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TAKE HOME MIDTERM SOLUTION

PASCAL J. THOMAS

This is a solution to a corrected version of the text. My apologies for the many mistakes which obscured the meaning of the questions.

The goal of this problem is to clarify the phrase "argument principle" and to show that if Ω is a domain, $f \in \mathcal{O}(\Omega)$ admits a holomorphic square root if and only if:

$$\text{(1)} \qquad \text{For any closed path } \gamma : [a,b] \longrightarrow \Omega \setminus f^{-1}(0), \quad \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta \in 2\mathbb{Z}.$$

1) Let φ be a differentiable map from $[a,b] \subset \mathbb{R}$ to $\mathbb{C}\setminus\{0\}$. Let arg stand for any local continuous determination of the argument of complex numbers in a neighborhood U of $\varphi(t_0)$, i.e. some function such that

$$\text{for } z \in U, z = e^{\log|z| + i \arg z}.$$

Prove that any two functions with this property must differ by a constant if U is connected.

Therefore $\frac{d}{dt}(\arg \varphi(t))$ is always well defined. Show that

$$\frac{d}{dt}(\arg \varphi(t)) = \operatorname{Im}\left(\frac{\varphi'(t)}{\varphi(t)}\right).$$

Hint: consider the derivative of $e^{\log|\varphi(t)| + i \arg \varphi(t)}$.

Suppose that $\arg_1$ *and* $\arg_2$ *are two continuous determinations of the argument in the connected open set* U*. Then for any* $z \in U$,

$$\exp(\log|z| + i \arg_1(z)) = \exp(\log|z| + i \arg_2(z)),$$

which implies that there exists $k_z \in \mathbb{Z}$ *such that* $\arg_1(z) = \arg_2(z) + 2k_z\pi$*. So the function* $z \mapsto k_z = \frac{1}{2\pi}(\arg_1(z) - \arg_2(z))$ *is continous, integer-valued, so it must be constant on the connected set* U *(because the connected components of* $\mathbb{Z}$ *are reduced to a point).*

So if we choose a continuous determination of the argument on U*, and if it is differentiable (in the real sense), all other determinations will also be differentiable and share the same derivative, since they differ by a constant.*

We will at the same time show that $\arg\varphi$ *is differentiable and compute its derivative. Consider*

$$
\begin{aligned}
(2)\quad \varphi(t+h) &= \exp\left(\log|\varphi(t+h)| + i\arg(\varphi(t+h))\right) \\
&= \exp\left(\log|\varphi(t)| + i\arg(\varphi(t))\right)\exp\left(\log\left|\frac{\varphi(t+h)}{\varphi(t)}\right| + i[\arg(\varphi(t+h)) - \arg(\varphi(t))]\right) \\
&= \varphi(t)\exp\left(\log\left|\frac{\varphi(t+h)}{\varphi(t)}\right| + i[\arg(\varphi(t+h)) - \arg(\varphi(t))]\right).
\end{aligned}
$$

Since φ *is differentiable,* $\varphi(t+h) = \varphi(t) + \varphi'(t)h + o(h)$*, and*

$$\log\left|\frac{\varphi(t+h)}{\varphi(t)}\right| + i[\arg(\varphi(t+h)) - \arg(\varphi(t))] = O(h).$$

Recall that $o(O(x)) = o(x)$*. Thus, using the fact that* $e^w = 1 + w + o(w)$*, equation* (2) *yields*

$$\varphi(t)+\varphi'(t)h+o(h) = \varphi(t)\left(1 + \log\left|\frac{\varphi(t+h)}{\varphi(t)}\right| + i[\arg(\varphi(t+h)) - \arg(\varphi(t))] + o(h)\right).$$

Divide by $\varphi(t)$ *on each side (recall that* $\varphi(t) \neq 0$ *by hypothesis) and take the imaginary part of everything. The* log *term is real and disappears, and we are left with*

$$\arg(\varphi(t+h)) - \arg(\varphi(t)) = Im\left(\frac{\varphi'(t)}{\varphi(t)}\right)h + o(h),$$

which proves differentiability and yields the derivative we wanted.

2) The quantity $\int_a^b \frac{d}{dt}(\arg\varphi(t))\,dt$ is called the *variation of the argument* along the curve φ.

If φ is a closed path, i.e. $\varphi(a) = \varphi(b)$, show that this quantity belongs to $2\pi\mathbb{Z}$.

Compute the variation of the argument when $a = 0$, $b = 2\pi$, $\varphi(t) = e^{imt}$, where $m \in \mathbb{Z}$ is a parameter (this corresponds to m turns around the unit circle, taking orientation into account).

