UNIVERSITÉ PAUL SABATIER : L3 PARCOURS SPÉCIAL
COURS INTENSIF : ADVANCED COMPLEX ANALYSIS

PASCAL J. THOMAS

Foreword. These notes are a condensed version of a class taught to third-year students in the "Parcours Spécial" of the Licence de Mathématiques of the Université Paul Sabatier, Toulouse. Some proofs may be only cursorily indicated. Interested readers should refer to the books by Ahlfors and Rudin given in the bibliographical references.

0. Prerequisites

We take it that you attended Jasmin Raissy's "Analyse Complexe" class in L2 Special, as well as a class in advanced calculus, covering at least the basics of differentiable maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and the Inverse Function Theorem.

We shall assume that the following topics are known.

0.1. **Definition of holomorphic functions.** As $\mathbb{C}$-differentiable maps, or as analytic functions (locally expendable as power series). Both are equivalent (major theorem).

The set of all holomorphic functions on an open set Ω is denoted by $\mathcal{O}(\Omega)$ (O stands for *olomorfa*, the Italian word, and we use it because there are too many other kinds of function spaces called H).

We usually consider holomorphic functions as defined on a *domain,* i.e. a connected open set.

0.2. **Path integration.** We recall that a *curve* in a topological space X is a continuous map γ from an interval $[a, b]$ to X. Intervals of definition can be modified by using strictly increasing bijections. Two curves γ_1, γ_2 defined respectively on $[a, b]$ and $[b, c]$ can be concatenated in the obvious way if $\gamma_1(b) = \gamma_2(b)$. We can still do it if γ_2 is defined on any interval $[c, d]$, provided that $\gamma_1(b) = \gamma_2(c)$, at the expense of translating the interval of definition of γ_2 by $b - c$.

When X happens to be a subset of a $\mathbb{R}$-vector space, a *path* is a piecewise-$\mathcal{C}^1$ curve, i.e. the concatenation of a finite number of curves which are $\mathcal{C}^1$. Given a bounded measurable function on X, we set

$$\int_\gamma f(\zeta)d\zeta := \int_a^b f(\gamma(t))\gamma'(t)dt,$$

where the last integral is to be understood as a finite sum on each of the differentiable "pieces" of γ.

0.3. **Cauchy's theorem.** Here is the elementary form of Cauchy's Theorem which you saw in second year. We recall that a (P)-domain is a bounded domain Δ such that its boundary is made up of a finite number of pairwise disjoint closed curves Γ_j, $1 \leq j \leq n$, each of them being the image of a closed path γ_j with no double points except for the origin and extremity, the γ_j being oriented so that Δ always lies to the left of its boundary.

Theorem 0.1. *If $f \in \mathcal{O}(\Omega)$ such that $\operatorname{Re} f, \operatorname{Im} f \in \mathcal{C}^1(\Omega)$ and Δ is a (P)-domain such that $\bar{\Delta} \subset \Omega$, then*

$$\int_{\partial\Delta} f(\zeta) d\zeta = 0.$$

line

0.4. **The Schwarz Lemma.** Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

Theorem 0.2. *Let $f \in \mathcal{O}(\mathbb{D})$ such that $f(\mathbb{D}) \subset \bar{\mathbb{D}}$ and $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$, and $|f'(0)| \leq 1$. If equality occurs in the first inequality for $z \neq 0$, or in the second one, then there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$.*

0.5. **Classification of isolated singularities.** Let $f \in \mathcal{O}(D(a,r) \setminus \{a\})$. If f is locally bounded near a, then f can be extended to a holomorphic function on the whole disc and we say that a is a *removable* singularity (theorem due to Riemann). If $\lim_{z \to a} |f(z)| = +\infty$, there exists $m \in \mathbb{N}^*$ such that $(z-a)^m f(z)$ extends holomorphically to the whole disc, and we say that a is a *pole* and call the smallest m as above the *order* of the pole. Finally, if none of those two cases happen, which also mean that the Laurent expansion near a

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$

admits non-zero coefficients a_k for arbitrarily large negative values of k, we say that a is an *essential* singularity.

0.6. **Complex logarithm.** The function $\operatorname{Log} z := \log |z| + i \operatorname{Arg} z$, where Arg is the unique determination of the argument of z belonging to $(-\pi, +\pi]$, is holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$, and verifies $\exp(\operatorname{Log} z) = z$ on its domain.

1. Conformal Maps

In this section, we consider holomorphic maps as maps of domains in the plane, from a geometric point of view.

1.1. Definition.

Proposition 1.1. Let f be a holomorphic function on an open set Ω. If $z_0 \in \Omega$ and $f'(z_0) \neq 0$, there exists a neighborhood U of z_0 such that f is a one-to-one (injective) map on U.

Proof. For $|h|$ small enough,

$$f(z_0 + h) = f(z_0) + f'(z_0)h + o(h),$$

so as map from $\mathbb{R}^2$ to $\mathbb{R}^2$, writing $h = h_1 + ih_2$ and $f(z) = \operatorname{Re} f(z) + i \operatorname{Im} f(z)$, the differential of f at z_0 is given by

$$Df(z_0) \cdot h = \begin{pmatrix} \operatorname{Re} f'(z_0) & -\operatorname{Im} f'(z_0) \\ \operatorname{Im} f'(z_0) & \operatorname{Re} f'(z_0) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

The determinant of the differential map is $|f'(z_0)|^2 \neq 0$. So applying the Inverse Function Theorem, we find a neighborhood U so that f is a bijection (one-to-one and onto) from U to $f(U)$. □

Another proof could be obtained using power series. We remark that the converse is true (exercise).

Notice that the computation shows that the differential map is a direct similarity, i.e. a rotation followed by a dilation. Such linear maps preserve angles between vectors. This is the property we are interested in.

Definition 1.2. Let U_1 and $U_2 \subset \mathbb{R}^2$ be open sets. A map $\Phi : U_1 \longrightarrow U_2$ is *conformal* at a point z_0 if it is differentiable there and for any $\mathcal{C}^1$ curves γ_1, γ_2 such that $\gamma_1(0) = \gamma_2(0) = z_0$, with non-vanishing derivative at 0, then

$$\text{angle } \left((\Phi \circ \gamma_2)'(0), (\Phi \circ \gamma_1)'(0)\right) = \text{angle } \left(\gamma_2'(0), \gamma_1'(0)\right),$$

where the angles are understood as oriented angles between nonzero vectors.

We say that Φ is conformal on U_1 if it is conformal at each point of U_1.

Remark. We could define a notion of conformality in higher dimension, using non-oriented angles. It turns out that this is much more restrictive than in dimension 2, giving rise to a very rigid class of mappings. This is a difficult theorem due to Liouville, which we will not discuss.

1.2. **Characterization.**

Proposition 1.3. Let $f \in \mathcal{C}^1(\Omega)$, and $z_0 \in \Omega$. Suppose that $Df(z_0) \neq 0$. Then f is conformal at z_0 if and only if $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$.

Therefore such an f is conformal on Ω if and only if $f \in \mathcal{O}(\Omega)$ and $f'(z_0) \neq 0$ for any $z_0 \in \Omega$.

Proof. We recall the complex notation for partial derivatives, with the usual notation $z = x + iy$:

$$\begin{aligned} Df(z_0) \cdot h &= \frac{\partial f}{\partial x}(z_0) h_1 + \frac{\partial f}{\partial y}(z_0) h_2 \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i\frac{\partial f}{\partial y}(z_0)\right)(h_1 + ih_2) + \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + i\frac{\partial f}{\partial y}(z_0)\right)(h_1 - ih_2) \\ &=: \frac{\partial f}{\partial z}(z_0) h + \frac{\partial f}{\partial \bar{z}}(z_0) \bar{h}. \end{aligned}$$

This implies the complex form of the Chain Rule:

$$(f \circ \gamma)'(t) = \frac{\partial f}{\partial z}(\gamma(t))\gamma'(t) + \frac{\partial f}{\partial \bar{z}}(\gamma(t))\bar{\gamma}'(t).$$

Thus, if we suppose that $\frac{\partial f}{\partial \bar z}(z_0)=0$, then
$$\frac{(f\circ\gamma_2)'(t)}{(f\circ\gamma_1)'(t)}=\frac{f'(\gamma_2(t))\gamma_2'(t)}{f'(\gamma_1(t))\gamma_1'(t)}.$$
When γ_1,γ_2 are as in Definition 1.2, then $f'(\gamma_2(0))=f'(z_0)=f'(\gamma_1(0))$, so the above ratio when $t=0$ is $=\frac{\gamma_2'(0)}{\gamma_1'(0)}$ and we are done.

Conversely, consider the curves $\gamma_1(t)=z_0+e^{i\alpha}t$, $\gamma_2(t)=z_0+e^{i(\alpha+\theta)}t$, where $\alpha,\theta\in[0,2\pi)$. The angle between those two curves is θ. We have
$$(f\circ\gamma_1)'(0)=\frac{\partial f}{\partial z}(z_0)e^{i\alpha}+\frac{\partial f}{\partial \bar z}(z_0)e^{-i\alpha},$$
$$(f\circ\gamma_2)'(0)=\frac{\partial f}{\partial z}(z_0)e^{i(\alpha+\theta)}+\frac{\partial f}{\partial \bar z}(z_0)e^{-i(\alpha+\theta)}.$$
If we had $\frac{\partial f}{\partial z}(z_0)=0$, the angle between the curves $f\circ\gamma_1$ and $f\circ\gamma_2$ at $f(z_0)$ would be $-\theta\neq\theta$ if we choose $\theta\neq\pi$.

