\left\{z_k \in Y_m \setminus b : h(z_k) > C(f,\delta)2^m/N_m\right\}.$$

The mean-value property for harmonic functions gives

$$h(0) = \int_0^{2\pi} h((1 - 2^{-m})e^{i\theta})\frac{d\theta}{2\pi} \geq C_1(\delta)^{-1} \cdot N'_m \cdot 2^{-m} \cdot C_2 C(f,\delta)2^m/N_m,$$

therefore

$$N'_m \leq (C_2 C(f,\delta))^{-1} C_1(\delta) h(0) N_m < N_m/4$$

for $C(f,\delta)$ large enough. At the remaining points z_k, of which there are at least $N_m/2$, we do have

$$|f(z_k)| \geq e^{-h(z_k)} \geq \exp\{-C(f,\delta)2^m/N_m\}.$$

$\square$

Proof of Proposition 2.3 (a). Under conditions of Proposition 2.3, we have

$$\sum_m N_m 2^{-m} = \infty.$$

Moreover, $\rho_\theta(1-|z|) = (1-|z|)e^{\theta(1-|z|)} > 2^{-m-1}e^{\theta(2^{-m})} = \frac{1}{2}\rho_\theta(2^{-m})$, $z \in Y_m$. Let $f \in H^\infty$, $f \not\equiv 0$, be given, and let $\mathcal{J}$ be the set from Lemma 2.2. If $\gamma > C(f, \delta)$, then by (2.2)

$$\sum_k \rho_\theta(1-|z_k|)|f(z_k)| \geq \frac{1}{2}\sum_{m \notin \mathcal{J}} \sum_{z_k \in Y_m} \rho_\theta(2^{-m})|f(z_k)|$$

$$\geq \frac{1}{4}\sum_{m \notin \mathcal{J}} N_m \rho_\theta(2^{-m})\exp\{-\gamma 2^m/N_m\} = \infty.$$

$\square$

Lemma 4.1. *Suppose that $P_1(t)$, $t \in (0,1)$, is a nondecreasing function, $P_1(t)(1-t)$ is a nonincreasing function tending to 0 as $t \to 1$, and $d_m \leq 1$ are given positive numbers. Then there are a sequence $\{z_k\} \subset \mathbb{D}$ and an absolute constant M with the following properties: all points z_k are situated on the circumferences $|z| = 1 - 2^{-m}$, $m = 1, 2, \ldots$;*

$$\min\{|z_j - z_k| : j \neq k, \ |z_j| = |z_k| = 1 - 2^{-m}\} \geq d_m;$$

(4.1) $\qquad N_z(1 - 2^{-m}) > M(d_m P_1(1 - 2^{-m}))^{-1} + 1, \quad m = 1, 2, \ldots,$

where $N_z(1 - 2^{-m})$ is the number of points z_k with $|z_k| = 1 - 2^{-m}$, and for every $T > 0$ there is a nontrivial function $f \in H^\infty$ such that

(4.2) $\qquad \log|f(z_k)| \leq -P_1(|z_k|)T, \quad k = 1, 2 \ldots$

We omit the proof, which is exactly the same as the proof of Lemma 4.3, part 2 in [EE] with $\phi(t_j) = 1/d_j$ and $t_j = 1 - 2^{-j}$. It follows from Lemma 4.3 in [EE] that Lemma 4.1 is sharp.

Proof of Proposition 2.3 (b). Pick $d_m = 2^{-m}$. By assumption, $N_m 2^{-m} \downarrow 0$, $\{N_m\}$ is a nondecreasing sequence and $N_m \to \infty$ on $\mathcal{L}$. Hence, there exists a nondecreasing function $P_1(t)$ such that

$$P_1(1 - 2^{-m}) = M 2^m/N_m, \quad m \in \mathcal{L},$$

where M is the constant from (4.1), $P_1(t) \uparrow \infty$, $P_1(t)(1-t) \downarrow 0$ as $t \to 1$. Let $\{z_k\}$ be the sequence from Lemma 4.1. By (4.1),

$$N_z(1 - 2^{-m}) > N_m, \quad m \in \mathcal{L}.$$

If we choose a sufficiently big number T, then by (4.2)

$$\log|f(z_k)| \leq -M 2^m T/N_m < -\gamma 2^m/N_m, \quad |z_k| = 1 - 2^{-m}, \quad m \in \mathcal{L}.$$