If $b_1 = \varphi(t_0) \neq 0$*, then the argument admits a continuous determination (and thus several) in the disc* $D(b_1, |b_1|)$*; by continuity of* φ*, there exists* $\delta > 0$ *such that for* $u \in (t_0 - \delta, t_0 + \delta)$*,* $\varphi(u) \in D(b_1, |b_1|)$*. The functions*

$$\int_a^u \frac{d}{dt}(\arg\varphi(t))\,dt \text{ and } \arg(\varphi(u))$$

have the same derivative, and thus must differ by a constant. This is true for any $t_0 \in [a, b]$*.*

To see that $\int_a^b \frac{d}{dt}(\arg\varphi(t))\,dt \in 2\pi\mathbb{Z}$ *when* $\varphi(a) = \varphi(b)$*, we need to be more precise. Choose any determination* $\arg\varphi(a)$ *of the argument of* $\varphi(a)$*. Let*

$$A(u) := \arg\varphi(a) + \int_a^u \frac{d}{dt}(\arg\varphi(t))\,dt.$$

Consider the set of u *such that* $A(u)$ *is a determination of the argument of* $\varphi(u)$*, i.e.* $\varphi(u)e^{-iA(u)} \in \mathbb{R}_+^*$*. Since* $\varphi(u) \neq 0$*, it is the same as* $\{u \in [a, b] : \varphi(u)e^{-iA(u)} \in \mathbb{R}_+\}$*, so it is closed. It is not empty since it contains* a*. Finally, it is also open by the argument given above, because in a neighborhood of* t_0*,* $\varphi(u) = |\varphi(u)|e^{i\arg(\varphi(u))}$*. So* *it must be the whole interval, and $A(b)$ is a determination of the argument of $\varphi(b)$, therefore $A(b) - A(a) = A(b) - \arg\varphi(a) = A(b) - \arg\varphi(b) \in 2\pi\mathbb{Z}$. In the example, $\varphi'(t) = ime^{imt}$, so Im $\left(\frac{\varphi'(t)}{\varphi(t)}\right) = m$, and $\int_0^{2\pi} m\,dt = 2\pi m$.*

3) Let γ be a differentiable map from $[a, b] \subset \mathbb{R}$ to $\mathbb{C}$, and f a differentiable map from an open set in $\mathbb{C}$ to $\mathbb{C}$. Using the definitions of $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$, prove the complex form of the Chain Rule:

$$(f \circ \gamma)'(t) = \frac{\partial f}{\partial z}(\gamma(t))\gamma'(t) + \frac{\partial f}{\partial \bar{z}}(\gamma(t))\bar{\gamma}'(t).$$

Write $\gamma(t) =: \gamma_1(t) + i\gamma_2(t)$, where the γ_j are real-valued. Then

$$(f \circ \gamma)'(t) = \frac{\partial f}{\partial x}(\gamma(t))\gamma_1'(t) + \frac{\partial f}{\partial y}(\gamma(t))\gamma_2'(t).$$

Using the definitions of $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$,

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}\right),$$
$$\frac{\partial f}{\partial y} = i\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right).$$

Collecting the terms, we get the formula.

4) If γ is a path in Ω, f is holomorphic on Ω, and if $\arg f$ is any determination of the argument of f, defined in a neighborhood of a where $f(a) \neq 0$, show that

$$\frac{d}{dt}(\arg f(\gamma(t))) = \mathrm{Im}\left(\frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\right).$$

Hint: consider the path $\varphi := f \circ \gamma$.

If γ is a closed path, show that the variation of the argument along $f \circ \gamma$ is $\frac{1}{i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta$.

Since f is holomorphic, we write $f'(z)$ for $\frac{\partial f}{\partial z}$, and the formula for the Chain Rule becomes $(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t)$ (note that the two derivatives in this formula have slightly different meanings). Then, using $\varphi = f \circ \gamma$,

$$\frac{d}{dt}(\arg f(\gamma(t))) = Im\left(\frac{(f \circ \gamma)'(t)}{f \circ \gamma(t)}\right) = Im\left(\frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}\right),$$

which yields the result. So, by the above computation, the variation of the argument along $f \circ \gamma$ is

$$\int_a^b Im\left(\frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\right)dt = Im\left(\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt\right) = Im\left(\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta\right).$$

If γ is a path from a fixed a to z, $\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta$ provides a determination of $\log f(z)$ (locally).

