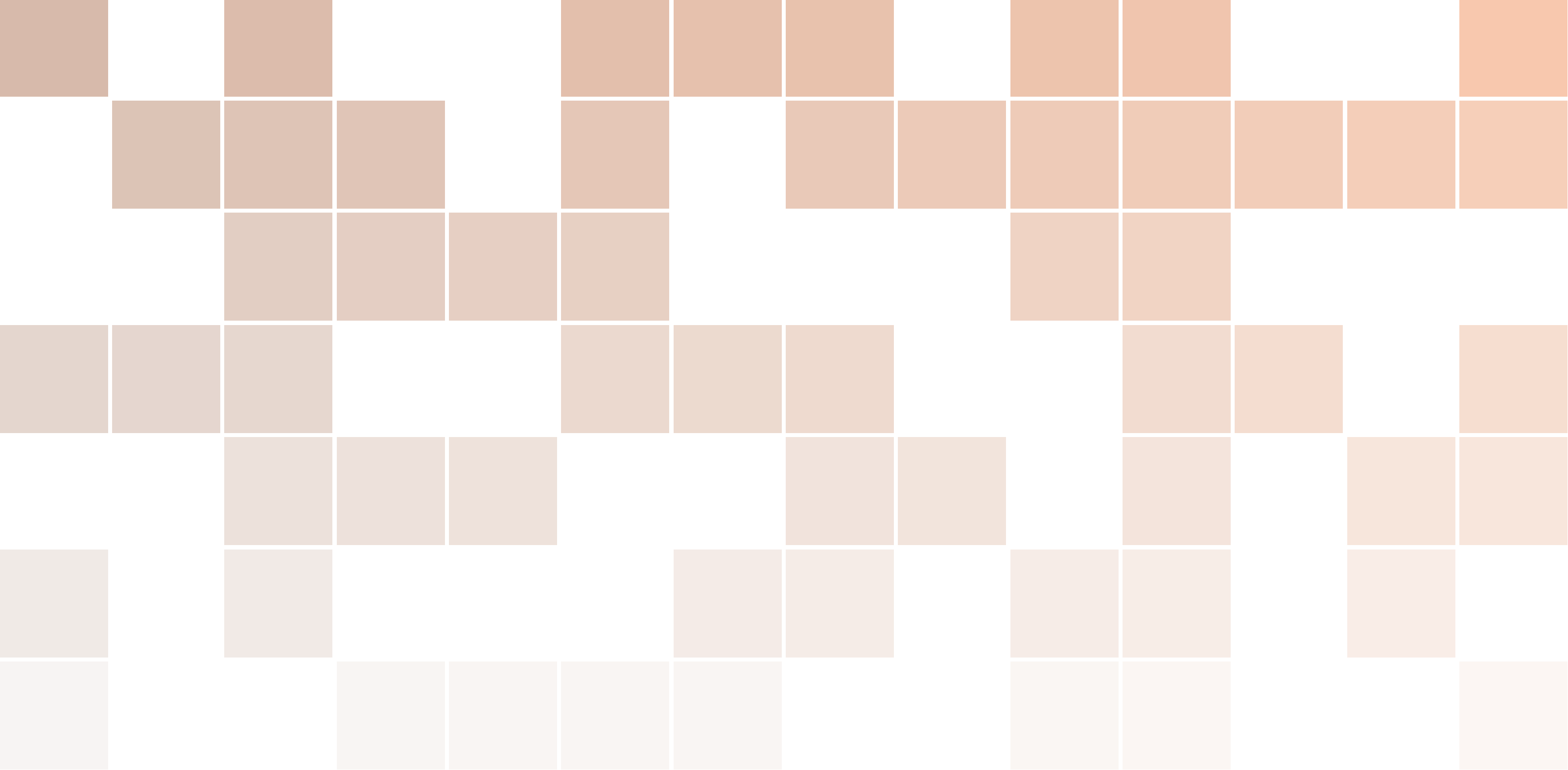
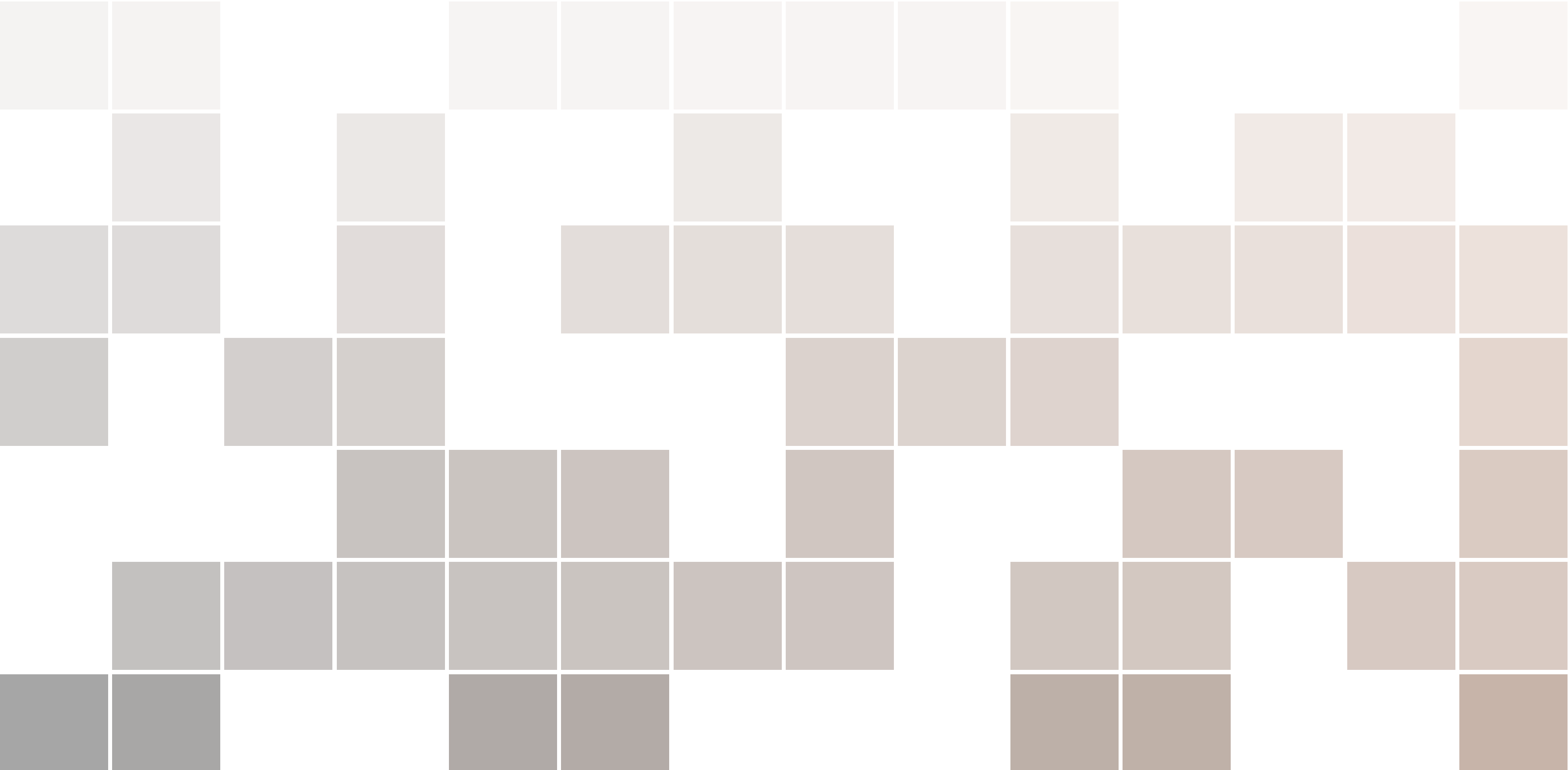