For every $m \in \mathcal{L}$ we choose N_m points from $\{z_k\}$ with $|z_k| = 1 - 2^{-m}$. We also denote this subsequence by $\{z_k\}$. Using (2.3), we have

$$\sum_k \rho_\theta(1-|z_k|)|f(z_k)| = \sum_{m \in \mathcal{L}} \sum_{z_k \in Y_m} \rho_\theta(1-|z_k|)|f(z_k)| \leq$$

$$\leq c\sum_{m \in \mathcal{L}} N_m \rho_\theta(2^{-m})\exp\{-\gamma 2^m/N_m\} < \infty.$$

Thus, $\{z_k\} \in \Lambda_\theta$, and the proof of Proposition 2.3 is complete. $\square$

5. Further properties of thin sequences

The following assertion was proved in [EE] (see Proposition 7.8). *Let N_m be given nonnegative integers, and let*

$$z_{m,k} = (1 - 2^{-m})\exp(2\pi ik/N_m), \quad 1 \le k \le N_m, \quad m = 1, 2, \ldots$$

for $N_m > 0$. Then $\{z_{m,k}\} \not\subseteq \Lambda_\theta$ for $\rho_\theta \equiv 1$ if and only if $\{z_{m,k}\}$ is a non-Blaschke sequence, that is $\sum_m N_m 2^{-m} = \infty$.

On the other hand, if we know only, that $\{z_{m,k}\}$ is a separated sequence, we need essentially stronger conditions on $\{N_m\}$ (see Proposition 2.3 with $\rho_\theta \equiv 1$). Thus, there is a certain connection between the dispersion of points over annuluses Y_m and the massivity of a sequence, i. e. numbers N_m (we also can interpret this relation as a connection between the dispersion of points and the possible decay of nontrivial bounded function over a sequence). In the present section we investigate this connection. In particular, we develop Lemma 2.2, Proposition 2.3 and some results in [EE], and give another approach to obtaining such assertions, which is based on estimates of subharmonic functions outside exceptional sets.

Let a sequence $\{z_k\}$ be given. For every m, for which Y_m contains at least six points z_k, and for every $z_k \in Y_m$ we set

$$d_{m,k} := \min\{|z_j - z_k| : j \ne k, \ z_j, z_k \in Y_m\}.$$

Let $\overline{d}_m$ be the mean value of $[N_m/6]$ smallest numbers $d_{m,k}$ (here $[\cdot]$ is the integral part of a number). Our main tool is the following lemma.

Lemma 5.1. *Let $\{z_k\}$ be a separated sequence. Let a set $\mathcal{K}$ of indices be such that $N_m \ge 6$ as $m \in \mathcal{K}$, and $\sum_{m \in \mathcal{K}} N_m 2^{-m} = \infty$. Then for every $f \in H^\infty$, $f \not\equiv 0$, there exists a set $\mathcal{J}$ of indices with the following properties: $\sum_{m \in \mathcal{J}} N_m 2^{-m} < \infty$, and*

$$(5.1) \qquad\qquad |f(z_k)| > \exp\{-C(f, \delta)(N_m \overline{d}_m)^{-1}\}$$

for at least $N_m/2$ points $z_k \in Y_m$, when $m \in \mathcal{K} \setminus \mathcal{J}$, where $C(f, \delta)$ is a positive constant and δ is the constant from (1.1).

Since $\overline{d}_m > \text{Const } 2^{-m}$, Lemma 5.1 refines Lemma 2.2. We shall prove Lemma 5.1 in Section 6.