Otherwise, choose α such that $\frac{\partial f}{\partial z}(z_0)e^{i\alpha}+\frac{\partial f}{\partial \bar z}(z_0)e^{-i\alpha}\neq 0$; since we assume that f is conformal at z_0, we must have for all θ
$$\frac{\frac{\partial f}{\partial z}(z_0)e^{i(\alpha+\theta)}+\frac{\partial f}{\partial \bar z}(z_0)e^{-i(\alpha+\theta)}}{\frac{\partial f}{\partial z}(z_0)e^{i\alpha}+\frac{\partial f}{\partial \bar z}(z_0)e^{-i\alpha}}=e^{i\theta}.$$
This easily implies $\frac{\partial f}{\partial \bar z}(z_0)=0$. □

Note that things change when $f'(z_0)=0$. This time, the tangent lines to the curves will be given by higher-order derivatives of f, and conformality will be lost (even when the images of curves still admit tangent lines). For instance, $f(z)=z^m$ multiplies all angles by m at the point 0.

1.3. **Examples: Linear Fractional Maps.**

Definition 1.4. The *Riemann Sphere* is the set $S:=\mathbb{C}\cup\{\infty\}$ endowed with the following topology. On $\mathbb{C}$, we take the usual topology, and a neighborhood basis of ∞ is given by the complements of closed discs centered at 0, i.e. $\big(\{\infty\}\cup(\mathbb{C}\setminus\bar D(0,R),R>0\big)$.

Consider the map $\mathcal{I}:z\mapsto\frac{1}{z}$, which can be extended by $\infty\mapsto 0$ and $0\mapsto\infty$. This map is called the *inversion* (with respect to the unit circle). We say that a function f is holomorphic in a neighborhood of ∞ iff f is holomorphic in $\mathbb{C}\setminus\bar D(0,R)$ for some $R>0$, $\lim_{|z|\to\infty}f(z)$ exists, and $f\circ\mathcal{I}$ is holomorphic in a neighborhood of 0.

Exercise 1.5. Prove that the Riemann sphere is homeomorphic to
$$S^2:=\left\{\xi\in\mathbb{R}^3:\|\xi\|=1\right\}\cong\left\{(z,t)\in\mathbb{C}\times\mathbb{R}:|z|^2+t^2=1\right\},$$
where $\|\xi\|^2=\|(x,y,t)\|^2=x^2+y^2+t^2$ is the Euclidean norm, using the map
$$\mathbb{C}\ni z\mapsto\left(\frac{2z}{|z|^2+1},\frac{|z|^2-1}{|z|^2+1}\right).$$
This map is called a stereographic projection (there are different versions of it). Is it conformal? (you have to define a notion of angle on the sphere first).

Definition 1.6. Given $(a, b, c, d) \in \mathbb{C}$ with $(c, d) \neq (0, 0)$, a *linear fractional map* is the map $\varphi : S \longrightarrow S$ defined by

$$\begin{aligned}\varphi(z) &= \frac{az+b}{cz+d} \text{ if } z \in \mathbb{C}, cz+d \neq 0,\\ \varphi(\infty) &= \frac{a}{c} \text{ if } c \neq 0, \quad \varphi(\infty) = \infty \text{ if } c = 0,\\ \varphi(-\frac{d}{c}) &= \infty \text{ if } c \neq 0.\end{aligned}$$

It is easy to check that φ is holomorphic in a neighborhood of any point $z_0 \in S$ such that $\varphi(z_0) \neq \infty$, and that $\mathcal{I} \circ \varphi$ is holomorphic in a neighborhood of a point $z_0 \in S$ such that $\varphi(z_0) = \infty$.

Proposition 1.7. To a matrix $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}$, associate the linear fractional map described in Definition 1.6, denoted by φ_M. Then

(1) $\varphi_M = \varphi_N$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $M = \lambda N$;
(2) $\varphi_M \circ \varphi_N = \varphi_{MN}$;
(3) φ is a bijection of S if and only if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, and in this case $\varphi_M^{-1}(w) = \varphi_{M^{-1}}(w) = \frac{dw-b}{-cw+a}$ when $w \in \mathbb{C} \setminus \{a/c\}$, and φ is a homeomorphism.

The (elementary) proof is left to the reader.

One also checks easily that $\varphi \in \mathcal{O}(\mathbb{C} \setminus \{-d/c\})$ and that

$$\varphi'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0, \quad \forall z \in \mathbb{C} \setminus \{-d/c\},$$

so, except in the degenerate case when it is constant, φ extends to a conformal bijection of S, with the angle at ∞ of two (generalized) curves γ_1, γ_2 tending to ∞ being defined as the angle of $\mathcal{I} \circ \gamma_1, \mathcal{I} \circ \gamma_2$ at 0.

Proposition 1.8. The non-constant linear fractional maps form a group under composition of maps, denoted by $\mathcal{H}(S)$. The map $M \mapsto \varphi_M$ is a group homomorphism $GL(2, \mathbb{C}) \longrightarrow \mathcal{H}(S)$, its kernel is made up of the scalar matrices. The group $\mathcal{H}(S)$ is generated by the rotations $z \mapsto e^{i\theta} z$, $\theta \in \mathbb{R}$, the dilations $z \mapsto \lambda z$, $\lambda > 0$, the translations $z \mapsto z + b$, $b \in \mathbb{C}$, and the inversion $\mathcal{I}$.

The proof is left to the reader. For the last statement, use partial fraction decomposition (which reduces here to one Euclidean division).

Proposition 1.9. The image of a line or circle under a linear fractional map is a line or circle.

The proof is left to the reader. Notice that the image or a line or circle of $\mathbb{C}$ under the stereographic projection is always a circle in the sphere of $\mathbb{R}^3$, and that lines are characterized by the fact that their image circle passes through the "North Pole" $(0, 0, 1)$.

Proposition 1.10. Given any distinct points $a_1, a_2, a_3 \in S$, there exists $\varphi \in \mathcal{H}(S)$ such that $\varphi(a_1) = 0$, $\varphi(a_2) = 1$, $\varphi(a_3) = \infty$.

The proof is left to the reader. Here are some hints: suppose for the moment that $a_1, a_3 \in \mathbb{C}$. We may write $\varphi(z) = c\frac{z-\alpha}{z-\beta}$. It is easy to choose α such that $\varphi(a_1) = 0$, and β such that $\varphi(a_3) = \infty$. Then all that is left is the computation of c. One must discuss the various special cases when one of the a_j is ∞.

The value $\varphi(z)$ is known as the *cross ratio* of (z, a_2, a_1, a_3).

1.4. **Automorphisms of the unit disc.** The domain $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ occurs in many places. The Riemann Mapping Theorem will shows us that it is actually the model for all simply connected domains in $\mathbb{C}$ distinct from the whole plane. It turns out that all the holomorphic bijective self maps of the disc are linear fractional.

Lemma 1.11. If $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$, let

$$\varphi_{a,\theta}(z) := e^{i\theta} \frac{a - z}{1 - \bar{a}z}.$$

Then $\varphi_{a,\theta}(z)(\mathbb{D}) = \mathbb{D}$ and $\varphi_{a,\theta}(z)(\partial\mathbb{D}) = \partial\mathbb{D}$.

Writing $\varphi_a := \varphi_{a,0}$, we have $\varphi_a \circ \varphi_a = id$ (the identity map of the disc).

Proof. When $|z| \leq 1$, $|\bar{a}z| \leq |a| < 1$, so $\varphi_{a,\theta}$ is holomorphic on a neighborhood of $\bar{\mathbb{D}}$.

Suppose first that $z \in \partial\mathbb{D}$, say $z = e^{i\alpha}$. Then

$$\varphi_a(z) = \frac{a - e^{i\alpha}}{1 - \bar{a}e^{i\alpha}} = e^{-i\alpha} \frac{a - e^{i\alpha}}{e^{-i\alpha} - \bar{a}} = -e^{-i\alpha} \frac{a - e^{i\alpha}}{\overline{a - e^{i\alpha}}},$$

so $|\varphi_a(z)| = 1$ and we have $\varphi_a(\partial\mathbb{D}) \subset \partial\mathbb{D}$. By the maximum principle, since φ_a is not constant, $\varphi_a(\mathbb{D}) \subset \mathbb{D}$.

A straightforward computation shows that $\varphi_a(\varphi_a(z)) = z$ for any $z \in \bar{\mathbb{D}}$. It follows that $\varphi_a(z)(\mathbb{D}) = \mathbb{D}$ and $\varphi_a(z)(\partial\mathbb{D}) = \partial\mathbb{D}$.

The general case is obtained by observing that the rotation r_θ of angle θ also preserves the unit disc and circle. We note that $\varphi_{a,\theta}^{-1} = \varphi_a \circ r_{-\theta}$. ☐

Proposition 1.12. Any holomorphic bijection of $\mathbb{D}$ is of the form $\varphi_{a,\theta}$ for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

Proof. Consider first the special case where f is a holomorphic bijection of $\mathbb{D}$ such that $f(0) = 0$. Then, by the Schwarz Lemma (Theorem 0.2), $|f'(0)| \leq 1$. This also applies to the inverse map f^{-1}: $|(f^{-1})'(0)| \leq 1$. But $(f^{-1})'(0) = 1/f'(0)$, so $|f'(0)| = 1$, and by the case of equality in the Schwarz Lemma, f must be a rotation.

