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AN EXISTENCE AND STABILITY RESULT FOR STANDING WAVES OF NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider a nonlinear Schrödinger equation with a nonlinearity of the form $V(x)g(u)$. Assuming that $V(x)$ behaves like $|x|^{-b}$ at infinity and $g(s)$ like $|s|^p$ around 0, we prove the existence and orbital stability of travelling waves if $1 < p < 1 + (4-2b)/N$.

1. Introduction

This paper concerns the existence and orbital stability of standing waves for the nonlinear Schrödinger equation

$$iu_t + \Delta u = V(x)g(u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \ N \geqslant 3. \tag{1.1}$$

Here $u \in H^1(\mathbb{R}^N, \mathbb{C})$, V is a real-valued potential and g is a nonlinearity satisfying $g(e^{i\theta}s) = e^{i\theta}g(s)$ for $s \in \mathbb{R}$.

A solution of the form $u(t,x) = e^{i\lambda t}\varphi(x)$, where $\lambda \in \mathbb{R}$, is called a standing wave. For solutions of this type with $\varphi \in H^1(\mathbb{R}^N, \mathbb{R})$, (1.1) is equivalent to

$$-\Delta\varphi + \lambda\varphi = V(x)g(\varphi), \quad \varphi \in H^1(\mathbb{R}^N, \mathbb{R}). \tag{1.2}$$

We are interested in the existence of positive solutions of (1.2) for small $\lambda > 0$. In addition we study the stability of the corresponding solutions of (1.1).

In the autonomous case, i.e., when V is a constant, we refer to the fundamental papers of Berestycki and Lions [2] where sufficient and almost necessary conditions are derived for the existence in $H^1(\mathbb{R}^N, \mathbb{R})$ of a solution of (1.2). When (1.2) is nonautonomous only partial results are known. A major difficulty to overcome is the lack of a priori bounds for the solutions. In contrast to the autonomous case, where using dilations and taking advantage of the Pohozaev identity is at the heart of the results of [2], no such

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device is available when V is nonconstant. Accordingly, most of the works dealing with existence require g to be of power type, i.e., $g(\varphi) = |\varphi|^{p-1}\varphi$ for a $p > 1$, or to satisfy the so-called Ambrosetti-Rabinowitz superquadraticity condition:

$$\exists \mu > 2 \text{ such that } G(s) \leqslant \mu g(s)s, \forall s \geqslant 0, \text{ where } G(s) = \int_0^s g(t)dt.$$

See nevertheless [16] for an existence result under general conditions on g but for a restricted class of potentials V.

In this paper we prove the existence of solutions of (1.2), for small $\lambda > 0$, under the following assumptions (H1)-(H4) where $0 < b < 2$ and $1 < p < 1 + \frac{4-2b}{N}$:

(H1) there exists $\gamma > 2N/\{(N+2) - (N-2)p\}$ such that $V \in L^{\gamma}_{loc}(\mathbb{R}^N)$;
(H2) $\lim_{|x|\to\infty} V(x)|x|^b = 1$;
(H3) there exists $\varepsilon > 0$ such that $g : [0, \varepsilon] \to \mathbb{R}$ is continuous;
(H4) $\lim_{s\to 0^+} \dfrac{g(s)}{s^p} = 1$.

Our approach is variational. Since only conditions around 0 are imposed on g, a first step will be to suitably extend g on all $\mathbb{R}$. This leads to studying a modified problem but, as we shall see, the solutions we obtain for the modified problem have the property of converging to zero in the $L^\infty(\mathbb{R}^N)$ norm as λ decrease to zero. Thus, for sufficiently small $\lambda > 0$, they correspond to solutions of (1.2).

To get a solution of the modified equation we still face a lack of a priori bounds. To overcome this difficulty we borrow and further develop a method introduced by Berti and Bolle in a paper [3] who studies nonlinear wave equations. This method, roughly, permits us to show the boundedness of a Palais-Smale sequence at the mountain pass level for a class of functionals having a geometry *sufficiently* close to the one of the functional corresponding to the case $g(\varphi) = |\varphi|^{p-1}\varphi$. It relies on penalizing the functional outside the region where one expect to find a critical point. Our existence result is the following.

Theorem 1. *Assume* (H1)-(H4). *Then, there exists* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0]$, (1.2) *has a nontrivial solution* φ_λ. *Furthermore,* φ_λ *has the following properties.*

(1) *For all* $x \in \mathbb{R}^N$, $\varphi_\lambda \geqslant 0$.
(2) *When* $\lambda \to 0$, $||\varphi_\lambda||_{H^1(\mathbb{R}^N)} \to 0$ *and* $||\varphi_\lambda||_{L^\infty(\mathbb{R}^N)} \to 0$.

Since our solutions converge to zero in $H^1(\mathbb{R}^N, \mathbb{R})$ as $\lambda \to 0$, 0 is a bifurcation point of (1.2). With our approach we can, see Remark 9, obtain sharp estimates on the $L^p(\mathbb{R}^N)$ bifurcation of our solutions as $\lambda \to 0$. We refer to [15, 23] for previous results of bifurcations from the infimum of the essential spectrum.

Once the existence of solutions of (1.2) is proved we consider the stability of the associated travelling waves. The study of the orbital stability of solutions of (1.1) has seen the contributions of many authors. It is of particular significance for physical reasons and we refer the reader to the introductions of [9, 22, 24] for motivations for studying this problem. In the case V constant and $g(u) = |u|^{p-1}u$, Cazenave and Lions [5] proved the stability of the ground state solutions of (1.2) when $1 < p < 1 + \frac{4}{N}$ and for any $\lambda > 0$. On the contrary when $1+\frac{4}{N} < p < 1+\frac{4}{N-2}$, Berestycki and Cazenave [1] showed the instability of bounded states of (1.2), and when $p = 1 + \frac{4}{N}$, Weinstein [26] proved that instability also holds. We also mention [12] for a general stability theory of solitary waves of Hamiltonian systems.

In [5] both the autonomous character of (1.2) and the fact that g is homogeneous are essential in the proofs. Also dealing with a homogeneous and to some extend autonomous nonlinearity seems essential to use directly the results of [12] (see nevertheless [19]). When (1.2) is nonautonomous only partial results are known so far (see [8, 9, 13, 22, 24] and the references therein). Directly related to our stability result is a recent work of De Bouard and Fukuizumi [6] where stability of positive ground states of (1.2) is obtained for $g(u) = |u|^{p-1}u$ under the following conditions on V:

(1) $V \geqslant 0$, $V \not\equiv 0$, $V \in \mathcal{C}(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$, $V \in L^{\theta^*}(|x| \leqslant 1)$, where $\theta^* = 2N/\{(N+2) - (N-2)p\}$;

(2) there exists $b \in (0,2)$, $C > 0$ and $a > \{(N+2) - (N-2)p\}/2 > b$ such that $\left|\left(V(x) - |x|^{-b}\right)\right| \leqslant C|x|^{-a}$ for all x with $|x| \geqslant 1$.

Under these assumptions and if $1 < p < 1 + (4-2b)/(N-2)$ the existence of ground state solutions follows immediately from the existing literature. In [6] De Bouard and Fukuizumi proved that the corresponding standing waves are stable if $1 < p < 1 + (4-2b)/N$ and $\lambda > 0$ is small.

Our stability result, Theorem 2, extends the result of [6]. If we do borrow some arguments from this paper, new ingredients are necessary to derive Theorem 2. In particular, the fact that we do not know if the solutions obtained in Theorem 1 are ground states is a new major difficulty. To state our stability result we need some definitions and preliminary results. First, to check that the local Cauchy problem is well posed for (1.1), in addition to (H1)-(H4), we require on g

(H5) $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$;

(H6) there exist $C > 0$ and $\alpha \in [0, \frac{4}{N-2})$ such that $\lim_{|s|\to\infty} \frac{|g'(s)|}{|s|^\alpha} \leqslant C$.

Clearly, (H5)-(H6) are sufficient to guarantee that the condition

$$|g(v) - g(u)| \leqslant C(1 + |v|^\alpha + |u|^\alpha)|v - u| \quad \text{for all } u, v \in \mathbb{R}$$

introduced in Remark 4.3.2 of [4] holds. By [4] we then know that the local Cauchy problem is well posed.

For $v \in H^1(\mathbb{R}^N, \mathbb{C})$ we write $v = v_1 + iv_2$. The space $H^1(\mathbb{R}^N, \mathbb{C})$ will be equipped with the norm

$$||v|| = \sqrt{||v||_2^2 + ||\nabla v||_2^2}$$

where $||v||_2^2 = |v_1|_2^2 + |v_2|_2^2$ and $||\nabla v||_2^2 = |\nabla v_1|_2^2 + |\nabla v_2|_2^2$. Here and elsewhere $|\cdot|_p$ denotes the usual norm on $L^p(\mathbb{R}^N, \mathbb{R})$. We also define on $H^1(\mathbb{R}^N, \mathbb{C})$ the scalar product

$$\langle u, v \rangle_2 = \int_{\mathbb{R}^N} \mathrm{Re}(u(x)\overline{v(x)})dx.$$

Finally, let the energy functional E and the charge Q on $H^1(\mathbb{R}^N, \mathbb{C})$ be given by

$$E(v) = \frac{1}{2}||\nabla v||_2^2 - \int_{\mathbb{R}^N} V(x)G(v)dx \quad \text{and} \quad Q(v) = \frac{1}{2}||v||_2^2,$$

where $G(z) = \int_0^{|z|} g(t)dt$, for all $z \in \mathbb{C}$. It follows from [4] that:

Proposition 1. *Assume* (H1)-(H6). *Then, for every $u_0 \in H^1(\mathbb{R}^N, \mathbb{C})$ there exist $T_{u_0} > 0$ and a unique solution $u(t) \in \mathcal{C}([0, T_{u_0}), H^1(\mathbb{R}^N, \mathbb{C}))$ with $u(0) = u_0$ satisfying*

$$E(u(t)) = E(u_0), \qquad Q(u(t)) = Q(u_0), \qquad \textit{for all } t \in [0, T_{u_0}).$$

Finally, we require

(H7) $\lim_{s \to 0^+} \dfrac{g'(s)}{ps^{p-1}} = 1$ where p is given in (H4).

Now by stability we mean:

Definition 2. *Let φ_λ be a solution of* (1.2). *We say that the travelling wave $u(x,t) = e^{i\lambda t}\varphi_\lambda(x)$ associated to φ_λ is* stable *in $H^1(\mathbb{R}^N, \mathbb{C})$ if for all $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If $u_0 \in H^1(\mathbb{R}^N, \mathbb{C})$ is such that $||u_0 - \varphi_\lambda|| < \delta$ and $u(t)$ is a solution of* (1.1) *in some interval $[0, T_{u_0})$ with $u(0) = u_0$, then $u(t)$ can be continued to a solution in $[0, +\infty)$ and*

$$\sup_{t \in [0, +\infty)} \inf_{\theta \in \mathbb{R}} ||u(t) - e^{i\theta}\varphi_\lambda|| < \varepsilon.$$

Our result is the following:

Theorem 2. *Assume* (H1)-(H7) *and let* (φ_λ) *be the family of solutions of* (1.2) *obtained in Theorem* 1. *Then there exists* $\lambda_1 > 0$ *such that for all* $\lambda \in (0, \lambda_1]$ *the travelling wave* $e^{i\lambda t}\varphi_\lambda(x)$ *is stable in* $H^1(\mathbb{R}^N, \mathbb{C})$.

