UNIVERSITÉ PAUL SABATIER :
L3 PARCOURS SPÉCIAL, 2016-17
TAKE HOME MIDTERM EXAM

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Handed out on Friday, January 27, 2017. Work to be returned on Monday, January 30, at 4 pm (beginning of class).

Many questions are independent. Do what you can!

The goal of this problem is to clarify the phrase "argument principle" and to show that if Ω is a domain, $f \in \mathcal{O}(\Omega)$ admits a holomorphic square root if and only if:

$$\text{(1)} \qquad \text{For any closed path } \gamma : [a,b] \longrightarrow \Omega \setminus f^{-1}(0), \quad \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta \in 2\mathbb{Z}.$$

1) Let φ be a differentiable map from $[a,b] \subset \mathbb{R}$ to $\mathbb{C}\setminus\{0\}$. Let arg stand for any local continuous determination of the argument of complex numbers in a neighborhood U of $\varphi(t_0)$, i.e. some function such that

$$\text{for } z \in U, z = e^{\log|z| + i \arg z}.$$

Prove that any two functions with this property must differ by a constant if U is connected.

Therefore $\frac{d}{dt}(\arg\varphi(t))$ is always well defined. Show that

$$\frac{d}{dt}(\arg\varphi(t)) = \mathrm{Im}\left(\frac{\varphi'(t)}{\varphi(t)}\right).$$

Hint: consider the derivative of $e^{\log|\varphi(t)| + i\arg\varphi(t)}$.

2) The quantity $\int_a^b \frac{d}{dt}(\arg\varphi(t))\,dt$ is called the *variation of the argument* along the curve φ.

If φ is a closed path, i.e. $\varphi(a) = \varphi(b)$, show that this quantity belongs to $2\pi\mathbb{Z}$.

Compute the variation of the argument when $a = 0$, $b = 2\pi$, $\varphi(t) = e^{imt}$, where $m \in \mathbb{Z}$ is a parameter (this corresponds to m turns around the unit circle, taking orientation into account).

3) Let γ be a differentiable map from $[a,b] \subset \mathbb{R}$ to $\mathbb{C}$, and f a differentiable map from an open set in $\mathbb{C}$ to $\mathbb{C}$. Using the definitions of $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar z}$, prove the complex form of the Chain Rule:

$$(f\circ\gamma)'(t) = \frac{\partial f}{\partial z}(\gamma(t))\gamma'(t) + \frac{\partial f}{\partial \bar z}(\gamma(t))\bar\gamma'(t).$$

4) If γ is a path in Ω, f is holomorphic on Ω, and if $\arg f$ is any determination of the argument of f, defined in a neighborhood of a where $f(a) \neq 0$, show that

$$\frac{d}{dt}(\arg f(\gamma(t))) = \mathrm{Im}\left(\frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\right).$$

Hint: consider the path $\varphi := f \circ \gamma$.

Show that the variation of the argument along $f \circ \gamma$ is $\frac{1}{i} \int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta$.

5) Prove that if f admits a holomorphic square root, i.e. if there exists a holomorphic function $g \in \mathcal{O}(\Omega)$ such that $f(z) = g(z)^2$, for all $z \in \Omega$, then (1) holds.

6) Suppose now that $f \in \mathcal{O}(\Omega)$ verifies (1).

Let $z_0 \in \Omega$, and for each $z \in \Omega$, with $f(z) \neq 0$, let γ_z be a path from z_0 to z in $\Omega \setminus f^{-1}(0)$. Prove that the function

$$z \mapsto \exp\left(\frac{1}{2} \int_{\gamma_z} \frac{f'(\zeta)}{f(\zeta)} d\zeta\right) =: h(z)$$

is well defined, i.e. it does not depend on the choice of γ_z, and continuous on $\Omega \setminus f^{-1}(0)$.

7) Prove that $g(z) := h(z)$ is holomorphic and extends to Ω, and that it verifies $g(z)^2 = C f(z)$, for a constant C.