In the general case, let $a := f^{-1}(0)$. Then $f_1 := f \circ \varphi_a$ is another holomorphic bijection of $\mathbb{D}$ and verifies $f_1(0) = 0$. By the first part of the proof, $f \circ \varphi_a = r_\theta$ for some $\theta \in \mathbb{R}$. Finally, $f = f \circ \varphi_a \circ \varphi_a = r_\theta \circ \varphi_a$. ☐

1.5. **Other Examples, and applications.**

- For $m \in \mathbb{N}^*$, $\theta \in \mathbb{R}$, $0 < \alpha \leq \frac{2\pi}{m}$, the map $z \mapsto z^m$ is a conformal bijection from the sector $\{z \in \mathbb{C} : \theta < \arg z < \theta + \alpha\}$ to the sector $\{z \in \mathbb{C} : m\theta < \arg z < m\theta + m\alpha\}$. When we write $\theta < \arg z < \theta + \alpha$, we mean that there exists some determination of the argument of z which satisfies that inequality.

  For instance, $z \mapsto z^2$ maps $\{\operatorname{Re} z > 0\}$ to $\mathbb{C} \setminus \mathbb{R}_-$. Its inverse map is often denoted $z^{1/2}$ (but be careful with that notation).

- For $0 < \alpha \le 2\pi$, $\theta \in \mathbb{R}$, $-\infty \le a < b \le \infty$, $z \mapsto e^z$ is a conformal bijection from the rectangle $\{a < \operatorname{Re} z < b, \theta < \operatorname{Im} z < \theta + \alpha\}$ to the region $\{e^a < |z| < e^b, \theta < \arg z < \theta + \alpha\}$.

  For instance, the strip $\{-\pi < \operatorname{Im} z < \pi\}$ is mapped by the exponential map to $\mathbb{C} \setminus \mathbb{R}_-$.
- The map Log is the inverse map of the exponential on the domains above.

Theorem 1.13. *(Casorati-Weierstrass) If w_0 is an essential singularity of $f \in \mathcal{O}(U \setminus \{w_0\}$, where U is a neighborhood of w_0, then for any $r > 0$ such that $D(w_0, r) \subset U$, $f(D(w_0, r) \setminus \{w_0\})$ is dense in $\mathbb{C}$.*

Proof. We only give an outline, to be filled by the reader.

Proceed by contradiction: if $f(D(w_0, r) \setminus \{w_0\})$ is not dense in $\mathbb{C}$, there exist $a \in \mathbb{C}$ and $r_1 > 0$ such that $D(a, r_1) \cap f(D(w_0, r) \setminus \{w_0\}) = \emptyset$.

Find a conformal map ψ from $\mathbb{C} \setminus \bar{D}(a, r_1)$ to $\mathbb{D}$.

Using Riemann's removable singularity theorem, prove that $\psi \circ f$ extends holomorphically to U. Conclude that w_0 is a pole or removable singularity for f. □

Proposition 1.14. Let $f \in \mathcal{O}(\mathbb{C})$ (we say that f is an entire function) such that its values omit a ray emanating from the origin, i.e. there exists $\theta \in \mathbb{R}$ such that for any $z \in \mathbb{C}$, $f(z) \notin e^{i\theta}\mathbb{R}_+$. Then f is constant.

Proof left to the reader. Hint: find a conformal bijection from $\mathbb{C} \setminus e^{i\theta}\mathbb{R}_+$ to $\mathbb{D}$, and apply Liouville's Theorem.

2. Path integrals and antiderivatives

We know what the derivative of a holomorphic function is. Now given $g \in \mathcal{O}(\Omega)$, does there exist $G \in \mathcal{O}(\Omega)$ such that $G' = g$? The answer to this question turns out to be expressed in topological terms, and we will have to take a closer look at Cauchy's Theorem.

2.1. Examples; the convex case.

Let us give two simple but fundamental examples. If $\Omega = \mathbb{D}$, then the power series expansion $g(z) = \sum_n a_n z^n$ is valid for any $z \in \mathbb{D}$, and we can take $G(z) = \sum_n \frac{a_n}{n+1} z^{n+1}$. Any two antiderivatives will differ by an additive constant (because of the familiar fact that a differentiable map on a connected open set, the differential of which vanishes identically, must be constant).

In general however we cannot represent a holomorphic function by a single power series. Think of the function $1/(1-z)$ on $\mathbb{C} \setminus \{1\}$, for instance. And there are cases where an antiderivative simply does not exist.

Proposition 2.1. Let $\Omega = \mathbb{C} \setminus \{0\}$ and $g(z) = 1/z$. There is no function $G \in \mathcal{O}(\Omega)$ such that $G' = g$.

Proof. Suppose such a G exists. Then

$$\frac{d}{dz}\left(\frac{1}{z}e^{G(z)}\right) = \frac{1}{z}\frac{1}{z}e^G(z) - \frac{1}{z^2}e^{G(z)} = 0,$$

so $e^{G(z)} = Cz$ for some constant $C \in \mathbb{C}$. Choosing $C_1 \in \mathbb{C}$ such that $e^{C_1} = C$, then $e^{G(z)-C_1} = z$, and taking $\operatorname{Im}(G(z) - C_1)$ we would have a continuous determination of the argument of z on $\mathbb{C} \setminus \{0\}$, which is known to be impossible. □

There is an easy way to solve our problem when Ω is convex, which will serve as template for the eventual proof.

Notation: whenever $a, b \in \mathbb{C}$, $[a; b]$ will stand both for the line segment between a and b and the oriented path given by $\gamma(t) = (1-t)a + tb$, $0 \le t \le 1$.

Proposition 2.2. Let Ω be a convex domain, $z_0 \in \Omega$, $g \in \mathcal{O}(\Omega)$. Let

$$G(z) := \int_{[z_0;z]} g(\zeta) d\zeta.$$

Then G is the unique antiderivative of g in Ω such that $G(z_0) = 0$.

Proof. Uniqueness is proved as above: if G_1, G_2 are two solutions, $(G_1 - G_2)' = g - g = 0$, so we have a constant function and $G_1(z) - G_2(z) = G_1(z_0) - G_2(z_0) = 0$.

To study the complex differentiability of G at $z \in \Omega$, let $h \in \mathbb{C}$ be such that $z + h \in \Omega$. By convexity, the triangle with vertices $z_0, z, z+h$ is a (P)-domain contained in Ω. So Cauchy's Theorem 0.1 tells us that

$$\int_{[z_0;z]} g(\zeta) d\zeta + \int_{[z;z+h]} g(\zeta) d\zeta - \int_{[z_0;z+h]} g(\zeta) d\zeta = 0.$$

In other words,

$$G(z+h) - G(z) = \int_{[z;z+h]} g(\zeta) d\zeta = hg(z) + \int_{[z;z+h]} (g(\zeta) - g(z))\, d\zeta.$$

Let $\varepsilon > 0$. There exists $\delta > 0$ such that for $|h| \le \delta$, for any $\zeta \in [z; z+h]$, $|g(\zeta) - g(z)| < \varepsilon$, so that the last integral is bounded in modulus by $\varepsilon |h|$. So the error term is an $o(h)$ and we have proved $G'(z) = g(z)$. □

Observe that if we tried to copy this proof in the case of $\mathbb{C} \setminus \{0\}$, taking for instance $z_0 = 1$ and $z = -1$, we could not take a straight path because 0 needs to be avoided, and our path integral would yields different results depending on whether the path goes above or below the origin (even without considering more complicated paths).

2.2. **Path integrals.** Let us suppose that we have a domain where this kind of difficulty doesn't arise. Then things work well.

Recall that when are given paths γ_1, γ_2, we define $\gamma_1 + \gamma_2$ and $-\gamma_1$ by

$$\int_{\gamma_1+\gamma_2} f(\zeta) d\zeta := \int_{\gamma_1} f(\zeta) d\zeta + \int_{\gamma_2} f(\zeta) d\zeta, \quad \int_{-\gamma_1} f(\zeta) d\zeta := -\int_{\gamma_1} f(\zeta) d\zeta.$$

When the origin of γ_2 happens to coincide with the endpoint of γ_1, taking $\gamma_1 + \gamma_2$ is the same as concatenating the paths; taking $-\gamma_1$ is the same as going along the same path in the reverse direction. And of course $\gamma_1 - \gamma_1 = 0$, that is to say, the constant path, over which any integral will vanish.

Theorem 2.3. *Let Ω be a domain such that for any closed path γ with range in Ω, any $f \in \mathcal{O}(\Omega)$, then $\int_\gamma f(\zeta) d\zeta = 0$. Let $z_0 \in \Omega$. Then any $f \in \mathcal{O}(\Omega)$ admits a unique antiderivative $F \in \mathcal{O}(\Omega)$ such that $F(z_0) = 0$.*

Proof. Since Ω is connected, uniqueness is proved as before.

