J. London Math. Soc. (2) 89 (2014) 623–639
doi:10.1112/jlms/jdt083

Multi-speed solitary wave solutions for nonlinear Schrödinger systems

Isabella Ianni and Stefan Le Coz

Abstract

We prove the existence of a new type of solutions to a nonlinear Schrödinger system. These solutions, which we call *multi-speeds solitary waves*, behave at large time as a couple of scalar solitary waves traveling at different speeds. The proof relies on the construction of approximations of the multi-speeds solitary waves by solving the system backward in time and using energy methods to obtain uniform estimates.

1. *Introduction*

We consider the following nonlinear Schrödinger system:

$$\begin{cases} i\partial_t u_1 + \Delta u_1 + \mu_1 |u_1|^2 u_1 + \beta |u_2|^2 u_1 = 0, \\ i\partial_t u_2 + \Delta u_2 + \mu_2 |u_2|^2 u_2 + \beta |u_1|^2 u_2 = 0, \end{cases} \tag{1.1}$$

where, for $j = 1, 2$, we have $u_j : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $d = 1, 2, 3$, $\mu_j > 0$, and $\beta \in \mathbb{R} \setminus \{0\}$.

This type of system appears in various physical settings, of which we now give three examples.

When $d = \mu_1 = \mu_2 = \beta = 1$, the system (1.1) is sometimes called *Manakov system*, as it was examined by Manakov [**21**] as an asymptotic model for the propagation of the electric field in a wageguide. With this specific choice of parameters, the system is completely integrable and can be solved by means of the inverse scattering transform. Such analysis is performed in detail in the book [**1**], which contains also many examples of physical situations where (1.1) is used.

Later on, (1.1) was derived to model the propagation of light in an optical fiber when taking into account polarization of light and birefringence of the fiber; see, for example, [**2**]. In this case, $d = \mu_1 = \mu_2 = 1$ and the parameter β, which measures the strength of the XPM (cross-phase modulation) interaction, varies depending on the nature of the fiber (for example, $\beta = 2$ for dual-core fibers or $\beta = \frac{2}{3}$ for single-core fibers).

In higher dimension $d = 3$, (1.1) can model the interaction of two Bose–Einstein condensates of atoms in different spin states (see, for example, [**14**]). In this case, if N denotes the number of atoms in the jst condensate and a_{jk} is a factor proportional to the scattering length between a j-species atom and a k-species atom (a_{jk} may be positive or negative, depending on whether the collision between particles results in an attractive or repulsive interaction), the parameters of (1.1) stands for $\mu_j = (N-1)a_{jj}$ and $\beta = N a_{jk}$. The trapping potential is turned off to model the expansion of the condensates in experiments.

From the mathematical point of view, there has been recently an increasing interest for (1.1) and its stationary versions. We give only a few samples of the mathematical studies around (1.1). As mentioned before, the system is completely integrable in the Manakov case, but any modification of the parameters breaks integrability and the analysis of the dynamics

Received 5 April 2013; revised 20 October 2013; published online 28 January 2014.

2010 *Mathematics Subject Classification* 35Q55 (primary), 35C08, 35Q51, 37K40 (secondary).

This research was supported in part by the French Agence Nationale de la Recherche through the project ESONSE and PRIN 2009-WRJ3W7 grant.

of (1.1) in non-integrable cases is largely open. A lot of recent studies (see, for example, [**3**, **17**, **19**, **27**, **30**, **31**]) are concerned with the existence of standing wave solutions for various ranges of parameters μ_1, μ_2, β. The stability of such standing waves was also investigated in various cases (see, among many others, [**9**, **20**, **24**, **25**]).

In this work, we want to investigate the existence of solutions to (1.1) where each component behaves like a soliton, as we explain precisely now.

When $u_1 \equiv 0$ or $u_2 \equiv 0$, the system (1.1) reduces to the scalar Schrödinger equation

$$i\partial_t u + \Delta u + \mu|u|^2 u = 0. \tag{1.2}$$

It is well known that (1.2) admits *solitary waves* (see [**16**, **26**]), which are solutions with a fixed profile, possibly rotating and traveling on a line (see Theorem 1 and Section 2 for more details). If R_1 denotes a solitary wave solution to (1.2), then $(R_1, 0)^\intercal$ is trivially a solution to (1.1). If R_2 is another solitary wave solution to (1.2), then, due to the non-trivial interaction $\beta \neq 0$, the couple $(R_1, R_2)^\intercal$ has no reason to be a solution to (1.1). Nevertheless, our goal in this paper is to exhibit solutions of (1.1) behave in large time like a couple of solitary waves $(R_1, R_2)^\intercal$, provided the relative speed of the solitary waves is large enough. We call such solutions *multi-speeds solitary waves*. To our knowledge, this is the first time that such solutions are exhibited for non-integrable Schrödinger systems. Our main result is the following.

THEOREM 1. *For $j = 1, 2$, let $\omega_j > 0, \gamma_j \in \mathbb{R}, x_j, v_j \in \mathbb{R}^d$, and $\Phi_j \in H^1(\mathbb{R}^d)$ be a solution to*

$$-\Delta\Phi_j + \Phi_j - |\Phi_j|^2\Phi_j = 0, \quad \Phi_j \in H^1(\mathbb{R}^d). \tag{1.3}$$

Define

$$R_j(t,x) := e^{i(\omega_j t - |v_j|^2 t/4 + (1/2)v_j\cdot x + \gamma_j)}\sqrt{\frac{\omega_j}{\mu_j}}\Phi_j(\sqrt{\omega_j}(x - v_j t - x_j)), \tag{1.4}$$

$$v_\star := |v_1 - v_2|, \quad \omega_\star := \tfrac{1}{4}\min\{\omega_1, \omega_2\}. \tag{1.5}$$

There exists $v_\sharp > 0$ such that if $v_\star > v_\sharp$, then there exists $T_0 \in \mathbb{R}$ and a multi-speeds solitary wave $(u_1, u_2)^\intercal$ solution of (1.1) defined on $[T_0, +\infty)$ such that, for all $t \in [T_0, +\infty)$, the following holds:

$$\left\| \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} - \begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} \right\|_{H^1\times H^1} \leqslant e^{-\sqrt{\omega_\star}v_\star t}. \tag{1.6}$$

The strategy of the proof of Theorem 1 is inspired by the one developed for the study of multi-solitons for scalar nonlinear Schrödinger equations in [**12**, **13**, **22**, **23**]. The idea is to solve (1.1) backward in time taking as final data at the final time T^n a couple of solitary waves, where T^n is an increasing sequence of time. In this way, we define a sequence of solutions to (1.1) which are approximated multi-speeds solitary waves. Then the proof relies on two main steps. First, we show that the approximate solutions satisfy the required estimate (1.6) on a sequence of time intervals $[T_0, T^n]$, *with T_0 independent of n* (see Proposition 1). Then we prove that the sequence of initial data obtained at T_0 is compact (see Proposition 2). Therefore, we can extract an initial data giving rise to a solution of (1.1) that satisfies the conclusion of Theorem 1.

Our approach is very flexible and can probably be extended to many other situations. We do not need many of the technical features present in [**12**, **13**, **22**, **23**] like modulation theory or localization procedures. Surprisingly, the presence of the coupling is helping us. Indeed, as we will see in Section 4, the coupling will act as a localizing factor. We do not require either any assumptions on the attractiveness ($\beta < 0$) or repulsiveness ($\beta > 0$) of the coupling, or on the strength of the nonlinearities μ_1, μ_2. Whereas, it is common when working with solitary waves to consider only ground state profiles Φ_j of (1.3), in our case, as in [**12**], the profiles can be ground states or excited states. Note that even if R_1 is a ground state of the scalar equation, $(R_1, 0)^\intercal$ may not be a ground state of the system, but fortunately we do not require such properties for our composing solitons.

Our only limitation is the assumption on large relative speed $v_\star$. This is due to technical restrictions with proving the uniform estimates (see Section 4). We use a coercivity property that holds up a finite number of L^2-scalar products, and we do not have enough information on the dynamics around the solitons to be able to control all these scalar products without the help of the high speed assumption. We conjecture that in fact our result holds without such assumption, as it is, for example, the case with scalar multisolitons when the composing solitons are ground states.

The rest of the paper is divided as follows. In Section 2, we gather some useful facts about scalar nonlinear Schrödinger equations and their solitary waves. Then, in Section 3 we prove the existence of multi-speeds solitary waves, assuming uniform estimates and a compactness result. The proof of the uniform estimates and the compactness result are given in Sections 4 and 5.

Notation. Before going further, we make precise some notation. The norms of $L^p(\mathbb{R}^d)$-spaces will be denoted by $\|\cdot\|_{L^p}$ and the norm of $H^1(\mathbb{R}^d)$ by $\|\cdot\|_{H^1}$. The spaces $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ are endowed with the norms

$$\left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_{L^2 \times L^2} = \sqrt{\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2}, \quad \left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_{H^1 \times H^1} = \sqrt{\|u_1\|_{H^1}^2 + \|u_2\|_{H^1}^2}.$$

When writing vectors inside the text, we will use the superscript $\intercal$ to denote the transpose of a vector, that is: $(u_1, u_2)^\intercal = \binom{u_1}{u_2}$. The derivative with respect to the time t will be denoted either by $\partial/\partial t$ or simply ∂_t. Throughout the paper, the letter C will denote various positive constants whose exact values may change from line to line, but are of no importance for the analysis.