Theoretical and numerical analysis of dispersive PDEs

Advanced Course 3 EIMAR4E1 M2 MAT-RI

Stefan LE COZ



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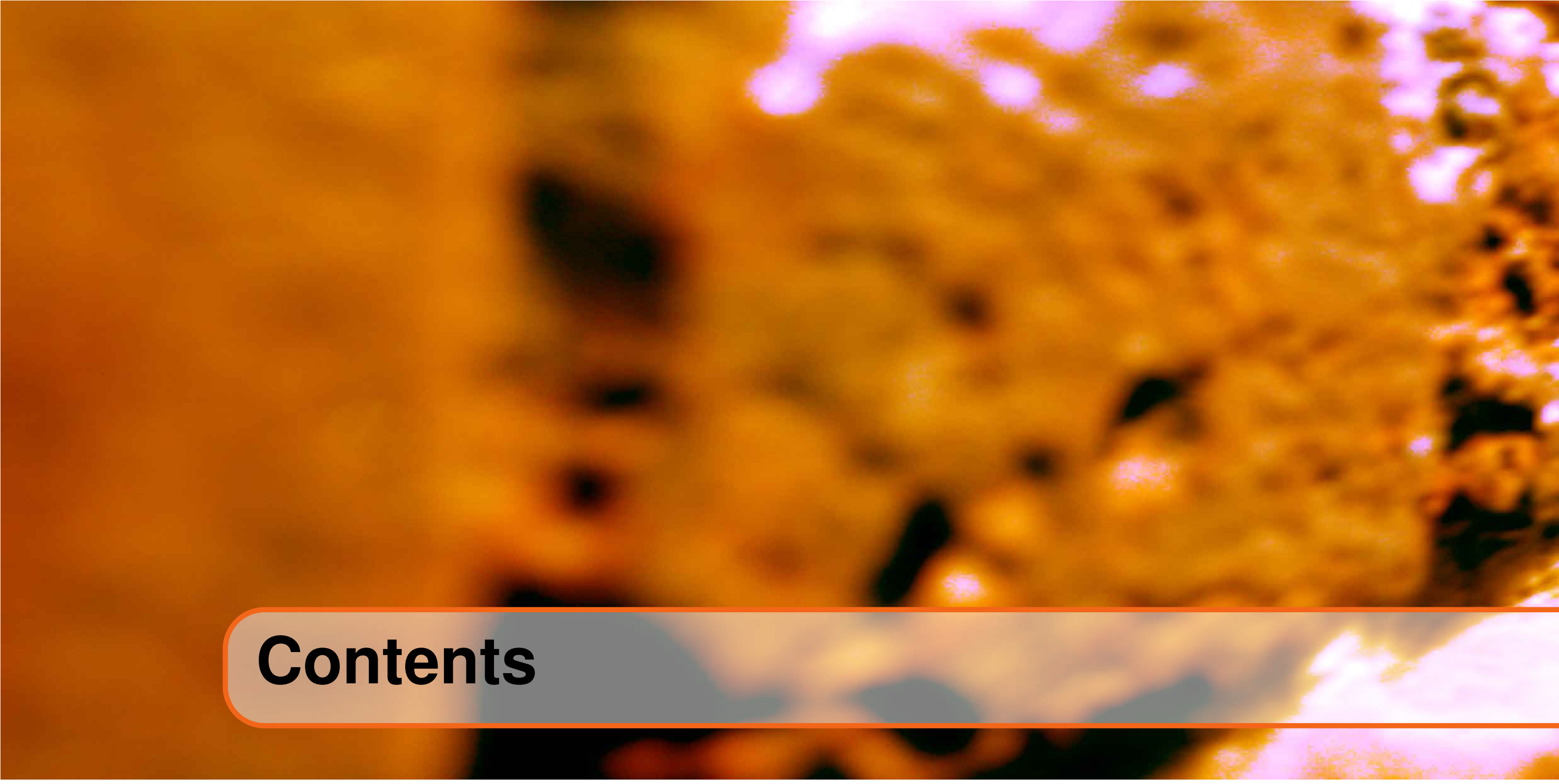
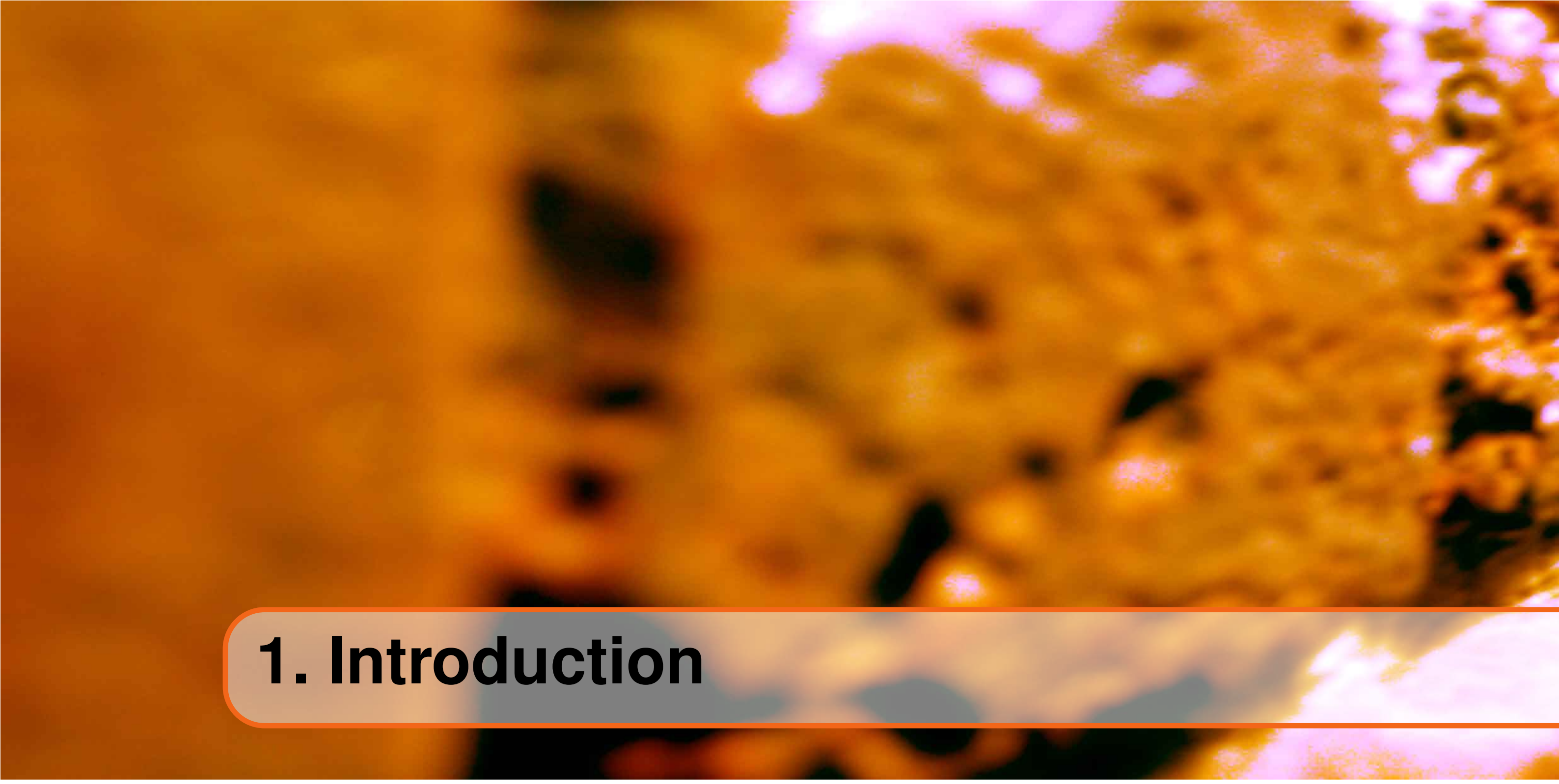


Contents

These notes have been written for the participants of the Master 2 course *Theoretical and numerical analysis of dispersive PDEs* at the University of Toulouse during the years 2018/2019 and 2019/2020. No claim to originality is made : these notes are mostly based on the existing literature on Schrödinger equations, in particular the book of Thierry Cazenave *Semilinear Schrödinger equations* [3] and the École Polytechnique lectures notes (in French) of Raphaël Danchin and Pierre Raphaël *Solitons, dispersion et explosion, une introduction à l'étude des ondes non linéaires* [5].



1. Introduction

The goal of this series of lectures is to present on a model case, the nonlinear Schrödinger equation, a variety of techniques developed in the last 40 years for the study of nonlinear dispersive PDE.

Before entering into the main matter of our topic, we give a few word of introduction.

1.1 Three examples

There are three main examples in the family of nonlinear dispersive PDE.

The first main example is the Korteweg-de Vries equation

$$u_t + u_{xxx} + 2uu_x = 0, \tag{KdV}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$. It was derived independently by Korteweg and de Vries [13] and Boussinesq [2, footnote on page 360], even though history retained only Korteweg and de Vries. The equation can model the propagation of water in a canal (see Picture 1.1). Assuming that u is small with respect to h and that l is large with respect to h, the Korteweg-de Vries equation is obtained by a series of (formal) approximations from the water-wave system.

The second main example is the nonlinear Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u + f(u) = 0, \tag{1.1}$$

where $u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$, $m \in \mathbb{R}$ and f is a nonlinearity, typically of power-type, for example

$$f(u) = |u|^{p-1}u, \quad p > 1.$$

One of the first appearance of this equation is in the specific case of the sine-Gordon equation (i.e. $d = 1$ and $f(u) = \sin(u)$) which was introduced in the framework of the

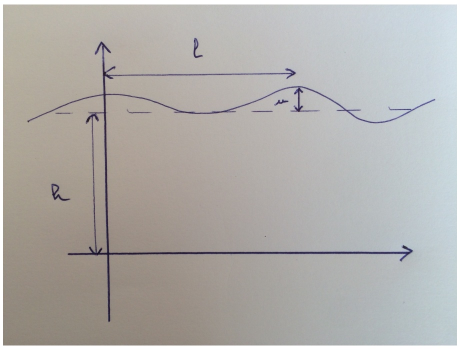


Figure 1.1: Propagation of water in a shallow canal

study of surfaces of constant negative curvature [1] and also appears in the study of crystal dislocations [8].

The third main example, which will be our principal object of study, is the nonlinear Schrödinger equation, given by

$$iu_t + \Delta u + f(u) = 0,$$

where $u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$ and f is a nonlinearity, typically of power-type. The nonlinear Schrödinger equations appear in a variety of physical settings, for example for the modelling of Bose-Einstein condensates [11, 17] or in nonlinear optics [25].

1.2 What is a dispersive equation ?

We say that a nonlinear PDE is dispersive when its linear part is dispersive, i.e. if its wave solutions spread out in space as they evolve in time. More precisely, assume that we are given a PDE such as the linear Schrödinger equation

$$iu_t + \Delta u = 0. \tag{1.2}$$

We look for a solution in the form of a *monochromatic* (or *harmonic*) plane wave

$$u(t,x) = Ae^{i(kx-\omega t)},$$

where $A > 0$ is the *amplitude* of the wave, $k \in \mathbb{R}^d$ is the *(angular) wave vector* and $\omega \in \mathbb{R}$ is the *angular frequency*. Substituting the ansatz in (1.2), we see that a plane wave is a solution when the *dispersion relation*

$$\omega = |k|^2$$

is satisfied. In that case, the frequency is a real valued function of the wave number (i.e. the norm of the wave vector). Moreover, denoting the *phase velocity* by

$$v = \frac{\omega k}{|k|^2},$$

we write the plane wave solution of (1.2) as

$$u(t,x) = Ae^{ik(x-v(k)t)}$$

and observe that the wave travels with velocity $v(k) = \frac{\omega k}{|k|^2} = k$. Therefore, the waves with large wave numbers travel faster than the waves with small ones. In general, we have the following definition.

Definition 1.2.1 A PDE is said to be *dispersive* if the function

$$\begin{aligned} g : \mathbb{R}^d &\to \mathbb{C} \\ k &\mapsto \frac{\omega(k)}{|k|} \end{aligned}$$

is real valued and not constant.

R The definition of what is a dispersive equation may vary from authors to authors. For example, one may also require that g is monotonic in $|k|$, or that $|g(k)| \to \infty$ as $|k| \to \infty$.

Exercise 1.1 Compute the dispersion relation for the following equations.

- The Airy equation (or linearized KdV)

$$\partial_t u + c\partial_x u + \partial_{xxx} u = 0, \quad u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad c \in \mathbb{R}.$$

- The Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u = 0, \quad u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}.$$

- The heat equation

$$u_t - u_{xx} = 0, \quad u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}.$$

- The transport equation

$$\partial_t u + v \cdot \nabla u = 0, \quad u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}, \quad v \in \mathbb{R}^d.$$

- The linearized Benjamin-Bona-Mahony equation

$$\partial_t u + c\partial_x u - \partial_{xx}\partial_t u = 0, \quad u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad c \in \mathbb{R}.$$

- Coupled mode equations (compare with the Klein-Gordon equation)

$$\begin{cases} \partial_t E_+ + \partial_x E_+ + \kappa E_- = 0, \\ \partial_t E_- + \partial_x E_+ + \kappa E_+ = 0; \end{cases}, \quad E_\pm : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad \kappa \in \mathbb{R}.$$

Which of these equations are dispersive ? ■

1.3 The soliton resolution conjecture

A large part of the interest for nonlinear dispersive equations stems from the ground breaking discovery made in the 60's for the Korteweg-de Vries equation: generically, a solution of the Korteweg-de Vries equation will decompose at large time as a sum of solitary waves and a dispersive remainder (see [27] for a preliminary numerical study, [9, 14, 15, 16, 22] for developments around the inverse scattering method and [7, 18] for the soliton resolution).