From Theorem 2 we see that, for $\lambda > 0$ small enough, stability only depends on the behaviour of V at infinity and of g around zero. Indeed, as it is shown in [10] when $V(x) = |x|^{-b}$, instability occurs for $g(u) = |u|^{p-1}u$ if $p > 1 + \frac{4-2b}{N}$. To our knowledge Theorem 2 is the first result to enlighten this fact.

For $v \in H^1(\mathbb{R}^N, \mathbb{C})$ and $\lambda > 0$ let

$$S_\lambda(v) = \frac{1}{2}(||\nabla v||_2^2 + \lambda ||v||_2^2) - \int_{\mathbb{R}^N} V(x)G(v)dx.$$

Under our assumptions it is standard to check that S_λ is C^2. Our proof of Theorem 2 relies on the following stability criterion established in [12].

Proposition 3. *Assume* (H1)-(H7) *and let* φ_λ *be a solution of* (1.2). *If there exists* $\delta > 0$ *such that for every* $v \in H^1(\mathbb{R}^N, \mathbb{C})$ *satisfying* $\langle \varphi_\lambda, v \rangle_2 = 0$ *and* $\langle i\varphi_\lambda, v \rangle_2 = 0$ *we have*

$$\left\langle S''_\lambda(\varphi_\lambda)v, v \right\rangle \geqslant \delta ||v||^2,$$

then the standing wave $e^{i\lambda t}\varphi_\lambda(x)$ *is stable in* $H^1(\mathbb{R}^N, \mathbb{C})$.

To check this criterion, following an approach laid down in [7], we first show, in Subsection 3.1, that our solutions (φ_λ), properly rescaled, converge in $H^1(\mathbb{R}^N, \mathbb{R})$ to the unique positive solution $\psi \in H^1(\mathbb{R}^N, \mathbb{R})$ of the limit equation

$$-\Delta u + u = \frac{1}{|x|^b}|u|^{p-1}u, \quad u \in H^1(\mathbb{R}^N, \mathbb{R}). \tag{1.3}$$

Then we derive, see Subsection 3.2, some properties of $\psi \in H^1(\mathbb{R}^N, \mathbb{R})$, in particular we show that it is nondegenerate. Some of the techniques we present in this subsection require little compactness in the underlying problem and could be useful in other contexts. Finally, in Subsection 3.3 we show that the conclusion of Proposition 3 holds.

The paper is organized as follows. In Section 2 we establish Theorem 1 and in Section 3 we prove Theorem 2. A uniqueness result which is necessary for the proof of Theorem 2 is established, using results of [28], in the Appendix.

Notation. Throughout the article the letter C will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem. Also we make the convention that when we take a subsequence of a sequence (u_n) we denote it again by (u_n).

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2. EXISTENCE

This section is devoted to the proof of Theorem 1. For this we use a variational approach and consequently a first step is to extend the nonlinearity g outside of $[0,\varepsilon]$. Let $H \equiv H^1(\mathbb{R}^N, \mathbb{R})$ be equipped with its standard norm $|\cdot|_H$. We consider the modified problem

$$-\Delta v + \lambda v = V(x) f(v), \ v \in H, \tag{2.1}$$

where

$$f(s) = \begin{cases} g(\varepsilon) \ \text{ if } s \geqslant \varepsilon \\ g(s) \ \text{ if } s \in [0,\varepsilon] \\ 0 \ \text{ if } s \leqslant 0. \end{cases}$$

It is convenient to write (2.1) as

$$-\Delta v + \lambda v = V(x)\left(v_+^p + r(v)\right), \ v \in H \tag{2.2}$$

with $v_+ = \max\{v, 0\}$ and $r(s) = f(s) - s_+^p$.

To develop our variational procedure we rescaled (2.2) in order to eliminate $\lambda > 0$ from the linear part. For $v \in H$, let $\tilde{v} \in H$ be such that

$$v(x) = \lambda^{\frac{2-b}{2(p-1)}} \tilde{v}(\sqrt{\lambda} x). \tag{2.3}$$

Clearly $v \in H$ satisfies (2.2) if and only if $\tilde{v} \in H$ satisfies

$$-\Delta \tilde{v} + \tilde{v} = V_\lambda(x)\tilde{v}_+^p + V(\frac{x}{\sqrt{\lambda}})\tilde{r}(\tilde{v}), \tag{2.4}$$

where

$$\tilde{r}(s) = \lambda^{-\frac{2-b}{2(p-1)}-1} r(\lambda^{\frac{2-b}{2(p-1)}} s) \ \text{ and } \ V_\lambda(x) = \lambda^{-b/2} V(x/\sqrt{\lambda}). \tag{2.5}$$

A solution of (2.4) will be obtained as a critical point of the functional $\tilde{S}_\lambda : H \to \mathbb{R}$ given by

$$\tilde{S}_\lambda(v) = \frac{1}{2}|v|_H^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} V_\lambda(x) v(x)_+^{p+1} dx - \tilde{R}_\lambda(v)$$

with

$$\tilde{R}_\lambda(s) = \int_{\mathbb{R}^N} \lambda^{b/2} V_\lambda(x)\Big(\int_0^{|s|} \tilde{r}(t)dt\Big)dx.$$

By (H1) we can fix a $p' \in (p, 1 + (4-2b)/(N-2))$ such that $2N/\{(N+2) - (N-2)p'\} < \gamma$. The following estimate will be crucial thorough the paper.

Lemma 4. *Assume* (H1)-(H4). *Then for any $q \in [1, p']$ there exists $C > 0$ such that for any $\lambda > 0$ sufficiently small and all $v \in H$,*

$$\Big|\int_{\mathbb{R}^N} V_\lambda(x)|v(x)|^{q+1}dx\Big| \leqslant C|v|_H^{q+1}.$$

Proof. By the assumptions (H1)-(H2) there exists $R > 0$ such that

$$|V(x)| \leqslant 2|x|^{-b},\ \forall\, |x| \geqslant R \ \text{ and } \ V \in L^\gamma(B(R)). \tag{2.6}$$

Here $B(R) = \{x \in \mathbb{R}^N : |x| < R\}$. We have

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} V_\lambda(x)|v(x)|^{q+1}dx\Big| \leqslant{}& \Big|\int_{B(R)} V_\lambda(x)|v(x)|^{q+1}dx\Big| \\ &+ \Big|\int_{\mathbb{R}^N\backslash B(R)} V_\lambda(x)|v(x)|^{q+1}dx\Big|.\end{aligned} \tag{2.7}$$

By Hölder's inequality,

$$\Big|\int_{B(R)} V_\lambda(x)|v(x)|^{q+1}dx\Big| \leqslant |V_\lambda|_{L^\theta(B(R))}\,|v|_{2^*}^{q+1} \tag{2.8}$$

with $\theta = 2N/\{(N+2) - (N-2)q\}$. But

$$|V_\lambda|^\theta_{L^\theta(B(R))} = |V_\lambda|^\theta_{L^\theta(B(\sqrt{\lambda}R))} + |V_\lambda|^\theta_{L^\theta(B(R)\backslash B(\sqrt{\lambda}R))} \tag{2.9}$$

and, since $|V_\lambda|^\theta_{L^\theta(B(\sqrt{\lambda}R))} = \lambda^{-b\theta/2+N/2}\,|V|_{L^\theta(B(R))}$ with $-b\theta/2 + N/2 > 0$, we can assume that

$$|V_\lambda|_{L^\theta(B(\sqrt{\lambda}R))} \leqslant 1. \tag{2.10}$$

Also, from (2.6) it follows that $V_\lambda(x) \leqslant 2|x|^{-b}$ on $\mathbb{R}^N\backslash B(\sqrt{\lambda}R))$. Thus

$$|V_\lambda|_{L^\theta(B(R)\backslash B(\sqrt{\lambda}R))} \leqslant \Big|\frac{2}{|x|^b}\Big|_{L^\theta(B(R))} \leqslant C, \tag{2.11}$$

and

$$\Big|\int_{\mathbb{R}^N\backslash B(R)} V_\lambda(x)|v(x)|^{q+1}dx\Big| \leqslant C|v|^{q+1}_{q+1}. \tag{2.12}$$

Now, combining (2.7)-(2.12) and using Sobolev's embeddings we get the required estimate. □

A first consequence of Lemma 4 is the following estimate on the "rest" $\tilde{R}_\lambda$ of the functional $\tilde{S}_\lambda$.

Lemma 5. *Assume* (H1)-(H4). *Then there exist* $C > 0$ *and* $\alpha > 0$ *such that for all* $a > 0$ *there exists* $A > 0$ *such that*

$$|\tilde{R}_\lambda(v)| + |\nabla \tilde{R}_\lambda(v)v| \leqslant C(a|v|_H^{p+1} + \lambda^\alpha A|v|_H^{p'+1}), \tag{2.13}$$

for all $\lambda > 0$ *sufficiently small and all* $v \in H$.

Proof. Clearly, from the definition of r, we see that for any $a > 0$ there exists $A > 0$ such that

$$|r(s)| \leqslant a|s|^p + A|s|^{p'}, \ \forall s \in \mathbb{R}. \tag{2.14}$$

This implies, see (2.5), that

$$|\tilde{r}(s)| \leqslant \lambda^{-b/2} a|s|^p + \lambda^{-b/2}\lambda^\alpha A|s|^{p'}, \ \forall s \in \mathbb{R}, \tag{2.15}$$

with $\alpha = \frac{(p'-p)(2-b)}{2(p-1)} > 0$. As a consequence, for any $v \in H$,

$$|\tilde{R}_\lambda(v)| \leqslant \frac{a}{p+1}\int_{\mathbb{R}^N} |V_\lambda(x)||v(x)|^{p+1}dx + \frac{\lambda^\alpha A}{p'+1}\int_{\mathbb{R}^N} |V_\lambda(x)||v(x)|^{p'+1}dx,$$

and using Lemma 4 we get that

$$|\tilde{R}_\lambda(v)| \leqslant C(a|v|_H^{p+1} + \lambda^\alpha A|v|_H^{p'+1}). \tag{2.16}$$

Analogously, we can prove that

$$|\nabla \tilde{R}_\lambda(v)v| \leqslant C(a|v|_H^{p+1} + \lambda^\alpha A|v|_H^{p'+1}). \tag{2.17}$$

Combining (2.16) and (2.17) finishes the proof. □

We shall obtain a critical point of $\tilde{S}_\lambda$ by a mountain pass type argument. However, even though it is likely that $\tilde{S}_\lambda$ has a mountain pass geometry, showing that the Palais-Smale sequences at the mountain pass level are bounded seems out of reach under our weak assumptions on g. To overcome this difficulty we develop an approach, inspired by [3], which consists in truncating the "rest" term of $\tilde{S}_\lambda$ outside of a ball centered at the origin and to showing that, as $\lambda > 0$ goes to zero, all Palais-Smale sequences at the mountain-pass level lie in this ball. Precisely, let $T > 0$ be the truncation radius (its value will be indicated later) and consider a smooth function $\nu : [0, +\infty) \to \mathbb{R}$ such that

$$\begin{cases} \nu(s) = 1 & \text{for} \quad s \in [0,1], \\ 0 \leqslant \nu(s) \leqslant 1 & \text{for} \quad s \in [1,2], \\ \nu(s) = 0 & \text{for} \quad s \in [2, +\infty), \\ |\nu'|_\infty \leqslant 2. \end{cases}$$

For $v \in H$, we define

$$\widehat{S}_\lambda(v) = \frac{1}{2}|v|_H^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} V_\lambda(x)v(x)_+^{p+1}dx - \widehat{R}_\lambda(v),$$

where $\widehat{R}_\lambda(v) = t(v)\tilde{R}_\lambda(v)$ with $t(v) := \nu(\frac{|v|_H^2}{T^2})$.