2. *Scalar solitary waves*

In this section, we summarize the results on scalar solitary waves that we will need for the proof of Theorem 1. For more on scalar Schrödinger equations, the reader is referred to [**6**, **28**, **29**] and the references cited therein. Consider the scalar Schrödinger equation

$$i\partial_t u + \Delta u + \mu_0 |u|^2 u = 0, \tag{2.1}$$

where μ_0 is a positive constant.

The energy, mass and momentum, defined as follows, are conserved along the flow of (2.1).

$$E(u, \mu_0) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{\mu_0}{4}\|u\|_{L^4}^4, \quad M(u) := \frac{1}{2}\|u\|_{L^2}^2, \quad P(u) := \frac{1}{2}\mathcal{Im}\int_{\mathbb{R}^d} u\nabla\bar{u}\, dx.$$

A basic solitary wave u is a solution of (2.1) of the form $u(t,x) = (e^{it}/\sqrt{\mu_0})\Phi(x)$, where Φ is a solution of

$$-\Delta\Phi + \Phi - |\Phi|^2\Phi = 0, \quad \Phi \in H^1(\mathbb{R}^d). \tag{2.2}$$

The existence and properties of solutions to equations of the type (2.2) are well-known (see, for example, the fundamental work of Berestycki and Lions [**4**, **5**]). All solutions to (2.2) are smooth and exponentially decreasing. Precisely, for all $\eta < 1$, for all solutions Φ to (2.2) and for all $x \in \mathbb{R}^d$, there exists $C_\Phi > 0$ such that the following estimate holds:

$$|\Phi(x)| + |\nabla\Phi(x)| \leqslant C_\Phi\, e^{-\eta|x|}.$$

Equation (2.2) admits a unique *ground state*, that is, a positive and radial solution that minimizes among all solutions the *action* $S := E(\cdot, 1) + M$. In dimension $d \geqslant 2$, there exist also infinitely many other solutions called *excited states*. Apart from when $d = 1$, the classification of solutions to (2.2) is still an active research area. Classification of radial solutions was completed recently in the works [**10**, **11**]. Among non-radial solutions we mention the vortices, which were first constructed by Lions [**18**]. In dimension 2, a vortex is a solution of (2.2) of the form $\Phi(\rho,\theta) = e^{im\theta}\Psi(\rho)$ where (ρ,θ) are polar coordinates and $m \in \mathbb{R}$. In this work, we treat any type of solutions to (2.2).

Invariances by scaling, translation, phase shift and Galilean transform generate a $(2d+2)$-parameters family of solitary wave solutions to (2.1). Precisely, let $\omega_0 > 0, \gamma_0 \in \mathbb{R}$, $x_0, v_0 \in \mathbb{R}^d$, and take $\Phi_0 \in H^1(\mathbb{R}^d)$ a solution to (2.2). Then, R_0 defined by

$$R_0(t,x) := e^{i(\omega_0 t - (|v_0|^2/4)t + (1/2)v_0\cdot x + \gamma_0)}\sqrt{\frac{\omega_0}{\mu_0}}\Phi_0(\sqrt{\omega_0}(x - v_0 t - x_0)), \tag{2.3}$$

is a solution to (2.1). Note that, for t fixed, $R_0(t,\cdot)$ is a critical point of the functional S_0 defined by

$$S_0 := E(\cdot,\mu_0) + \left(\omega_0 + \frac{|v_0|^2}{4}\right)M + v_0\cdot P. \tag{2.4}$$

Coercivity properties of linearizations of S_0-like functionals will play an important role in our analysis. We define the linearized action H_0 for $t \in \mathbb{R}$ and $\varepsilon \in H^1(\mathbb{R}^d)$ by

$$H_0(t,\varepsilon) := \langle S_0''(R_0(t))\varepsilon,\varepsilon\rangle. \tag{2.5}$$

Lemma 1 (Scalar Coercivity). *Take $\omega_0 > 0, \gamma_0 \in \mathbb{R}$, $x_0, v_0 \in \mathbb{R}^d$, $\Phi_0 \in H^1(\mathbb{R}^d)$ be a solution to (2.2), R_0 be the solitary wave solution of (2.1) given by (2.3), and S_0 and H_0 be the functionals given by (2.4)–(2.5). Then there exist $c_0 > 0$, $\nu_0 \in \mathbb{N}$, and a family of normalized functions $\{\xi_0^k \in L^2(\mathbb{R}^d); \|\xi_0^k\|_{L^2} = 1, k = 1,\dots,\nu_0\}$ such that, for all $t \in \mathbb{R}$ and for all $\varepsilon \in H^1(\mathbb{R}^d)$, we have*

$$c_0\|\varepsilon\|_{H^1}^2 \leqslant H_0(t,\varepsilon) + \sum_{k=1}^{\nu_0}\left(\varepsilon,\xi_0^k(t)\right)_2^2,$$

where by $\xi_0^k(t)$ we denote the functions defined by

$$\xi_0^k(t)(x) := e^{i(\omega_0 t - (|v_0|^2/4)t + (1/2)v_0\cdot x + \gamma_0)}\sqrt{\frac{\omega_0}{\mu_0}}\xi_0^k(\sqrt{\omega_0}(x - v_0 t - x_0)).$$

Proof. The result being classical, we only recall the main arguments. Consider Φ to be a real solution of (2.2). Then Φ is a critical point of the functional $S = E(\cdot,1) + M$. For $\varepsilon \in H^1(\mathbb{R}^d)$, the functional $\langle S''(\Phi)\varepsilon,\varepsilon\rangle$ can be decomposed by writing

$$\langle S''(\Phi)\varepsilon,\varepsilon\rangle = \langle L_+\mathcal{Re}(\varepsilon),\mathcal{Re}(\varepsilon)\rangle + \langle L_-\mathcal{Im}(\varepsilon),\mathcal{Im}(\varepsilon)\rangle,$$

where L_+, L_- are two self-adjoint linear operators defined by

$$L_+ = -\Delta + 1 - 3|\Phi|^2,$$
$$L_- = -\Delta + 1 - |\Phi|^2.$$

The operators L_+ and L_- are self-adjoint compact perturbations of $-\Delta + 1$, hence their spectrums lie on the real line and consist of essential spectrum on $[1,+\infty)$ and a finite number of eigenvalues on $(-\infty,\eta]$ for any $\eta < 1$. Hence, there exist $c_0 > 0$, $\nu_0 \in \mathbb{N}$ corresponding to the number of non-positive eigenvalues of L_+ and L_- (counted with multiplicity) and a family of normalized eigenfunctions $\{\xi_0^k \in L^2(\mathbb{R}^d); \|\xi_0^k\|_{L^2} = 1, k = 1,\dots,\nu_0\}$ such that

$$c_0\|\varepsilon\|_{H^1}^2 \leqslant \langle S''(\Phi)\varepsilon,\varepsilon\rangle + \sum_{k=1}^{\nu_0}\left(\varepsilon,\xi_0^k\right)_2^2.$$

The conclusion of the lemma follows by extending the arguments to complex-valued Φ and applying scaling, phase shift, translations and Galilean transform (see [**12**, **22**] for details). $\square$

3. *Construction of the solution*

Starting from now and for the rest of the paper, we fix for $j=1,2$, a set of parameters $\omega_j > 0, \gamma_j \in \mathbb{R}$, $x_j, v_j \in \mathbb{R}^d$, and $\Phi_j \in H^1(\mathbb{R}^d)$ solution to (1.3). Let R_j denote the corresponding solitary wave defined in (1.4), $v_\star$ be the relative speed and $\omega_\star$ be the minimal frequency, the latter two defined in (1.5).

Before starting the proof, we need some preliminaries on the local well-posedness of (1.1). In our setting, local well-posedness follows from classical arguments of the local Cauchy theory for Schrödinger equations (see, for example, [**6**, Remark 3.3.12, **8**]). Precisely, for any $0<\sigma\leqslant 1$ such that $2<4/(d-2\sigma)$ or $\sigma=1$ and for any initial data $(u_1^0,u_2^0)^\intercal \in H^\sigma(\mathbb{R}^d)\times H^\sigma(\mathbb{R}^d)$, there exists $T_\star, T^\star>0$ and there exists a solution to (1.1) $(u_1,u_2)^\intercal \in \mathcal{C}((-T_\star,T^\star),H^\sigma(\mathbb{R}^d)\times H^\sigma(\mathbb{R}^d))$ such that $(u_1(0),u_2(0))^\intercal=(u_1^0,u_2^0)^\intercal$. If, in addition, $(u_1^0,u_2^0)^\intercal\in H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$, then the solution also belongs to $\mathcal{C}^1((-T_\star,T^\star),H^{-1}(\mathbb{R}^d)\times H^{-1}(\mathbb{R}^d))$ and the *blow-up alternative* holds, that is, if $T^\star<+\infty$ (respectively $T_\star<+\infty$), then

$$\lim_{t\to T^\star}\left\|\begin{pmatrix}u_1(t)\\u_2(t)\end{pmatrix}\right\|_{H^1\times H^1}=+\infty,\qquad \left(\text{resp. }\lim_{t\to -T_\star}\left\|\begin{pmatrix}u_1(t)\\u_2(t)\end{pmatrix}\right\|_{H^1\times H^1}=+\infty\right).$$