In order to give the reader a taste of what *soliton resolution* means without having to go through lengthy preliminaries, we consider the following toy model, the *Box-Ball model*.

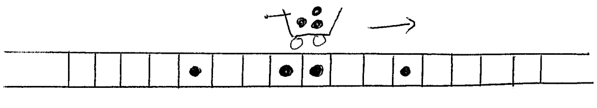


Figure 1.2: Illustration of the Box-Ball model

The Box-Ball model is a nonlocal cellular automaton working in the following way (see also Figure 1.2). We imagine an infinite row of boxes. Each box contain either one or zero ball. At each time step, a cart runs above the row of boxes from left to right. When the cart passes above a box, the following actions are taken. If the box contains a ball, the ball is loaded in the cart (which has an infinite capacity), leaving the box empty. If the box is empty, a ball is removed from the cart and dropped off in the box, provided the cart contains at least one ball to do so (if not the box is left empty).

Mathematically, the box-ball model can be represented in the following way : the row of boxes is mapped to $\mathbb{Z}$ and the fact that the box contains a ball or not is represented by a 1 or a 0. For the evolution, starting from an initial data $u_0 : \mathbb{Z} \to \{0, 1\}$, we apply the discrete evolution rule

$$u(t = 0, z) = u_0(z),$$

$$u(t+1, z) = \begin{cases} 1 & \text{if } u(t,z) = 0 \text{ and } \displaystyle\sum_{k=-\infty}^{z-1} u(t,k) > \sum_{k=-\infty}^{z-1} u(t+1,k), \\ 0 & \text{otherwise.} \end{cases}$$

Let us start by a simple example of evolution of an initial data by the box-ball dynamics.

$\cdots$00001111000000000000000000000000000000000$\cdots t = 0$
$\cdots$0000000011110000000000000000000000000000$\cdots t = 1$
$\cdots$0000000000001111000000000000000000000000$\cdots t = 2$
$\cdots$0000000000000000111100000000000000000000$\cdots t = 3$
$\cdots$0000000000000000000011110000000000000000$\cdots t = 4$
$\cdots$0000000000000000000000001111000000000000$\cdots t = 5$
$\cdots$0000000000000000000000000000111100000000$\cdots t = 6$
$\cdots$0000000000000000000000000000000011110000$\cdots t = 7$

In this example, we see that an initial data containing only a sequence of 1 leads to a very simple evolution where the 1s are simply translated of four boxes at each step of time. This behavior is typical of the behavior of solitary waves or solitons[1]. The following result can easily be proved.

---

[1] The original definition of solitons was much more restrictive that the one of solitary waves, but in the field of PDE both are nowadays synonyms

Proposition 1.3.1 — Solitons. If there exist $z_0 \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$u_0(z) = \begin{cases} 1 & \text{if } z = z_0 + k, \quad 0 \leq k < n, \\ 0 & \text{otherwise,} \end{cases}$$

then the evolution of the Box-Ball model is given by

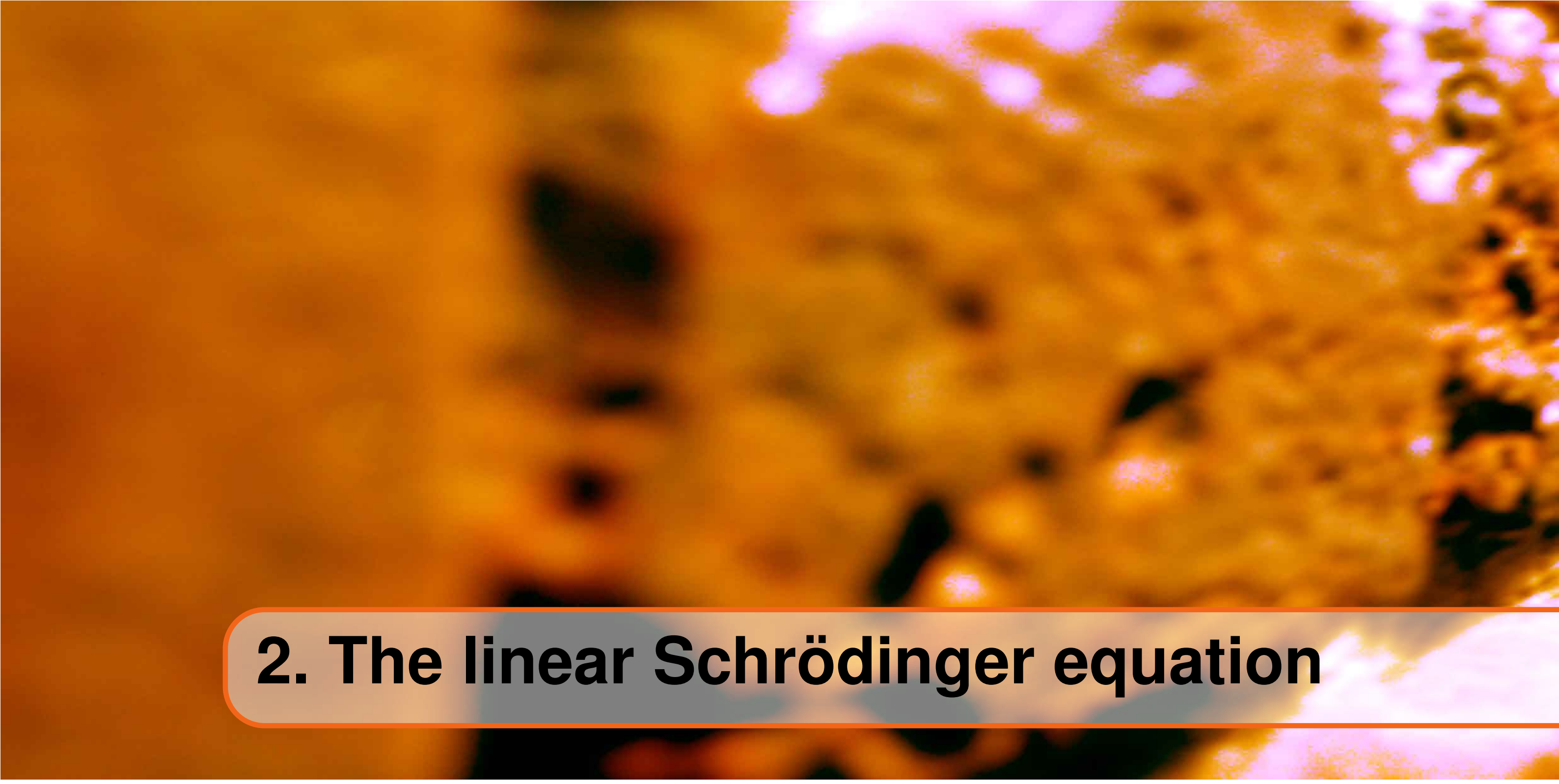
$$u(t, z) = u_0(z - nt).$$

Let us now consider another example, with a slightly more complicated initial data.

$$\begin{array}{ll}
\cdots 0000111100000011100010000000000000000000\cdots & t = 0 \\
\cdots 0000000011110000011101000000000000000000\cdots & t = 1 \\
\cdots 0000000000001111000010111000000000000000\cdots & t = 2 \\
\cdots 0000000000000000111101000111000000000000\cdots & t = 3 \\
\cdots 0000000000000000000010111000111100000000\cdots & t = 4 \\
\cdots 0000000000000000000001000111000011110000\cdots & t = 5
\end{array}$$

In this example, we observe that after some interaction, a pattern emerges from the evolution. A large soliton made of four 1 and travelling at speed four comes in front and is followed by slower soliton of three 1 traveling at speed three, himself followed by a slower 1 soliton. Such behavior is called *soliton resolution*. The following result was proved in [23].

Theorem 1.3.2 — Solitons Resolution [23]. Given any initial data u_0 containing a finite number of 1, the associated solution of the Box-Ball model decomposes into a sum of solitons at large time.



2. The linear Schrödinger equation

As much (but not all) of the analysis of the nonlinear Schrödinger equations is done by perturbation of the linear case, we study in this chapter the linear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = 0, \\ u(0,x) = u_0, \end{cases} \qquad u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}. \tag{2.1}$$

2.1 Explicit solution in the Schwartz space

We start by considering the equation for initial data in the Schwartz space. With the help of Fourier analysis, we can obtain an explicit solution.

Lemma 2.1.1 If $u_0 \in \mathcal{S}(\mathbb{R}^d)$, then there exists a unique solution $u \in C^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$ to (2.1) which is given by

$$u(t) = S(t)u_0 = S_t * u_0 = \mathcal{F}^{-1}\left(e^{-it|\xi|^2}\hat{u}_0(\xi)\right), \tag{2.2}$$

where we have defined the *Schrödinger kernel* S_t by

$$S_t = \frac{1}{(4\pi i t)^{\frac{d}{2}}} e^{i\frac{|x|^2}{4t}} \text{ if } t \neq 0, \qquad S_0 = \delta_{x=0}.$$

In these notes, the powers of complex numbers are understood in the principal value sense, i.e. given $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$ the number z^α is defined by

$$z^\alpha = |z|^\alpha e^{i\alpha\theta} \text{ where } z = |z|e^{i\theta},\ \theta \in (-\pi, \pi].$$

Proof of Lemma 2.1.1. We only give some elements of the proof. Assume that $u \in C^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$ is a solution to (2.1), and take the Fourier transform of (2.1) in space x to obtain for all $\xi \in \mathbb{R}^d$ the differential equation

$$i\partial_t \hat{u}(t,\xi) - |\xi|^2 \hat{u}(t,\xi) = 0, \qquad \hat{u}(0,\xi) = \hat{u}_0(\xi).$$

We can solve these equations explicitly and get the expression of u in Fourier variable

$$\hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{u}_0(\xi).$$

The formula for the convolution in the space variable is then a direct consequence of the formula for the Fourier transform of complex Gaussians of Lemma 2.1.2. ■

Lemma 2.1.2 For all $z \in \mathbb{C} \setminus \{0\}$ such that $\Re(z) \geq 0$, we have

$$\mathcal{F}\left(e^{-z|\cdot|^2}\right)(\xi) = \left(\frac{\pi}{z}\right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z}}.$$

Proof. Reformulating the statement of the lemma, we need to show that, given any $\xi \in \mathbb{R}^d$, the functions

$$z \mapsto \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-z|x|^2} dx, \quad z \mapsto \left(\frac{\pi}{z}\right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z}}$$

are well defined and coincide on $i\mathbb{R}$. Note that the integral in the first function is an oscillatory integral on $i\mathbb{R}$ and is not absolutely convergent. Hence the Fourier transform in the statement of Lemma 2.1.2 can be taken in the L^1 sense for $\Re(z) > 0$, but it has to be understood in the distributional sense for $\Re(z) = 0$. Define the right half-complex plane

$$D = \{z \in \mathbb{C} : \Re(z) > 0\}.$$

The above defined functions are both well-defined and holomorphic on D. Moreover, recall that we know (e.g. from a probability course) that the two functions coincide for z in $\mathbb{R}$. From the principle of isolated zeros of holomorphic functions, we infer that the two functions also coincide on D.