Since Ω is open and connected, for any $z \in \Omega$, there exists a broken line path γ_z such that $\gamma_z(0) = z_0$ and $\gamma_z(1) = z$. Define

$$F(z) := \int_{\gamma_z} f(\zeta) d\zeta.$$

This seems to depend on the choice of the path γ_z, but it doesn't. Indeed, take any other piecewise-$\mathcal{C}^1$ path $\tilde{\gamma}_z$ with the same endpoints. Then $\gamma_z - \tilde{\gamma}_z$ is a closed path in Ω, so

$$\int_{\gamma_z} f(\zeta) d\zeta - \int_{\tilde{\gamma}_z} f(\zeta) d\zeta = 0.$$

To prove that $F'(z) = f(z)$, choose r such that $D(z, r) \subset \Omega$. For $|h| < r$, $\gamma_z + [z; z+h]$ is a path from z_0 to $z + h$, so by the above

$$F(z+h) - F(z) = \int_{\gamma_z + [z;z+h]} f(\zeta) d\zeta - \int_{\gamma_z} f(\zeta) d\zeta = \int_{[z;z+h]} f(\zeta) d\zeta.$$

The same calculation as in the proof of Proposition 2.2 proves that F is complex-differentiable and $F'(z) = f(z)$. □

We observe that the hypothesis in Theorem 2.3 was necessary.

Proposition 2.4. If $f = F'$ where $f, F \in \mathcal{O}(\Omega)$, and γ is a path in Ω, parametrized by $[0, 1]$, then

$$\int_\gamma f(\zeta) d\zeta = F(\gamma(1)) - F(\gamma(0)).$$

Proof. The complex Chain Rule implies that $\frac{d}{dt}(F \circ \gamma(t)) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$, so that applying the Fundamental Theorem of Calculus,

$$\int_0^1 f(\gamma(t))\gamma'(t) dt = F \circ \gamma(1) - F \circ \gamma(0).$$

□

In particular, the integral of a derivative of a holomorphic function on a closed path is always 0.

2.3. **Simply connected domains.** It remains to see which domains have the good property in the hypothesis of Theorem 2.3. They turn out to be the simply connected domains, which we will define forthwith.

Definition 2.5. A domain Ω is said to be *simply connected* if given any two continuous curves γ_0, γ_1 such that $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$, then there exists a *homotopy* with fixed endpoints between those curves, i.e. a continuous map

$$\begin{array}{rll} H: & [0,1] \times [0,1] & \longrightarrow \Omega \\ \text{such that} & H(0,t) = \gamma_0(t), & H(1,t) = \gamma_1(t), 0 \le t \le 1 \\ \text{and} & H(\theta,0) = \gamma_0(0), & H(\theta,1) = \gamma_0(1), 0 \le \theta \le 1. \end{array}$$

When such an H exists between two given curves γ_0, γ_1 (with common endpoints) we say that γ_0 and γ_1 are *homotopic* (with fixed endpoints). Note that H is only assumed to be continuous because we want this notion to be invariant under homeomorphism. So even when γ_0, γ_1 are paths, for $0 < \theta < 1$, the map $H(\theta, \cdot)$ will only be a continuous curve, which could be very irregular.

Theorem 2.6. *If γ_0, γ_1 are homotopic paths in Ω (with fixed endpoints), and $f \in \mathcal{O}(\Omega)$, then*

$$\int_{\gamma_0} f(\zeta)d\zeta = \int_{\gamma_1} f(\zeta)d\zeta.$$

Corollary 2.7. If Ω is a simply connected domain, any $f \in \mathcal{O}(\Omega)$ admits an antiderivative.

Proof of Theorem 2.6. Let H be a homotopy between γ_0 and γ_1, as in Definition 2.5. Since $[0,1]\times[0,1]$ is compact, its image is too, and $\delta := \text{dist } (H([0,1]\times[0,1]), S \setminus \Omega) > 0$. By Heine's Theorem, H is uniformly continuous, so there exists $N \in \mathbb{N}^*$ such that

$$\max(|\theta - \theta'|, |t - t'|) \leq \frac{1}{N} \Rightarrow |H(\theta, t) - H(\theta', t')| < \delta.$$

Define $\hat{\gamma}_\theta$ to be the broken line path with vertices $H(\theta, \frac{j}{N})$, $0 \leq j \leq N$, parametrized by $0 \leq t \leq 1$, so that $\hat{\gamma}_\theta(\frac{j}{N}) = H(\theta, \frac{j}{N})$. In particular, if $\theta = 0$ or 1, then $\hat{\gamma}_\theta(\frac{j}{N}) = \gamma_\theta(\frac{j}{N})$, $0 \leq j \leq N$. So we have created, for each of our paths, an approximation by a broken line; and $N-1$ intermediate broken line paths for $\theta = \frac{k}{N}$, $1 \leq k \leq N-1$.

Claim. For $\theta = 0$ or 1,

$$\int_{\hat{\gamma}_\theta} f(\zeta)d\zeta = \int_{\gamma_\theta} f(\zeta)d\zeta.$$

Proof. It will be enough to show that for $0 \leq j \leq N-1$,

$$\int_{\hat{\gamma}_\theta|_{[\frac{j}{N},\frac{j+1}{N}]}} f(\zeta)d\zeta = \int_{\gamma_\theta|_{[\frac{j}{N},\frac{j+1}{N}]}} f(\zeta)d\zeta.$$

The closed path $\hat{\gamma}_\theta|_{[\frac{j}{N},\frac{j+1}{N}]} - \gamma_\theta|_{[\frac{j}{N},\frac{j+1}{N}]}$ lies inside the disk $D(\gamma_\theta(\frac{j}{N}), \delta)$ which is included in Ω by our choice of δ. In a disc, by our argument on power series or by Theorem 2.2, f admits an antiderivative, so its integral on closed paths must vanish. □

Now it will be enough to prove that

$$\int_{\hat{\gamma}_0} f(\zeta)d\zeta = \int_{\hat{\gamma}_1} f(\zeta)d\zeta,$$

and by an immediate induction, this reduces to proving, for $0 \leq k \leq N-1$,

$$\int_{\hat{\gamma}_{\frac{k}{N}}} f(\zeta)d\zeta = \int_{\hat{\gamma}_{\frac{k+1}{N}}} f(\zeta)d\zeta. \tag{1}$$

In order to do this, we will prove by induction on j, for $0 \leq j \leq N-1$, that

$$\int_{\hat{\gamma}_{\frac{k}{N}}|_{[0,\frac{j}{N}]}} f(\zeta)d\zeta + \int_{\left[\hat{\gamma}_{\frac{k}{N}}(\frac{j}{N}), \hat{\gamma}_{\frac{k+1}{N}}(\frac{j}{N})\right]} f(\zeta)d\zeta = \int_{\hat{\gamma}_{\frac{k+1}{N}}|_{[0,\frac{j}{N}]}} f(\zeta)d\zeta. \tag{2}$$

Suppose the property is true at rank j. To simplify notations, let

$$A := \hat\gamma_{\frac{k}{N}}(\frac{j}{N}), \quad B := \hat\gamma_{\frac{k}{N}}(\frac{j+1}{N}), \quad C := \hat\gamma_{\frac{k+1}{N}}(\frac{j+1}{N}), \quad D := \hat\gamma_{\frac{k+1}{N}}(\frac{j}{N}).$$

Then

$$\begin{aligned}\int_{\hat\gamma_{\frac{k+1}{N}}|_{[0,\frac{j+1}{N}]}} f(\zeta)d\zeta &= \int_{\hat\gamma_{\frac{k+1}{N}}|_{[0,\frac{j}{N}]}} f(\zeta)d\zeta + \int_{[D,C]} f(\zeta)d\zeta \\ &= \int_{\hat\gamma_{\frac{k}{N}}|_{[0,\frac{j}{N}]}} f(\zeta)d\zeta + \int_{[A,D]} f(\zeta)d\zeta + \int_{[D,C]} f(\zeta)d\zeta,\end{aligned}$$

Ω is a simply connected domain by the induction hypothesis (2). The tetragon (A,B,C,D) and its inside are contained in the disc $D(A,\delta) \subset \Omega$, so once again using the fact that any holomorphic function admits an antiderivative on a disc,

$$\int_{[A,D]} f(\zeta)d\zeta + \int_{[D,C]} f(\zeta)d\zeta = \int_{[A,B]} f(\zeta)d\zeta + \int_{[B,C]} f(\zeta)d\zeta.$$

Finally

$$\begin{aligned}\int_{\hat\gamma_{\frac{k+1}{N}}|_{[0,\frac{j+1}{N}]}} f(\zeta)d\zeta &= \int_{\hat\gamma_{\frac{k}{N}}|_{[0,\frac{j}{N}]}} f(\zeta)d\zeta + \int_{[A,B]} f(\zeta)d\zeta + \int_{[B,C]} f(\zeta)d\zeta \\ &= \int_{\hat\gamma_{\frac{k}{N}}|_{[0,\frac{j+1}{N}]}} f(\zeta)d\zeta + \int_{\left[\hat\gamma_{\frac{k}{N}}(\frac{j+1}{N}),\hat\gamma_{\frac{k+1}{N}}(\frac{j+1}{N})\right]} f(\zeta)d\zeta.\end{aligned}$$

The property (2) is proved, so we apply it with $j = N$. Then the fact that the homotopy fixes the endpoints implies that $\hat\gamma_{\frac{k}{N}}(1) = \hat\gamma_{\frac{k+1}{N}}(1)$, so the integral term on a line segment disappears and we have proved that when we integrate over the whole interval $[0,1]$, (1) holds. □

Corollary 2.8. Let Ω be a simply connected domain. Let $f \in \mathcal{O}(\Omega)$ satisfy $f(z) \neq 0$, for any $z \in \Omega$. Then there exist $g \in \mathcal{O}(\Omega)$ and $h_m \in \mathcal{O}(\Omega)$ such that $e^{g(z)} = f(z)$ and $h_m^m(z) = f(z)$, for $z \in \Omega$, $m \in \mathbb{N}^*$.