We have the following bounds on $\widehat{R}_\lambda(v)$ and $\nabla\widehat{R}_\lambda(v)v$:

Lemma 6. *Assume* (H1)-(H4). *Then there exists $C > 0$ such that for all $a > 0$, there exists $A > 0$, satisfying for all $v \in H$*

$$|\widehat{R}_\lambda(v)| \leqslant C(aT^{p+1} + \lambda^\alpha AT^{p'+1}), \tag{2.18}$$

$$|\nabla\widehat{R}_\lambda(v)v| \leqslant C(1+T^2)(aT^{p+1} + \lambda^\alpha AT^{p'+1}). \tag{2.19}$$

Proof. Since $t(v) = 0$ for $|v|_H > \sqrt{2}T$, (2.18) follows directly from Lemma 5. Also $\nabla\widehat{R}_\lambda(v) = t(v)\nabla\tilde{R}_\lambda(v) + \tilde{R}_\lambda(v)\nabla t(v)$ with $\nabla t(v)v = 2\nu'(\dfrac{|v|_H^2}{T^2})|v|_H^2$ and thus we also have (2.19). $\square$

Lemma 7. *Assume* (H1)-(H4). *Then there exists $\overline{\lambda} > 0$ such that for all $\lambda \in (0, \overline{\lambda}]$, $\widehat{S}_\lambda$ has a mountain pass geometry. Also $\widehat{S}_\lambda$ admits at the mountain pass level $c(\lambda) > 0$ a critical point $\tilde{\varphi}_\lambda \in H \setminus \{0\}$ satisfying $\tilde{S}_\lambda(\tilde{\varphi}_\lambda) = \widehat{S}_\lambda(\tilde{\varphi}_\lambda)$. Moreover, there exists $C > 0$ such that $|\tilde{\varphi}_\lambda|_H \leqslant C$, $\forall \lambda \in (0, \overline{\lambda}]$.*

Proof. Let us prove that $\widehat{S}_\lambda$ has a mountain pass geometry for any $\lambda > 0$ sufficiently small. Obviously, we have $\widehat{S}_\lambda(0) = 0$. Let $a > 0$. From Lemma 4 (used with $q = p$) and Lemma 5 there exists $A > 0$ such that for $v \in H$

$$\widehat{S}_\lambda(v) \geqslant \frac{1}{2}|v|_H^2 - C((1+a)|v|_H^{p+1} + \lambda^\alpha A|v|_H^{p'+1}).$$

Thus taking $\delta > 0$ small enough there exists $m \geqslant 0$ such that $\widehat{S}_\lambda(v) > m > 0$ if $|v|_H = \delta$, uniformly for $\lambda > 0$ sufficiently small.

Now let $\psi \in C_0^\infty \backslash \{0\}$ with $\psi \geqslant 0$ and $\psi = 0$ on $B(1)$. Because of (H2), there exists $R > 0$ such that

$$V(x) \geqslant \frac{1}{2|x|^b} \text{ if } |x| \geqslant R.$$

Thus, for $\lambda > 0$ small enough

$$\int_{\mathbb{R}^N} V_\lambda(x)\psi(x)^{p+1}dx \geqslant \int_{\mathbb{R}^N} \frac{1}{2|x|^b}\psi(x)^{p+1}dx.$$

Defining $\psi_B := B\psi$ we observe that for $B > 0$ large enough $\widehat{R}_\lambda(\psi_B) = 0$. Thus letting $D = \frac{|\psi|_H^2}{2}$ and $E = \int_{\mathbb{R}^N} \frac{1}{2|x|^b}\psi(x)^{p+1}dx$ we have, for $B > 0$ large enough,

$$\widehat{S}_\lambda(\psi_B) \leqslant DB^2 - EB^{p+1} < 0$$

for any $\lambda > 0$ sufficiently small.

Since $\widehat{S}_\lambda$ has a mountain pass geometry defining

$$c(\lambda) := \inf_{\gamma \in \Gamma} \sup_{s \in [0,1]} \widehat{S}_\lambda(\gamma(s))$$

where $\Gamma := \{\gamma \in \mathcal{C}([0,1], H) : \gamma(0) = 0, \widehat{S}_\lambda(\gamma(1)) < 0\}$, Ekeland's principle gives the existence of a Palais-Smale sequence at the mountain pass level $c(\lambda)$, namely of a sequence $(v_n) \subset H$ such that

$$\nabla \widehat{S}_\lambda(v_n) \quad \to \quad 0, \tag{2.20}$$

$$\widehat{S}_\lambda(v_n) \quad \to \quad c(\lambda). \tag{2.21}$$

Let us show that, if $\lambda > 0$ is small enough, this Palais-Smale sequence lies, for $n \in \mathbb{N}$ large, in the ball of H where $\widehat{S}_\lambda$ and $\tilde{S}_\lambda$ coincide. We begin by an estimate on the mountain pass level. For every $t \in [0,1]$ we have

$$\widehat{S}_\lambda(t\psi_B) \leqslant DB^2 t^2 - EB^{p+1} t^{p+1} + |\widehat{R}_\lambda(t\psi_B)|.$$

Thanks to (2.18) and the definition of $c(\lambda)$ this gives

$$c(\lambda) \leqslant W + C(aT^{p+1} + A\lambda^\alpha T^{p'+1}) \tag{2.22}$$

with $W = D(\frac{2D}{(p+1)E})^{\frac{2}{p-1}} - E(\frac{2D}{(p+1)E})^{\frac{p+1}{p-1}}$. Note that the constants W, C are independent of $T > 0$ and of $\lambda > 0$ sufficiently small.

To prove that $\limsup_{n \to \infty} |v_n|_H < T$ we first show that (v_n) is bounded on H. Seeking a contradiction, we assume that, up to a subsequence, $|v_n|_H \to \infty$. Therefore, for $n \in \mathbb{N}$ large enough, we have $|v_n|_H^2 > 2T^2$ and thus $\widehat{R}_\lambda(v_n) = \nabla \widehat{R}_\lambda(v_n) v_n = 0$. It follows that

$$2\widehat{S}_\lambda(v_n) - \nabla \widehat{S}_\lambda(v_n) v_n = \Big(1 - \frac{2}{p+1}\Big) \int_{\mathbb{R}^N} V_\lambda(x)(v_n(x))_+^{p+1} dx.$$

Furthermore, since $\widehat{S}_\lambda(v_n) \to c(\lambda)$, we can assume that $\widehat{S}_\lambda(v_n) \leqslant 2c(\lambda)$ and we get

$$\Big(1 - \frac{2}{p+1}\Big) \int_{\mathbb{R}^N} V_\lambda(x)(v_n(x))_+^{p+1} dx \leqslant 4c(\lambda) + \|\nabla \widehat{S}_\lambda(v_n)\| |v_n|_H.$$

Consequently we have

$$\begin{aligned} |v_n|_H^2 &= \nabla \widehat{S}_\lambda(v_n) v_n + \int_{\mathbb{R}^N} V_\lambda(x)(v_n(x))_+^{p+1} dx \\ &\leqslant \Big(1 + \frac{p+1}{p-1}\Big) \|\nabla \widehat{S}_\lambda(v_n)\| |v_n|_H + 4\Big(\frac{p+1}{p-1}\Big) c(\lambda) \end{aligned}$$

and therefore

$$|v_n|_H \leqslant \Big(1+\frac{p+1}{p-1}\Big)\|\nabla \widehat{S}_\lambda(v_n)\| + 4\Big(\frac{p+1}{p-1}\Big)c(\lambda)|v_n|_H^{-1}.$$

Since the right member tends to 0 as $n \to \infty$ we have a contradiction. Thus (v_n) stays bounded in H and, in particular, $\nabla \widehat{S}_\lambda(v_n)v_n \to 0$.

Let us now show that $|v_n|_H < T$ for $n \in \mathbb{N}$ large. Still arguing by contradiction, we assume that $\lim_{n\to\infty} |v_n|_H \in [T, +\infty)$. We have

$$\widehat{S}_\lambda(v_n) - \frac{1}{p+1}\nabla \widehat{S}_\lambda(v_n)v_n = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)|v_n|_H^2 - \widehat{R}_\lambda(v_n) + \frac{1}{p+1}\nabla \widehat{R}_\lambda(v_n)v_n. \tag{2.23}$$

Then using (2.18)-(2.22) and passing to the limit in (2.23), we obtain

$$\Big(\frac{1}{2} - \frac{1}{p+1}\Big)T^2 \leqslant W + C(3+T^2)(aT^{p+1} + A\lambda^\alpha T^{2^\star}).$$

At this point, choosing $a > 0$ sufficiently small, we see that if $T^2 > \frac{2(p+1)}{p-1}W$ we obtain a contradiction when $\lambda > 0$ is small enough. This proves that (v_n) lies in the region where $\tilde{S}_\lambda$ and $\widehat{S}_\lambda$ coincide.

Now since $(v_n) \subset H$ is bounded, without loss of generality we can assume that $v_n \rightharpoonup v_\infty$ weakly in H. To end the proof we just need to show that $v_n \to v_\infty$ strongly in H. The condition $\nabla \widehat{S}_\lambda(v_n) \to 0$ is just

$$-\Delta v_n + v_n - V_\lambda(x)(v_n)_+^p - V(\frac{x}{\sqrt{\lambda}})\tilde{r}(v_n) \to 0 \text{ in } H^{-1}. \tag{2.24}$$

Because of the decrease of V to 0 at infinity we have, in a standard way, that

$$V_\lambda(x)(v_n)_+^p - V(\frac{x}{\sqrt{\lambda}})\tilde{r}(v_n) \to V_\lambda(x)(v_\infty)_+^p - V(\frac{x}{\sqrt{\lambda}})\tilde{r}(v_\infty) \text{ in } H^{-1}. \tag{2.25}$$

Now let $L : H \to H^{-1}$ be defined by

$$\langle Lu, v\rangle = \int_{\mathbb{R}^N}(\nabla u \nabla v + uv)dx.$$

The operator L is invertible, therefore, from (2.24)-(2.25),

$$v_n \to L^{-1}\Big(V_\lambda(x)(v_\infty)_+^p - V(\frac{x}{\sqrt{\lambda}})\tilde{r}(v_\infty)\Big).$$

By uniqueness of the limit, we have $v_n \to v_\infty$ in H and by continuity v_∞ is a solution of (2.4) at the mountain pass level $c(\lambda)$. We set $\tilde{\varphi}_\lambda = v_\infty$. At this point the lemma is proved. $\square$

Lemma 8. *Assume* (H1)-(H4). *The solutions of* (2.4), *obtained in Lemma* 7, *have, in addition, the following properties:*

(i) $|\tilde{\varphi}_\lambda|_\infty \leqslant C$, *for a* $C > 0$ *independent of* $\lambda \in (0, \overline{\lambda}]$,
(ii) *for all* $x \in \mathbb{R}^N$, $\tilde{\varphi}_\lambda(x) \geqslant 0$.