Take $t \in \mathbb{R}$, $t \neq 0$. There exists a sequence $(z_n) \subset D$ converging towards it. Using the definition of the Fourier transform of a distribution, for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}\left\langle \mathcal{F}\left(e^{-it|\cdot|^2}\right), \phi\right\rangle &= \left\langle e^{-it|\cdot|^2}, \hat{\phi}\right\rangle = \lim_{n\to\infty}\left\langle e^{-z_n|\cdot|^2}, \hat{\phi}\right\rangle = \lim_{n\to\infty}\left\langle \mathcal{F}\left(e^{-z_n|\cdot|^2}\right), \phi\right\rangle \\ &= \lim_{n\to\infty}\left(\frac{\pi}{z_n}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4z_n}}\phi(\xi)d\xi = \left(\frac{\pi}{it}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4it}}\phi(\xi)d\xi = \left\langle \left(\frac{\pi}{it}\right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4it}}, \phi\right\rangle\end{aligned}$$

where the second to last equality is due to the dominated convergence theorem. ■

The Duhamel formula provides the solutions for the inhomogeneous linear Schrödinger equation.

Lemma 2.1.3 — Duhamel Formula. Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $F \in C(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$. Then the solution $u \in C^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$ of the inhomogeneous linear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = F, \\ u_{|t=0} = u_0, \end{cases}$$

is given by the Duhamel representation formula

$$u(t) = S(t)u_0 - i\int_0^t S(t-s)F(s)ds. \tag{2.3}$$

Exercise 2.1 Let $F \in C(\mathbb{R}, \mathcal{S}(\mathbb{R}^d)) \cap L^1_t L^2_x(\mathbb{R}, \mathbb{R}^d)$ and consider the inhomogeneous linear Schrödinger equation

$$iu_t + \Delta u = F.$$

Construct a solution $u \in C^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$ such that

$$\lim_{t \to \infty} \|u(t)\|_{L^2} = 0.$$

■

Proof. The spatial Fourier transform of u verifies for any given $\xi \in \mathbb{R}^d$ the ODE

$$\partial_t \hat{u}(t,\xi) - |\xi|^2 \hat{u}(t,\xi) = \hat{F}(t,\xi), \quad \hat{u}(0,\xi) = \hat{u}_0(\xi).$$

This ODE can be explicitly integrated to find

$$\hat{u}(t,\xi) = e^{-it|\xi|^2} \hat{u}_0(\xi) - i \int_0^t e^{-i(t-s)|\xi|^2} \hat{F}(s,\xi) ds.$$

Taking the reverse Fourier transform, one gets the desired formula. ■

2.2 The Schrödinger group in $H^s(\mathbb{R}^d)$

The explicit representation (2.2) is making sense for $u_0 \in H^s(\mathbb{R}^d)$ (and even for $u_0 \in \mathcal{S}'(\mathbb{R}^d)$).

Definition 2.2.1 — Schrödinger group. Let $s \in \mathbb{R}$. The *Schrödinger group* S is defined for any $u_0 \in H^s(\mathbb{R}^d)$ and for any $t \in \mathbb{R}$ by the formula

$$S(t)u_0 = S_t * u_0 = \mathcal{F}^{-1}\left(e^{-it|\xi|^2} \hat{u}_0\right).$$

The following proposition is a direct consequence of the Fourier representation formula for S and Plancherel's identity.

Proposition 2.2.1 Let $s \in \mathbb{R}$. The Schrödinger group S is a strongly continuous group of unitary operators on $H^s(\mathbb{R}^d)$, i.e. the following properties are satisfied for any $u_0 \in H^s(\mathbb{R}^d)$.

- Regularity. We have $t \mapsto S(t)u_0 \in C(\mathbb{R}, H^s(\mathbb{R}^d))$.
- Isometry. For any $t \in \mathbb{R}$, we have $\|S(t)u_0\|_{H^s} = \|u_0\|_{H^s}$.
- Group. For any $(t,s) \in \mathbb{R}^2$ we have $S(s)S(t)u_0 = S(s+t)u_0$ and $S(0)u_0 = u_0$.
- Adjoint. For the Hilbert structure of $H^s(\mathbb{R}^d)$ we have $S(t)^* = S(-t)$.

An essential observation stemming from the explicit formula for the Schrödinger group is the dispersive estimate.

Proposition 2.2.2 — Dispersive estimate. Let $t \in \mathbb{R} \setminus \{0\}$, $p \in [2,\infty]$ and p' the conjugate exponent of p (i.e. $1/p + 1/p' = 1$). Then $S(t)$ is a continuous operator from $L^{p'}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ and we have

$$\|S(t)u_0\|_{L^p} \le \frac{1}{|4\pi t|^{\frac{d}{2}\left(\frac{1}{p'} - \frac{1}{p}\right)}} \|u_0\|_{L^{p'}}.$$

Proof. By density of $\mathcal{S}(\mathbb{R}^d)$ into L^p-spaces, it is enough to prove the statement for $u_0 \in \mathcal{S}(\mathbb{R}^d)$. In that case, using the explicit representation formula (2.2) and Young's inequality, we have

$$\|S(t)u_0\|_{L^\infty} \le \|S_t\|_{L^\infty}\|u_0\|_{L^1} = \frac{1}{|4\pi t|^{\frac{d}{2}}}\|u_0\|_{L^1}.$$

On the other hand, as S is an isometry on L^2, we have

$$\|S(t)u_0\|_{L^2} = \|u_0\|_{L^2}.$$

The conclusion then follows from Riesz-Thorin interpolation theorem[1]. ■

As a corollary, we have the following observation on the local dispersion of the mass.

Corollary 2.2.3 — Local dispersion of the mass. Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $R > 0$. Then

$$\int_{|x|<R} |S(t)u_0|^2 \lesssim R^d\|S(t)u_0\|_{L^\infty}^2 \lesssim \frac{R^d}{|t|^d} \to 0 \quad \text{as } |t| \to \infty.$$

2.3 Distributional solutions

Definition 2.3.1 — Weak solutions. Let $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ and $F \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{S}'(\mathbb{R}^d))$. We say that a distribution $u \in C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^d))$ is a *weak solution* of the inhomogeneous linear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = F, \\ u_{|t=0} = u_0, \end{cases} \tag{2.4}$$

if for any $\phi \in C^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$ and for any $t \in \mathbb{R}$ we have

$$\int_0^t \langle u(s), \Delta\phi(s) + i\partial_t\phi(s)\rangle\, ds = i\,\langle u_0, \phi(0)\rangle - i\,\langle u(t), \phi(t)\rangle + \int_0^t \langle F(s), \phi(s)\rangle\, ds,$$

where $\langle\cdot,\cdot\rangle$ is the duality product between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$.

---

[1] We recall the Riesz-Thorin interpolation theorem.

Theorem — Riesz–Thorin interpolation theorem. . Suppose $1 \le p_0 \le p_1 \le \infty$, $1 \le q_0 \le q_1 \le \infty$ and let $T : L^{p_0}(\mathbb{R}^d) + L^{q_0}(\mathbb{R}^d) \to L^{p_1}(\mathbb{R}^d) + L^{q_1}(\mathbb{R}^d)$ be a linear operator that maps $L^{p_0}(\mathbb{R}^d)$ (resp. $L^{p_1}(\mathbb{R}^d)$) boundedly into $L^{p_1}(\mathbb{R}^d)$ (resp. $L^{q_1}(\mathbb{R}^d)$). For $0 < \theta < 1$, let p_θ and q_θ be defined by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T maps $L^{p_\theta}(\mathbb{R}^d)$ boundedly into $L^{q_\theta}(\mathbb{R}^d)$ and satisfies the operator norm estimate

$$\|T\|_{L^{p_\theta}\to L^{q_\theta}} \le \|T\|_{L^{p_0}\to L^{q_0}}^{1-\theta}\|T\|_{L^{p_1}\to L^{q_1}}^{\theta}.$$

Proposition 2.3.1 If $u_0 \in \mathcal{S}'(\mathbb{R}^d)$, then the distribution defined by

$$S(t)u_0 = S_t * u_0 = \mathcal{F}^{-1}\left(e^{-it|\xi|^2}\hat{u}_0\right)$$

belongs to $C^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^d))$ and is a weak solution of the linear Schrödinger equation (2.1).

As a consequence of Proposition 2.3.1, we can observe the following *infinite speed of propagation property* of the linear Schrödinger equation. Indeed, choose as initial data $u_0 = \delta_{x=0}$. The (weak) solution of (2.1) is then given for $t \neq 0$ by

$$u(t) = S_t = \frac{1}{(4\pi i t)^{\frac{d}{2}}} e^{i\frac{|x|^2}{4t}}.$$

In particular, $u(t)$ is nowhere 0, even thought the support of the initial data was restricted to a point.

Proof of Proposition 2.3.1. Take $\phi \in C^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$. By definition of u, we have

$$\begin{aligned}
&\int_0^t \langle u(s), \Delta\phi(s) + i\partial_t\phi(s)\rangle\, ds = \int_0^t \left\langle \mathcal{F}^{-1}\left(e^{-is|\xi|^2}\hat{u}_0\right), \Delta\phi(s) + i\partial_t\phi(s)\right\rangle ds \\
&= \int_0^t \left\langle e^{-is|\xi|^2}\hat{u}_0, \mathcal{F}^{-1}\left(\Delta\phi(s) + i\partial_t\phi(s)\right)\right\rangle ds \\
&= -(2\pi)^{-d} \int_0^t \left\langle \hat{u}_0, e^{is|\xi|^2}\left(|\xi|^2\hat{\phi}(s,-\xi) - i\partial_t\hat{\phi}(s,-\xi)\right)\right\rangle ds \\
&= (2\pi)^{-d} \int_0^t \left\langle \hat{u}_0, \partial_s\left(e^{is|\xi|^2} i\hat{\phi}(s,-\xi)\right)\right\rangle ds \\
&= (2\pi)^{-d} \left\langle \hat{u}_0, \int_0^t \partial_s\left(e^{is|\xi|^2} i\hat{\phi}(s,-\xi)\right) ds\right\rangle \\
&= (2\pi)^{-d} \left\langle \hat{u}_0, e^{it|\xi|^2} i\hat{\phi}(t,-\xi) - i\hat{\phi}(0,-\xi)\right\rangle \\
&= -i(2\pi)^{-d}\left(\left\langle \hat{u}_0, e^{it|\xi|^2}\hat{\phi}(t,-\xi)\right\rangle - \left\langle \hat{u}_0, \hat{\phi}(0,-\xi)\right\rangle ds\right) \\
&\qquad\qquad = -i(2\pi)^{-d}\left(\left\langle \hat{u}(t), \mathcal{F}^{-1}\phi(t)\right\rangle - \left\langle \hat{u}_0, \mathcal{F}^{-1}\phi(0)\right\rangle ds\right).
\end{aligned}$$

This shows that u is indeed a weak solution of the homogeneous linear Schrödinger equation (2.1). ■

The Duhamel formula can be extended to the case of low regularity solutions. We give the following result without proof.