If γ is a closed path, the whole integral is the difference between two determinations of the logarithm, therefore a multiple of $2\pi i$; in particular,

$$Im\left(\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta\right) = \frac{1}{i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta.$$

5) Prove that if f admits a holomorphic square root, i.e. if there exists a holomorphic function $g \in \mathcal{O}(\Omega)$ such that $f(z) = g(z)^2$, for all $z \in \Omega$, then (1) holds.

Since $f(z) = g(z)^2$, $f'(z) = 2g(z)g'(z)$, so

$$\frac{1}{i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta = 2\frac{1}{i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)}d\zeta \in 2\cdot 2\pi\mathbb{Z},$$

by the previous question applied to g.

6) Suppose now that $f \in \mathcal{O}(\Omega)$ verifies (1).

Let $z_0 \in \Omega$, and for each $z \in \Omega$, with $f(z) \neq 0$, let γ_z be a path from z_0 to z in $\Omega \setminus f^{-1}(0)$. Prove that the function

$$z \mapsto \exp\left(\frac{1}{2}\int_{\gamma_z} \frac{f'(\zeta)}{f(\zeta)}d\zeta\right) =: h(z)$$

is well defined, i.e. it does not depend on the choice of γ_z, and continuous on $\Omega\setminus f^{-1}(0)$.
Suppose that $\gamma_{1,z}$ and $\gamma_{2,z}$ are two paths from z_0 to z. Then $\tilde\gamma = \gamma_{1,z} - \gamma_{2,z}$ (i.e. the path $\gamma_{1,z}$, followed by the path $\gamma_{2,z}$ in the reverse direction) is a closed path, so

$$\frac{1}{2}\int_{\gamma_{1,z}} \frac{f'(\zeta)}{f(\zeta)}d\zeta - \frac{1}{2}\int_{\gamma_{2,z}} \frac{f'(\zeta)}{f(\zeta)}d\zeta = \frac{1}{2}\int_{\tilde\gamma_z} \frac{f'(\zeta)}{f(\zeta)}d\zeta = \frac{1}{2}4k\pi i = 2k\pi i.$$

Passing to the exponentials, since $\exp(2k\pi i) = 1$, we find

$$\exp\left(\frac{1}{2}\int_{\gamma_{1,z}} \frac{f'(\zeta)}{f(\zeta)}d\zeta - \frac{1}{2}\int_{\gamma_{2,z}} \frac{f'(\zeta)}{f(\zeta)}d\zeta)\right) = 1.$$

To prove continuity, if $z' \in \bar D(z,r)$ with $r>0$ chosen so that $\bar D(z,2r) \subset \Omega\setminus f^{-1}(0)$, we can replace $\gamma_{z'}$ by $\gamma_z + [z,z']$, where $[z,z']$ is the oriented line segment from z to z', so that

$$\frac{h(z')}{h(z)} = \exp\left(\frac{1}{2}\int_{[z,z']} \frac{f'(\zeta)}{f(\zeta)}d\zeta\right).$$

If $m := \min_{\bar D(z,r)} |f| > 0$ and $M := \max_{\bar D(z,2r)} |f|$, then $\max_{\bar D(z,r)} |f'| \le M/r$ and the integral inside the exponential is bounded above by $M|z'-z|/(rm) < \varepsilon$ for r small enough. Then use the fact that h is locally bounded by M', and $|h(z')-h(z)| \le M'\left|\frac{h(z')}{h(z)} - 1\right|$.

7) Prove that $h(z)$ is holomorphic and extends to Ω, and that it verifies $h(z)^2 = Cf(z)$, for a constant C.

The function h is holomorphic because it is the exponential of a function that is holomorphic, by the usual proof (once we know it is well defined). Furthermore,

$\frac{(h^2)'(z)}{h^2(z)} = \frac{f'(z)}{f(z)}$. *So we have*

$$\frac{d}{dz}\left(\frac{h(z)^2}{f(z)}\right) = \frac{1}{f(z)}\frac{f'(z)}{f(z)}h(z)^2 - \frac{h(z)^2 f'(z)}{f(z)} = 0,$$

therefore $h(z)^2 = Cf(z)$. *Choosing* C_1 *such that* $C_1^2 = C$, *we find that* $g(z) := h(z)/C_1$ *provides a holomorphic square root for* f *on* $\Omega \setminus f^{-1}(0)$.

Since g *is bounded (because* $|g| = \sqrt{|f|}$*), and the zeroes of* f *are isolated, Riemann's Removable Singularity Theorem implies that* g *extends to a holomorphic function on* Ω. *By continuity, it is still equal to a square root of* f.