Corollary 5.2. *Let $\theta(t)$ be a positive nonincreasing function and $\rho_\theta(t) = te^{\theta(t)}$.*

a) Suppose that $\{z_k\}$ is a separated sequence. Let $\mathcal{K}$ be a set of indices satisfying conditions of Lemma 5.1 and such that

$$(5.2) \qquad \sum_{m \in \mathcal{K} \setminus \mathcal{J}} N_m \rho_\theta(2^{-m})\exp\{-\gamma(N_m \overline{d}_m)^{-1}\} = \infty \quad \text{for every } \gamma > 0$$

and for every set $\mathcal{J}$ of indices with $\sum_{m \in \mathcal{J}} N_m 2^{-m} < \infty$. Then $\{z_k\} \not\subseteq \Lambda_\theta$.

b) Suppose that a set $\mathcal{L}$ of indices, a sequence $\{N_m\}$ of nonnegative integers and a sequence $\{d_m\}$ of positive numbers are such that $1 \ge d_m \ge 2^{-m}$, $\{d_m 2^m\}$ is a nondecreasing sequence, $N_m \uparrow \infty$, $N_m d_m \downarrow 0$ as $m \in \mathcal{L}$, $m \to \infty$, and

$$(5.3) \qquad \sum_{m \in \mathcal{L}} N_m \rho_\theta(2^{-m})\exp\{-\gamma(N_m d_m)^{-1}\} < \infty \quad \text{for some } \gamma > 0.$$

Then there exists a separated θ-thin sequence $\{z_k\}$ with $|z_k| = 1 - 2^{-m}$ as $z_k \in Y_m$, for which N_m are the given numbers and $\overline{d}_m \ge d_m$.

Proof. (a) The proof is essentially the same as the proof of Proposition 2.3 (a) with the obvious corrections: we use Lemma 5.1 instead of Lemma 2.2.

(b) We argue by analogy with the proof of Proposition 2.3 (b). Let $P_1(t)$ be a nondecreasing function such that $P_1(t) \uparrow \infty$, $P_1(t)(1-t) \downarrow 0$ as $t \to 1$ and

$$P_1(1 - 2^{-m}) = M(N_m d_m)^{-1}, \quad m \in \mathcal{L},$$

where M is the constant from (4.1). Let $\{z_k\}$ be the sequence from Lemma 4.1. By (4.1),

$$N_z(1 - 2^{-m}) \geq N_m, \quad m \in \mathcal{L}.$$

By (4.2), for sufficiently big numbers T we have

$$\log |f(z_k)| \leq -MT(N_m d_m)^{-1} < -\gamma(N_m d_m)^{-1}, \quad |z_k| = 1 - 2^{-m}, \quad m \in \mathcal{L}.$$

For every $m \in \mathcal{L}$ we choose N_m points from $\{z_k\}$ with $|z_k| = 1 - 2^{-m}$. The obtained sequence we also denote by $\{z_k\}$. Using (5.3), we have

$$\begin{aligned}
\sum_k \rho_\theta(1 - |z_k|)|f(z_k)| &= \sum_{m \in \mathcal{L}} \sum_{z_k \in Y_m} \rho_\theta(1 - |z_k|)|f(z_k)| \leq \\
&\leq c \sum_{m \in \mathcal{L}} N_m \rho_\theta(2^{-m}) \exp\{-\gamma(N_m d_m)^{-1}\} < \infty.
\end{aligned}$$

$\square$

Corollary 5.2 shows the connection between the dispersion of points over annuluses Y_m and the massivity of a sequence $\{z_k\}$. It is clear that Corollary 5.2 is a direct generalization of Proposition 2.3. Another special case gives the following assertion.

Corollary 5.3. *Let $N_m \overline{d}_m \geq c > 0$ as $m \in \mathcal{K}$ and $\rho_\theta(t) \geq c_1 t$, $c_1 > 0$. Then a separated sequence $\{z_k\}$ is θ-thick if and only if*

$$(5.4) \qquad\qquad \sum_{m \in \mathcal{K}} N_m 2^{-m} = \infty.$$

Proof. Clearly, if $\{z_k\} \not\subseteq \Lambda_\theta$, then (5.4) holds.