Proof. Since f does not vanish, then $\frac{f'}{f} \in \mathcal{O}(\Omega)$. Choose $z_0 \in \Omega$ and $\alpha \in \mathbb{C}$ such that $e^\alpha = f(z_0)$ (this is possible since $f(z_0) \neq 0$). Let g be the unique antiderivative of $\frac{f'}{f}$ such that $g(z_0) = \alpha$. Then $e^{g(z_0)} = f(z_0)$, and

$$\left(\frac{e^g}{f}\right)' = \frac{(g'f - f')e^g}{f^2} = 0,$$

so $\frac{e^g}{f}$ is constant, and this constant must be $1 = \frac{e^{g(z_0)}}{f(z_0)}$.

Then it is enough to take $h_m(z) := e^{\frac{1}{m}g(z)}$. □

2.4. **Another look at Cauchy's Theorem.**

You will recall that in Jasmin Raissy's course, Cauchy's theorem on the boundary of a (P)-domain was proved under the assumption that the function f was of class $\mathcal{C}^1$. This is stronger than just assuming that the complex derivative exists. In order to close the gap in the (long) proof that any complex-differentiable function is indeed analytic (i.e. locally expandable as a power series), it is enough to prove Cauchy's theorem on a polygonal contour when the whole inside of it is contained in the domain where the function f is holomorphic, and this can be reduced to the case of a triangle (by decomposing into triangles).

Theorem 2.9. *Let f be complex-differentiable on a domain Ω which contains the closed triangle T (boundary and inside). Then*

$$\int_{\partial T} f(\zeta) d\zeta = 0.$$

Here ∂T must be understood as the sum of the three line segments making up the boundary of T, oriented so that it is run through in the trigonometric direction.

Proof. Given any triangle Δ with vertices A, B, C, let A', B', C' be the midpoints of $[B, C]$, $[C, A]$, $[A, B]$ respectively. Then we have four subtriangles $\Delta^{(1)}$ with vertices (A, C', B'), $\Delta^{(2)}$ with vertices (B, A', C'), $\Delta^{(3)}$ with vertices (C, B', A'), $\Delta^{(4)}$ with vertices (A', B', C'), each of which has a perimeter half of that of Δ and a diameter half of that of Δ.

Note, too, that for any integrable function defined on the whole of Δ, because of cancellations along the line segments $[A', B']$, $[B', C']$, $[C', A']$,

$$\int_{\partial \Delta} f(\zeta) d\zeta = \sum_{i=1}^{4} \int_{\partial \Delta^{(i)}} f(\zeta) d\zeta.$$

We define a sequence of triangles T_n in the following way: $T_0 := T$; given T_n, applying the triangle inequality to the relation above with $\Delta = T_n$, there exists $i_0 \in \{1, 2, 3, 4\}$ such that

$$\left| \int_{\partial T_n^{(i_0)}} f(\zeta) d\zeta \right| \geq \frac{1}{4} \left| \int_{\partial T} f(\zeta) d\zeta \right|.$$

We choose $T_{n+1} := T_n^{(i_0)}$. An immediate induction shows that

$$\left| \int_{\partial T_n^{(i_0)}} f(\zeta) d\zeta \right| \geq 4^{-n} \left| \int_{\partial T} f(\zeta) d\zeta \right|.$$

Also, the perimeter of T_n, i.e. the length of ∂T_n, verifies $\ell(\partial T_n) = 2^{-n} \ell(\partial T)$, and $\operatorname{diam}(T_n) = 2^{-n} \operatorname{diam}(T)$.

Since T is compact and $\operatorname{diam} T_n \to 0$, $\bigcap_n T_n = \{z_0\}$ for some $z_0 \in T$. The function f is complex differentiable at z_0, so

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + r(z), \text{ with } r(z) = o(|z - z_0|).$$

Since $f(z_0) + f'(z_0)(z - z_0)$ is a polynomial, with an explicit antiderivative,

$$\int_{\partial T_n} f(\zeta) d\zeta = \int_{\partial T_n} r(\zeta) d\zeta.$$

Let $\varepsilon > 0$. There exists n_0 such that for $n \geq n_0$, $\zeta \in T_n$, $|r(\zeta)| \leq \varepsilon |\zeta - z_0| \leq \varepsilon \operatorname{diam} T_n \leq 2^{-n} \operatorname{diam}(T) \varepsilon$. Finally

$$\left| \int_{\partial T} f(\zeta) d\zeta \right| \leq 4^n \left| \int_{\partial T_n} r(\zeta) d\zeta \right| \leq 4^n \ell(\partial T_n) 2^{-n} \operatorname{diam}(T) \varepsilon \leq 4^n 2^{-n} \ell(\partial T) 2^{-n} \operatorname{diam}(T) \varepsilon \leq C \varepsilon.$$

Since this is true for any $\varepsilon > 0$, $\int_{\partial T} f(\zeta) d\zeta = 0$. ∎

3. Sequences of holomorphic functions

3.1. **Weierstrass' Theorem.** We know that a uniform limit of differentiable functions (or even real-analytic functions) has no reason to be differentiable. Take for instance $f_n(x) := \sqrt{x^2 + \frac{1}{n^2}}$ on the real line.

The situation is strikingly different for holomorphic functions. First recall the notion of convergence that we will be using.

Definition 3.1. We say that a sequence $(f_n)_n \subset \mathcal{O}(\Omega)$ converges *uniformly on compact sets* if for any $K \Subset \Omega$, $(f_n)_n$ converges uniformly on K.

We sometimes talk about "compact convergence".

Observe that the sequence $(z^n)_n$ converges to 0 uniformly on compact sets of $\mathbb{D}$, but not uniformly on $\mathbb{D}$. What are the sequence of entire functions $(f_n)_n$ which converge uniformly on $\mathbb{C}$? (hint: there are very few). Those two examples should convince you that uniform convergence on the whole of the domain Ω is not the correct notion for convergence of sequences of holomorphic functions.

Note that we can restrict attention to a countable sequence of compact sets.

Proposition 3.2. For any domain $\Omega \subset \mathbb{C}$, there exists a countable *exhaustion* by compact sets, i.e. a sequence of compact sets $(K_n)_{n\geq 1}$ with

- $K_n \subset \Omega$;
- $K_n \subset K_{n+1}^\circ$ (the interior of K_{n+1});
- $\bigcup_n K_n = \Omega$;
- for any compact subset $K \subset \Omega$, there exists n such that $K \subset K_n$.

Proof. We can choose

$$K_n := \left\{ z \in \Omega : \operatorname{dist}(z, \mathbb{C} \setminus \Omega) \geq \frac{1}{n}, |z| \leq n \right\}.$$

The properties are easily checked (exercise). The last one can be obtained by considering the covering of K by the open sets K_n°, or by using the fact that $\operatorname{dist}(K, \mathbb{C}\setminus\Omega) > 0$. □

Theorem 3.3. *(Weierstrass)*

If $(f_n)_n \subset \mathcal{O}(\Omega)$ converges on compact sets to f, then $f \in \mathcal{O}(\Omega)$.

Proof. Let $a \in \Omega$, and $r > 0$ such that $\overline{D}(a,r) \subset \Omega$. By Cauchy's formula, for any $z \in D(a,r)$, any $n \in \mathbb{N}$,

$$f_n(z) - \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f_n(\zeta)}{\zeta - z} d\zeta = 0.$$

Since $\partial D(a,r)$ is compact in Ω, and $|\zeta - z| \geq r - |z_a| > 0$ for any $\zeta \in \partial D(a,r)$, we can pass to the limit under the integral sign, and we get

$$f(z) - \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Since

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{(\zeta - a)} \sum_{n\geq 0} \left(\frac{(z-a)}{(\zeta - a)} \right)^n,$$

we easily prove that f is expandable in power series on the open disc $D(a, r)$, and therefore holomorphic on it. □

3.2. **Normal families.** First we need to recall a general fact about uniform convergence. We need a definition.

Definition 3.4. Let $(X, d_X), (Y, d_Y)$ be two metric spaces. Let $\mathcal{F} \subset \mathcal{C}(X, Y)$ be a family of continuous map from X to Y. We say that $\mathcal{F}$ is *equicontinuous* if for any $x \in X$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $f \in \mathcal{F}$, $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$.

Comment: uniform continuity is when, for a given ε, the δ doesn't depend on the point x where continuity is tested. Here δ may depend on x, but it must be independent of $f \in \mathcal{F}$. In practice, when X is a compact set, δ won't depend on x either.

Of course any subfamily of an equicontinuous family is also equicontinuous.

You will see a proof of the following theorem in Pascale Roesch's topology class.

Theorem 3.5. *(Ascoli)*

If X is a compact metric space, Y a metric space, if $\mathcal{F} \subset \mathcal{C}(X, Y)$ is an equicontinuous family such that for any $x \in X$, the set $\{f(x) : f \in \mathcal{F}\}$ is relatively compact in Y, then for any sequence $(f_n)_n \subset \mathcal{F}$, there exists a subsequence $(f_{n_k})_k$ which is uniformly convergent on X.

Quite often, the family $\mathcal{F}$ which we consider is itself a sequence of functions.