In what follows, we shall mainly work with the scalar energy $E(\cdot,\mu)$, momentum P and masse M defined in Section 2, but we remark here that the system (1.1) admits its own conservation laws. Precisely, the total energy $\mathcal{E}$, the total momentum $\mathcal{P}$ (defined as follows) and the masses M of each component are conserved quantities for the $H^1(\mathbb{R}^d)$-flow of (1.1):

$$\mathcal{E}\begin{pmatrix}u_1(t)\\u_2(t)\end{pmatrix}:=E(u_1(t),\mu_1)+E(u_2(t),\mu_2)-\frac{\beta}{2}\int_{\mathbb{R}^d}|u_1(t)|^2|u_2(t)|^2=\mathcal{E}\begin{pmatrix}u_1^0\\u_2^0\end{pmatrix}, \tag{3.1}$$

$$\mathcal{P}\begin{pmatrix}u_1(t)\\u_2(t)\end{pmatrix}:=P(u_1(t))+P(u_2(t))=\mathcal{P}\begin{pmatrix}u_1^0\\u_2^0\end{pmatrix}, \tag{3.2}$$

$$M(u_1(t))=M(u_1^0),\quad M(u_2(t))=M(u_2^0). \tag{3.3}$$

We can now define a sequence of approximated multi-speeds solitary waves. Let $T^n\in\mathbb{R}$ be an increasing sequence of times such that $\lim_{n\to+\infty}T^n=+\infty$. For each $n\in\mathbb{N}$, let $(u_1^n,u_2^n)^\intercal$ be the solution of (1.1) defined on the interval $(T_n,T^n]$ and such that the *final data* satisfy $(u_1^n(T^n),u_2^n(T^n))^\intercal=(R_1(T^n),R_2(T^n))^\intercal$. We will prove that there exists some T_0 independent of n such that, for every n large enough, $(u_1^n,u_2^n)^\intercal$ is defined on $[T_0,T^n]$ and is close to $(R_1,R_2)^\intercal$. More precisely, we have the following proposition, which will be proved in Section 4.

Proposition 1 (Uniform estimates). *There exists $v_\sharp$ such that if $v_\star>v_\sharp$, then the following holds. There exist $T_0\in\mathbb{R}$ and $n_0\in\mathbb{N}$ such that, for all $n\geqslant n_0$ and for all $t\in[T_0,T^n]$, the following estimate is satisfied*:

$$\left\|\begin{pmatrix}u_1^n(t)\\u_2^n(t)\end{pmatrix}-\begin{pmatrix}R_1(t)\\R_2(t)\end{pmatrix}\right\|_{H^1\times H^1}\leqslant e^{-\sqrt{\omega_\star}v_\star t}. \tag{3.4}$$

As T^n goes to $+\infty$, the sequence $(u_1^n,u_2^n)^\intercal$ provides a better approximation of a multi-speeds solitary wave. What remains to show is the convergence of this sequence. Owing to local well-posedness and uniform estimates, the main issue is to obtain the convergence of the sequence of initial data $(u_1^n(T_0), u_2^n(T_0))^\intercal$. This is the object of the following proposition, which will be proved in Section 5.

PROPOSITION 2 (Compactness). *There exists $(u_1^0, u_2^0)^\intercal \in H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ such that, possibly for a subsequence only, $(u_1^n(T_0), u_2^n(T_0))^\intercal \to (u_1^0, u_2^0)^\intercal$ strongly in $H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ for any $s \in [0,1)$ when $n \to +\infty$.*

We can now prove Theorem 1.

Proof of Theorem 1. Let $(u_1^0, u_2^0)^\intercal$ be the initial data given by Proposition 2 and let $(u_1, u_2)^\intercal$ be the solution to (1.1) on $[T_0, T^\infty)$ with initial data $(u_1(T_0), u_2(T_0))^\intercal = (u_1^0, u_2^0)^\intercal$. We show that $T^\infty = +\infty$ and that $(u_1, u_2)^\intercal$ fulfills the conclusions of Theorem 1. From Proposition 2, the local well-posedness theory for (1.1) and the boundedness in $H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ (implied by Proposition 1), we have, for $t \in [T_0, T^\infty)$, the convergences

$$\begin{pmatrix} u_1^n(t) \\ u_2^n(t) \end{pmatrix} \underset{H^1\times H^1}{\overset{H^s\times H^s}{\rightrightarrows}} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},$$

where the convergence is taken strongly in $H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ for any $0 \leqslant s < 1$ and weakly in $H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$. Consequently, we can estimate, for all $t \in [T_0, T^\infty)$,

$$\left\| \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} - \begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} \right\|_{H^1\times H^1} \leqslant \liminf_{n\to+\infty} \left\| \begin{pmatrix} u_1^n(t) \\ u_2^n(t) \end{pmatrix} - \begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} \right\|_{H^1\times H^1} \leqslant e^{-\sqrt{\omega_\star} v_\star t}.$$

In particular, this implies that $(u_1(t), u_2(t))^\intercal$ is bounded in $H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ on $[T_0, T^\infty)$. Hence, the blow-up alternative implies that $T^\infty = +\infty$ and therefore $(u_1, u_2)^\intercal$ satisfies the conclusions of Theorem 1. □

4. *Uniform estimates*

In this section, we prove Proposition 1. From the local well-posedness theory, estimate (3.4) always holds on some short interval around T^n. The goal of the following lemma is to allow us to stretch this interval up to the interval $[T_0, T^n]$.

LEMMA 2 (Bootstrap). *There exists $v_\sharp$ such that if $v_\star > v_\sharp$, then there exists $T_0 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$, the following property is satisfied for any $t_0 \in [T_0, T^n]$.*

If, for all $t \in [t_0, T^n]$, we have

$$\left\| \begin{pmatrix} u_1^n(t) \\ u_2^n(t) \end{pmatrix} - \begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} \right\|_{H^1\times H^1} \leqslant e^{-\sqrt{\omega_\star} v_\star t},$$

then, for all $t \in [t_0, T^n]$, we have

$$\left\| \begin{pmatrix} u_1^n(t) \\ u_2^n(t) \end{pmatrix} - \begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} \right\|_{H^1\times H^1} \leqslant \frac{1}{2} e^{-\sqrt{\omega_\star} v_\star t}.$$

Before going further, we indicate how Lemma 2 is used to prove Proposition 1.

Proof of Proposition 1. Let T_0, n_0, $v_\sharp$ be given by Lemma 2, fix $n > n_0$ and assume $v_\star > v_\sharp$. Define

$$t_\sharp := \inf\{t_\dagger \text{ such that (3.4) holds for all } t \in [t_\dagger, T^n]\}.$$

From the local well-posedness theory we know that $t_\sharp < T^n$. We prove by contradiction that $t_\sharp = T_0$. Assume that $t_\sharp > T_0$. By Lemma 2, for all $t \in [t_\sharp, T^n]$ we have

$$\left\| \begin{pmatrix} u_1^n(t) \\ u_2^n(t) \end{pmatrix} - \begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} \right\|_{H^1 \times H^1} \leqslant \frac{1}{2} e^{-\sqrt{\omega_\star} v_\star t}.$$

Therefore, by continuity of $(u_1^n, u_2^n)^\intercal$, there exists $t_\ddagger < t_\sharp$ such that (3.4) holds on $[t_\ddagger, T^n]$, hence contradicting the minimality of $t_\sharp$. As a consequence, $t_\sharp = T_0$ and the proposition is proved. □

Before proving Lemma 2, we need some preparation. We will work for fixed n, hence dependency in n will only be understood, except for T^n. In particular, we shall denote u_1^n by u_1, etc. Let $(\varepsilon_1, \varepsilon_2)^\intercal \in H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ be such that

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}. \tag{4.1}$$

Take $t_0 < T^n$ and assume the following *bootstrap hypothesis*:

$$\left\| \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix} \right\|_{H^1 \times H^1} \leqslant e^{-\sqrt{\omega_\star} v_\star t} \quad \text{for all } t \in [t_0, T^n]. \tag{4.2}$$

For $j = 1, 2$, we denote by S_j and H_j the functionals defined for the solitary wave R_j in the same way as S_0 and H_0 were for R_0 in (2.4) and (2.5), respectively. Note that, conversely to what was happening in the works [**12**, **13**, **22**], we do not need to localize the functionals around each solitary wave, since in our case the coupling will act as a localizing factor. Let $\mathcal{S}$ be the functional defined for $(w_1, w_2)^\intercal \in H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ by

$$\mathcal{S}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} := S_1(w_1) + S_2(w_2), \tag{4.3}$$

and $\mathcal{H}$ be the functional defined for $(t, (\varpi_1, \varpi_2)^\intercal) \in \mathbb{R} \times H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ by

$$\mathcal{H}\left(t, \begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix}\right) := H_1(t, \varpi_1) + H_2(t, \varpi_2).$$

A direct consequence of Lemma 1 on $\mathcal{H}$ is the following result.

LEMMA 3 (Vectorial Coercivity). *There exists $c_\star > 0$ such that, for all $t \in \mathbb{R}$ and for all $(\varpi_1, \varpi_2)^\intercal \in H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$, we have*

$$c_\star \left\| \begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix} \right\|_{H^1 \times H^1}^2 \leqslant \mathcal{H}\left(t, \begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix}\right) + \sum_{j=1,2} \sum_{k=1}^{\nu_j} \left(\varpi_j, \xi_j^k(t)\right)_2^2, \tag{4.4}$$

where (ξ_j^k) are given for $j = 1, 2$ by Lemma 1.