Conversely, the condition (5.4) and the inequality $\rho_\theta(t) \geq c_1 t$ imply (5.2), and our assertion follows from Corollary 5.2. $\square$

Corollary 5.3 is a generalization of Proposition 7.8 in [EE], quoted at the beginning of this section. It is easy to see that Corollary 5.3 is not correct for $\rho_\theta(t)$ such that $\liminf_{t \to 0} \rho_\theta(t)/t = 0$. Indeed, in this case there exist numbers $t_j \in (0, 1)$ for which $t_{j+1}/t_j < 1/2$ and $\sum_{j=1}^\infty \rho_\theta(t_j)/t_j < \infty$. Let $\mathcal{K} = \{m_j\}$, where $2^{-m_j-1} < t_j \leq 2^{-m_j}$, and let $N_{m_j} = 2^{m_j}$, $z_{m_j,k} = (1 - t_j)\exp(2\pi i k/N_{m_j})$, $1 \leq k < N_{m_j}$. Then $\{z_{m_j,k}\}$ is a separated θ-thin sequence satisfying (5.4).

6. Proof of Lemma 5.1.

We need certain preliminary definitions and results. Let f be a meromorphic function in $\mathbb{D}$, and let $\{a_k\}$ and $\{b_k\}$ be zeros and poles of f, respectively. We assume

that 0 is neither a zero nor a pole of f. The Nevanlinna characteristic of f is defined by (cf. for example Hayman [Ha], p. 4)

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\varphi})|\, d\varphi + \sum_{|b_i| < r} \log \frac{r}{|b_i|}.$$

Since $T_f(r)$ is a nondecreasing function of r, $\lim_{r\uparrow 1} T_f(r) =: T_f(1)$ exists for functions f in the Nevanlinna class N, i. e., functions of bounded characteristic.

To $f \in H^\infty(\mathbb{D})$ with zeros $\{a_n\}$, $a_n \neq 0$, we associate a non-negative measure η on the closed unit disc $\overline{\mathbb{D}}$ defined as follows: for every set $E \subset \overline{\mathbb{D}}$ for which $E \cap \partial\mathbb{D}$ is measurable, we define

$$(6.1) \qquad \eta(E) = \lim_{r\to 1} \frac{1}{2\pi} \int_{e^{i\theta} \in E} \log^- |f(re^{i\theta})|\, d\theta + \sum_{a_n \in E} \log \frac{1}{|a_n|}.$$

We note that $\eta(\overline{\mathbb{D}}) = T_{1/f}(1)$. For the proof that this limit exists, we refer to [EE, Appendix 1]. We quote the following theorem in the form given in [EE] (see Theorem 2.10 in [EE] and the remark after it).

Theorem 6.1. *(Govorov–Grishin) Suppose that $f \in H^\infty(\mathbb{D})$, $f(z) \not\equiv 0$. For every $P > 1$ there exists a system of discs $D^k(w_k, r_k)$ such that*

$$(6.2) \qquad \log |f(z)| > -A_1 P T_{1/f}(1), \quad z \in \mathbb{D} \setminus \bigcup_k D^k,$$

$$(6.3) \qquad \sum_k r_k < 1/P,$$

and every $z \in \bigcup_k D^k$ belongs to at most A_2 discs D^k (we shall say that A_2 is the multiplicity of this covering). Moreover,

$$(6.4) \qquad \eta(D^k) > A_3 P T_{1/f}(1) r_k.$$

Here A_1, A_2 and A_3 are absolute positive constants and η is the non-negative measure on the closed unit disc $\overline{\mathbb{D}}$ defined by (6.1).

Proof of Lemma 5.1. Let

$$\mathcal{M}_0 = \{m : N_m \overline{d}_m > 1/2\},$$

$$\mathcal{M}_j = \{m : 2^{-j-1} < N_m \overline{d}_m \leq 2^{-j}\},\ j = 1, 2, \ldots,$$

$$G_j = \bigcup_{m \in \mathcal{M}_j} Y_m, \quad P_j = c(\delta) 2^j,$$

where $c(\delta)$ is a positive constant which will be chosen later, depending only on δ, and δ is the constant from (1.1). Let a function $f \in H^\infty$, $f \not\equiv 0$, be given and let $D_j^k(w_{j,k}, r_{j,k})$ be a system of disks from Theorem 6.1 with $P = P_j$. We introduce the

following sets of disks:

$$\mathcal{D}_{j,1} = \{D_j^k : D_j^k \cap G_j \neq \varnothing, \quad r_{j,k} < \tfrac{1}{8}(1 - |w_{j,k}|)\}, \quad \mathcal{D}_1 = \bigcup_j \mathcal{D}_{j,1};$$

$$\mathcal{D}_{j,2} = \{D_j^k : D_j^k \cap G_j \neq \varnothing, \quad r_{j,k} \geq \tfrac{1}{8}(1 - |w_{j,k}|)\}, \quad \mathcal{D}_2 = \bigcup_j \mathcal{D}_{j,2}.$$