Proof. Starting from (2.1) and the change of variables (2.3) we see that our solutions $\tilde{\varphi}_\lambda$ satisfy

$$-\Delta\tilde{\varphi}_\lambda + \tilde{\varphi}_\lambda = \lambda^{-\frac{2-b}{2(p-1)}-1} V(\frac{x}{\sqrt{\lambda}}) f(\lambda^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_\lambda). \tag{2.26}$$

We see from (H4) that $|f(s)| \leqslant C|s|^p$ for a $C > 0$, for all $s \in \mathbb{R}$. Thus

$$\Big|\lambda^{-\frac{2-b}{2(p-1)}-1} V(\frac{x}{\sqrt{\lambda}}) f(\lambda^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_\lambda)\Big| \leqslant C|V_\lambda(x)||\tilde{\varphi}_\lambda|^p \tag{2.27}$$

with a $C > 0$, independent of $\lambda \in (0, \overline{\lambda}]$. To obtain (i) we follow a bootstrap argument. The crucial point is to insure that the estimates we get are independent of $\lambda \in (0, \overline{\lambda}]$.

Let $\theta = 2N/\{(N+2) - (N-2)p\}$. Assuming that $\tilde{\varphi}_\lambda \in L^q(\mathbb{R}^N)$ we claim that

$$V_\lambda |\tilde{\varphi}_\lambda|^p \in L^r(\mathbb{R}^N) \text{ with } r = \frac{\theta q}{\theta p + q} \text{ and is bounded in } L^r(\mathbb{R}^N)$$
$$\text{as a function of } |\tilde{\varphi}_\lambda|_q \text{ only.}$$

To see this we choose $R > 0$ such that $|V(x)| \leqslant 2|x|^{-b}$, for all $|x| \geqslant R$ and we write $\mathbb{R}^N = B(\sqrt{\lambda}R) \cup (B(R)\backslash B(\sqrt{\lambda}R)) \cup (\mathbb{R}^N \backslash B(R))$.

On $\mathbb{R}^N\backslash B(R)$ since $|V_\lambda(x)| \leqslant C$, for a $C > 0$, we directly have

$$|V_\lambda||\tilde{\varphi}_\lambda|^p \in L^{\frac{q}{p}}(\mathbb{R}^N\backslash B(R))$$

and thus, since $(\tilde{\varphi}_\lambda)$ is bounded in $L^p(\mathbb{R}^N)$,

$$|V_\lambda||\tilde{\varphi}_\lambda|^p \in L^r(\mathbb{R}^N\backslash B(R)).$$

On $B(R)\backslash B(\sqrt{\lambda}R)$ we have $|V_\lambda(x)| \leqslant 2|x|^{-b}$ with $|x|^{-b} \in L^\theta(B(R))$. Thus

$$\begin{aligned}\int_{B(R)\backslash B(\sqrt{\lambda}R)} |V_\lambda(x)|^r |\tilde{\varphi}_\lambda|^{rp} dx &\leqslant \Big(\int_{B(R)} \frac{1}{|x|^{b\theta}} dx\Big)^{\frac{q}{q+\theta p}} \Big(\int_{B(R)} |\tilde{\varphi}_\lambda|^q dx\Big)^{\frac{\theta p}{q+\theta p}} \\ &\leqslant C|\tilde{\varphi}_\lambda|_q^{\frac{\theta q p}{q+\theta p}}.\end{aligned}$$

On $B(\sqrt{\lambda}R)$ we have

$$\int_{B(\sqrt{\lambda}R)} |V_\lambda(x)|^r |\tilde{\varphi}_\lambda|^{rp} dx \leqslant \Big(\int_{B(\sqrt{\lambda}R)} |V_\lambda(x)|^\theta dx\Big)^{\frac{q}{q+\theta p}} \Big(\int_{B(\sqrt{\lambda}R)} |\tilde{\varphi}_\lambda|^q dx\Big)^{\frac{\theta p}{q+\theta p}}$$

with

$$|V_\lambda|^\theta_{L^\theta(B(\sqrt{\lambda}R))} = \lambda^{-b\theta/2+N/2} |V|^\theta_{L^\theta(B(R))} \to 0$$

and this proves our claim. Now since $V_\lambda|\tilde{\varphi}_\lambda|^p \in L^r(\mathbb{R}^N)$ we have $\tilde{\varphi}_\lambda \in W^{2,r}(\mathbb{R}^N)$ and thus $\tilde{\varphi}_\lambda \in L^t(\mathbb{R}^N)$ with $t = \frac{Nr}{N-2r}$.

It is now easy to check that, choosing $q = 2^*$, we have $t > q$ and that the bootstrap will give, in a finite number of steps, $r > \frac{N}{2}$ so that $\tilde{\varphi}_\lambda \in W^{2,r}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$. In addition since for a $C > 0$, $|\tilde{\varphi}_\lambda|_H \leqslant C$, for all $\lambda \in (0, \overline{\lambda}]$ we have, for a $C > 0$, $|\tilde{\varphi}_\lambda|_{2^*} \leqslant C, \forall \lambda \in (0, \overline{\lambda}]$ and by our claim the various constants of the Sobolev's embeddings are independent of $\lambda \in (0, \overline{\lambda}]$. This proves (i).

For (ii), we argue as follows. Let $\varphi = \varphi_+ - \varphi_-$ where $\varphi_+ = \max\{\varphi, 0\}$ and $\varphi_- = \max\{-\varphi, 0\}$ and suppose that φ satisfies

$$-\Delta\varphi + \varphi = V\Big(\frac{x}{\sqrt{\lambda}}\Big)\tilde{f}(\varphi)$$

with $\tilde{f} = 0$ if $s \leqslant 0$. We know that $\varphi_+, \varphi_- \in H$. Then, by multiplying by φ_- and integrating, we obtain

$$-\int_{\mathbb{R}^N} |\nabla\varphi_-|^2 - \varphi_-^2 = 0.$$

Therefore, $\varphi_- = 0$. □

Now we can give the

Proof of Theorem 1. Taking into account Lemmas 7 and 8 all that remains to show is that $|\varphi_\lambda|_H \to 0$ and $|\varphi_\lambda|_\infty \to 0$, as $\lambda \to 0$, when φ_λ is given by

$$\varphi_\lambda(x) = \lambda^{\frac{2-b}{2(p-1)}}\tilde{\varphi}_\lambda(\sqrt{\lambda}x).$$

Since $b < 2$ we immediately get, from Lemma 8, that $|\varphi_\lambda|_\infty \to 0$ and this proves, in particular, that φ_λ is a solution of (1.2) when $\lambda > 0$ is small enough. Now, since $p < 1+\frac{4-2b}{N}$ we see from direct calculations that $|\varphi_\lambda|_H \to 0$. □

Remark 9. We deduce from the proof of Theorem 1 that (1.2) admits solutions $\varphi_\lambda \in H$ which satisfy, for any $\lambda > 0$ small enough,

$$|\varphi_\lambda|_q \leqslant C|\lambda|^{\frac{2-b}{2(p-1)} - \frac{N}{2q}} \text{ if } 1 \leqslant q < \infty \text{ and } |\varphi_\lambda|_\infty \leqslant C|\lambda|^{\frac{2-b}{2(p-1)}}.$$

These decay estimates should be compared with the ones obtained in Theorem 5.9 of [23]. The comparison suggests that using a rescaling approach, as in the present paper, is fruitful to get the sharpest bifurcation estimates.

3. Stability

In this section we prove Theorem 2. The proof is divided into three steps. First we prove the convergence in H of the solutions $(\tilde{\varphi}_\lambda)$ of the rescaled problem to the unique positive solution $\psi \in H$ of the limit problem

$$-\Delta\varphi + \varphi = \frac{1}{|x|^b}|\varphi|^{p-1}\varphi,\ \varphi \in H. \tag{3.1}$$

Existence for (3.1) is standard because of the compactness of the nonlinear term and can, for example, be obtained by minimizing S under the constraint $I(v) = 0$ for $v \in H\backslash\{0\}$ where

$$\begin{aligned} S(v) &= \frac{1}{2}|v|_H^2 - \frac{1}{p+1}\int_{\mathbb{R}^N}\frac{1}{|x|^b}|v(x)|^{p+1}dx, & (3.2)\\ I(v) &= |v|_H^2 - \int_{\mathbb{R}^N}\frac{1}{|x|^b}|v(x)|^{p+1}dx. & (3.3)\end{aligned}$$

We know from [11] that positive solutions of (3.1) are radial. They also decay exponentially at infinity. The uniqueness of $\psi \in H$ follows from [28].

Secondly, we establish some additional properties of the limit problem. In particular we prove that $\psi \in H$ is nondegenerate.

In the third step after having translated the stability criterion in the rescaled variables, we prove that it holds.

Notation. Since in addition to (H1)-(H4) we now assume (H5)-(H7), we are somehow in the case of the modified problem, and therefore we will use the same notation. In particular, r will be now defined by

$$r(s) = g(s) - |s|^{p-1}s.$$

3.1. A convergence lemma.

We start with a key technical result.

Lemma 10. *Assume* (H1)-(H4). *Let* $(v_\lambda) \subset H$ *be a bounded sequence in* H *and* $q \in [1, p']$. *Then we have, as* $\lambda \to 0$,

$$\int_{\mathbb{R}^N}\left|\frac{1}{|x|^b} - V_\lambda(x)\right||v_\lambda(x)|^{q+1}dx \to 0.$$

Proof. For $R > 0$ we write

$$\begin{aligned}\int_{\mathbb{R}^N}\left|\frac{1}{|x|^b} - V_\lambda(x)\right||v_\lambda(x)|^{q+1}dx &\leqslant \int_{B(\sqrt{\lambda}R)}\left|\frac{1}{|x|^b} - V_\lambda(x)\right||v_\lambda(x)|^{q+1}dx\\ &+ \int_{\mathbb{R}^N\backslash B(\sqrt{\lambda}R)}\left|\frac{1}{|x|^b} - V_\lambda(x)\right||v_\lambda(x)|^{q+1}dx.\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Fixing $R > 0$ large enough we have

$$\Big|\frac{1}{|x|^b} - V_\lambda(x)\Big| \leqslant \frac{\varepsilon}{|x|^b} \quad \text{for } x \in \mathbb{R}^N \backslash B(\sqrt{\lambda}R).$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^N\backslash B(\sqrt{\lambda}R)} \Big|\frac{1}{|x|^b} - V_\lambda(x)\Big| |v_\lambda(x)|^{q+1}dx \quad &\leqslant \quad \varepsilon \int_{B(1)\backslash B(\sqrt{\lambda}R)} \frac{1}{|x|^b}|v_\lambda(x)|^{q+1}dx \\ &+ \quad \varepsilon \int_{\mathbb{R}^N\backslash B(1)} |v_\lambda(x)|^{q+1}dx\end{aligned}$$

with, for $\theta = 2N/\{(N+2)-(N-2)q\}$,

$$\int_{B(1)\backslash B(\sqrt{\lambda}R)} \frac{1}{|x|^b}|v_\lambda(x)|^{q+1}dx \leqslant |\frac{1}{|x|^b}|_{L^\theta(B(1))}|v_\lambda|_{2^*}^{q+1} \leqslant C$$

and

$$\int_{\mathbb{R}^N\backslash B(1)} |v_\lambda(x)|^{q+1}dx \leqslant |v_\lambda|_{q+1}^{q+1} \leqslant C.$$

Now,

$$\begin{aligned}&\int_{B(\sqrt{\lambda}R)} \Big|\frac{1}{|x|^b} - V_\lambda(x)\Big||v_\lambda(x)|^{q+1}dx \\ \leqslant \quad &\Big(|\frac{1}{|x|^b}|_{L^\theta(B(\sqrt{\lambda}R))} + |V_\lambda|_{L^\theta(B(\sqrt{\lambda}R))}\Big)|v_\lambda|_{2^*}^{q+1}\end{aligned}$$

and since

$$|\frac{1}{|x|^b}|_{L^\theta(B(\sqrt{\lambda}R))} \to 0 \quad \text{and} \quad |V_\lambda|_{L^\theta(B(\sqrt{\lambda}R))} = \lambda^{-b\theta/2+N/2}\,|V|_{L^\theta(B(R))} \to 0$$

as $\lambda \to 0$, this ends the proof. ☐

Now the main result of this subsection is

Lemma 11. *Assume* (H1)-(H4). *Then the solutions* $(\tilde{\varphi}_\lambda)_\lambda$ *of the rescaled equation* (2.4) *satisfy*

$$\lim_{\lambda\to 0} |\tilde{\varphi}_\lambda - \psi|_H = 0.$$

Proof. We divide the proof into two steps. First, we prove that there exists $(\mu(\lambda)) \subset \mathbb{R}$ such that $\mu(\lambda) \to 1$ and $(\mu(\lambda)\tilde{\varphi}_\lambda)$ is a minimizing sequence for

$$\min\{S(v) : v \in H \setminus \{0\}, I(v) = 0\}. \tag{3.4}$$

Secondly, using this information, we prove the convergence of $(\tilde{\varphi}_\lambda)$ to ψ.