Note that the use of coercivity properties is reminiscent from the stability theory for standing waves of scalar nonlinear Schrödinger equation developed in [**7**, **15**, **32**, **33**]. However, in this theory, the functional equivalent to $\mathcal{S}$ is a conserved quantity, which is not the case for $\mathcal{S}$ (remark that $\mathcal{S}$ is build upon the conserved quantities of the scalar problem and not upon those of (1.1) given in (3.1)–(3.3)). However, we will still be able to estimate the RHS of (4.4) owing to an $L^2(\mathbb{R}^d)$-control (to deal with the scalar products) and owing to the fact that $\mathcal{S}$ is almost a conservation law (to deal with $\mathcal{H}$).

Lemma 4 ($L^2(\mathbb{R}^d)$-control). *Let $(\varepsilon_1,\varepsilon_2)^\intercal$ be given by (4.1) and assume (4.2). Then there exists $C>0$ independent of $v_\star$ such that, for all $t\in[t_0,T^n]$, the following estimate holds*:

$$\left\|\begin{pmatrix}\varepsilon_1(t)\\ \varepsilon_2(t)\end{pmatrix}\right\|_{L^2\times L^2}\leqslant\frac{C}{\sqrt{\omega_\star}v_\star}\,e^{-\sqrt{\omega_\star}v_\star t}.$$

Lemma 5 (Almost Conservation Law). *Assume (4.2). There exists $T_0>0$ depending only on v_1,v_2 such that if $t_0>T_0$, then there exists $C>0$ independent of n and of $v_\star$ such that, for all $t\in[t_0,T^n]$, the following estimate holds*:

$$\left|\mathcal{S}\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix}-\mathcal{S}\begin{pmatrix}u_1(T^n)\\ u_2(T^n)\end{pmatrix}\right|\leqslant\frac{C}{\sqrt{\omega_\star}v_\star}\,e^{-2\sqrt{\omega_\star}v_\star t}.\tag{4.5}$$

Before showing Lemmas 4 and 5, we prove Lemma 2.

Proof of Lemma 2. Let $(\varepsilon_1,\varepsilon_2)^\intercal$ be given by (4.1), assume (4.2) and assume also that $t_0>T_0$ where T_0 is given by Lemma 5. Let $t\in[t_0,T^n]$. By Lemma 3, we have the following estimate:

$$c_\star\left\|\begin{pmatrix}\varepsilon_1(t)\\ \varepsilon_2(t)\end{pmatrix}\right\|^2_{H^1\times H^1}\leqslant\mathcal{H}\left(t,\begin{pmatrix}\varepsilon_1(t)\\ \varepsilon_2(t)\end{pmatrix}\right)+\sum_{j=1,2}\sum_{k=1}^{\nu_j}\left(\varepsilon_j(t),\xi_j^k(t)\right)_2^2.\tag{4.6}$$

Using that R_1 and R_2 are critical points of S_1 and S_2, we have

$$\begin{aligned}\mathcal{S}\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix}&=\mathcal{S}\begin{pmatrix}R_1(t)+\varepsilon_1(t)\\ R_2(t)+\varepsilon_2(t)\end{pmatrix}\\&=\mathcal{S}\begin{pmatrix}R_1(t)\\ R_2(t)\end{pmatrix}+\mathcal{H}\left(t,\begin{pmatrix}\varepsilon_1(t)\\ \varepsilon_2(t)\end{pmatrix}\right)+O\left(\left\|\begin{pmatrix}\varepsilon_1(t)\\ \varepsilon_2(t)\end{pmatrix}\right\|^3_{H^1\times H^1}\right).\end{aligned}\tag{4.7}$$

By Lemma 5, we have

$$\left|\mathcal{S}\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix}-\mathcal{S}\begin{pmatrix}u_1(T^n)\\ u_2(T^n)\end{pmatrix}\right|\leqslant\frac{C}{\sqrt{\omega_\star}v_\star}\,e^{-2\sqrt{\omega_\star}v_\star t}.\tag{4.8}$$

By definition of $(u_1,u_2)^\intercal$ and since $\mathcal{S}$ is made of conserved quantities for R_1 and R_2, we have:

$$\mathcal{S}\begin{pmatrix}u_1(T^n)\\ u_2(T^n)\end{pmatrix}=\mathcal{S}\begin{pmatrix}R_1(T^n)\\ R_2(T^n)\end{pmatrix}=\mathcal{S}\begin{pmatrix}R_1(t)\\ R_2(t)\end{pmatrix}.\tag{4.9}$$

From the bootstrap assumption (4.2) we have

$$O\left(\left\|\begin{pmatrix}\varepsilon_1(t)\\ \varepsilon_2(t)\end{pmatrix}\right\|^3_{H^1\times H^1}\right)=Ce^{-3\sqrt{\omega_\star}v_\star t}.\tag{4.10}$$

Combining (4.7)–(4.10), we infer that, possibly increasing T_0, we have

$$\left|\mathcal{H}\left(t,\begin{pmatrix}\varepsilon_1(t)\\ \varepsilon_2(t)\end{pmatrix}\right)\right|\leqslant\frac{C}{\sqrt{\omega_\star}v_\star}\,e^{-2\sqrt{\omega_\star}v_\star t}.\tag{4.11}$$

Hence, to control the $H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$-norm, it remains to control the $L^2(\mathbb{R}^d)$-scalar products in the RHS of (4.6). This is done using Lemma 4 and remembering that the ξ_j^k are bounded in $L^2(\mathbb{R}^d)$:

$$\sum_{j=1,2}\sum_{k=1}^{\nu_j}\left(\varepsilon_j(t),\xi_j^k(t)\right)_2^2\leqslant C\left\|\begin{pmatrix}\varepsilon_1(t)\\ \varepsilon_2(t)\end{pmatrix}\right\|^2_{L^2\times L^2}\leqslant\frac{C}{\sqrt{\omega_\star}v_\star}\,e^{-2\sqrt{\omega_\star}v_\star t}.\tag{4.12}$$

Combining (4.6), (4.11), and (4.12), we get

$$\left\| \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix} \right\|_{H^1\times H^1}^2 \leqslant \frac{C}{\sqrt{\omega_\star} v_\star}\, e^{-2\sqrt{\omega_\star} v_\star t}.$$

Therefore, there exists $v_\sharp$ such that if $v_\star > v_\sharp$, then we have

$$\left\| \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix} \right\|_{H^1\times H^1} \leqslant \frac{1}{2}\, e^{-\sqrt{\omega_\star} v_\star t},$$

which is the desired conclusion. □

The following estimate on the interaction of the two solitary waves will be central in the proofs of Lemmas 4 and 5.

LEMMA 6 (Solitary Waves Interaction). *There exists $C > 0$ depending on Φ_1, Φ_2, ω_1, ω_2, μ_1, μ_2, but not on v_1, v_2 such that, for all $x \in \mathbb{R}^d$, we have*

$$\Big\| |R_1(t)||R_2(t)| \Big\|_{L^2} \leqslant Ce^{-(3/2)\sqrt{\omega_\star} v_\star t},$$

$$\Big\| (|R_1(t)| + |\nabla R_1(t)|)(|R_2(t)| + |\nabla R_2(t)|) \Big\|_{L^2} \leqslant C(1+|v_1|+|v_2|)^2\, e^{-(3/2)\sqrt{\omega_\star} v_\star t}.$$

Proof. Take $0 < \eta < 1$. Each Φ_j verifies

$$|\Phi_j(x)| + |\nabla \Phi_j(x)| \leqslant Ce^{-\eta|x|},$$

where $C = C(\Phi_j)$. Using the definition (1.4) of a solitary wave, we have, for each R_j, the estimate

$$|R_j(t,x)| + |\nabla R_j(t,x)| \leqslant C(1+|v_j|)\, e^{-\eta\sqrt{\omega_j}|x-v_jt-x_j|},$$

where $C = C(\Phi_j, \omega_j, \mu_j)$. Therefore,

$$\begin{aligned}&(|R_1(t,x)| + |\nabla R_1(t,x)|)(|R_2(t,x)| + |\nabla R_2(t,x)|)\\ &\quad \leqslant C(1+|v_1|+|v_2|)^2\, e^{-\eta\sqrt{\min_{j=1,2}\{\omega_j\}}(|x-v_1t-x_1|+|x-v_2t-x_2|)},\end{aligned}$$

where C depends on Φ_1, Φ_2, ω_1, ω_2 and μ_1, μ_2. Let $0 < \delta < \eta$. Since

$$|(v_1 - v_2)t| \leqslant |x - v_1t| + |x - v_2t|,$$

we infer that

$$\begin{aligned}&(|R_1(t,x)| + |\nabla R_1(t,x)|)(|R_2(t,x)| + |\nabla R_2(t,x)|)\\ &\quad \leqslant C(1+|v_1|+|v_2|)^2 e^{-\delta\sqrt{\min_{j=1,2}\{\omega_j\}}(|x-v_1t-x_1|+|x-v_2t-x_2|)}\\ &\qquad \cdot e^{-(\eta-\delta)\sqrt{\min_{j=1,2}\{\omega_j\}}|(v_1-v_2)t|},\end{aligned}$$

where now C depends also on x_1, x_2. Choosing $\eta = \frac{7}{8}$, $\delta = \frac{1}{8}$ and remembering that $\omega_\star = \frac{1}{4}\min\{\omega_1, \omega_2\}$ and $v_\star = |v_1 - v_2|$, we obtain

$$\begin{aligned}&(|R_1(t,x)| + |\nabla R_1(t,x)|)(|R_2(t,x)| + |\nabla R_2(t,x)|)\\ &\quad \leqslant C(1+|v_1|+|v_2|)^2 e^{-(1/2)\sqrt{\omega_\star}(|x-v_1t-x_1|+|x-v_2t-x_2|)}\, e^{-(3/2)\sqrt{\omega_\star} v_\star t}.\end{aligned}$$

Taking the $L^2(\mathbb{R}^d)$-norm and using the Cauchy–Schwartz inequality, we get

$$\begin{aligned}&\Big\|(|R_1(t)|+|\nabla R_1(t)|)(|R_2(t)|+|\nabla R_2(t)|)\Big\|_{L^2}\\&\quad\leqslant C(1+|v_1|+|v_2|)^2 e^{-(3/2)\sqrt{\omega_\star}v_\star t}\|e^{-(1/2)\sqrt{\omega_\star}|x|}\|_{L^2}\\&\quad\leqslant C(1+|v_1|+|v_2|)^2 e^{-(3/2)\sqrt{\omega_\star}v_\star t},\end{aligned}$$

which is the desired conclusion. □

To prove the $L^2(\mathbb{R}^d)$-control Lemma 4, as in [**12**] we adopt the following strategy. We first write the system satisfied by $(\varepsilon_1,\varepsilon_2)^\intercal$. Then, we differentiate in time the $L^2(\mathbb{R}^d)$-masses of ε_1 and ε_2, and estimate the result with $e^{-2\sqrt{\omega_\star}v_\star t}$. Integrating in time finally allows us to gain the extra factor $1/\sqrt{\omega_\star}v_\star$.