We also recall a special case of Cauchy's inequalities, which you saw in Jasmin Raissy's second year complex analysis class.

Proposition 3.6. Let $f \in \mathcal{O}(\Omega)$, and $a \in \Omega$, $r > 0$ such that $\overline{D}(a, r) \subset \Omega$. Then

$$|f'(a)| \leq \frac{1}{r} \max_{\overline{D}(a,r)} |f|.$$

Proof. Derivating the Cauchy formula under the integral sign, we have

$$f'(a) = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(\zeta)}{(\zeta - a)^2} d\zeta,$$

and we bound the modulus of the integral from above in the usual way, by the product of the length of the path by the maximum of the modulus of the integrand. □

Theorem 3.7. *(Montel)*

Let $\mathcal{F} \subset \mathcal{O}(\Omega)$ verify

$$\forall K \Subset \Omega, \sup_{\mathcal{F}} \max_{z \in K} |f(z)| =: c_K < \infty.$$

Then for any sequence $(f_n)_n \subset \mathcal{F}$, there exists a subsequence $(f_{n_k})_k$ which is uniformly convergent on compact sets of Ω.

This conclusion is usually expressed by saying that $\mathcal{F}$ is a normal family.

Proof. Let $(f_n)_n \subset \mathcal{F}$. We first show that given any compact subset $K \subset \Omega$, we can find a subsequence $(f_{n_k})_k$ which is uniformly convergent on K. A priori, it depends on K.

We verify the hypotheses of Ascoli's Theorem (3.5) in this case. For any $a \in K$, $\{f(a) : f \in \mathcal{F}\} \subset D(0, c_{\{a\}})$, so that set of values is relatively compact in $\mathbb{C}$.

Now we prove equicontinuity. Let $\delta := \operatorname{dist}(K, \mathbb{C} \setminus \Omega) > 0$ (if $\Omega = \mathbb{C}$, we take $\delta = 1$). Let

$$K_1 := \bigcup_{z \in K} \overline{D}(z, \frac{\delta}{2}) = \left\{ z \in \mathbb{C} : \operatorname{dist}(z, K) \leq \frac{\delta}{2} \right\}.$$

Then K_1 is closed as the pre-image of a closed interval under the continuous function $\operatorname{dist}(\cdot, K)$, and bounded since $\operatorname{diam} K_1 \leq \delta + \operatorname{diam} K$, so it is compact. Let

$$K_2 := \bigcup_{z \in K} \overline{D}(z, \frac{\delta}{4}) = \left\{ z \in \mathbb{C} : \operatorname{dist}(z, K) \leq \frac{\delta}{4} \right\}.$$

For $z \in K_2$, $\overline{D}(z, \frac{\delta}{4}) \subset K_1$.

By Cauchy's inequality (Proposition 3.6), for $f \in \mathcal{F}$, $z \in K_2$,

$$|f'(z)| \leq \frac{4}{\delta} \max_{z \in \overline{D}(z, \frac{\delta}{4})} |f(z)| \leq \frac{4}{\delta} \max_{z \in K_1} |f(z)| \leq \frac{4c_{K_1}}{\delta}.$$

On the other hand, if $z_1, z_2 \in K$ and $|z_1 - z_2| \leq \frac{\delta}{4}$, then $[z_1, z_2] \subset \overline{D}(z_1, \frac{\delta}{4}) \subset K_2$, so for $f \in \mathcal{F}$,

$$|f(z_1) - f(z_2)| \leq |z_1 - z_2| \max_{[z_1, z_2]} |f'| \leq \frac{4c_{K_1}}{\delta} |z_1 - z_2|.$$

Finally, for any $\varepsilon > 0$, if $|z_1 - z_2| \leq \min\left(\frac{\delta}{4}, \frac{\delta}{4c_{K_1}} \varepsilon\right)$, then $|f(z_1) - f(z_2)| \leq \varepsilon$, so equicontinuity is proved. By Ascoli's Theorem, there exists a subsequence $(f_{n_k})_k$ which is uniformly convergent on K.

Now we must use this fact to obtain one subsequence that is unformly convergent on *all* compact subsets of Ω. Consider the exhaustion $(K_n)_n$ of Ω by compact sets given by Proposition 3.2. We proceed by "diagonal extraction" (as in the proof of Ascoli's Theorem). In what follows, φ_j always denotes a strictly increasing map from $\mathbb{N}$ to $\mathbb{N}$.

By the previous argument, there is a subsequence $(f_{\varphi_1(n)})_n$ which is uniformly convergent on K_1. Suppose now that we have found a subsequence $(f_{\varphi_m(n)})_n$ which is uniformly convergent on K_m. By the previous argument, there is a subsequence of it, $(f_{\psi \circ \varphi_m(n)})_n$ which is uniformly convergent on K_{m+1}. We write $\varphi_{m+1} := \psi \circ \varphi_m$, and proceed by induction.

Finally we set $n_k := \varphi_k(k)$. If we omit the finite set $\{0, \ldots, m-1\}$ and consider the terms with $k \geq m$, we see that the sequence $(f_{n_k})_k$ is a subsequence of $(f_{\varphi_m(n)})_n$, so it must be uniformly convergent on K_m. This is true for any m. If we take an arbitrary compact set $K \subset \Omega$, then there exists m such that $K \subset K_m$, so $(f_{n_k})_k$ is uniformly convergent on K. □

Exercise 3.8. Let $\mathcal{F} \subset \mathcal{O}(\Omega)$ verify $\bigcup_{f \in \mathcal{F}} f(\Omega)$ is not dense in $\mathbb{C}$. Then $\mathcal{F}$ is a normal family in the following extended sense: for any sequence $(f_n)_n \subset \mathcal{F}$, there exists a subsequence $(f_{n_k})_k$ which is uniformly convergent on compact sets of Ω, or a subsequence $(f_{n_k})_k$ which is uniformly convergent to ∞ — i.e. $(1/f_{n_k})_k$ is uniformly convergent to 0 — on compact sets of Ω.

Hint: this begins as the proof of the Theorem 1.13 (Casorati-Weierstrass).

3.3. **Rouché's and Hurwitz's Theorems.** We recall a simple form of a result known as the *argument principle.* Recall that the *multiplicity* of a zero of f at $a \in \Omega$ is

$$m_a(f) := \min \left\{ n : f^{(n)}(a) = 0 \right\}.$$

If $m_a(f) = 0$, f doesn't have a zero at a, if $m_a(f) = 1$, f has a simple zero, etc.

Proposition 3.9. Suppose that $f \in \mathcal{O}(\Omega)$ and $\overline{D}(a,r) \subset \Omega$. Assume that $f(z) \neq 0$ for all $z \in \partial D(a,r)$. Then

$$\frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \sum_{z \in D(a,r)} m_a(f).$$

The sum on the right hand side only has a finite number of nonzero terms, and is usually described as "the number of zeros of f counted with multiplicities" inside $D(a,r)$. The equality is proved by applying the Residue Theorem (exercise).

Theorem 3.10. *(Rouché)*
Let $f, g \in \mathcal{O}(\Omega)$ and $\overline{D}(a,r) \subset \Omega$. Suppose that $|g(z)| < |f(z)|$ for all $z \in \partial D(a,r)$. Then f and $f+g$ have the same number of zeros (counted with multiplicities) in $D(a,r)$.

Proof. For $0 \leq t \leq 1$, $z \in \partial D(a,r)$, then $|f(z) + tg(z)| \geq |f(z)| - |g(z)| > 0$. So the function

$$N(t) := \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f'(\zeta) + tg'(\zeta)}{f(\zeta) + tg(\zeta)} d\zeta$$

is well defined and continuous on $[0,1]$. But it's integer-valued, so it must be constant; $N(0)$ is the number of zeros of f in $D(a,r)$, and $N(1)$ is the number of zeros of $f+g$ in $D(a,r)$. □

Theorem 3.11. *(Hurwitz)*
Let Ω be a domain. Let $(f_n)_n \subset \mathcal{O}(\Omega)$ a sequence which converges to f uniformly on compact sets. Suppose that $f_n(z) \neq 0$ for all $z \in \Omega$. Then either $f \equiv 0$ or $f(z) \neq 0$ for all $z \in \Omega$.

Proof. Suppose that f does not vanish identically, but that $f(a) = 0$. Then, by the Theorem of Isolated Zeros, there exists $r > 0$ such that $f(z) \neq 0$ on $\partial D(a,r)$. By compactness, $|f(z)| \geq \delta > 0$ on $\partial D(a,r)$. For n large enough, by uniform convergence on the compact set $\partial D(a,r)$, $|f_n(z) - f(z)| < \delta$ on $\partial D(a,r)$. We then apply Rouché's Theorem with $g = f_n - f$ to conclude that $f_n = f + g$ must have at least one zero inside $D(a,r)$: contradiction. □

Corollary 3.12. Let Ω be a domain. Let $(f_n)_n \subset \mathcal{O}(\Omega)$ a sequence which converges to f uniformly on compact sets. Suppose that f_n is a one-to-one map on Ω for each n. Then either f is constant or it is a one-to-one map on Ω.