Proposition 2.3.2 Let $u_0 \in L^2(\mathbb{R}^d)$ and $F \in L^1_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^d))$. Then the inhomogeneous linear Schrödinger equation (2.4) admits a *unique* weak solution $u \in C(\mathbb{R}, L^2)$, which is given by the Duhamel formula (2.3). Moreover, the evolution of the mass is given for all $t \in \mathbb{R}$ by

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + 2\Im \int_0^t \int_{\mathbb{R}^d} F(s,x)\bar{u}(s,x)\,dx\,ds.$$

2.4 Strichartz Estimates

In this section, we present the Strichartz estimates, which are a fundamental tool for the study of linear and nonlinear dispersive equations.

The idea behind Strichartz estimates is to use the fixed time dispersive estimate to obtain more general inequalities by trading time-averaging for space-integrability. More precisely, we aim to prove inequalities of the type

$$\|S(t)u_0\|_{L_t^q L_x^r} \leq C\|u_0\|_{L^2},$$

where we have denoted by $\|\cdot\|_{L_t^q L_x^r}$ the space-time norm

$$\|u\|_{L_t^q L_x^r} = \left(\int_{\mathbb{R}} \|u(t,\cdot)\|_{L_x^r}^q\right)^{\frac{1}{q}}$$

if q and r are finite, with obvious modifications if q or r is ∞.

By a homogeneity argument, one sees that such estimate can be valid only for certain couples. More precisely, for $\lambda \in \mathbb{R} \setminus \{0\}$, define u^λ by $u^\lambda(x) = u_0(\lambda x)$. Then we have

$$(S(t)u^\lambda)(x) = (S(\lambda^2 t)u_0)(\lambda x).$$

As a consequence, we see that the above space-time estimate can be true only if (q, r) verify

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$

This motivates the following definition.

Definition 2.4.1 — Admissible pairs. We say that $(q, r) \in [2, \infty] \times [2, \infty]$ is a *(Schrödinger)-admissible pair* if it satisfies

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \qquad (q, r, d) \neq (2, \infty, 2).$$

We say that the pair is *strictly admissible* if in addition $(q, r) \neq \left(2, \frac{2d}{(d-2)}\right)$. The point $\left(2, \frac{2d}{(d-2)}\right)$ is called the *endpoint*.

Exercise 2.2
1. Represent the set of admissible pair on the $(\frac{1}{r}, \frac{1}{q})$ frame for $d = 1$, $d = 2$, $d = 3$.
2. For $d = 3$, compute the endpoint.
3. In which case do we have $q = r$?

■

Theorem 2.4.1 — Strichartz estimates. For any admissible pairs (q_1, r_1), (q_2, r_2) there exists $C > 0$ such that the following hold.

- Homogeneous estimate. For any $u_0 \in L^2(\mathbb{R}^d)$ we have

$$\|S(t)u_0\|_{L_t^{q_1} L_x^{r_1}} \leq C\|u_0\|_{L^2}.$$

- Inhomogeneous estimate. For $F \in L_t^{q_2'} L_x^{r_2'}(\mathbb{R} \times \mathbb{R}^d)$, we have

$$\left\| \int_0^t S(t-s)F(s)ds \right\|_{L_t^{q_1} L_x^{r_1}} \leq C \|F\|_{L_t^{q_2'} L_x^{r_2'}}.$$

Strichartz estimates were originally studied by Strichartz [21] for abstract considerations (see also [19, 24] for pioneering studies). See [10] for the homogeneous estimates, [4, 26] for extensions of inhomogeneous estimates and [12] for the endpoints.

Before giving the proof of Strichartz estimates, we introduce the two main ingredients of the proof, i.e. the TT^* lemma and Hardy-Littlewood-Sobolev inequality.

Lemma 2.4.2 — TT^*. Let $T : H \to B$ be a continuous operator from the Hilbert space H to the Banach space B. Define $T^* : B' \to H$ the adjoint of T from the dual B' of B to H by

$$(T^* x, y)_H = \langle x, Ty \rangle_{B',B} \, .$$

Then we have

$$\|TT^*\|_{\mathcal{L}(B',B)} = \|T\|^2_{\mathcal{L}(H,B)} = \|T^*\|^2_{\mathcal{L}(B',H)}.$$

Lemma 2.4.3 — Hardy-Littlewood-Sobolev inequality. Let $\alpha, \beta, \gamma \in (1, \infty)$, $\beta < \gamma$, and

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1 + \frac{1}{\gamma}.$$

Define the kernel $\phi_\alpha : \mathbb{R}^d \to \mathbb{R}$ by

$$\phi_\alpha(y) = \frac{1}{|y|^{\frac{d}{\alpha}}}.$$

Then the *Riesz potential*

$$u \to u * \phi_\alpha$$

is a continuous operator from $L^\beta(\mathbb{R}^d)$ to $L^\gamma(\mathbb{R}^d)$.

Note the mnemotechnic relation

$$1 - \frac{1}{\alpha} + 1 - \frac{1}{\beta} = 1 - \frac{1}{\gamma}.$$

The Hardy-Littlewood-Sobolev inequality says that even if ϕ_α does not belong to $L^\alpha(\mathbb{R}^d)$, the convolution can be treated as if it were the case.

The proof of this inequality is out of the scope of these notes, the interested reader might refer to [20, p. 119].

Proof of Lemma 2.4.2. Given $x \in B'$, we have

$$\begin{aligned} \|T^* x\|_H = \sup_{\|y\|_H = 1} |(T^* x, y)_H| &= \sup_{\|y\|_H = 1} |\langle x, Ty \rangle_{B',B}| \\ &\leq \|x\|_{B'} \sup_{\|y\|_H = 1} \|Ty\|_B = \|x\|_{B'} \|T\|_{\mathcal{L}(H,B)}. \end{aligned}$$

Therefore, we have

$$\|T^*\|_{\mathcal{L}(B',H)} \leq \|T\|_{\mathcal{L}(H,B)}.$$

Following the same line of reasoning, we also have

$$\|T\|_{\mathcal{L}(H,B)} \leq \|T^*\|_{\mathcal{L}(B',H)}.$$

By composition, we have

$$\|TT^*\|_{\mathcal{L}(B',B)} \leq \|T\|_{\mathcal{L}(H,B)}\|T^*\|_{\mathcal{L}(B',H)}.$$

Finally, using again the Hilbert structure, for any $x \in B'$, we have

$$\|T^*x\|_H^2 = (T^*x, T^*x) = \langle x, TT^*x\rangle_{B',B} \leq \|x\|_{B'}^2\|TT^*\|_{\mathcal{L}(B',B)},$$

which implies that

$$\|T^*\|^2_{\mathcal{L}(B',H)} \leq \|TT^*\|_{\mathcal{L}(B',B)}.$$

Combining the previous inequalities gives the desired conclusion. ■

Proof of Theorem 2.4.1. We restrict ourself to the strictly admissible pairs, the endpoint case being much more involved and out of the scope of these notes (see [12] for the proof of Strichartz estimates in the endpoint case).

Let (q,r) be an admissible pair. To place ourself in the framework of the TT^* lemma, we set

$$H = L^2(\mathbb{R}^d), \quad B = L^q(\mathbb{R}, L^r(\mathbb{R}^d)), \quad B' = L^{q'}(\mathbb{R}, L^{r'}(\mathbb{R}^d)), \quad T : u_0 \to (t \to S(t)u_0).$$

The homogeneous Strichartz inequality is equivalent to having $\|T\|_{\mathcal{L}(H,B)} < \infty$.

The following arguments are valid for functions in $\mathcal{S}(\mathbb{R}^d)$ and can be extended by density to the desired spaces. Since $S^*(t) = S(-t)$, for any $G \in B'$ we have

$$\begin{aligned}\langle G, Tu_0\rangle_{B',B} &= \int_{\mathbb{R}\times\mathbb{R}^d} G(s,x)\overline{S(s)u_0(x)}dxds\\ &= \int_{\mathbb{R}} (G(s), S(s)u_0)_H\, ds = \int_{\mathbb{R}} (S(-s)G(s), u_0)_H\, ds = \left(\int_{\mathbb{R}} S(-s)G(s)ds, u_0\right)_H.\end{aligned}$$

Therefore, the adjoint of T and the composition TT^* are given by

$$T^* : G \to \int_{\mathbb{R}} S(-s)G(s)ds, \quad TT^* : G \to \left(t \to \int_{\mathbb{R}} S(t-s)G(s)ds\right).$$

We remark that TT^* is related to the Duhamel term of the inhomogeneous linear Schrödinger equation.

We start by proving the homogenous estimate. For $G \in L^{q'}(\mathbb{R}, L^{r'}(\mathbb{R}^d))$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}\|TT^*G(t)\|_{L^r_x} &= \left\|\int_{\mathbb{R}} S(t-s)G(s)ds\right\|_{L^r_x} \leq \int_{\mathbb{R}} \|S(t-s)G(s)\|_{L^r_x}ds\\ &\leq \int_{\mathbb{R}} \frac{1}{|4\pi(t-s)|^{\frac{d}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)}}\|G(s)\|_{L^{r'}_x}ds = \frac{1}{(4\pi)^{\frac{2}{q}}}\int_{\mathbb{R}}\frac{1}{|t-s|^{\frac{2}{q}}}\|G(s)\|_{L^{r'}_x}ds\\ &= \frac{1}{(4\pi)^{\frac{2}{q}}}\frac{1}{|t|^{\frac{2}{q}}} * \|G(t)\|_{L^{r'}_x}\end{aligned}$$

where we have used the dispersive estimate Proposition 2.2.2 and the relation

$$\frac{d}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)=\frac{d}{2}\left(1-\frac{2}{r}\right)=\frac{2}{q}.$$

Assuming that $2<q<\infty$, we now use the Hardy-Littlewood-Sobolev inequality Lemma 2.4.3 in dimension 1 with[2] $\alpha=\frac{q}{2}$, $\beta=q$ and $\gamma=q$ to obtain

$$\|TT^*G\|_{L_t^qL_x^r}\leq C\|G\|_{L_t^{q'}L_x^{r'}}.$$

From the previous inequality and the TT^* argument Lemma 2.4.2, we have

$$\|TT^*\|_{\mathcal{L}(B',B)}=\|T\|^2_{\mathcal{L}(H,B)}=\|T^*\|^2_{\mathcal{L}(B',H)}<\infty.$$

This proves the homogeneous Strichartz inequality.