We shall show that

$$\sum_{z_l \in \mathcal{D}_1} (1 - |z_l|) < \infty.$$

By (1.1), each disk $D_j^k \in \mathcal{D}_1$ contains at most $N = N(\delta)$ points z_l. Moreover, disks from $\mathcal{D}_1$ do not intersect $\partial \mathbb{D}$, and by (6.1)

$$\eta(D_j^k) = \sum_{a_i \in D_j^k} \log \frac{1}{|a_i|}, \quad D_j^k \in \mathcal{D}_1,$$

where a_i are zeros of f. By (6.4), $\eta(D_j^k) > 0$. Hence, every disc $D_j^k \in \mathcal{D}_1$ contains at least one zero of f. Each disk in $\mathcal{D}_1$ intersects at most two sets Y_m. For each j the multiplicity of the family $\{D_j^k\}$ is the absolute constant A_2. Hence, the multiplicity of the covering $\mathcal{D}_1$ is at most $3A_2$. Thus,

$$\sum_{z_l \in \mathcal{D}_1} (1 - |z_l|) \leq \sum_{D_j^k \in \mathcal{D}_1} \sum_{z_l \in D_j^k} (1 - |z_l|) \leq$$

$$\leq c \sum_{D_j^k \in \mathcal{D}_1} \sum_{a_i \in D_j^k} (1 - |a_i|) \leq 3A_2 c \sum_i (1 - |a_i|) < \infty,$$

that what we need.

Let $\mathcal{J}_1$ be the set of indices m such that $\#\{z_l : z_l \in Y_m \cap \mathcal{D}_1\} \geq N_m/6$. Then

$$\sum_{m \in \mathcal{J}_1} N_m 2^{-m} \leq c \sum_{z_l \in \mathcal{D}_1} (1 - |z_l|) < \infty.$$

We set $\mathcal{K}_1 = \mathcal{K} \setminus \mathcal{J}_1$. Then at least $\tfrac{5}{6} N_m$ points z_l do not belong to $\mathcal{D}_1$ for every $m \in \mathcal{K}_1$.

Fix $j \geq 0$. We split the points $z_l \in \mathcal{D}_{j,2} \cap G_j$, into two subsets $Z_{j,1}$ and $Z_{j,2}$ in the following way. Let $z_l \in Y_m$. Then $z_l \in Z_{j,1}$, if every disk $D_j^k \in \mathcal{D}_{j,2}$ containing z_l, does not contain other points $z_{l'} \in Y_m$; $z_l \in Z_{j,2}$, if there exists a disk $D_j^k \in \mathcal{D}_{j,2}$ containing at least one other point $z_{l'} \in Y_m$ besides z_l. Let

$$N_m^{j,i} = \#\{z_l : z_l \in Y_m \cap Z_{j,i}\}, \quad i = 1, 2;$$

$$\mathcal{J}_{j,2} = \{m \in \mathcal{M}_j : N_m^{j,1} \geq N_m/6\}, \quad \mathcal{J}_2 = \bigcup_j \mathcal{J}_{j,2}.$$

