UNIVERSITÉ PAUL SABATIER : L3 PARCOURS SPÉCIAL, 2016-17 EXERCISE SHEET # 3: SEQUENCES OF FUNCTIONS, RIEMANN MAPPING THEOREM

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Exercises or questions marked with a * are not mandatory. Do them later.

1.1. a) We recall a topological fact: if A is a relatively compact set in a metric space, a sequence $(a_n)_n \subset A$ converges to a limit l if and only if every convergent subsequence $(a_{n_k})_k$ converges to l.

b) Let Ω be a domain and let $A \subset \Omega$ be a subset with at least one cluster point in Ω (in other words, A is not discrete). Let $(f_n)_n \subset \mathcal{O}(\Omega)$ be a locally bounded sequence of holomorphic functions.

We assume that there exists $f \in \mathcal{O}(\Omega)$ such that for any $a \in A$, $f_n(a) \to f(a)$. Show that $f_n \to f$, uniformly on compact subsets of Ω (this is known as Vitali's Theorem).

1.2. We have already remarked (Exercise 1.5 in Sheet #2) that $S \setminus [-1, +1]$, is simply connected. Write an explicit holomorphic bijection from that domain to $S \setminus \overline{\mathbb{D}}$. Use this to deduce a holomorphic bijection between $\mathbb{C} \setminus [-1, +1]$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ (which are not simply connected).

1.3. Show that if $\mathcal{F} \subset \mathcal{O}(\Omega)$ is a normal family, then $\mathcal{F}' \subset \mathcal{O}(\Omega)$, where $\mathcal{F}' := \{f' : f \in \mathcal{F}\}$.

Prove by a simple counter-example that the converse fails. What additional hypothesis could you add to the hypothesis "$\mathcal{F}'$ is a normal family" to imply that $\mathcal{F}$ is a normal family?

1.4. * Let Ω be a domain and $(f_n)_n \subset \mathcal{O}(\Omega)$. We assume that $(f_n)_n$ avoids an open disc $D(a, r)$, i.e. for all $n \in \mathbb{N}$, $z \in \Omega$, $f_n(z) \notin D(a, r)$.

Prove that there exists a subsequence $(f_{n_k})_k$ such that either

(1) $(f_{n_k})_k$ converges uniformly on compact subsets of Ω;
(2) or $|f_{n_k}| \to \infty$, uniformly on compact subsets of Ω, in the sense that for any compact set $K \subset \Omega$, for any $A > 0$, there exists $N \in \mathbb{N}$ such that $k \geq N, z \in K \Rightarrow |f_{n_k}(z)| > A$.

Hints: first find $\varphi \in \mathcal{H}(S)$ such that $\varphi(S \setminus \overline{D}(a, R) = \mathbb{D}$. Then work with $\varphi \circ f_n$. Then come back to the original sequence. To deal with the case of convergence to ∞, you will have to use Hurwitz's Theorem.

1.5. Prove that there is no one-to-one holomorphic bijection between $\mathbb{D} \setminus \{0\}$ and $\Omega := \{z : r < |z| < R\}$, where $r > 0$ (even though those two domains are homeomorphic).

Hints: proceed by contradiction; you will need Riemann's Removable Singularity Theorem and the Open Mapping Theorem.

1.6. Let Ω_1 and Ω_2 be two simply connected domains, both distinct from $\mathbb{C}$, and $z_0 \in \Omega_1$. Let f be a holomorphic bijection from Ω_1 to Ω_2. Prove that for any holomorphic map $h : \Omega_1 \longrightarrow \Omega_2$ with $h(z_0) = f(z_0)$, then $|h'(z_0)| \leq |f'(z_0)|$.

1.7. We recall that we say that f is holomorphic in a neighborhood of $\infty \in S$ if and only if $f \circ \mathcal{I}$ is holomorphic in a neighborhood of 0.

We would like to know when the class of holomorphic functions $\mathcal{O}(\Omega, \overline{\mathbb{D}})$ (the holomorphic functions bounded by 1 in modulus) is non-trivial, for a domain $\Omega \subset S$. We consider the class trivial if it contains only constant functions.

a) If $\Omega_1 \subset \Omega_2$ are domains, and the class $\mathcal{O}(\Omega_1, \overline{\mathbb{D}})$ is trivial, prove that the class $\mathcal{O}(\Omega_2, \overline{\mathbb{D}})$ is trivial.

If $\Omega = \mathbb{C}$ or $\Omega = S$, prove that the class is trivial.

b) If $\Omega = S \setminus A$, where A is a finite set, prove that the class is trivial. Hint: you can reduce yourselves to the case where $\infty \in A$.

c) If $\Omega = S \setminus [-1, +1]$, prove that $\mathcal{O}(\Omega, \overline{\mathbb{D}})$ is not trivial.

Hint: use exercise 1.2.

*d) We now consider a more complicated situation. Let $A \subset [-1, 1]$ be a closed set.

d-i) We recall from measure theory that a compact set $F \subset \mathbb{R}$ is of measure 0 if and only if for any $\varepsilon > 0$, there exists an open set U_ε such that $F \subset U_\varepsilon$ and $\lambda_1(U_\varepsilon) < \varepsilon$, where λ_1 denotes the Lebesgue measure on $\mathbb{R}$.

Show that $\lambda_1(A) = 0$ if and only if for any $\varepsilon > 0$, there exists a finite collection $I_1, \ldots, I_N$ of disjoint open intervals such that $I_j = (a_j, b_j)$ and

$$\sum_{j=1}^{N} (b_j - a_j) < \varepsilon.$$

d-ii) Show that for any $\varepsilon > 0$, there is a path γ made up of a finite union of rectangles with sides parallel to the axes, run through in the trigonometric direction, such that each point in A is inside exactly one of those rectangles, and for each $a \in A$, $\mathrm{Ind}_a(\gamma) = 1$ (the index of γ with respect to a), and that the total length of γ is $< \varepsilon$.

d-iii) Suppose that $f \in \mathcal{O}(\Omega, \overline{\mathbb{D}})$. We want to prove that f is constant. Show that you can reduce yourselves to the case where $f(\infty) = 0$.

d-iv) Let $z \in \mathbb{C}$ and $R > |z|$. Let $C(0, R)$ be the circle of center 0, radius R, run through once in the trigonometric direction. Using the Cauchy formula, show that we can find γ as in question d-ii) such that

$$2\pi i f(z) = \int_{C(0,R)} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

d-v) Deduce from the above formula that $f(z) = 0$.

Comment: this is a simplified form of Painlevé's Theorem (Painlevé lived in the days when major political figures could also be first-rate mathematicians, look him up on Wikipedia). More refined versions of the problem have kept some mathematicians busy until the early years of the 21st century, until they were solved by the Catalan mathematican Xavier Tolsa Domènech (born 1966).