We now prove the inhomogeneous Strichartz inequality. We first treat the case where $q_1=q_2=q$ and $r_1=r_2=r$. In that case, the inhomogeneous estimate is in fact given by TT^*F restricted to $[0,t]$. Indeed, define the cut-off function

$$\chi(t,s)=\begin{cases}1 \text{ if } 0\leq s\leq t \text{ or } t\leq s\leq 0,\\ 0 \text{ otherwise.}\end{cases}$$

Then we have

$$\int_0^t S(t-s)F(s)ds=\int_{\mathbb{R}}\chi(t,s)S(t-s)F(s)ds=(TT^*(\chi F))(t)$$

As before, we have

$$\|TT^*(\chi F)\|_{L_t^qL_x^r}\leq C\|\chi F\|_{L_t^{q'}L_x^{r'}}\leq C\|F\|_{L_t^{q'}L_x^{r'}},$$

which is precisely the inhomogeneous estimate with the same pair.

To obtain the full inhomogeneous estimate with different pairs, we first prove that we have

$$\|TT^*(\chi F)\|_{L_t^\infty L_x^2}\leq\|F\|_{L_t^{q_2'}L_x^{r_2'}}$$

and then proceed by interpolation. Using the group structure of S, we have

$$\int_{\mathbb{R}}\chi(t,s)S(t-s)F(s)ds=S(t)\int_{\mathbb{R}}S(-s)\chi(t,s)F(s)ds=S(t)T^*\chi(t,\cdot)F.$$

Using the conservation of L^2-norm by S, for all $t\in\mathbb{R}$ we obtain

$$\left\|\int_{\mathbb{R}}\chi(t,s)S(t-s)F(s)ds\right\|_{L_x^2}=\|T^*\chi(t,\cdot)F\|_{L_x^2}\leq C\|\chi(t,\cdot)F\|_{L_t^{q_2'}L_x^{r_2'}}\leq C\|F\|_{L_t^{q_2'}L_x^{r_2'}}.$$

---

[2] Here, we use the fact that the pair is strictly admissible because we need $\alpha>1$. If $\alpha=\infty$, i.e. if $q=\infty$, then $r=2$ and the Strichartz inequality is simply the conservation of the L^2-norm.

In other words, the mapping

$$\Phi : F \to \int_0^t S(t-s)F(s)ds$$

is bounded from $L_t^{q_2'}L_x^{r_2'}$ to $L_t^\infty L_x^2$. Moreover, it is also bounded from $L_t^{q_2'}L_x^{r_2'}$ to $L_t^{q_2}L_x^{r_2}$. Therefore, from the generalized Riesz-Thorin interpolation Theorem, the mapping Φ is also bounded from $L_t^{q_2'}L_x^{r_2'}$ to $L_t^{q_1}L_x^{r_1}$ for any admissible pair (q_1, r_1), provided $q_1 \geq q_2$.

The case $q_2 \geq q_1$ is treated by duality. Indeed, if we prove that the mapping Φ is bounded from $L_t^1L_x^2$ to $L_t^{q_1}L_x^{r_1}$ for any strictly admissible pair, then, knowing that Φ is also bounded from $L_t^{q_1'}L_x^{r_1'}$ to $L_t^{q_1}L_x^{r_1}$, the result will follow for any strictly admissible pairs (q_1, r_1) and (q_2, r_2) such that $q_1 \leq q_2$.

We have

$$\|\Phi(F)\|_{L_t^{q_1}L_x^{r_1}} = \sup_{\|\psi\|_{L_t^{q_1'}L_x^{r_1'}}} \left| \int_{\mathbb{R}\times\mathbb{R}^d} \Phi(F)\bar{\psi}dtdx \right|.$$

As before, we may assume by density that ψ is smooth and rapidly decaying. We have

$$\int_{\mathbb{R}\times\mathbb{R}^d} \Phi(F)\bar{\psi}dtdx = \int_{\mathbb{R}\times\mathbb{R}^d}\int_{\mathbb{R}} \chi(t,s)S(t-s)F(s)ds\bar{\psi}(t)dtdx$$

$$= \int_{\mathbb{R}\times\mathbb{R}} (S(t)S(-s)\chi(t,s)F(s), \psi(t))_{L_x^2}\, dsdt$$

$$= \int_{\mathbb{R}} \left(S(-s)F(s), \int_{\mathbb{R}} S(-t)\chi(t,s)\psi(t)dt \right)_{L_x^2} ds.$$

From Cauchy-Schwartz inequality, we obtain

$$\left| \int_{\mathbb{R}\times\mathbb{R}^d} \Phi(F)\bar{\psi}dtdx \right| \leq \int_{\mathbb{R}} \|S(-s)F(s)\|_{L_x^2} \|T^*\chi(\cdot, s)\psi\|_{L_x^2} ds.$$

Since $T : L^2 \to L_t^{q_1}L_x^{r_1}$ bounded implies $T^* : L_t^{q_1'}L_x^{r_1'} \to L^2$ bounded, for any $s \in \mathbb{R}$ we have

$$\|T^*\chi(\cdot, s)\psi\|_{L_x^2} \leq C\|\chi(\cdot, s)\psi\|_{L_t^{q_1'}L_x^{r_1'}} \leq C\|\psi\|_{L_t^{q_1'}L_x^{r_1'}}.$$

Moreover, S is unitary on L^2 and we get

$$\left| \int_{\mathbb{R}\times\mathbb{R}^d} \Phi(F)\bar{\psi}dtdx \right| \leq \|F\|_{L_t^1L_x^2} \|\psi\|_{L_t^{q_1'}L_x^{r_1'}},$$

which implies that Φ is bounded from $L_t^1L_x^2$ to $L_t^{q_1}L_x^{r_1}$ and concludes the proof.

■

Theorem 2.4.4 — Generalized Riesz-Thorin Theorem. Consider $(m_j, p_j), (q_j, r_j) \in [1, \infty]^2$, $j = 0, 1$. Let T be a linear operator

$$T : L_t^{m_0} L_x^{p_0} + L_t^{m_1} L_x^{p_1} \mapsto L_t^{q_0} L_x^{r_0} + L_t^{q_1} L_x^{r_1}.$$

Assume that

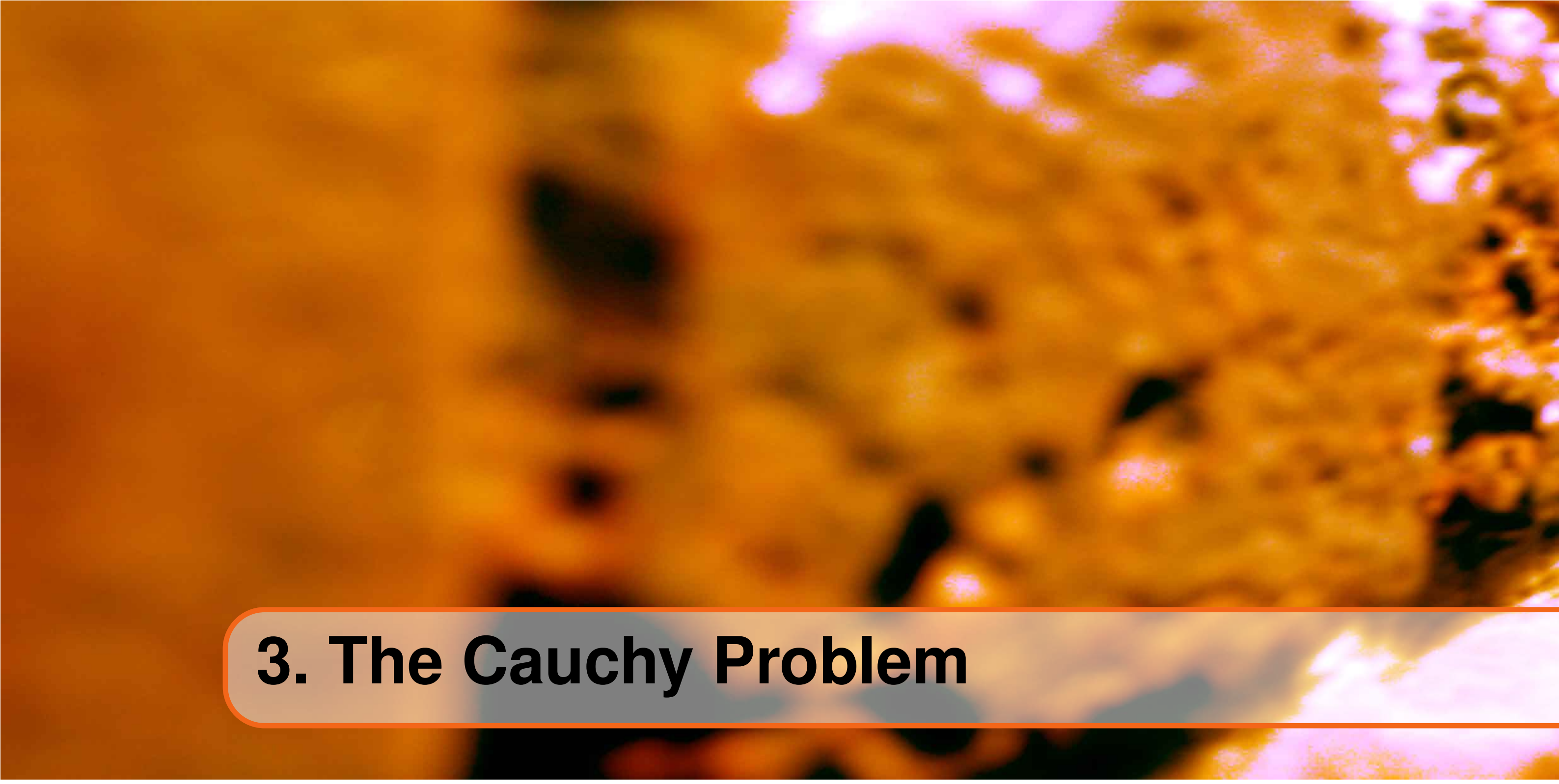
$$T : L_t^{m_0} L_x^{p_0} \mapsto L_t^{q_0} L_x^{r_0}, \quad T : L_t^{m_1} L_x^{p_1} \mapsto L_t^{q_1} L_x^{r_1}$$

are bounded. Then for all $\theta \in [0, 1]$ the operator

$$T : L_t^{m_\theta} L_x^{p_\theta} \mapsto L_t^{q_\theta} L_x^{r_\theta},$$
$$\frac{1}{m_\theta} = \frac{\theta}{m_0} + \frac{1-\theta}{m_1}, \quad \frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}, \quad \frac{1}{r_\theta} = \frac{\theta}{r_0} + \frac{1-\theta}{r_1}.$$

is also bounded. Moreover, we have

$$\|T\|_{\mathcal{L}(L_t^{m_\theta} L_x^{p_\theta}, L_t^{q_\theta} L_x^{r_\theta})} \le \|T\|^{\theta}_{\mathcal{L}(L_t^{m_0} L_x^{p_0}, L_t^{q_0} L_x^{r_0})} \|T\|^{1-\theta}_{\mathcal{L}(L_t^{m_1} L_x^{p_1}, L_t^{q_1} L_x^{r_1})}.$$



3. The Cauchy Problem

In this chapter, we will discuss the Cauchy Problem for the nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u \pm |u|^{p-1}u = 0, \\ u(t=0) = u_0, \end{cases} \tag{NLS}$$

where $p \in \mathbb{R}$, $p > 1$ and $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $d \geq 1$. If the sign in front of the nonlinearity is $+$, we say that it is *focusing* and if it is $-$ we say that it is *defocusing*. The terminology echos the physical origin of the equation where the medium can either enforce or oppose the dispersion of the beam.