Proof of Lemma 4. The couple $(\varepsilon_1,\varepsilon_2)^\intercal$ satisfies the equation

$$i\partial_t\begin{pmatrix}\varepsilon_1\\\varepsilon_2\end{pmatrix}+\mathcal{L}\begin{pmatrix}\varepsilon_1\\\varepsilon_2\end{pmatrix}+\mathcal{N}\begin{pmatrix}\varepsilon_1\\\varepsilon_2\end{pmatrix}+\mathcal{F}=0$$

where $\mathcal{L}$ denotes the linear part in $(\varepsilon_1,\varepsilon_2)^\intercal$, $\mathcal{N}$ the nonlinear part and $\mathcal{F}$ the source term. Precisely, we set

$$\mathcal{L}\begin{pmatrix}\varepsilon_1\\\varepsilon_2\end{pmatrix}:=\begin{pmatrix}L_1(\varepsilon_1,\varepsilon_2)\\L_2(\varepsilon_1,\varepsilon_2)\end{pmatrix},\quad \mathcal{N}\begin{pmatrix}\varepsilon_1\\\varepsilon_2\end{pmatrix}:=\begin{pmatrix}N_1(\varepsilon_1,\varepsilon_2)\\N_2(\varepsilon_1,\varepsilon_2)\end{pmatrix},\quad \mathcal{F}:=\beta\begin{pmatrix}|R_1|^2R_2\\|R_2|^2R_1\end{pmatrix},$$

where

$$\begin{aligned}\begin{pmatrix}L_1(\varepsilon_1,\varepsilon_2)\\L_2(\varepsilon_1,\varepsilon_2)\end{pmatrix}&=\begin{pmatrix}\Delta\varepsilon_1+(2\mu_1|R_1|^2+\beta|R_2|^2)\varepsilon_1+\mu_1R_1^2\bar\varepsilon_1+\beta(R_1\bar R_2\varepsilon_2+R_1R_2\bar\varepsilon_2)\\\Delta\varepsilon_2+(2\mu_2|R_2|^2+\beta|R_1|^2)\varepsilon_2+\mu_2R_2^2\bar\varepsilon_2+\beta(\bar R_1R_2\varepsilon_1+R_1R_2\bar\varepsilon_1)\end{pmatrix},\\\begin{pmatrix}N_1(\varepsilon_1,\varepsilon_2)\\N_2(\varepsilon_1,\varepsilon_2)\end{pmatrix}&=\begin{pmatrix}\mu_1\left(\bar R_1\varepsilon_1^2+2R_1|\varepsilon_1|^2+|\varepsilon_1|^2\varepsilon_1\right)\\\mu_2\left(\bar R_2\varepsilon_2^2+2R_2|\varepsilon_2|^2+|\varepsilon_2|^2\varepsilon_2\right)\end{pmatrix}\\&\quad+\beta\begin{pmatrix}R_2\bar\varepsilon_2\varepsilon_1+\bar R_2\varepsilon_2\varepsilon_1+R_1|\varepsilon_2|^2+|\varepsilon_2|^2\varepsilon_1\\R_1\bar\varepsilon_1\varepsilon_2+\bar R_1\varepsilon_1\varepsilon_2+R_2|\varepsilon_1|^2+|\varepsilon_1|^2\varepsilon_2\end{pmatrix}.\end{aligned}$$

We make the computations for ε_1, the case of ε_2 being exactly symmetric.

$$\begin{aligned}\frac{\partial}{\partial t}M(\varepsilon_1)&=\frac{1}{2}\frac{\partial}{\partial t}\left(\|\varepsilon_1(t)\|_{L^2}^2\right)\\&=-\mathcal{Im}\int_{\mathbb{R}^d}(L_1(\varepsilon_1,\varepsilon_2)\bar\varepsilon_1+N_1(\varepsilon_1,\varepsilon_2)\bar\varepsilon_1+\beta|R_1|^2R_2\bar\varepsilon_1)\,dx.\end{aligned}\tag{4.13}$$

Using the bootstrap assumption (4.2), we immediately obtain the following estimate:

$$\begin{aligned}\left|\mathcal{Im}\int_{\mathbb{R}^d}L_1(\varepsilon_1,\varepsilon_2)\bar\varepsilon_1\,dx\right|&=\left|\mathcal{Im}\int_{\mathbb{R}^d}\mu_1R_1^2\bar\varepsilon_1^2+\beta(R_1\bar R_2\bar\varepsilon_1\varepsilon_2+R_1R_2\bar\varepsilon_1\bar\varepsilon_2)\,dx\right|,\\&\leqslant C(\|R_1\|_{L^\infty}^2+\|R_2\|_{L^\infty}^2)(\|\varepsilon_1\|_{H^1}^2+\|\varepsilon_2\|_{H^1}^2),\\&\leqslant Ce^{-2\sqrt{\omega_\star}v_\star t}.\end{aligned}\tag{4.14}$$

Here, and in the rest of the proof, the constant C may depend on $\beta,\mu_1,\mu_2,\Phi_1,\Phi_2,x_1,x_2$, but *not* on v_1,v_2. This is due to the fact that $\|R_j\|_{L^\infty}=\sqrt{\omega_j/\mu_j}\|\Phi_j\|_{L^\infty}$ for $j=1,2$. We consider now the nonlinear part. Since $d\leqslant 3$, we have the embedding of $H^1(\mathbb{R}^d)$ into $L^3(\mathbb{R}^d)$ and $L^4(\mathbb{R}^d)$

and therefore we can prove that

$$\begin{aligned}\left|\mathcal{I}m\int_{\mathbb{R}^d} N_1(\varepsilon_1,\varepsilon_2)\bar{\varepsilon}_1\,dx\right| &\leqslant \left|\int_{\mathbb{R}^d}\mu_1(\bar{R}_1\varepsilon_1^2+2R_1|\varepsilon_1|^2+|\varepsilon_1|^2\varepsilon_1)\bar{\varepsilon}_1\,dx\right| \\ &\quad+\left|\int_{\mathbb{R}^d}\beta|(R_2\bar{\varepsilon}_2\varepsilon_1+\bar{R}_2\varepsilon_2\varepsilon_1+R_1|\varepsilon_2|^2+|\varepsilon_2|^2\varepsilon_1)\bar{\varepsilon}_1\,dx\right| \\ &\leqslant C(\|R_1\|_{L^\infty}+\|R_2\|_{L^\infty})(\|\varepsilon_1\|_{H^1}^2\|\varepsilon_2\|_{H^1}^2+\|\varepsilon_1\|_{H^1}^3+\|\varepsilon_1\|_{H^1}^4) \\ &\leqslant Ce^{-3\sqrt{\omega_\star}v_\star t}. \end{aligned} \tag{4.15}$$

Last, in order to estimate the source term, we need to use also Lemma 6 in combination with the bootstrap assumption (4.2).

$$\left|\mathcal{I}m\int|R_2|^2R_1\bar{\varepsilon}_1\right|\leqslant\|R_2\|_{L^\infty}\,\||R_1||R_2|\|_{L^2}\,\|\varepsilon_1\|_{H^1}\leqslant Ce^{-(5/2)\sqrt{\omega_\star}v_\star t}. \tag{4.16}$$

Combining (4.13)–(4.16) we get,

$$\left|\frac{\partial}{\partial t}M(\varepsilon_1)\right|\leqslant Ce^{-2\sqrt{\omega_\star}v_\star t}.$$

Integrating in time and recalling that by definition we have $\varepsilon_1(T^n)=0$, we obtain

$$M(\varepsilon_1(t))\leqslant\int_t^{T^n}\left|\frac{\partial}{\partial s}M(\varepsilon_1(s))\right|ds\leqslant\frac{C}{\sqrt{\omega_\star}v_\star}\;e^{-2\sqrt{\omega_\star}v_\star t},$$

which is the desired conclusion for ε_1. As already said, the calculations for ε_2 are perfectly symmetric, hence the lemma is proved. □

Recall that $\mathcal{S}$ is built with scalar energies, masses, and momentums. To prove Lemma 5, the idea is, as for the proof of Lemma 4, to differentiate in time the various quantities involved in $\mathcal{S}$ (see (4.3) and (2.4)), control the result with $e^{-2\sqrt{\omega_\star}v_\star t}$ and then integrate to gain the extra factor $1/\sqrt{\omega_\star}v_\star$.