Proof. Let $a \in \Omega$. Then $\Omega \setminus \{a\}$ is also a connected open set. Apply Hurwitz's Theorem to the sequence $f_n - f_n(a)$ on $\Omega \setminus \{a\}$. Either the limit vanishes identically, thus f is constant, or it has no zero, so f doesn't take the value $f(a)$, except at a. □

4. Riemann's Mapping Theorem

4.1. The Mapping Theorem.

Theorem 4.1. *(Riemann)*
Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and $z_0 \in \Omega$, then there exists a unique conformal bijection from Ω to $\mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Proof. Uniqueness is easy to prove, using what we know about holomorphic bijections of the disc. Let f_1, f_2 be two maps as in the theorem. Then $\varphi := f_1 \circ f_2^{-1}$ is a holomorphic bijection of the disc such that $\varphi(0) = 0$, so $\varphi(z) = e^{i\alpha} z$ for some $\alpha \in \mathbb{R}$. But $\varphi'(0) = f_1'(z_0)/f_2'(f_2^{-1}(0)) = f_1'(z_0)/f_2'(z_0) > 0$, so $\varphi(z) = z$.

Notice that the argument above relies on the Schwarz Lemma. The proof of existence we are going to give (due to Koebe) also exploits the idea of the Schwarz Lemma which can be partly restated in the following way: if $f : \mathbb{D} \longrightarrow \mathbb{D}$ is holomorphic with $f(0) = 0$, then $|f'(0)| \leq 1$, and if $|f'(0)|$ turns out to be maximal, then f is actually a bijection of the disc.

We define a set of maps over which we will want to maximize a derivative.

$$\mathcal{S}(\Omega, \mathbb{D}) := \{f \in \mathcal{O}(\Omega, \mathbb{D}) : f \text{ is one-to-one on } \Omega, f(z_0) = 0, f'(z_0) > 0\}.$$

Notice that if $\Omega = \mathbb{C}$ this set is empty by Liouville's Theorem (all functions in $\mathcal{O}(\Omega, \mathbb{D})$ are bounded, thus constants, thus not one-to-one).

The proof is organized in three steps:

(1) Prove that $\mathcal{S}(\Omega, \mathbb{D}) \neq \emptyset$.
(2) Prove that the upper bound of $f'(z_0)$ over $\mathcal{S}(\Omega, \mathbb{D})$ exists and is attained by a function of the class.
(3) Prove that that function is onto $\mathbb{D}$.

Step 1. Let $a \in \mathbb{C} \setminus \Omega$. Because Ω is simply connected, and $z - a \neq 0$ for $z \in \Omega$, there exists $h \in \mathcal{O}(\Omega)$ such that $h(z)^2 = z - a$.

If $z_1, z_2 \in \Omega$ verify $h(z_1) = h(z_2)$ or $h(z_1) = -h(z_2)$, then $h(z_1)^2 = h(z_2)^2$, so $z_1 - a = z_2 - a$, so $z_1 = z_2$. In particular h is one-to-one. By the open mapping theorem for holomorphic functions (or even the easier version for holomorphic functions with nonvanishing derivatives, which are local diffeomorphisms), there exists $\rho > 0$ such that $D(h(z_0), \rho) \subset h(\Omega)$. In particular $\rho < |h(z_0)|$ since $0 \notin h(\Omega)$.

If we take $w \in D(-h(z_0), \rho)$, then $-w \in D(h(z_0), \rho)$, so letting $-w = h(z_1)$, if we had $w = h(z_2)$, it would follow that $z_1 = z_2$, so $w = -w$, which is excluded. So $h(\Omega) \cap D(-h(z_0), \rho) = \emptyset$. Therefore

$$f(z) := \frac{\rho}{h(z) + h(z_0)} \in \mathcal{S}(\Omega, \mathbb{D}).$$

Let $a := f(z_0) = \frac{\rho}{2h(z_0)}$, then $\varphi_a \circ f \in \mathcal{O}(\Omega, \mathbb{D})$, is a one-to-one map by composition, and $\varphi_a \circ f(z_0) = 0$, where as usual $\varphi_a(z) := \frac{a-z}{1-\bar{a}z}$. We can choose $\alpha \in \mathbb{R}$ such that $e^{i\alpha}(\varphi_a \circ f)'(z_0) > 0$, then $f_0 := e^{i\alpha}\varphi_a \circ f \in \mathcal{S}(\Omega, \mathbb{D})$.

Step 2. Let $r := \mathrm{dist}(z_0, \mathbb{C} \setminus \Omega$, then by Cauchy's inequalities $|f'(z_0)| \leq 1/r$ for any $f \in \mathcal{S}(\Omega, \mathbb{D})$. So the set of values of $f'(z_0)$ when $f \in \mathcal{S}(\Omega, \mathbb{D})$, being bounded above, admits an upper bound $s \in \mathbb{R}_+^*$, so there exists a sequence $(f_n)_n \subset \mathcal{S}(\Omega, \mathbb{D})$ such that $f_n'(z_0) \to s$.

Since it is a bounded sequence, Montel's Theorem applies. Let $f := \lim_n f_n$. Then $f(z_0) = \lim_n f_n(z_0) = 0$. Because $f'(z_0) = s > 0$, f is not constant, so $f(\Omega) \subset \mathbb{D}$ (and not $\overline{\mathbb{D}}$). By Hurwitz's Theorem, f must be one-to-one. So $f \in \mathcal{S}(\Omega, \mathbb{D})$.

Step 3. We proceed by contradiction. Suppose that $w_0 \in \mathbb{D} \setminus f(\Omega)$. Because Ω is simply connected, $\varphi_{w_0} \circ f$ admits a holomorphic square root in Ω, with values in $\mathbb{D}$. Let $F \in \mathcal{O}(\Omega)$ verify $F^2 = \varphi_{w_0} \circ f$, write $\sqrt{w_0} := F(z_0)$.

Let $G := e^{i\beta}\varphi_{\sqrt{w_0}} \circ F$, where β is chosen so that $G'(z_0) > 0$. As before, we can check that F is one-to-one, and so is G, $G(z_0) = e^{i\beta}\varphi_{\sqrt{w_0}}(F(z_0)) = 0$, so $G \in \mathcal{S}(\Omega, \mathbb{D})$. Reversing the compositions of maps, and recalling that all φ_a are involutions,

$$f = \varphi_{w_0} \circ F^2 = \varphi_{w_0} \circ \sigma \circ \varphi_{\sqrt{w_0}} \circ r_{-\beta} \circ G =: \psi \circ G,$$

where σ denoes the map $w \mapsto w^2$, and $r_{-\beta}$ the rotation of angle $-\beta$, in the unit disc. The map ψ verifies $\psi(\mathbb{D}) \subset \mathbb{D}$ and $\psi(0) = \psi(G(z_0)) = f(z_0) = 0$, so by the Schwarz Lemma either $|\psi'(0)| < 1$ or ψ is a rotation. But σ is a two-to-one map outside of 0 and all the other maps involved in the composition product that yields ψ are bijections, so ψ cannot be a bijection, therefore

$$|f'(z_0)| = |\psi'(0)G'(z_0)| < |G'(z_0)|,$$

which contradict the maximality of $f'(z_0)$: there cannot exist a w_0 as assumed. □

4.2. **Topological consequences.** In the case of a domain in the plane, being simply connected is equivalent to a simpler topological property, i.e. having a connected complement in the sphere. Let us show why this property can be interesting from the point of view of complex analysis with a simple fact.

Proposition 4.2. Suppose $\Omega \subset \mathbb{C}$ is a domain such that $S \setminus \Omega$ is connected. Then for any $a \in \mathbb{C} \setminus \Omega$, the index with respect to a of any closed path γ in Ω vanishes, i.e.

$$\mathrm{Ind}_a(\gamma) := \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - a} = 0.$$

Proof. The function $a \mapsto \mathrm{Ind}_a(\gamma)$ is continuous on $\mathbb{C} \setminus \Omega$ and integer-valued, so that it is constant on connected components. Letting a tend to ∞, we see that $\mathrm{Ind}_a(\gamma) \to 0$, so this function can be extended by continuity to $S \setminus \Omega$. This is connected, so the function must be constant on it, and the constant must be the value at ∞, i.e. 0. □

One must be careful in the above proof that the number of connected components of $\mathbb{C} \setminus \Omega$ may differ from that of $S \setminus \Omega$. Take for instance the strip $\Omega = \{-\pi < \mathrm{Im}\, z < \pi\}$, then $S \setminus \Omega$ is connected, but $\mathbb{C} \setminus \Omega$ is made up of two closed half-planes joined only at ∞.

About the result : notice that $f_a(\zeta) := \frac{1}{\zeta - a}$ defines a function of $\mathcal{O}(\Omega)$. If we could replace it by any $f \in \mathcal{O}(\Omega)$, we would have the property we need. In some sense, all such functions can indeed be represented by limits of combinations of the elementary functions f_a, as we will see below. But first we prove the converse.

Proposition 4.3. Suppose $\Omega \subset \mathbb{C}$ is a domain such that $S \setminus \Omega$ has more than one connected component. Then there exist $a \in S \setminus \Omega$ and a closed path γ in Ω such that

$$\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - a} \neq 0.$$

Corollary 4.4. If $\Omega \subset \mathbb{C}$ is simply connected, then $S \setminus \Omega$ is connected.

Proof. Indeed, if the conclusion was false, then Proposition 4.3 shows that the function $f(\zeta) = \frac{1}{\zeta - a}$ is holomorphic in Ω and admits no antiderivative, since its integral over a closed path does not vanish. □

Proof of Proposition 4.3.