We begin by showing that $\limsup_{\lambda\to 0} S(\tilde{\varphi}_\lambda) \leqslant S(\psi)$. Let $\gamma_0 : [0,1] \to H$ be such that $\gamma_0(t) := Ct\psi$, for a $C > 0$. Then, fixing $C > 0$ large enough, we have $S(\gamma_0(1)) < 0$ and $S(\psi) = \max_{t\in[0,1]} S(\gamma_0(t))$ as is easily seen from the simple "radial" behaviour of S.

Let $\varepsilon > 0$ be arbitrary. From Lemmas 5 and 10 we see that, for any $\lambda > 0$ small enough,

$$|\widehat{S}_\lambda(\gamma_0(s)) - S(\gamma_0(s))| \leqslant \varepsilon,\ \forall s \in [0,1],$$

and since $\widehat{S}_\lambda(\tilde{\varphi}_\lambda) = c(\lambda)$ it follows that

$$\tilde{S}_\lambda(\tilde{\varphi}_\lambda) = \widehat{S}_\lambda(\tilde{\varphi}_\lambda) \leqslant \max_{s\in[0,1]} \widehat{S}_\lambda(\gamma_0(s)) \leqslant \max_{s\in[0,1]} S(\gamma_0(s)) + \varepsilon = S(\psi) + \varepsilon.$$

Thus $\limsup_{\lambda\to 0} \tilde{S}_\lambda(\tilde{\varphi}_\lambda) \leqslant S(\psi)$. Now, using Lemmas 5 and 10, we have

$$\lim_{\lambda\to 0} |S(\tilde{\varphi}_\lambda) - \tilde{S}_\lambda(\tilde{\varphi}_\lambda)| = 0$$

and we deduce that $\limsup_{\lambda\to 0} S(\tilde{\varphi}_\lambda) \leqslant S(\psi)$.

Let us now show the existence of a sequence $(\mu(\lambda))$ such that $\mu(\lambda) \to 1$ and $I(\mu(\lambda)\tilde{\varphi}_\lambda) = 0$. Since $\nabla\tilde{S}_\lambda(\tilde{\varphi}_\lambda)\tilde{\varphi}_\lambda = 0$ we have

$$I(\tilde{\varphi}_\lambda) = -\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^b} - V_\lambda(x)\Big)|\tilde{\varphi}_\lambda|^{p+1}dx + \nabla\tilde{R}_\lambda(\tilde{\varphi}_\lambda)\tilde{\varphi}_\lambda.$$

Thus by Lemmas 5 and 10, $I(\tilde{\varphi}_\lambda) \to 0$. Let $\mu(\lambda) := \Big(\frac{|\tilde{\varphi}_\lambda|_H^2}{\int_{\mathbb{R}^N} \frac{1}{|x|^b}|\tilde{\varphi}_\lambda|^{p+1}dx}\Big)^{\frac{1}{p-1}}$. Then $I(\mu(\lambda)\tilde{\varphi}_\lambda) = 0$ and we have

$$|\mu(\lambda)^{p-1} - 1| = \frac{|I(\tilde{\varphi}_\lambda)|}{\int_{\mathbb{R}^N} \frac{1}{|x|^b}|\tilde{\varphi}_\lambda|^{p+1}dx}.$$

From the mountain pass geometry and since $\nabla\tilde{S}_\lambda(\tilde{\varphi}_\lambda)\tilde{\varphi}_\lambda = 0$ the denominator stays bounded away from 0 and since $I(\tilde{\varphi}_\lambda) \to 0$ we deduce that $\lim_{\lambda\to 0} \mu(\lambda) = 1$. Thus, by continuity of S, we have

$$\limsup_{\lambda\to 0} S(\mu(\lambda)\tilde{\varphi}_\lambda) = \limsup_{\lambda\to 0} S(\tilde{\varphi}_\lambda) \leqslant S(\psi)$$

and since $I(\mu(\lambda)\tilde{\varphi}_\lambda) = 0$, $(\mu(\lambda)\tilde{\varphi}_\lambda)$ is a minimizing sequence for (3.4).

Now, using this information, we show the convergence of $(\tilde{\varphi}_\lambda)$ to ψ in H. Since $(\mu(\lambda)\tilde{\varphi}_\lambda)$ is bounded, there exists $\tilde{\varphi}_0$ such that, up to a subsequence, $\mu(\lambda)\tilde{\varphi}_\lambda \rightharpoonup \tilde{\varphi}_0$ weakly in H. Clearly, the minimizing sequences of (3.4) are the minimizing sequences of

$$\min\{|v|_H^2 : v \in H \setminus \{0\}, I(v) = 0\},$$

and since for $v \in H$ such that $I(v) < 0$ there exists $0 < t < 1$ such that $I(tv) = 0$, (3.4) is also equivalent to

$$\min\{|v|_H^2 : v \in H \setminus \{0\}, I(v) \leqslant 0\}.$$

If we assume that

$$|\tilde{\varphi_0}|_H^2 < \limsup_{\lambda \to 0} |\mu(\lambda)\tilde{\varphi}_\lambda|_H^2 = |\psi|_H^2 \tag{3.5}$$

since, as can be proved in a standard way,

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} \frac{1}{|x|^b} |\mu(\lambda)\tilde{\varphi}_\lambda|^{p+1} dx = \int_{\mathbb{R}^N} \frac{1}{|x|^b} |\tilde{\varphi_0}|^{p+1} dx,$$

we get that

$$I(\tilde{\varphi_0}) < \limsup_{\lambda \to 0} I(\mu(\lambda)\tilde{\varphi}_\lambda) = 0.$$

Thus (3.5) contradicts the variational characterization of $\psi \in H$. We deduce that $\mu(\lambda)\tilde{\varphi}_\lambda \to \tilde{\varphi_0}$ strongly in H. In particular $\tilde{\varphi_0}$ is a minimizer of (3.4) and thus, by uniqueness, $\tilde{\varphi_0} = \psi$. □

3.2. **Further properties of the limit problem.** We define the self adjoint operator $L_1 : D(L_1) \to L^2(\mathbb{R}^N)$ by

$$L_1 = -\Delta + 1 - p\frac{1}{|x|^b}\psi^{p-1},$$

where $D(L_1) = \{v \in H^2(\mathbb{R}^N) : |x|^{-b}\psi^{p-1}v \in L^2(\mathbb{R}^N)\}$.

Proposition 12. *If $v \in D(L_1)$ satisfies $L_1 v = 0$, then $v = 0$.*

In the same spirit as Theorem 2.5 in [17], we performed a reduction of the problem by proving that the kernel of L_1 contains only radial functions.

Lemma 13. *If $v \in D(L_1)$ satisfies $L_1 v = 0$, then $v \in H^1_{rad}(\mathbb{R}^N)$.*

Before proving Lemma 13, we introduce some notation and recall some properties of spherical harmonics.

Let $\mathcal{H}_k$ be the space of spherical harmonics of degree k with $\dim \mathcal{H}_k = a_k = \begin{pmatrix} k \\ N+k-1 \end{pmatrix} - \begin{pmatrix} k-2 \\ N+k-3 \end{pmatrix}$ for $k \geqslant 2$, $a_1 = N$, $a_0 = 1$. For each k let $\{Y_1^k, \dots, Y_{a_k}^k\}$ be an orthonormal basis of $\mathcal{H}_k$. It is known that any function $v \in L^2(\mathbb{R}^N)$ can be decomposed as follows:

$$v = \sum_{k=0}^{+\infty} \sum_{i=1}^{a_k} v_{k,i}(|x|) Y_1^k\Big(\frac{x}{|x|}\Big),$$

where $v_{k,i}(r) := \int_{S^{N-1}} v(r\theta) Y_i^k(\theta) d\theta$.

Proof. Our proof follows a method due to [20] which has also been used in [14].

Let $v \in D(L_1)$ be such that $L_1 v = 0$ and consider its decomposition by spherical harmonics $\sum_{k=0}^{+\infty} \sum_{i=1}^{a_k} v_{k,i}(|x|)Y_1^k(\frac{x}{|x|})$. Since $L_1 v = 0$, the functions $v_{k,i}$ satisfy

$$v_{k,i}'' + \frac{N-1}{r} v_{k,i}' + (-1 + \frac{p}{r^b}\psi^{p-1})v_{k,i} - \mu_k v_{k,i} = 0, \tag{3.6}$$

where $\mu_k = k(k+N-2)$. In particular $v_{k,i} \in W^{1,\infty}(0,+\infty) \cap \mathcal{C}^2(0,+\infty)$. To prove the lemma it suffices to show that $v_{k,i} \equiv 0$, for all $k \geqslant 1$.

The function $\psi(r) := \psi(|x|)$ satisfies

$$\psi'' + \frac{N-1}{r}\psi' - \psi + \frac{1}{r^b}\psi^p = 0, \tag{3.7}$$

thus $\psi \in \mathcal{C}^3(0,+\infty)$ and differentiating (3.7) we get

$$\psi''' + \frac{N-1}{r}\psi'' - \frac{N-1}{r^2}\psi' - \psi' + \frac{p}{r^b}\psi^{p-1}\psi' - \frac{b}{r^{b+1}}\psi^p = 0. \tag{3.8}$$

Let $0 < a < b < +\infty$. Multiplying (3.6) by $\psi' r^{N-1}$ and integrating over (a,b) it follows that

$$\int_a^b v_{k,i} r^{N-1}(\psi''' + \frac{N-1}{r}\psi'' - \psi' + \frac{p}{r^b}\psi^{p-1}\psi') - \mu_k v_{k,i} r^{N-3}\psi' dr + g(b) - g(a) = 0,$$

where $g(r) := \psi' r^{N-1} v_{k,i}' - \psi'' r^{N-1} v_{k,i}$. Using (3.8), we get

$$(N-1-\mu_k)\int_a^b v_{k,i} r^{N-3}\psi' dr + \int_a^b v_{k,i} r^{N-1}\frac{b}{r^{b+1}}\psi^p dr + g(b) - g(a) = 0. \tag{3.9}$$

Because ψ', ψ'' decay exponentially at infinity (see the Appendix) we have $g(r) \to 0$ as $r \to \infty$. Since $N \geqslant 2$ and $v_{k,i} \in W^{1,\infty}(\mathbb{R}^+)$ we also have $g(r) \to 0$ as $r \to 0$.