Proof of Lemma 5. Since the scalar masses are conserved by the flow of (1.1) and $(u_1,u_2)^\intercal$ is a solution of (1.1), it follows immediately that

$$|M(u_1(t))-M(u_1(T^n))|+|M(u_2(t))-M(u_2(T^n))|=0. \tag{4.17}$$

For the momentum part, we need to estimate

$$|v_1\cdot(P(u_1(t))-P(u_1(T^n)))+v_2\cdot(P(u_2(t))-P(u_2(T^n)))|\,.$$

In fact, since the total momentum (3.2) is a conserved quantity, we have to estimate

$$|(v_1-v_2)\cdot(P(u_1(t))-P(u_1(T^n)))|=v_\star\,|P(u_1(t))-P(u_1(T^n))|\,. \tag{4.18}$$

Hence, we differentiate at time t the scalar momentum P_1. Using the system (1.1) satisfied by $(u_1,u_2)^\intercal$ and integrations by parts, we obtain

$$\frac{\partial}{\partial t}P(u_1)=-\mathcal{I}m\int_{\mathbb{R}^d}\partial_t u_1\nabla\bar{u}_1\,dx=-\frac{1}{2}\int_{\mathbb{R}^d}|u_2|^2\nabla|u_1|^2\,dx.$$

We recall that $u_1=R_1+\varepsilon_1$ and $u_2=R_2+\varepsilon_2$, and replace in the previous equation to get

$$\begin{aligned}\frac{\partial}{\partial t}P(u_1)=-\frac{1}{2}\int_{\mathbb{R}^d}&|R_2|^2\nabla|R_1|^2+2|R_2|^2\nabla(\mathcal{R}e(\bar{R}_1\varepsilon_1))+|R_2|^2\nabla|\varepsilon_1|^2 \\ &+2\mathcal{R}e(\bar{R}_2\varepsilon_2)\nabla|R_1|^2+4\mathcal{R}e(\bar{R}_2\varepsilon_2)\nabla(\mathcal{R}e(\bar{R}_1\varepsilon_1))+2\mathcal{R}e(\bar{R}_2\varepsilon_2)\nabla|\varepsilon_1|^2 \\ &+|\varepsilon_2|^2\nabla|R_1|^2+2|\varepsilon_2|^2\nabla(\mathcal{R}e(\bar{R}_1\varepsilon_1)+|\varepsilon_2|^2\nabla|\varepsilon_1|^2\,dx.\end{aligned} \tag{4.19}$$

We treat the various products appearing differently depending on their order in R_j and ε_j. When there is a product of R_1 and R_2 or of their derivatives, we use Lemma 6, as for the following term.

$$\left|\int_{\mathbb{R}^d} |R_2|^2\nabla|R_1|^2\,dx\right| \leqslant C\|(|R_1|+|\nabla R_1|)|R_2|\|_{L^2}^2 \leqslant C(1+|v_1|+|v_2|)^4 e^{-3\sqrt{\omega_\star}v_\star t}.$$

To deal with the ε_j, we use the bootstrap assumption (4.2). With the help of Cauchy–Schwartz and Hölder inequalities and Sobolev embeddings, we get

$$\left|\int_{\mathbb{R}^d} |\varepsilon_2|^2\nabla|\varepsilon_1|^2\,dx\right| \leqslant C\|\nabla\varepsilon_1\|_{L^2}\|\varepsilon_1\|_{L^6}\|\varepsilon_2\|_{L^6}^2 \leqslant Ce^{-4\sqrt{\omega_\star}v_\star t}. \tag{4.20}$$

We possibly combine the two arguments as follows:

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \mathcal{R}e(\bar{R}_2\varepsilon_2)\nabla(\mathcal{R}e(\bar{R}_1\varepsilon_1))\,dx\right| &\leqslant \||R_2|(|R_1|+|\nabla R_1|)\|_{L^2}\||\varepsilon_2|(|\varepsilon_1|+|\nabla\varepsilon_1|)\|_{L^2}\\ &\leqslant C(1+|v_1|+|v_2|)^2 e^{-\frac{7}{2}\sqrt{\omega_\star}v_\star t}.\end{aligned} \tag{4.21}$$

When there is an extra R_j that we cannot use with Lemma 6, we just take its $L^\infty(\mathbb{R}^d)$-norm:

$$\begin{aligned}&\left|\int_{\mathbb{R}^d} |R_2|^2\nabla(\mathcal{R}e(\bar{R}_1\varepsilon_1))\,dx\right| + \left|\int_{\mathbb{R}^d} \mathcal{R}e(\bar{R}_2\varepsilon_2)\nabla|R_1|^2\,dx\right|\\ &\quad\leqslant C(\|R_1\|_{L^\infty}+\|R_2\|_{L^\infty})\|(|R_1|+|\nabla R_1|)|R_2|\|_{L^2}(\||\varepsilon_1|+|\nabla\varepsilon_1|\|_{L^2}+\|\varepsilon_2\|_{L^2})\\ &\quad\leqslant C(1+|v_1|+|v_2|)^2 e^{-(5/2)\sqrt{\omega_\star}v_\star t}.\end{aligned} \tag{4.22}$$

The following estimate is obtained with similar arguments:

$$\left|\int_{\mathbb{R}^d} \mathcal{R}e(\bar{R}_2\varepsilon_2)\nabla|\varepsilon_1|^2\,dx\right| \leqslant \|R_2\|_{L^\infty}\|\varepsilon_1\|_{L^4}\|\nabla\varepsilon_1\|_{L^2}\|\varepsilon_2\|_{L^4} \leqslant Ce^{-3\sqrt{\omega_\star}v_\star t}. \tag{4.23}$$

After an integration by parts, the next product can be treated as in (4.23):

$$\left|\int_{\mathbb{R}^d} |\varepsilon_2|^2\nabla(\mathcal{R}e(\bar{R}_1\varepsilon_1))\,dx\right| = \left|\int_{\mathbb{R}^d} \nabla|\varepsilon_2|^2(\mathcal{R}e(\bar{R}_1\varepsilon_1))\,dx\right| \leqslant Ce^{-3\sqrt{\omega_\star}v_\star t}. \tag{4.24}$$

Before estimating the remaining two terms, we make a remark about $\|\nabla|R_j|^2\|_{L^\infty}$. From the definition of a solitary wave (1.4), we have

$$\begin{aligned}\nabla(|R_j|^2) &= \frac{\omega_j}{\mu_j}\nabla|\Phi_j(\sqrt{\omega_j}(x-v_jt-x_j))|^2\\ &= \frac{2\omega_j^{3/2}}{\mu_j}\mathcal{R}e(\bar{\Phi}_j(\sqrt{\omega_j}(x-v_jt-x_j))\nabla\Phi_j(\sqrt{\omega_j}(x-v_jt-x_j))).\end{aligned}$$

This implies that

$$\|\nabla|R_j|^2\|_{L^\infty} \leqslant \frac{2\omega_j^{3/2}}{\mu_j}\|\Phi_j\|_{L^\infty}\|\nabla\Phi_j\|_{L^\infty},$$

and, in particular, $\|\nabla|R_j|^2\|_{L^\infty}$ *does not depend on* v_j. We can now write

$$\left|\int_{\mathbb{R}^d} |\varepsilon_2|^2\nabla|R_1|^2\,dx\right| \leqslant \|\nabla|R_1|^2\|_{L^\infty}\|\varepsilon_2\|_{L^2}^2 \leqslant C\|\varepsilon_2\|_{L^2}^2,$$

Here, if we use directly the bootstrap assumption (4.2), we will miss the correct estimate by a factor $1/v_\star$ because of the $v_\star$ appearing in (4.18). However, remembering that we already improved (4.2) at the $L^2(\mathbb{R}^d)$-level in Lemma 4, we can conclude that

$$\left|\int_{\mathbb{R}^d} |\varepsilon_2|^2\nabla|R_1|^2\,dx\right| \leqslant \frac{C}{\sqrt{\omega_\star}v_\star}\,e^{-2\sqrt{\omega_\star}v_\star}. \tag{4.25}$$

The last term is treated in a similar fashion after an integration by parts.

$$\left|\int_{\mathbb{R}^d} |R_2|^2\nabla|\varepsilon_1|^2\,dx\right| = \left|\int_{\mathbb{R}^d} \nabla|R_2|^2|\varepsilon_1|^2\,dx\right| \leqslant \frac{C}{\sqrt{\omega_\star}v_\star}\, e^{-2\sqrt{\omega_\star}v_\star}. \tag{4.26}$$

Take now T_0 large enough so that $v_\star(1+|v_1|+|v_2|)^4 e^{-(1/2)\sqrt{\omega_\star}v_\star T_0} < 1$. With this assumption and the fact that $t_0 > T_0$, we can now combine (4.19)–(4.26), and argue in the same fashion for the scalar momentum of u_2, to finally find

$$\left|v_\star \frac{\partial}{\partial t} P(u_1)\right| \leqslant Ce^{-2\sqrt{\omega_\star}v_\star t} \tag{4.27}$$

for C depending on $\Phi_1, \Phi_2, \omega_1, \omega_2, \mu_1, \mu_2$, but not on v_1, v_2. Therefore, we obtain the following control on scalar momentums:

$$\begin{aligned}
&|v_1\cdot(P(u_1(t)) - P(u_1(T^n))) + v_2\cdot(P(u_2(t)) - P(u_2(T^n)))| \\
&\quad = v_\star\,|P(u_1(t)) - P(u_1(T^n))| \leqslant \int_t^{T^n} \left|v_\star\frac{\partial}{\partial s}P(u_1(s))\right| ds \\
&\quad \leqslant \frac{C}{\sqrt{\omega_\star}v_\star}\, e^{-2\sqrt{\omega_\star}v_\star t}.
\end{aligned} \tag{4.28}$$