Since $S \setminus \Omega$ is not connected and closed, it can be written as the disjoint union of two nonempty closed sets, say A and B. Without loss of generality, assume that $\infty \in B$, so that there is a neighborhood U of ∞ such that $A \cap U = \emptyset$. This means that there exists $R > 0$ such that $A \subset \bar{D}(0, R)$, therefore A is actually compact in $\mathbb{C}$.

Choose $\delta > 0$ such that $\delta\sqrt{2} < \text{dist}(A, B)$. For any $x_0, y_0 \in \mathbb{R}$, $j, k \in \mathbb{Z}$, we define

$$\Delta_{j,k} := \{x + iy : x_0 + j\delta \le x \le x_0 + (j+1)\delta, y_0 + k\delta \le y \le y_0 + (k+1)\delta\}. \tag{3}$$

Let $a \in A$. We choose x_0, y_0 such that there exists a (j_0, k_0) such that $a \in \Delta^\circ_{j_0,k_0}$. This square is unique since the open squares are disjoint.

For each (j, k), $\partial\Delta_{j,k}$ is the path made up of the four sides of the square, oriented in the trigonometric direction. Clearly,

$$\int_{\partial\Delta_{j_0,k_0}} \frac{d\zeta}{\zeta - a} = 2\pi i, \quad \int_{\partial\Delta_{j,k}} \frac{d\zeta}{\zeta - a} = 0 \text{ when } (j,k) \neq (j_0, k_0).$$

Consider the cycle

$$\Gamma := \sum_{(j,k):\Delta_{j,k}\cap A \neq \emptyset} \partial\Delta_{j,k}.$$

Then $\int_\Gamma \frac{d\zeta}{\zeta - a} = 2\pi i$.

We claim that Γ is equivalent as a cycle (i.e. in its effect when you take a path integral along it) to a union of closed paths $\gamma \subset \Omega$. By the choice of δ, whenever $\Delta_{j,k} \cap A \neq \emptyset$, then $\Delta_{j,k} \cap B = \emptyset$, so the support of Γ does not meet B, where we call *support* of a cycle the union of all the images of the paths that make it up.

On the other hand, consider Γ as a sum of oriented segments (the sides of the squares $\Delta_{j,k}$). Let $\mathcal{A}$ be the family $\{\Delta_{j,k} : \Delta_{j,k} \cap A \neq \emptyset\}$.

Whenever a side $[a, b]$ meets A, then each of the two squares are in $\mathcal{A}$, so they both enter the sum that defines Γ. But the orientations given by the two squares on their common side are opposite to each other, because the squares are on each side of the line segment $[a, b]$; so their sum as a cycle is equal to 0. We remove line segments whenever their are bordered by two squares which meet A (this may also happen in some cases when the line segment itself does not meet A).

Finally, Γ reduces to a cycle γ with support included in

$$\bigcup \{\sigma \text{ side of } \Delta_{j,k} \text{ s.t. } \Delta_{j,k} \cap A \neq \emptyset \text{ and } \sigma \cap A = \emptyset\}.$$

In particular, $\operatorname{supp} \gamma \cap (A \cup B) = \emptyset$, so $\operatorname{supp} \gamma \subset \Omega$.

Now we need to see that γ is a union of closed paths. For this, it is enough to see that each extremity of a line segment $[a, b] \subset \operatorname{supp} \gamma$, say a, must be contained in another such line segment. Since $[a, b]$ is included in the boundary of only one of the squares which meet A, a can be contained in exactly one, two or three of the squares in $\mathcal{A}$. In each case it is easy to find another line segment in the support of γ that has its extremity at a. In the case where there are more than one of them, which is exactly when a belongs to two squares with sides in the support of γ intersecting only at a, then choose the other line segment that lies in the boundary of the same square as $[a, b]$. In this way, we decompose γ in a finite union of polygonal paths that have no endpoint, so closed polygonal paths. □

Theorem 4.5. *A domain $\Omega \subset \mathbb{C}$ is simply connected if and only if $S \setminus \Omega$ is connected.*

Proof. Because of Corollary 4.4, we only need to prove that if $S \setminus \Omega$ is connected, Ω is simply connected. If $\Omega = \mathbb{C}$, there is nothing left to prove (it is convex, for instance), so we may assume $\Omega \neq \mathbb{C}$.

We will prove that any holomorphic function in Ω admits an antiderivative, therefore any nonvanishing holomorphic function admits a logarithm, thus a holomorphic square root, and the proof of Riemann's Mapping Theorem shows that Ω is in holomorphic bijection with the unit disc, so in particular homeomorphic to the unit disc, and thus simply connected.

To have an antiderivative to any holomorphic function, it is enough to show that for any closed path $\gamma : [0, 1] \longrightarrow \Omega$ and any $f \in \mathcal{O}(\Omega)$, $\int_\gamma f(\zeta)d\zeta = 0$.

Claim. *Given γ, there exists a union of closed paths $\Gamma \subset \Omega \setminus \operatorname{supp}\gamma$ such that for any $f \in \mathcal{O}(\Omega)$ and any $\zeta \in \operatorname{supp}\gamma$,*

$$f(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - \zeta}d\xi, \tag{4}$$

and for any $\xi \in \operatorname{supp}\Gamma$, $Ind_\xi(\gamma) := \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta-\xi}d\zeta = 0$.

If we accept the Claim, then since $\operatorname{dist}(\operatorname{supp}\gamma, \operatorname{supp}\Gamma) > 0$, we have a bounded integrand and can apply Fubini's Theorem:

$$\int_\gamma f(\zeta)d\zeta = \int_\gamma \left(\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - \zeta}d\xi\right)d\zeta = \frac{1}{2\pi i}\int_\Gamma f(\xi)\left(\int_\gamma \frac{1}{\xi - \zeta}d\zeta\right)d\xi = 0.$$

Now we prove the Claim, in more or less the same way as we had proved Proposition 4.3.

Let $\delta > 0$ be chosen so that $\delta\sqrt{2} < \operatorname{dist}(\operatorname{supp}\gamma, S \setminus \Omega)$.
Choose $R > \max_t |\gamma(t)| + \delta\sqrt{2}$. We let

$$\mathcal{A} := \{\Delta_{j,k} : \Delta_{j,k} \subset \Omega \cap D(0, R)\}, \quad \Gamma_1 := \sum_{\Delta_{j,k}\in\mathcal{A}} \partial\Delta_{j,k}.$$

For each $\zeta \in \operatorname{supp}\gamma \setminus \bigcup_{j,k}\partial\Delta_{j,k}$, the choice of δ implies that $\zeta \in \Delta^\circ_{j_0,k_0}$, with $\Delta_{j_0,k_0} \subset \Omega \cap D(0, R)$. So, as before, for any $f \in \mathcal{O}(\Omega)$,

$$f(\zeta) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(\xi)}{\xi - \zeta}d\xi. \tag{5}$$

By construction, $\operatorname{supp}\Gamma_1 \subset \Omega \cap D(0, R)$. We prove that Γ_1 is equivalent to a cycle Γ with a smaller support that does not intersect $\operatorname{supp}\gamma$ in exactly the same way as in the proof of Proposition 4.3, by removing the sides that belong to two adjacent squares.

We want to choose x_0, y_0 such that (5) holds for $\zeta = \gamma(t)$ with t in a dense subset of $[0, 1]$. For $x \in \mathbb{R}$, let $U_x := \gamma_1^{-1}(x)^\circ$ and $B := \{x \in \mathbb{R} : U_x \neq \emptyset\}$. Then $(U_x, x \in B)$ is a family of disjoint nonempty open sets in $[0, 1]$, therefore it is at most countable, and so is $B + \delta\mathbb{Z} := \cup_{k\in\mathbb{Z}}(B + \delta k)$. Thus we can choose $x_0 \notin B + \delta\mathbb{Z}$, which guarantees that $\{t \in [0,1] : \gamma_1(t) \neq x_0 + \delta k, \forall k \in \mathbb{Z}\}$ is dense in $[0,1]$. We do the same thing for y_0 and γ_2.

For this choice of x_0, y_0, (5) holds for a dense subset of $\zeta \in \operatorname{supp}\gamma$, and both sides of the equation are continuous, since $\operatorname{supp}\gamma \cap \operatorname{supp}\Gamma = \emptyset$, so the equality holds for all $\zeta \in \operatorname{supp}\gamma$.

Now we must show that $\operatorname{Ind}_\xi(\gamma)$ vanishes when $\xi \in \operatorname{supp}\Gamma$. By Proposition 4.2, when $a \in S \setminus \Omega$, $\operatorname{Ind}_a(\gamma) = 0$. When $a \in S \setminus D(0,R)$, the same property is proved by considering $\lim_{t\to\infty, t>1} \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - ta} d\zeta = 0$.

Since $\xi \in \operatorname{supp}\Gamma$, $\xi \in \Delta_{j,k}$ with $\Delta_{j,k} \setminus \Omega \neq \emptyset$ or $\Delta_{j,k} \setminus D(0,R) \neq \emptyset$. Let then $a \in \Delta_{j,k} \setminus (\Omega \cap D(0,R))$; the line segment $[\xi; a] \subset \Delta_{j,k} \subset \mathbb{C} \setminus \operatorname{supp}\gamma$, so the function $t \mapsto \operatorname{Ind}_{(1-t)\xi + ta}(\gamma)$ is continuous and integer-valued on $[0,1]$. Since it vanishes for $t = 1$, it vanishes identically, and we are done. □

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