Arguing by contradiction, we suppose $v_{k,i} \not\equiv 0$. Then, considering $-v_{k,i}$ instead of $v_{k,i}$ if necessary, there exist $0 \leqslant \alpha < \beta \leqslant +\infty$ such that

(i) $v_{k,i}(r) > 0$ in (α, β),
(ii) $v_{k,i}(\alpha) = v_{k,i}(\beta) = 0$,
(iii) $v_{k,i}'(\alpha) \geqslant 0$ if $\alpha \neq 0$ and $v_{k,i}'(\beta) \leqslant 0$ if $\beta \neq 0$.

It is standard to show that $\psi' < 0$ (see [11]), thus we have $g(\alpha) \leqslant 0$ and $g(\beta) \geqslant 0$. Therefore $g(\beta) - g(\alpha) \leqslant 0$ and thanks to (3.9) we have

$$(N-1-\mu_k)\int_a^b v_{k,i} r^{N-3}\psi' dr + \int_a^b v_{k,i} r^{N-1}\frac{b}{r^{b+1}}\psi^b \leqslant 0.$$

However, since $\psi' < 0$ and $N - 1 - \mu_k \leqslant 0$, we should have

$$(N-1-\mu_k)\int_a^b v_{k,i} r^{N-3}\psi' dr + \int_a^b v_{k,i} r^{N-1}\frac{b}{r^{b+1}}\psi^b > 0.$$

This contradiction proves that $v_{k,i} \equiv 0$ for all $k \geqslant 1$. $\square$

We are now in position to prove Proposition 12

Proof of Proposition 12. Our proof borrows some elements from [14] and [17]. Thanks to Lemma 13, it is enough to prove Proposition 12 for radial functions, therefore we work in $H^1_{rad}(\mathbb{R}^N)$.

For $\delta > 0$ small, we consider the following perturbation of (3.1)

$$-\Delta v + (1 + \delta e^{-|x|}\psi^{p-1})v = \Big(\frac{1}{|x|^b} + \delta e^{-|x|}\Big) v_+^p, \ v \in H^1_{rad}(\mathbb{R}^N). \tag{3.10}$$

Solutions of (3.10) are positive and can be obtained by minimizing the functional S_δ under the natural constraint $I_\delta(v) = 0$, for $v \in H^1_{rad}(\mathbb{R}^N) \setminus \{0\}$, where

$$\begin{aligned} S_\delta(v) &= \frac{1}{2}|v|_H^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} \frac{1}{|x|^b} v_+^{p+1} dx \\ &\quad -\delta\Big(\frac{1}{p+1}\int_{\mathbb{R}^N} e^{-|x|} v_+^{p+1} dx - \frac{1}{2}\int_{\mathbb{R}^N} e^{-|x|}\psi^{p-1} v^2 dx\Big), \\ I_\delta(v) &= |v|_H^2 - \int_{\mathbb{R}^N} \frac{1}{|x|^b} v_+^{p+1} dx \\ &\quad -\delta\Big(\int_{\mathbb{R}^N} e^{-|x|} v_+^{p+1} dx - \int_{\mathbb{R}^N} e^{-|x|}\psi^{p-1} v^2 dx\Big). \end{aligned}$$

Here both S_δ and I_δ are defined on $H^1_{rad}(\mathbb{R}^N)$ and it is standard to show that they are of class C^2.

We shall see in the Appendix that (3.10) has a unique positive radial solution for $\delta > 0$ small, and since $\psi \in H$ satisfies (3.10), it is this unique solution. In particular, $\psi \in H$ solves the following problem:

minimize $S_\delta(v)$ under the constraint $I_\delta(v) = 0$ for $v \in H^1_{rad}(\mathbb{R}^N) \setminus \{0\}$.

We recall that the Morse index of S_δ at ψ is given by

$$\begin{aligned} \text{Index}\, S''_\delta(\psi) &= Max\{\dim V : V \subset H^1_{rad}(\mathbb{R}^N) \text{ is a subspace such that} \\ &\qquad \big\langle S''_\delta(\psi)h, h\big\rangle < 0 \text{ for all h } \in V \setminus \{0\}\}. \end{aligned}$$

We claim that Index $S''_\delta(\psi) \leqslant 1$. To see this let us show that $\langle S''_\delta(\psi)v, v\rangle \geqslant 0$ on the subspace of co-dimension one $\{v \in H : \nabla I_\delta(\psi)v = 0\}$.

Let $v \in H^1_{rad}(\mathbb{R}^N)$ be such $\nabla I_\delta(\psi)v = 0$. Using the implicit function theorem, we see that there exist $\varepsilon > 0$ and a $\mathcal{C}^2$-curve $\phi : (-\varepsilon, \varepsilon) \to H^1_{rad}(\mathbb{R}^N)$ such that

$$\phi(0) = \psi, \ \phi'(0) = v \text{ and } I_\delta(\phi(t)) = 0.$$

Thanks to the variational characterization of ψ, 0 is a local minimum of $t \mapsto S_\delta(\phi(t))$, and therefore $\frac{d^2}{d^2t}S_\delta(\phi(t))|_{t=0} \geqslant 0$. But, since $\nabla S_\delta(\psi) = 0$, we have

$$0 \leqslant \frac{d^2}{d^2t}S_\delta(\phi(t))|_{t=0} = \left\langle S''_\delta(\psi)v, v\right\rangle.$$

At this point our claim is established. Now seeking a contradiction we assume the existence of $v_0 \in H^1_{rad}(\mathbb{R}^N) \setminus \{0\}$ such that $L_1 v_0 = 0$. Let $V := \operatorname{span}\{v_0, \psi\}$. Since

$$\langle L_1\psi, \psi\rangle = -(p-1)\int_{\mathbb{R}^N} \frac{1}{|x|^b}\psi^{p+1}dx < 0$$

and $\langle L_1 v_0, v\rangle = 0$ for all $v \in H^1_{rad}(\mathbb{R}^N)$, we see that V is of dimension 2 and that, for all $h \in V$, $\langle L_1 h, h\rangle \leqslant 0$. Thus we have, for all $h \in V \setminus \{0\}$,

$$\left\langle S''_\delta(\psi)h, h\right\rangle \;=\; \langle L_1 h, h\rangle - \delta(p-1)\int_{\mathbb{R}^N} \psi^{p-1}h^2 dx < 0,$$

which implies that Index $S''_\delta(\psi) \geqslant 2$. This contradiction ends the proof. $\square$

Lemma 14. [spectral properties] *The spectrum $\sigma(L_1)$ of L_1 contains a simple first eigenvalue $-\lambda_1 < 0$ and $\sigma(L_1) \setminus \{\lambda_1\} \subset (0, +\infty)$. Thus if $e_1 > 0$ denotes an eigenvector associated to $-\lambda_1$, such that $|e_1|_2 = 1$, then $e_1 \in H$ and H can be decomposed as $H = E_1 \oplus E_+$ where $E_1 = \operatorname{span}\{e_1\}$, E_+ is the eigenspace corresponding to the positive part of $\sigma(L_1)$ restricted to H and $E_1 \perp E_+$.*

Proof. Since $\langle L_1\psi, \psi\rangle < 0$, the first eigenvalue $-\lambda_1$ is negative, and it is standard to show that $-\lambda_1$ is simple and that the corresponding eigenvectors are in H. From Weyl's theorem, we see that the essential spectrum of L_1 is in $[1, +\infty)$ and that the spectrum in $(-\lambda_1, \frac{1}{2}]$ contains only a finite number of eigenvalues. Thanks to Proposition 12, the null space of L_1 is empty. Therefore to prove the lemma it just remains to show that $\lambda_2 \geqslant 0$ if it exists. Using the min-max principle for eigenvalues (see, for example, [21], page 75), it is sufficient to find an $f_0 \in H$ such that

$$\inf_{v \in f_0^\perp \setminus \{0\}} \langle L_1 v, v\rangle \geqslant 0,$$

where $f_0^\perp := \{v \in H : (v, f_0)_2 = 0\}$. We note that $\{v \in H : \nabla I(\psi)v = 0\} = \{v \in H : v \in f_0^\perp\}$ for an $f_0 \in H$. Indeed, it readily follows, using the fact that $\psi \in H$ solves (3.1), that

$$\nabla I(\psi)v = -p\int_{\mathbb{R}^N} \frac{1}{|x|^b}\psi^p v dx = (f_0, v)_2 \text{ with } f_0 = -p\frac{1}{|x|^b}\psi^p.$$

Since $\psi \in C^2(\mathbb{R}^N)$, ψ is exponentially decreasing, and $b+1<N$, we have $f_0 \in H$. Now, proving that $\langle L_1 v, v\rangle \geqslant 0$ for $v \in H$ such that $(v, f_0)_2 = 0$ can be done in the same way as showing that $\langle S''_\delta(\psi)v, v\rangle \geqslant 0$ for $v \in H$ such that $\nabla I_\delta(\psi)v = 0$ in the proof of Proposition 12. This ends the proof. $\square$

Lemma 15. *If $v \in H$ satisfies $(v,\psi)_2 = 0$ and $\langle L_1 v, v\rangle \leqslant 0$, then $v \equiv 0$. Here $(\cdot,\cdot)_2$ is the standard scalar product on $L^2(\mathbb{R}^N)$.*

Proof. We introduce $\psi_\lambda := \lambda^{\frac{2-b}{2(p-1)}}\psi(\sqrt{\lambda}x)$. Since ψ is a solution of (3.1), $\psi_\lambda \in H$ satisfies

$$-\Delta\psi_\lambda + \lambda\psi_\lambda - \frac{1}{|x|^b}\psi_\lambda^p = 0. \tag{3.11}$$

Differentiating (3.11) with respect to λ gives for $\lambda = 1$

$$-\Delta w + w - \frac{p}{|x|^b}\psi^{p-1}w = -\psi \ \text{ where } \ w = \frac{2-b}{2(p-1)}\psi + \frac{1}{2}x\cdot\nabla\psi. \tag{3.12}$$

That is, $L_1 w = -\psi$.

Let $v \in H$ be such that $v \not\equiv 0$ and $(v,\psi)_2 = 0$. To prove Lemma 15 it suffices to show that $\langle L_1 v, v\rangle > 0$.