Before considering the local and global well-posedness of the Cauchy Problem for (NLS), we discuss a number of *formal* aspects of the equation.

3.1 Formal aspects

First, the equation can be written in Hamiltonian formulation

$$u_t = JE'(u),$$

where the *Hamiltonian* (or *energy*) E is given by

$$E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1},$$

and the symplectic form J is just the multiplication by i. We refer to [6] for further theoretical discussions on the Hamiltonian formalism and its consequences for Schrödinger and other equations. In particular, the relationships between symmetries and conservations laws are rigorously studied in [6], and a *Noether's Theorem* is proved in the framework of infinite dimensional Hamiltonian systems. In this document, we will remain at a basic level and only observe these consequences without trying to put them in a more abstract framework.

The Schrödinger equation (NLS) enjoys a lot of symmetries. Precisely, we have the following proposition.

Proposition 3.1.1 — Symmetries. Given u a solution of (NLS), the functions given by the following expressions also are solutions of (NLS):

- *time translation* $u(t-s,x)$ for any $s\in\mathbb{R}$,
- *space translation* $u(t,x-y)$ for any $y\in\mathbb{R}^d$,
- *time reversal* $\bar{u}(-t,x)$,
- *phase shift* $e^{i\theta}u(t,x)$ for any $\theta\in\mathbb{R}$,
- *Galilean invariance* $e^{i\left(\frac{v}{2}\cdot(x-vt)+\frac{|v|^2t}{4}\right)}u(t,x-vt)$ for any $v\in\mathbb{R}^d$,
- *scaling* $\lambda^{\frac{2}{p-1}}u(\lambda^2t,\lambda x)$ for any $\lambda>0$.

Since (NLS) is a Hamiltonian system and enjoys compatible symmetries, by Noether Theorem (see [6]) corresponding quantities are (at least formally) conserved along the evolution in time. We first have the Hamiltonian E, then the *mass*

$$M(u)=\frac{1}{2}\|u\|_{L^2}^2,$$

which is linked to the phase shift invariance, and finally the *momentum*

$$P(u)=\frac{1}{2}\Im\int_{\mathbb{R}^d}u\nabla\bar{u}dx,$$

which is linked to the translation invariance. Remark that the momentum is a vector quantity. The fact that these quantities are conserved can be formally verified by direct calculations.

Exercise 3.1 Define the hamiltonian density $e(u)$, the mass density $m(u)$ and the momentum density $p(u)$ by

$$e(u)=\frac{1}{2}|\partial_xu|^2\mp\frac{1}{p+1}|u|^{p+1},\quad m(u)=\frac{1}{2}|u|^2,\quad p(u)=\frac{1}{2}\Im(u\partial_x\bar{u}).$$

Assuming that u verifies the one dimensional nonlinear Schrödinger equation

$$iu_t+u_{xx}\pm|u|^{p-1}u=0,$$

write in differential form the conservation laws associated to these quantities, i.e. show that

$$\partial_tm(u)=\partial_x(\dots).$$

Generalize these results to the higher dimensional setting. ■

All symmetries given in Proposition 3.1.1 hold in fact for any type of Gauge invariant nonlinearities (i.e. of the type $f(u)=g(|u|^2)u$, except for the scaling symmetry which is specific to power-type nonlinearities. In the case of power-type nonlinearities, we can classify the equations depending on which homogeneous Sobolev norm is preserved by the scaling symmetry. More precisely, define the *scaling parameter* s_c by being the only index such that

$$\|u_\lambda\|_{\dot{H}^{s_c}} = \|u\|_{\dot{H}^{s_c}}, \quad \text{where } u_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x), \ \lambda > 0.$$

In the setting of (NLS), we have

$$s_c = \frac{d}{2} - \frac{2}{p-1}.$$

We usually say that the equation (NLS) is H^{s_c}*-critical.* Two cases are of particular interest: $s_c = 0$ and $s_c = 1$, as they correspond to the regularity level required by the mass and energy conservation law. For example, when $p = 1 + \frac{4}{d}$, then $s_c = 0$ and we say that the equation is L^2-critical or *mass-critical.* If $p < 1 + \frac{4}{d}$, then $s_c < 0$ and we say that the equation is mass-subcritical. As we will see, the behavior of the solutions of (NLS) changes drastically when going from mass-subcritical to mass-supercritical. The energy-supercritical case being essentially uncharted territory, we will limit ourselves in these notes to the energy subcritical setting, i.e. we will assume for the rest of these notes that

$$1 < p < 1 + \frac{4}{(d-2)_+},$$

where by a_+ we denote $a_+ = \max(a, 0)$.

3.2 The Local Cauchy Problem

If, as suggested by the Hamiltonian formulation, one considers the equation (NLS) as a differential equation for the function u of the time variable t with values in an infinite dimensional function space X, the first question to answer is how to choose the function space X. In fact, several choices are possible. For example, one could look for a space in which the conservation laws are well-defined. In this case, we would restrict the exponent p to $1 < p < 1 + \frac{4}{d-2}$ (in such a way that $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$) and chose as function space X the space $H^1(\mathbb{R}^d)$. The space $H^1(\mathbb{R}^d)$ is often referred to as the *energy space*. On the other, one may try to solve (NLS) in spaces $H^s(\mathbb{R}^d)$ having the weakest possible regularity index s (with possibly $s < 0$) such that the local Cauchy problem remains well-posed (in some sense which includes not only local solvability for each initial data but also uniqueness, continuous dependence of the initial data, etc., see the discussion in [3]). In these notes, we will focus on the well-posedness in the energy space $H^1(\mathbb{R}^d)$. The main result of this chapter is the following.

Theorem 3.2.1 — Local Well-Posedness of the Cauchy Problem. Let $d \geq 1$ and $u_0 \in H^1(\mathbb{R}^d)$. Assume that $p \in \mathbb{R}$ verifies

$$1 < p < 1 + \frac{4}{(d-2)_+}.$$

Then there exists $T > 0$ such that the Cauchy Problem (NLS) admits a unique maximal solution $u \in C([0, T), H^1(\mathbb{R}^d))$. Moreover, there exist two constants $C, \alpha > 0$ depending only on p and d and such that

$$T \geq \frac{C}{\|u_0\|_{H^1}^{\alpha}}.$$

For any $t \in [0, T)$, we have

$$E(u(t)) = E(u_0), \quad M(u(t)) = M(u_0), \quad P(u(t)) = P(u_0).$$

Finally, we have the *blow-up alternative*:

$$\text{either } T = \infty \quad \textit{or} \quad \lim_{t \to T} \|u(t)\|_{H^1} = \infty.$$

In other words, for not too strong nonlinearities, we have local well-posedness of the Cauchy Problem (NLS) in the sense of the ODE in the infinite dimensional space $H^1(\mathbb{R}^d)$, with a blow-up alternative reminiscent from the blow-up alternative of the ODE case. Remark that the local well-posedness of the Cauchy Problem is independent of the nature (focusing or defocusing) of the nonlinearity.

A full proof of Theorem 3.2.1 can be found in [3, Section 4.4]. In these notes, for the sake of simplicity, we will restrict ourselves to the model case

$$d = 2, \quad p = 3$$

and we devote the rest of this section to the proof of Theorem 3.2.1 in that case.

As for the classical Cauchy-Lipschitz theorem, the idea is to use the Banach fixed-point theorem for contraction mapping. Indeed, by Duhamel formula, having a solution of (NLS) is (formally) equivalent to having a fixed point of the functional Φ defined by

$$\Phi(u)(t, x) = S(t)u_0(x) \pm i \int_0^t S(t - s) \left(|u(s, x)|^2 u(s, x) \right) ds.$$

The name of the game is to find a suitable function space in which Φ is a contraction mapping. In the present setting, a function space based on $H^1(\mathbb{R}^d)$ cannot be used. Indeed, $H^1(\mathbb{R}^d)$ is not an algebra, hence it may very well be that $|u|^2 u$ does not belong to $H^1(\mathbb{R})$ even though u does. Therefore, a more subtle strategy should be adopted. Strichartz estimates suggest us to work on $L^q L^r$ functions spaces with (q, r) admissible pairs chosen to fit the power of the nonlinearity $p = 3$. We introduce the following notation to indicate space-time norms where the time interval is not the entire line but the interval $(0, T)$ for some $T > 0$ and space integration is done in a Banach space E (e.g. L^r_x or H^1_x):

$$\|u\|_{L^q_T E} = \left(\int_0^T \|u(t, \cdot)\|_E^q dt \right)^{\frac{1}{q}}.$$

Lemma 3.2.2 — Generalized Hölder Inequality. In $L^q_T L^r_x$ spaces, we have the generalized Hölder inequality given for $J \in \mathbb{N}$, $J \geq 2$, $1 \leq q, r, q_j, r_j \leq \infty$, $j = 1, \ldots, J$ by

$$\left\| \prod_{j=1}^J u_j \right\|_{L^q_T L^r_x} \leq \prod_{j=1}^J \|u_j\|_{L^{q_j}_T L^{r_j}_x}, \quad \sum_{j=1}^J \frac{1}{q_j} = \frac{1}{q}, \quad \sum_{j=1}^J \frac{1}{r_j} = \frac{1}{r}.$$

Exercise 3.2 Prove Lemma 3.2.2. ▪

In dimension $d = 2$, the following are Strichartz admissible pairs:

$$(\infty, 2), \quad (3, 6).$$

We define the *Strichartz norm* adapted to these pairs by

$$\|u\|_{S_T} = \max\{\|u\|_{L_T^\infty L_x^2}, \|u\|_{L_T^3 L_x^6}\}$$

and the Banach space X_T by

$$X_T = \{u : (0, T) \times \mathbb{R}^d \to \mathbb{C} : \|u\|_{X_T} = \|u\|_{S_T} + \|\nabla u\|_{S_T} < \infty\}.$$

Using X_T, we can obtain a contraction mapping property for Φ.