We shall prove that

$$\sum_{m \in \mathcal{J}_2} N_m 2^{-m} < \infty.$$

Assume that $D_j^k \in \mathcal{D}_{j,2}$ and $D_j^k \cap Y_m \neq \varnothing$. Then

$$(6.5) \qquad\qquad r_{j,k} > \tfrac{1}{9} 2^{-m-1}.$$

Indeed, if $r_{j,k} \leq \frac{1}{9} 2^{-m-1}$, then by the inequality $|w_{j,k}| < 1 - 2^{-m-1} + r_{j,k}$ we have

$$r_{j,k} \geq \frac{1}{8}(1 - |w_{j,k}|) > \frac{1}{8}(2^{-m-1} - r_{j,k}) \geq \frac{1}{8} \cdot 8 r_{j,k} = r_{j,k},$$

and we come to a contradiction. From (6.5) we deduce that

$$\sum_{m: D_j^k \cap Y_m \neq \varnothing} 2^{-m} < 2 \cdot 2 \cdot 9 r_{j,k} = 36 r_{j,k}.$$

Moreover, the number of disks $D_j^k \in \mathcal{D}_{j,2}$ intersecting Y_m is at least $N_m^{j,1}$. Hence,

$$\sum_{m \in \mathcal{J}_{j,2}} N_m 2^{-m} \leq 6 \sum_{m \in \mathcal{J}_{j,2}} N_m^{j,1} 2^{-m}$$

$$\leq 6 \sum_{k: D_j^k \in \mathcal{D}_{j,2}} \sum_{m: D_j^k \cap Y_m \neq \varnothing} 2^{-m} < 6 \cdot 36 \sum_k r_{j,k} < \frac{c}{P_j} = c_1 2^{-j}$$

(in the last inequality we used (6.3)). We have

$$\sum_{m \in \mathcal{J}_2} N_m 2^{-m} = \sum_j \sum_{m \in \mathcal{J}_{j,2}} N_m 2^{-m} < c \sum_j 2^{-j} < \infty.$$

Now we shall show that

(6.6) $$N_m^{j,2} \leq \frac{1}{6} N_m \text{ for all } m \in \mathcal{K} \cap \mathcal{M}_j.$$

For fixed m we consider disks D_j^k containing more than one point $z_l \in Y_m \cap Z_{j,2}$. Then $2 r_{j,k} > d_{m,l}$. The separation condition (1.1) implies that $d_{m,l} \geq c(\delta) 2^{-m}$, where δ is the constant from (1.1); and since the disks $D(z_l, d_{m,l}/2)$ are disjoint, denoting by λ_2 the area measure,

$$
\begin{aligned}
C(\delta) 2^{-m} r_{j,k} &\geq \lambda_2(D(w_{j,k}, 2 r_{j,k}) \cap Y_m) \\
&\geq \sum_{l: z_l \in D_j^k \cap Y_m \cap Z_{j,2}} \lambda_2(D(z_l, d_{m,l}/2) \cap Y_m) \\
&\geq c'(\delta) 2^{-m} \sum_{l: z_l \in D_j^k \cap Y_m \cap Z_{j,2}} d_{m,l}.
\end{aligned}
$$

Thus

$$\sum_{l: z_l \in D_j^k \cap Y_m \cap Z_{j,2}} d_{m,l} < c_2(\delta) r_{j,k}.$$

If (6.6) does not hold, then

$$\frac{1}{6} N_m \overline{d}_m < N_m^{j,2} \overline{d}_m \leq \sum_{l: z_l \in Y_m \cap Z_{j,2}} d_{m,l}$$

$$\leq \sum_k \sum_{l: z_l \in D_j^k \cap Y_m \cap Z_{j,2}} d_{m,l} \leq \sum_k c_2(\delta) r_{j,k} < \frac{c_2(\delta)}{P_j} = \frac{c_2(\delta)}{c(\delta)} 2^{-j} < \frac{1}{6} N_m \overline{d}_m,$$

if $c(\delta)$ is sufficiently big. That contradiction proves the validity of (6.6). Thus, if $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$, then for every $m \in \mathcal{K} \setminus \mathcal{J}$ the number of points z_l lying outside of

$\bigcup_k D_j^k$ is at least

$$\tfrac{5}{6}N_m - N_m^{j,1} - N_m^{j,2} > \tfrac{5}{6}N_m - \tfrac{1}{6}N_m - \tfrac{1}{6}N_m = \tfrac{1}{2}N_m.$$

The inequality (5.1) follows from (6.2). $\qquad\qquad\qquad\qquad\qquad\square$

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