Using the orthogonal spectral decomposition $H = E_1 \oplus E_+$ we write v and w as

$$\begin{array}{rcl} v &=& \alpha e_1 + \xi \\ w &=& \beta e_1 + \zeta \end{array} \quad \text{where } \xi, \zeta \in E_+.$$

Therefore, we have

$$\begin{array}{rcl} \langle L_1 v, v\rangle &=& -\alpha^2\lambda_1^2 + \langle L_1\xi, \xi\rangle \\ \langle L_1 w, w\rangle &=& -\beta^2\lambda_1^2 + \langle L_1\zeta, \zeta\rangle . \end{array} \tag{3.13}$$

If $\alpha = 0$, then $\xi \not\equiv 0$ and $\langle L_1 v, v\rangle > 0$ is satisfied. In the sequel, we suppose $\alpha \neq 0$. From the expression for w, we have

$$\langle L_1 w, w\rangle = -\frac{1}{2}\left(\frac{2-b}{2(p-1)} - \frac{N}{2}\right)|\psi|_2^2 < 0. \tag{3.14}$$

Also from (3.12) and the fact that $(v,\psi)_2 = 0$, it follows that

$$\langle L_1\zeta, \xi\rangle = \alpha\beta\lambda_1^2 + \langle L_1 w, v\rangle = \alpha\beta\lambda_1^2. \tag{3.15}$$

Consequently, $\zeta \not\equiv 0$ since otherwise (3.15) would give $\beta = 0$, which leads to a contradiction in (3.14). Since $L_1 > 0$ on E_+, the inequality $\langle L_1\zeta, \xi\rangle^2 \leqslant \langle L_1\zeta, \zeta\rangle \langle L_1\xi, \xi\rangle$ holds. Combining (3.12)–(3.14) we obtain

$$\langle L_1 v, v\rangle = -\alpha^2\lambda_1^2 \quad + \quad \langle L_1\xi, \xi\rangle \geqslant -\alpha^2\lambda_1^2 + \frac{\langle L_1\xi, \zeta\rangle}{\langle L_1\zeta, \zeta\rangle}$$

$$
\begin{aligned}
&= -\alpha^2\lambda_1^2 + \frac{\alpha^2\beta^2\lambda_1^4}{\beta^2\lambda_1^2 + \langle L_1 w, w\rangle} \\
&= \frac{-\langle L_1 w, w\rangle \alpha^2\lambda_1^2}{\langle L_1\zeta, \zeta\rangle} > 0.
\end{aligned}
$$

This ends the proof. □

Remark 16. Our proof of Lemma 15 is inspired by the work [13], which was indicated to us by R. Fukuizumi. In Lemma 2.1 of [6] (see also Proposition 2.7 of [27]) an alternative proof of Lemma 15 is given. Another proof of Lemma 15 relying on the fact that ψ is a local minimum of S on the sphere of corresponding L^2-norm can also be performed [18].

3.3. **Verification of the stability criterion.** To prove Theorem 2 we shall use Proposition 3. Since the convergence result holds in the rescaled variables it is convenient to express Proposition 3 in these variables. For $v \in H^1(\mathbb{R}^N, \mathbb{C})$, let $\tilde{v} \in H^1(\mathbb{R}^N, \mathbb{C})$ be defined by

$$v(x) = \lambda^{\frac{2-b}{2(p-1)}}\tilde{v}(\sqrt{\lambda}x).$$

Then we have

$$
\begin{aligned}
\left\langle S_\lambda''(\varphi_\lambda)v, v\right\rangle &= \lambda^{1+\frac{2-b}{p-1}-\frac{N}{2}}\left\langle \tilde{S}_\lambda''(\tilde{\varphi}_\lambda)\tilde{v}, \tilde{v}\right\rangle, \\
\|\nabla v\|_2^2 + \lambda\|v\|_2^2 &= \lambda^{1+\frac{2-b}{p-1}-\frac{N}{2}}||\tilde{v}||_2^2, \\
\langle \varphi_\lambda, v\rangle_2 &= \lambda^{1+\frac{2-b}{p-1}-\frac{N}{2}}\langle \tilde{\varphi}_\lambda, \tilde{v}\rangle_2, \\
\langle i\varphi_\lambda, v\rangle_2 &= \lambda^{1+\frac{2-b}{p-1}-\frac{N}{2}}\langle i\tilde{\varphi}_\lambda, \tilde{v}\rangle_2,
\end{aligned}
$$

where now by $\tilde{S}_\lambda$ we denote the extension of $\tilde{S}_\lambda$ from H to $H^1(\mathbb{R}^N;\mathbb{C})$. Therefore, if there exists $\delta > 0$ such that for any $v \in H^1(\mathbb{R}^N, \mathbb{C})$ satisfying $\langle \tilde{\varphi}_\lambda, \tilde{v}\rangle_2 = \langle i\tilde{\varphi}_\lambda, \tilde{v}\rangle_2 = 0$ we have

$$\left\langle \tilde{S}_\lambda''(\tilde{\varphi}_\lambda)\tilde{v}, \tilde{v}\right\rangle \geqslant \delta||\tilde{v}||^2, \tag{3.16}$$

we have, for any $v \in H^1(\mathbb{R}^N, \mathbb{C})$ satisfying $\langle \varphi_\lambda, v\rangle_2 = \langle i\varphi_\lambda, v\rangle_2 = 0$,

$$\left\langle S_\lambda''(\varphi_\lambda)v, v\right\rangle \geqslant \delta(\|\nabla v\|_2^2 + \lambda\|v\|_2^2). \tag{3.17}$$

Clearly, for $v \in H^1(\mathbb{R}^N, \mathbb{C})$ the norm $\sqrt{\|\nabla v\|_2^2 + \lambda\|v\|_2^2}$ is equivalent to the norm $||v||$ and thus for proving (3.16) it suffices to check the assumptions of Proposition 3.

For $v \in H^1(\mathbb{R}^N, \mathbb{C})$, let $v_1 = \mathrm{Re}v$ and $v_2 = \mathrm{Im}v$. Then we have, after some calculations,

$$\left\langle \tilde{S}_\lambda''(\tilde{\varphi}_\lambda)v, v\right\rangle = \left\langle \tilde{L}_{1,\lambda}v_1, v_1\right\rangle + \left\langle \tilde{L}_{2,\lambda}v_2, v_2\right\rangle,$$

with

$$\begin{aligned}\left\langle \tilde{L}_{1,\lambda} v_1, v_1 \right\rangle &= |v_1|_H^2 - p \int_{\mathbb{R}^N} V_\lambda(x) \tilde{\varphi}_\lambda^{p-1} |v_1|^2 dx \\ &\quad - \int_{\mathbb{R}^N} V_\lambda(x) \lambda^{-1+\frac{b}{2}} r' \left(\lambda^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_\lambda \right) |v_1|^2 dx, \\ \left\langle \tilde{L}_{2,\lambda} v_2, v_2 \right\rangle &= |v_2|_H^2 - \int_{\mathbb{R}^N} V_\lambda(x) \tilde{\varphi}_\lambda^{p-1} |v_2|^2 dx \\ &\quad - \int_{\mathbb{R}^N} V_\lambda(x) \lambda^{\frac{b}{2}} \Big(\frac{\tilde{r}(\tilde{\varphi}_\lambda(x))}{\tilde{\varphi}_\lambda(x)} \Big) |v_2|^2 dx.\end{aligned}$$

In addition $\langle \tilde{\varphi}_\lambda, v \rangle_2 = (\tilde{\varphi}_\lambda, v_1)_2$ and $\langle i\tilde{\varphi}_\lambda, v \rangle_2 = (\tilde{\varphi}_\lambda, v_2)_2$. Thus, to end the proof of Theorem 2 it is enough to prove the following lemma.

Lemma 17. *Assume* (H1)-(H7). *There exists* $\lambda_0 > 0$ *such that*

(i) there exists $\delta_1 > 0$ *such that* $\left\langle \tilde{L}_{1,\lambda} v, v \right\rangle \geqslant \delta_1 |v|_H^2$ *for all* $v \in H$ *satisfying* $(v, \tilde{\varphi}_\lambda)_2 = 0$, *for all* $\lambda \in (0, \lambda_0]$;

(ii) there exists $\delta_2 > 0$ *such that* $\left\langle \tilde{L}_{2,\lambda} v, v \right\rangle \geqslant \delta_2 |v|_H^2$ *for all* $v \in H$ *satisfying* $(v, \tilde{\varphi}_\lambda)_2 = 0$, *for all* $\lambda \in (0, \lambda_0]$.

Proof. Seeking a contradiction for part (i), we assume that there exist $(\lambda_j) \subset \mathbb{R}^+$ with $\lambda_j \to 0$ and $(v_j) \in H$ such that

$$\begin{aligned}&\lim_{j \to \infty} \left\langle \tilde{L}_{1,\lambda_j} v_j, v_j \right\rangle \leqslant 0, \\ &|v_j|_H = 1, \qquad (v_j, \tilde{\varphi}_{\lambda_j})_2 = 0.\end{aligned}$$

Since $(v_j) \subset H$ is bounded, there exists $v_\infty \in H$ such that $v_j \rightharpoonup v_\infty$ weakly in H. Let us prove that

$$\lim_{j \to \infty} \int_{\mathbb{R}^N} V_{\lambda_j}(x) \lambda_j^{-1+\frac{b}{2}} r' \Big(\lambda_j^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_{\lambda_j} \Big) |v_j|^2 dx = 0, \tag{3.18}$$

$$\lim_{j \to \infty} \int_{\mathbb{R}^N} V_{\lambda_j}(x) \tilde{\varphi}_{\lambda_j}^{p-1} |v_j|^2 dx = \int_{\mathbb{R}^N} \frac{1}{|x|^b} \psi^{p-1} |v_\infty|^2 dx. \tag{3.19}$$

To prove (3.18) let $\varepsilon > 0$ be arbitrary. By (H7), we have $\lim_{s \to 0^+} \frac{r'(s)}{s^{p-1}} = 0$. Moreover, $(|\tilde{\varphi}_{\lambda_j}|_\infty)$ is bounded and therefore, for any $\lambda > 0$ sufficiently small, $r'(\lambda_j^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_{\lambda_j}) \leqslant C\varepsilon \lambda_j^{1-\frac{b}{2}}$. Thus,

$$\Big| \int_{\mathbb{R}^N} V_{\lambda_j}(x) \lambda_j^{-1+\frac{b}{2}} r' \Big(\lambda_j^{\frac{2-b}{2(p-1)}} \tilde{\varphi}_{\lambda_j} \Big) |v_j|^2 dx \Big| \leqslant \varepsilon C \Big| \int_{\mathbb{R}^N} V_{\lambda_j}(x) |v_j|^2 dx \Big|$$

and we conclude by Lemma 4. Clearly proving (3.19) is equivalent to showing that, as $\lambda \to 0$,

$$\int_{\mathbb{R}^N} \Big(V_{\lambda_j}(x) - \frac{1}{|x|^b}\Big)\tilde{\varphi}_{\lambda_j}^{p-1}|v_j|^2 dx \to 0, \tag{3.20}$$

$$\int_{\mathbb{R}^N} \frac{1}{|x|^b}\left(\tilde{\varphi}_{\lambda_j}^{p-1}|v_j|^2 - \psi^{p-1}|v_\infty|^2\right) dx \to 0. \tag{3.21}$$

Since $(|\tilde{\varphi}_{\lambda_j}|_\infty)$ is bounded, Lemma 10 shows that (3.20) holds. Now since $|x|^{-b} \to 0$ as $|x| \to \infty$, to show (3.21) it suffices to show that, for all $R > 0$,

$$\int_{B(R)} \frac{1}{|x|^b}\left(\tilde{\varphi}_{\lambda_j}^{p-1}|v_j|^2 - \psi^{p-1}|v_\infty|^2\right) dx \to 0. \tag{3.22}$$

We write

$$\begin{aligned}\int_{B(R)} \frac{1}{|x|^b}\tilde{\varphi}_{\lambda_j}^{p-1}|v_j|^2 dx &= \int_{B(R)} \frac{1}{|x|^b}(\tilde{\varphi}_{\lambda_j}^{p-1} - \psi^{p-1})|v_j|^2 dx \\ &+ \int_{B(R)} \frac{1}{|x|^b}\psi^{p-1}|v_j|^2 dx.\end{aligned}$$

Since $\tilde{\varphi}_{\lambda_j} \to \psi$ in H, we have, up to a subsequence, $|x|^{-b}\tilde{\varphi}_{\lambda_j}^{p-1} \to |x|^{-b}\psi^{p-1}$ almost everywhere and since

$$||x|^{-b}\tilde{\varphi}_{\lambda_j}^{p-1}| \leqslant C|x|^{-b} \in L^{\frac{N}{2}}(B(R)),$$

Lebesgue's theorem gives $|x|^{-b}\tilde{\varphi}_{\lambda_j}^{p-1} \to |x|^{-b}\psi^{p-1}$ in $L^{\frac{N}{2}}(B(R))$. Also we have $|v_j|^2 \rightharpoonup |v_\infty|^2$ weakly in $L^{\frac{N}{N-2}}(B(R))$. At this point (3.22) follows easily.