Now, we treat the energy part. The direct approach consisting in trying to differentiate in time the energies $E(u_j,\mu_j)$ and then argue as for the momentums is bound to fail because of the appearance of terms like

$$\int_{\mathbb{R}^d} \mathcal{Im}(\varepsilon_1\nabla\bar{\varepsilon}_1)\mathcal{Re}(\varepsilon_2\nabla\bar{\varepsilon}_2)\,dx,$$

which, unless $d=1$, we cannot treat with an $H^1(\mathbb{R}^d)$-information like (4.2). However, if we use the conservation of the total energy $\mathcal{E}$, we remark that,

$$\begin{aligned}
&E(u_1(t),\mu_1) - E(u_1(T^n),\mu_1) + E(u_2(t),\mu_2) - E(u_2(T^n),\mu_2) \\
&\quad = \mathcal{E}\begin{pmatrix} u_1(t) \\ u_2(t)\end{pmatrix} - \mathcal{E}\begin{pmatrix} u_1(T^n) \\ u_2(T^n)\end{pmatrix} - \beta\int_{\mathbb{R}^d}(|u_1(t)|^2|u_2(t)|^2 - |u_1(T^n)|^2|u_2(T^n)|^2)\,dx \\
&\quad = -\beta\int_{\mathbb{R}^d}(|u_1(t)|^2|u_2(t)|^2 - |u_1(T^n)|^2|u_2(T^n)|^2)\,dx.
\end{aligned} \tag{4.29}$$

Therefore, it is enough to prove that

$$\int_{\mathbb{R}^d}(|u_1(t)|^2|u_2(t)|^2 + |u_1(T^n)|^2|u_2(T^n)|^2)\,dx \leqslant \frac{C}{\sqrt{\omega_\star}v_\star}\, e^{-2\sqrt{\omega_\star}v_\star t}. \tag{4.30}$$

To obtain (4.30), we do not differentiate in time the LHS, but instead we try to obtain the estimate directly. First, note that by definition of $(u_1,u_2)^\intercal$ and Lemma 6 we have

$$\int_{\mathbb{R}^d}|u_1(T^n)|^2|u_2(T^n)|^2\,dx = \int_{\mathbb{R}^d}|R_1(T^n)|^2|R_2(T^n)|^2\,dx \leqslant Ce^{-3\sqrt{\omega_\star}v_\star T^n} \leqslant Ce^{-3\sqrt{\omega_\star}v_\star t}.$$

As before, for the other part, we replace u_j by $R_j+\varepsilon_j$ and develop.

$$\begin{aligned}
\int_{\mathbb{R}^d}|u_1|^2|u_2|^2\,dx = \int_{\mathbb{R}^d}(&|R_1|^2|R_2|^2 + 2|R_1|^2\mathcal{Re}(\bar{R}_2\varepsilon_2) + |R_1|^2|\varepsilon_2|^2 \\
&+ 2\mathcal{Re}(\bar{R}_1\varepsilon_1)|R_2|^2 + 4\mathcal{Re}(\bar{R}_1\varepsilon_1)\mathcal{Re}(\bar{R}_2\varepsilon_2) + 2\mathcal{Re}(\bar{R}_1\varepsilon_1)|\varepsilon_2|^2 \\
&+ |\varepsilon_1|^2|R_2|^2 + |\varepsilon_1|^2\mathcal{Re}(\bar{R}_2\varepsilon_2) + |\varepsilon_1|^2|\varepsilon_2|^2)\,dx.
\end{aligned} \tag{4.31}$$

The following estimates are obtained using the same arguments as in the momentum case, in particular Lemma 6 and the bootstrap assumption (4.2).

$$\left|\int_{\mathbb{R}^d} |R_1|^2|R_2|^2\,dx\right| \leqslant Ce^{-3\sqrt{\omega_\star}v_\star t}, \tag{4.32}$$

$$\left|\int_{\mathbb{R}^d} 2|R_1|^2\mathcal{R}e(\bar{R}_2\varepsilon_2)\,dx\right| \leqslant C\|R_1\|_{L^\infty}\||R_1||R_2|\|_{L^2}\|\varepsilon_2\|_{L^2} \leqslant Ce^{-(5/2)\sqrt{\omega_\star}v_\star t}, \tag{4.33}$$

$$\left|\int_{\mathbb{R}^d} \mathcal{R}e(\bar{R}_1\varepsilon_1)|R_2|^2\,dx\right| \leqslant C\|R_2\|_{L^\infty}\||R_1||R_2|\|_{L^2}\|\varepsilon_1\|_{L^2} \leqslant Ce^{-(5/2)\sqrt{\omega_\star}v_\star t}, \tag{4.34}$$

$$\left|\int_{\mathbb{R}^d} \mathcal{R}e(\bar{R}_1\varepsilon_1)\mathcal{R}e(\bar{R}_2\varepsilon_2)\,dx\right| \leqslant C\||R_1||R_2|\|_{L^2}\||\varepsilon_1||\varepsilon_2|\|_{L^2} \leqslant Ce^{-(7/2)\sqrt{\omega_\star}v_\star t}, \tag{4.35}$$

$$\left|\int_{\mathbb{R}^d} \mathcal{R}e(\bar{R}_1\varepsilon_1)|\varepsilon_2|^2\,dx\right| \leqslant C\|R_1\|_{L^\infty}\|\varepsilon_1\|_{L^2}\|\varepsilon_2\|_{L^4}^2 \leqslant Ce^{-3\sqrt{\omega_\star}v_\star t}, \tag{4.36}$$

$$\left|\int_{\mathbb{R}^d} |\varepsilon_1|^2\mathcal{R}e(\bar{R}_2\varepsilon_2)\,dx\right| \leqslant \|R_2\|_{L^\infty}\|\varepsilon_1\|_{L^4}^2\|\varepsilon_2\|_{L^2} \leqslant Ce^{-3\sqrt{\omega_\star}v_\star t}, \tag{4.37}$$

$$\left|\int_{\mathbb{R}^d} |\varepsilon_1|^2|\varepsilon_2|^2\,dx\right| \leqslant \|\varepsilon_1\|_{L^4}^2\|\varepsilon_2\|_{L^4}^2 \leqslant Ce^{-4\sqrt{\omega_\star}v_\star t}. \tag{4.38}$$

We need an extra argument for the two remaining terms. Indeed, we have

$$\left|\int_{\mathbb{R}^d} |\varepsilon_1|^2|R_2|^2\,dx\right| \leqslant \|R_2\|_{L^\infty}^2\|\varepsilon_1\|_{L^2}^2 \leqslant C\|\varepsilon_1\|_{L^2}^2,$$
$$\left|\int_{\mathbb{R}^d} |R_1|^2|\varepsilon_2|^2\,dx\right| \leqslant \|R_1\|_{L^\infty}^2\|\varepsilon_2\|_{L^2}^2 \leqslant C\|\varepsilon_2\|_{L^2}^2.$$

As for the momemtum part, if we use (4.2) here, we miss the correct estimate by a factor $1/\sqrt{\omega_\star}v_\star$. However, using Lemma 4, we can conclude that

$$\left|\int_{\mathbb{R}^d} |\varepsilon_1|^2|R_2|^2\,dx\right| \leqslant \frac{C}{\sqrt{\omega_\star}v_\star}e^{-2\sqrt{\omega_\star}v_\star t}, \tag{4.39}$$

$$\left|\int_{\mathbb{R}^d} |R_1|^2|\varepsilon_2|^2\,dx\right| \leqslant \frac{C}{\sqrt{\omega_\star}v_\star}\, e^{-2\sqrt{\omega_\star}v_\star t}. \tag{4.40}$$

Putting together (4.31)–(4.40) and assuming T_0 large enough implies the desired estimate (4.30).

To conclude the proof, we combine (4.17), (4.28)–(4.30) to obtain (4.5). □

5. *Compactness of the sequence of initial data*

In this section, we prove Proposition 2. The proof is similar to the one given in [**12**, **22**] and we repeat it here for the sake of completeness. We again use the superscript n to indicate the dependency in n.

From Proposition 1, we know that $(u_1^n(T_0), u_2^n(T_0))^\intercal$ is bounded in $H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$. Hence, there exist $(u_1^0, u_2^0)^\intercal \in H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$ such that

$$\begin{pmatrix} u_1^n(T_0) \\ u_2^n(T_0)\end{pmatrix} \overset{H^1}{\rightharpoonup} \begin{pmatrix} u_1^0 \\ u_2^0\end{pmatrix}. \tag{5.1}$$

We now prove that convergence in (5.1) holds also strongly in $L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$, the result of Proposition 2 then readily following by interpolation. Take $\delta > 0$, let n be large enough and let $T_\delta \in [T_0, T^n]$ be such that $e^{-\sqrt{\omega_\star}v_\star t} < \sqrt{\delta/4}$. Then, by Proposition 1,

$$\left\|\begin{pmatrix} u_1^n(T_\delta) \\ u_2^n(T_\delta)\end{pmatrix} - \begin{pmatrix} R_1(T_\delta) \\ R_2(T_\delta)\end{pmatrix}\right\|_{H^1\times H^1} \leqslant \sqrt{\frac{\delta}{4}}. \tag{5.2}$$

Take $\rho_\delta > 0$ such that

$$\int_{|x|>\rho_\delta} |R_1(T_\delta)|^2 + |R_2(T_\delta)|^2 \, dx \leqslant \frac{\delta}{4}. \tag{5.3}$$

Then we infer from (5.2) that

$$\int_{|x|>\rho_\delta} |u_1^n(T_\delta)|^2 + |u_2^n(T_\delta)|^2 \, dx \leqslant \frac{\delta}{2}.$$

Our goal is to transfer this smallness up to T_0. Let $\tau : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ cut-off function such that

$$\tau(s) = 0 \quad \text{for } s < 0, \quad \tau(s) = 1 \quad \text{for } s > 1, \quad \tau(s) \in [0,1] \quad \text{for } s \in \mathbb{R}, \quad \|\tau'\|_{L^\infty} \leqslant 2.$$

Take $\kappa_\delta > 0$ and define

$$V(t) := \frac{1}{2}\int_{\mathbb{R}^d} (|u_1^n|^2 + |u_2^n|^2)\tau\left(\frac{|x|-\rho_\delta}{\kappa_\delta}\right) dx.$$

Then we have

$$V'(t) = \mathcal{R}e\int_{\mathbb{R}^d} (\bar{u}_1^n \partial_t u_1^n + \bar{u}_2^n \partial_t u_2^n)\tau\left(\frac{|x|-\rho_\delta}{\kappa_\delta}\right) dx.$$

Using the equation satisfied by u_1 and after an integration by part, we obtain

$$\begin{aligned} \mathcal{R}e\int_{\mathbb{R}^d} \bar{u}_1^n \partial_t u_1^n \tau\left(\frac{|x|-\rho_\delta}{\kappa_\delta}\right) dx &= \mathcal{I}m\int_{\mathbb{R}^d} \bar{u}_1^n \Delta u_1^n \tau\left(\frac{|x|-\rho_\delta}{\kappa_\delta}\right) dx, \\ &= \frac{1}{\kappa_\delta}\mathcal{I}m\int_{\mathbb{R}^d} \bar{u}_1^n \frac{x}{|x|}\cdot \nabla u_1^n \tau'\left(\frac{|x|-\rho_\delta}{\kappa_\delta}\right) dx. \end{aligned} \tag{5.4}$$

From Proposition 1 we know that there exists n_0 such that

$$\sup_{n>n_0} \sup_{t\in[T_0,T^n]} \left\| \begin{pmatrix} u_1^n(t) \\ u_2^n(t) \end{pmatrix} \right\|_{H^1\times H^1} \leqslant 1.$$

Therefore, we infer from (5.4) and similar computations for u_2 that

$$|V'(t)| \leqslant \frac{1}{\kappa_\delta}.$$

Choose now κ_δ such that $(T_\delta - T_0)/\kappa_\delta < \delta/2$. Then

$$V(T_0) - V(T_\delta) = \int_{T_\delta}^{T_0} V'(t)\, dt \leqslant \frac{T_\delta - T_0}{\kappa_\delta} \leqslant \frac{\delta}{2}. \tag{5.5}$$

Set $r_\delta := \kappa_\delta + \rho_\delta$ (note that r_δ is independent of n). Since from (5.3) and the definition of τ we have $V(T_\delta) < \delta/2$, we deduce from (5.5) that

$$\int_{|x|>r_\delta} |u_1^n(T_0)|^2 + |u_2^n(T_0)|^2 \, dx \leqslant V(T_0) \leqslant \delta.$$

Therefore, the sequence $(u_1^n(T_0), u_2^n(T_0))^\intercal$ is $L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ compact, which concludes the proof.

References

1. M. J. Ablowitz, B. Prinari and A. D. Trubatch, *Discrete and continuous nonlinear Schrödinger systems*, London Mathematical Society Lecture Note Series 302 (Cambridge University Press, Cambridge, 2004).
2. G. Agrawal, *Nonlinear fiber optics*, Optics and Photonics (Academic Press, New York, 2007).
3. A. Ambrosetti and E. Colorado, 'Standing waves of some coupled nonlinear Schrödinger equations', *J. London Math. Soc.* (2) 75 (2007) 67–82.

4. H. Berestycki and P.-L. Lions, 'Nonlinear scalar field equations I', *Arch. Rat. Mech. Anal.* 82 (1983) 313–346.

5. H. Berestycki and P.-L. Lions, 'Nonlinear scalar field equations II', *Arch. Rat. Mech. Anal.* 82 (1983) 347–375.

6. T. Cazenave, *Semilinear Schrödinger equations* (New York University — Courant Institute, New York, 2003).

7. T. Cazenave and P.-L. Lions, 'Orbital stability of standing waves for some nonlinear Schrödinger equations', *Comm. Math. Phys.* 85 (1982) 549–561.

8. T. Cazenave and F. B. Weissler, 'The Cauchy problem for the critical nonlinear Schrödinger equation in H^s', *Nonlinear Anal.* 14 (1990) 807–836.

9. M. Colin, T. Colin and M. Ohta, 'Stability of solitary waves for a system of nonlinear Schrödinger equations with three wave interaction', *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26 (2009) 2211–2226.

10. C. Cortázar, M. García-Huidobro and C. S. Yarur, 'On the uniqueness of the second bound state solution of a semilinear equation', *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26 (2009) 2091–2110.

11. C. Cortázar, M. García-Huidobro and C. S. Yarur, 'On the uniqueness of sign changing bound state solutions of a semilinear equation', *Ann. Inst. H. Poincaré Anal. Non Linéaire* 28 (2011) 599–621.

12. R. Côte and S. Le Coz, 'High-speed excited multi-solitons in nonlinear Schrödinger equations', *J. Math. Pures Appl.* (9) 96 (2011) 135–166.

13. R. Côte, Y. Martel and F. Merle, 'Construction of multi-soliton solutions for the L^2-supercritical gKdV and NLS equations', *Rev. Mat. Iberoam.* 27 (2011) 273–302.

14. B. Esry, C. H. Greene, J. P. Burke Jr and J. L. Bohn, 'Hartree–Fock theory for double condensates', *Phys. Rev. Lett.* 78 (1997) 3594–3597.

15. M. Grillakis, J. Shatah and W. A. Strauss, 'Stability theory of solitary waves in the presence of symmetry I', *J. Funct. Anal.* 74 (1987) 160–197.

16. G. L. Lamb Jr, *Elements of soliton theory* (Wiley, New York, 1980).

17. T.-C. Lin and J. Wei, 'Ground state of N coupled nonlinear Schrödinger equations in $\mathbf{R}^n$, $n \leqslant 3$', *Comm. Math. Phys.* 255 (2005) 629–653.

18. P.-L. Lions, 'Solutions complexes d'équations elliptiques semilinéaires dans $\mathbf{R}^N$', *C. R. Acad. Sci. Paris Sér. I Math.* 302 (1986) 673–676.

19. L. A. Maia, E. Montefusco and B. Pellacci, 'Positive solutions for a weakly coupled nonlinear Schrödinger system', *J. Differential Equations* 229 (2006) 743–767.

20. L. A. Maia, E. Montefusco and B. Pellacci, 'Orbital stability property for coupled nonlinear Schrödinger equations', *Adv. Nonlinear Stud.* 10 (2010) 681–705.

21. S. V. Manakov, 'On the theory of two-dimensional stationary self-focusing of electromagnetic waves', *J. Exp. Theoret. Phys.* 38 (1974) 24–253.

22. Y. Martel and F. Merle, 'Multi solitary waves for nonlinear Schrödinger equations', *Ann. Inst. H. Poincaré Anal. Non Linéaire* 23 (2006) 849–864.

23. F. Merle, 'Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity', *Comm. Math. Phys.* 129 (1990) 223–240.

24. E. Montefusco, B. Pellacci and M. Squassina, 'Energy convexity estimates for non-degenerate ground states of nonlinear 1D Schrödinger systems', *Commun. Pure Appl. Anal.* 9 (2010) 867–884.

25. M. Ohta, 'Stability of solitary waves for coupled nonlinear Schrödinger equations', *Nonlinear Anal.* 26 (1996) 933–939.

26. P. C. Schuur, *Asymptotic analysis of soliton problems*, Lecture Notes in Mathematics 1232 (Springer, Berlin, 1986).

27. B. Sirakov, 'Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$', *Comm. Math. Phys.* 271 (2007) 199–221.

28. C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation*, Applied Mathematical Sciences 139 (Springer, New York, 1999). Self-focusing and wave collapse.

29. T. Tao, *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. Local and global analysis.

30. S. Terracini and G. Verzini, 'Multipulse phases in k-mixtures of Bose–Einstein condensates', *Arch. Rat. Mech. Anal.* 194 (2009) 717–741.

31. J. Wei and T. Weth, 'Radial solutions and phase separation in a system of two coupled Schrödinger equations', *Arch. Rat. Mech. Anal.* 190 (2008) 83–106.

32. M. I. Weinstein, 'Modulational stability of ground states of nonlinear Schrödinger equations', *SIAM J. Math. Anal.* 16 (1985) 472–491.

33. M. I. Weinstein, 'Lyapunov stability of ground states of nonlinear dispersive evolution equations', *Comm. Pure Appl. Math.* 39 (1986) 51–67.

Isabella Ianni
Dipartimento di Matematica e Fisica
Seconda Università di Napoli
Viale Lincoln 5
81100 Caserta
Italia

isabella.ianni@unina2.it

Stefan Le Coz
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse cedex 9
France

slecoz@math.univ-toulouse.fr