Now, on one hand, from (3.18)-(3.19) we have

$$\lim_{j\to\infty} \left\langle \tilde{L}_{1,\lambda_j} v_j, v_j \right\rangle = 1 - p\int_{\mathbb{R}^N} \frac{1}{|x|^b}\psi^{p-1}|v_\infty|^2 dx. \tag{3.23}$$

On the other hand, still by (3.18)-(3.19) and the weak convergence $v_j \rightharpoonup v_\infty$ in H, we have $(v_\infty, \psi)_2 = 0$ and,

$$\langle L_1 v_\infty, v_\infty\rangle \leqslant \lim_{j\to\infty} \left\langle \tilde{L}_{1,\lambda_j} v_j, v_j \right\rangle \leqslant 0 \text{ (by assumption)}$$

which implies, according to Lemma 15, that $v_\infty \equiv 0$. But this leads to a contradiction in (3.23) and finishes the proof of (i). To prove (ii), since (i) holds, it suffices to show that, for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} |V_\lambda(x)|\lambda^{\frac{b}{2}}\Big(\frac{\tilde{r}(\tilde{\varphi}_\lambda)}{\tilde{\varphi}_\lambda}\Big)|v|^2 dx \leqslant \varepsilon$$

when $|v|_H = 1$ and $\lambda > 0$ is sufficiently small. Let $\varepsilon > 0$ be arbitrary. Since $(|\tilde{\varphi}_\lambda|_\infty)$ is bounded, for $\lambda > 0$ small enough, we have, from (2.5), that

$$\frac{\tilde{r}(\tilde{\varphi}_\lambda)}{\tilde{\varphi}_\lambda} \leqslant \varepsilon \lambda_j^{-\frac{b}{2}} |\tilde{\varphi}_\lambda|^{p-1}.$$

Thus

$$\int_{\mathbb{R}^N} |V_\lambda(x)| \lambda^{\frac{b}{2}} \Big(\frac{\tilde{r}(\tilde{\varphi}_\lambda)}{\tilde{\varphi}_\lambda} \Big) |v|^2 dx \leqslant \varepsilon C \int_{\mathbb{R}^N} |V_\lambda(x)| |v|^2 dx \leqslant \varepsilon C$$

by Lemma 4 and we conclude. □

4. Appendix

Here, we prove the uniqueness of the nonzero solutions of (3.10). For this we use results of [28].

We know that the nonzero solutions of (3.10) are positive and by standard regularity arguments that they are in $\mathcal{C}^2(\mathbb{R}^N)$ and decay exponentially at infinity. Also, setting $v = v(r)$, $r = |x|$ they satisfy the ordinary differential equation

$$v'' + \frac{N-1}{r} v' + g(r)v + h(r)v_+^p = 0 \tag{4.1}$$

where $g(r) = -(1 + \delta e^{-r}\psi(r)^{p-1})$ and $h(r) = r^{-b} + \delta e^{-r}$. For $m \in [0, N-2]$ we define

$$\begin{aligned} G(r,m) &= -r^{m+2}\delta f' - \alpha_1 r^{m+1}(1+\delta f) + \alpha_2 r^{m-1}, \\ H(r,m) &= -\Big(\beta + \frac{2b}{p+1}\Big) r^{m+1-b} - \frac{2\delta}{p+1} r^{m+2} e^{-r} - \beta r^{m+1} \delta e^{-r}, \end{aligned}$$

where $f := e^{-r}\psi^{p-1}$, $\alpha_1 := -2(N-3-m)$, $\alpha_2 := m(N-2-m)(2N-4-m)/2$ and $\beta := 2N - 4 - m - 2(m+2)/(p+1)$.

According to Theorem 2.2 of [28], to establish the uniqueness of the positive solution of (4.1) it suffices to check the following conditions.

- (A1) g and h are in $\mathcal{C}^1((0,\infty))$,
- (A2) $r^{2-\sigma} g(r) \to 0$ and $r^{2-\sigma} h(r) \to 0$ as $r \to 0^+$ for some $\sigma > 0$,
- (C1) $h(r) \geqslant 0$ for all $r \in (0,\infty)$ and there exists $r_0 > 0$ such that $h(r_0) > 0$,
- (C2) $G(r, N-2) \leqslant 0$ for all $r \in (0,\infty)$,
- (C3) for each $m \in [0, N-2)$, there exists $\alpha(m) \in [0,\infty]$ such that $G(r,m) \geqslant 0$ for $r \in (0,\alpha(m))$ and $G(r,m) \leqslant 0$ for $r \in (\alpha(m),\infty)$,
- (C4) $H(r,0) \leqslant 0$ for all $r \in (0,\infty)$,
- (C5) for each $m \in (0, N-2]$, there exists $\beta(m) \in [0,\infty)$ such that $H(r,m) \geqslant 0$ for $r \in (0,\beta(m))$ and $H(r,m) \leqslant 0$ for $r \in (\beta(m),\infty)$.

In (C3), by $\alpha(m) = 0$ and $\alpha(m) = \infty$ we mean that $G(s,m) \leqslant 0$ and $G(s,m) \geqslant 0$, respectively, for all $s \in (0,\infty)$. The analogous convention holds for (C5).

The following lemma is useful to check (C1)-(C5). It was provided to us by K. Tanaka [25].

Lemma 18. *Let $f(r) = e^{-r}\psi(r)^{p-1}$. Then $f(r)$, $f_r(r)$ and $f_{rr}(r)$ are exponentially decaying at infinity.*

Proof. First, we prove that there exist constants $R_0 > 0$ and $C > 0$ such that

$$0 \leqslant -\psi_r(r) \leqslant C_2\psi(r) \ \text{ for all } r \in [R_0, \infty). \tag{4.2}$$

Let $W(r) = 1 - r^{-b}\psi(r)^{p-1}$. Then $\psi(r)$ satisfies

$$-\psi_{rr}(r) - \frac{N-1}{r}\psi_r(r) + W(r)\psi(r) = 0, \tag{4.3}$$

and defining $R(r)$ and $\theta(r)$ by

$$r^{N-1}\psi(r) = R(r)\sin\theta(r), \qquad r^{N-1}\psi'(r) = R(r)\cos\theta(r)$$

it follows that $\theta(r)$ satisfies

$$\theta_r(r) = \cos^2\theta(r) - W(r)sin^2\theta(r) + \frac{N-1}{r}\sin\theta(r)\cos\theta(r). \tag{4.4}$$

It is standard (see [11]) that $\psi_r(r) < 0$, for all $r \in (0,\infty)$. Thus $\theta(r) \subset [\pi/2, \pi]$. In addition, since $W(r) \to 1$ as $r \to \infty$, the right-hand side of (4.4) is negative in a neighbourhood of $\pi/2^+$ and positive in a neighbourhood of π^-, for $r > 0$ sufficiently large. This shows that $\theta(r)$ stays, for $r > 0$ large, confined in an interval $[a,b] \subset (\pi/2, \pi)$. This implies (4.2). Now we have, for $r > 0$ large,

$$|\frac{\partial}{\partial r}\psi(r)^{p-1}| = (p-1)\psi(r)^{p-2}|\psi_r(r)| \leqslant (p-1)C\psi(r)^{p-1},$$

and we can easily deduce that $f_r(r)$ is exponentially decaying. Also, we have

$$\frac{\partial^2}{\partial r^2}\psi(r)^{p-1} = (p-1)\psi(r)^{p-2}\psi_{rr}(r) + (p-1)(p-2)\psi(r)^{p-3}\psi_r(r)^2.$$

The term $(p-1)(p-2)\psi(r)^{p-3}\psi_r(r)^2$ can be treated as previously and thanks to (4.3) we have

$$\psi(r)^{p-2}\psi_{rr}(r) = -\frac{N-1}{r}\psi(r)^{p-2}\psi_r(r) + W(r)\psi(r)^{p-1},$$

which allows us to conclude that $f_{rr}(r)$ is also exponentially decaying. $\square$

The conditions (A1), (A2) and (C1) are clearly satisfied. For (C2), we have

$$G(r, N-2) = -r^{N-1}(\delta r f_r(r) + 2\delta f(r) + 2).$$

Since $f(r)$ and $f_r(r)$ are continuous on $[0,\infty)$ and exponentially decaying at ∞, $(rf_r(r) + 2f(r))$ is bounded, therefore, for $\delta > 0$ small enough (C2) is satisfied. For (C3), we distinguish two cases. If $N-3-m > 0$, then $\alpha_1 < 0$, $\alpha_2 > 0$ and we have

$$G(r, m) = r^{m+1}(-r\delta f_r(r) - \alpha_1\delta f(r) - \alpha_1) + \alpha_2 r^{m-1}.$$

Since $f(r)$ and $f_r(r)$ are exponentially decaying, $-r\delta f_r(r) - \alpha_1\delta f(r) - \alpha_1 > 0$ for $\delta > 0$ small enough, and consequently $G(r, m) \geqslant 0$ for all $r \in (0,\infty)$. If $N-3-m \leqslant 0$ then $\alpha_1 \geqslant 0$, $\alpha_2 > 0$ and thus we have

$$\frac{\partial}{\partial r}\Big(\frac{G(r,m)}{r^{m+1}}\Big) = -\delta f_r(r) - r\delta f_{rr}(r) - \alpha_1\delta f_r(r) - 2\alpha_2 r^{-3} < 0$$

for $\delta > 0$ sufficiently small. Thus (C3) also holds. Now

$$H(r, 0) = -\frac{2b}{p+1}r^{-b+1} - \beta r^{1-b} - \frac{2\delta}{p+1}r^2 e^{-r} - \beta r\delta e^{-r}$$

and since, because $p > 1$, $\beta > 0$, we see that (C4) holds. Let $m \in (0, N-2]$. We have

$$\frac{H(r,m)}{r^{m+1-b}} = -\Big(\beta + \frac{2b}{p+1}\Big) - \delta\Big(\frac{2r}{p+1} + \beta\Big)r^b e^{-r}.$$

Since the function $r \mapsto [2r/(p+1) + \beta]r^b e^{-r}$ is bounded, for $\delta > 0$ small enough and $\beta + 2b/(p+1) \neq 0$ the sign of $H(r,m)$ is constant. When $\beta + 2b/(p+1) = 0$ we see that the function $r \to -\delta\left(\frac{2r}{p+1} + \beta\right) r^b e^{-r}$ is positive on $(0, \beta(m))$ and negative on $(\beta(m), \infty)$ for $\beta(m) = 2b/(p+1)$. Therefore, in both cases $H(r,m)$ satisfies (C5).

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