Lemma 3.2.3 — Contraction mapping property. There exist $C_1, C_2 > 0$ such that for any $u_0 \in H^1(\mathbb{R}^d)$ the following property is satisfied. Let $T > 0$ be such that

$$0 < T < \frac{C_1}{\|u_0\|_{H^1}^6},$$

and define

$$\bar{B}_T = \{u \in X_T : \|u\|_{X_T} \le C_2 \|u_0\|_{H^1}\}$$

Then the mapping $\Phi : \bar{B}_T \to \bar{B}_T$ is a contraction mapping.

Proof. As usual, we will prove at the same time that Φ indeed maps $\bar{B}_T$ into $\bar{B}_T$ and that it is a contraction.

Let $T > 0$ and $u, v \in X_T$. We have

$$\Phi(u(t) - \Phi(v)(t) = \pm i \int_0^t S(t-s)(|u(s)|^2 u(s) - |v(s)|^2 v(s)) ds.$$

From inhomogeneous Strichartz estimates and the generalized Hölder inequality with $(p, p_1, p_2) = (1, 3, 3/2)$ and $(q, q_1, q_2) = (2, 3, 6)$, we get

$$\|\Phi(u) - \Phi(v)\|_{S_T} \lesssim \left\| |u|^2 u - |v|^2 v \right\|_{L_T^1 L_x^2} \lesssim \left\| (u - v)(|u|^2 + |v|^2) \right\|_{L_T^1 L_x^2}$$
$$\lesssim \|u - v\|_{L_T^3 L_x^6} \left(\|u\|_{L_T^3 L_x^6}^2 + \|v\|_{L_T^3 L_x^6}^2 \right).$$

Using the fact that ∇ and $S(t)$ commute, for the gradient of Φ we have

$$\nabla \Phi(u)(t) = S(t) \nabla u_0 \pm i \int_0^t S(t-s) \nabla \left(|u(s)|^2 u(s) \right) ds.$$

Using again inhomogeneous Strichartz estimates and the generalized Hölder inequality, but this time with $(p, p_1, p_2, p_3) = (1, 3, 3, 3)$ and $(q, q_1, q_2, q_3) = (2, 6, 6, 6)$, we get

$$\|\nabla \Phi(u) - \nabla \Phi(v)\|_{S_T} \lesssim \left\| \nabla \left(|u|^2 u - |v|^2 v \right) \right\|_{L_T^1 L_x^2}$$
$$\lesssim \left\| \nabla (u - v)(|u|^2 + |v|^2) \right\|_{L_T^1 L_x^2} + \| |u - v| (|\nabla u| + |\nabla v|)(|u| + |v|) \|_{L_T^1 L_x^2}$$
$$\lesssim \|\nabla (u - v)\|_{L_T^3 L_x^6} \left(\|u\|_{L_T^3 L_x^6}^2 + \|v\|_{L_T^3 L_x^6}^2 \right)$$
$$+ \|u - v\|_{L_T^3 L_x^6} \left(\|u\|_{L_T^3 L_x^6} + \|v\|_{L_T^3 L_x^6} \right) \left(\|\nabla u\|_{L_T^3 L_x^6} + \|\nabla v\|_{L_T^3 L_x^6} \right).$$

As a consequence, using the $L^3_T L^6_x$ part of the X_T-norm, we have the estimate

$$\|\Phi(u)-\Phi(v)\|_{X_T} \lesssim \|u-v\|_{X_T}\left(\|(u,\nabla u)\|_{L^3_T L^6_x} + \|(v,\nabla v)\|_{L^3_T L^6_x}\right)\left(\|u\|_{L^3_T L^6_x} + \|v\|_{L^3_T L^6_x}\right).$$

Now, using the injection $H^1(\mathbb{R}^2) \hookrightarrow L^6(\mathbb{R}^2)$ and the $L^1_T L^2_x$-part of the X_T norm we have

$$\|u\|_{L^3_T L^6_x} \lesssim \|u\|_{L^3_T H^1_x} \lesssim T^{\frac{1}{3}}\|u\|_{L^\infty_T H^1_x} \lesssim T^{\frac{1}{3}}\|u\|_{X_T}.$$

Getting back to Φ, there exists C (independent of u_0, u and v) such that

$$\|\Phi(u)-\Phi(v)\|_{X_T} \leq CT^{\frac{1}{3}}\|u-v\|_{X_T}\left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right). \tag{3.1}$$

With the preliminary estimate (3.1) in hand, we are now in position to prove that Φ sends B_T into B_T. Indeed, using the homogeneous Strichartz estimate, we have

$$\|\Phi(0)\|_{X_T} = \|S(t)u_0\|_{X_T} \lesssim \|u_0\|_{X_T}.$$

Therefore, specifying $v = 0$ in (3.1), there exists $\tilde{C}$ (independent of u_0) such that for all $u \in X_T$ we have

$$\|\Phi(u)\|_{X_T} \leq \tilde{C}\left(\|u_0\|_{H^1} + T^{\frac{1}{3}}\|u\|^3_{X_T}\right)$$

In view of the definition of X_T, we choose $C_2 = 2\tilde{C}$ and T such that

$$8\tilde{C}^3 T^{\frac{1}{3}}\|u_0\|^2_{H^1} \leq 1$$

to have

$$\|\Phi(u)\|_{X_T} \leq C_2\|u_0\|_{H^1}.$$

With this choice and (3.1), the functional Φ is Lipschitz with Lipschitz constant

$$k = 2CT^{\frac{1}{3}}C_2^2\|u_0\|^2_{H^1}.$$

We now choose C_1 such that $k < 1$, i.e.

$$T > \frac{1}{(2CC_2^2\|u_0\|^2_{H^1})^3},$$

so that $C_1 = (2CC_2^2)^3$. This concludes the proof. ■

We are now in position to prove the local well-posedness result of Theorem 3.2.1. The proof of conservation of energy, mass and momentum relies on further arguments involving in particular continuous dependance of the solution on the initial data and is out of the scope of these notes.

Proof of the existence, uniqueness and and blow-up alternative in Theorem 3.2.1. We first show the existence. The contraction mapping property Lemma 3.2.3 and Banach fixed point Theorem ensure the existence of $u \in \bar{B}_T$ such that $u = \Phi(u)$. We now show that $u \in C([0,T], H^1(\mathbb{R}^2))$. Since S is an isometry on $H^1(\mathbb{R}^2)$, if $v \in C([0,T], H^1(\mathbb{R}^2))$ then we also have $S(t)v \in C([0,T], H^1(\mathbb{R}^2))$. Therefore, writing

$$u = \Phi(u) = S(t)(u_0 \pm i\tilde{\Phi}(u)), \quad \tilde{\Phi}(u) = \int_0^t S(-s)\left(|u(s)|^2 u(s)\right) ds,$$

we see that to prove that $u \in C([0,T], H^1(\mathbb{R}^2))$ it is enough to prove $\tilde{\Phi}(u) \in C([0,T], H^1(\mathbb{R}^2))$. For any $\tau, \sigma \in [0,T]$, we have

$$\begin{aligned} \|\tilde{\Phi}(u)(\tau) - \tilde{\Phi}(u)(\sigma)\|_{L^2_x} &= \left\| \int_\sigma^\tau S(-s)\left(|u(s)|^2 u(s)\right) ds \right\|_{L^2_x} \\ &\leq \int_\sigma^\tau \| |u(s)|^2 u(s)\|_{L^2_x} ds \lesssim |\tau - \sigma| \|u\|^3_{L^\infty_T H^1_x} \lesssim |\tau - \sigma| \|u\|^3_{X_T}. \end{aligned}$$

Similarly, for the gradient we have

$$\begin{aligned} \|\nabla(\tilde{\Phi}(u)(\tau) - \tilde{\Phi}(u)(\sigma))\|_{L^2_x} &\leq \int_\sigma^\tau \|\nabla(|u(s)|^2 u(s))\|_{L^2_x} ds \leq \int_\sigma^\tau \|\nabla u(s)\|_{L^6_x} \|u(s)\|^2_{L^6_x} ds \\ &\lesssim |\tau - \sigma|^{\frac{2}{3}} \|u\|^2_{L^\infty_T H^1_x} \|\nabla u\|_{L^3_T L^6_x} \lesssim |\tau - \sigma|^{\frac{2}{3}} \|u\|^3_{X_T}. \end{aligned}$$

Therefore $\tilde{\Phi}(u) \in C([0,T], H^1(\mathbb{R}^2))$ and the same is true for u itself.

We now show uniqueness in $C([0,T], H^1(\mathbb{R}^2))$. Let $v \in C([0,T], H^1(\mathbb{R}^2))$ be another solution of (NLS) with $v(0) = u_0$. Denote

$$M = \max\{\|u\|_{L^\infty_T H^1_x}, \|v\|_{L^\infty_T H^1_x}\}.$$

From Sobolev embeddings, we know that $|v|^2 v \in L^1_T L^2_x$. From Duhamel formula in weak regularity (Proposition 2.3.2), we have

$$v = \Phi(v).$$

Moreover, $v \in L^\infty_T H^1_X \subset L^3_T L^6_x$. Therefore, as in the proof of the contraction mapping property Lemma 3.2.3, for any $\tilde{T} \in (0,T]$ we have

$$\begin{aligned} \|u - v\|_{L^3_{\tilde{T}} L^6_x} &= \|\Phi(u) - \Phi(v)\|_{L^3_{\tilde{T}} L^6_x} \lesssim \|u - v\|_{L^3_{\tilde{T}} L^6_x} \left(\|u\|^2_{L^3_{\tilde{T}} L^6_x} + \|v\|^2_{L^3_{\tilde{T}} L^6_x} \right) \\ &\lesssim \tilde{T}^{\frac{2}{3}} \|u - v\|_{L^3_{\tilde{T}} L^6_x} \left(\|u\|^2_{L^\infty_{\tilde{T}} H^1_x} + \|v\|^2_{L^\infty_{\tilde{T}} H^1_x} \right) \leq \tilde{T}^{\frac{2}{3}} M^2 \|u - v\|_{L^3_{\tilde{T}} L^6_x}. \end{aligned}$$

Hence if $\tilde{T}$ has been chosen sufficiently small, i.e. $\tilde{T} \lesssim M^{-2}$ then we have

$$\|u - v\|_{L^3_{\tilde{T}} L^6_x} < \|u - v\|_{L^3_{\tilde{T}} L^6_x},$$

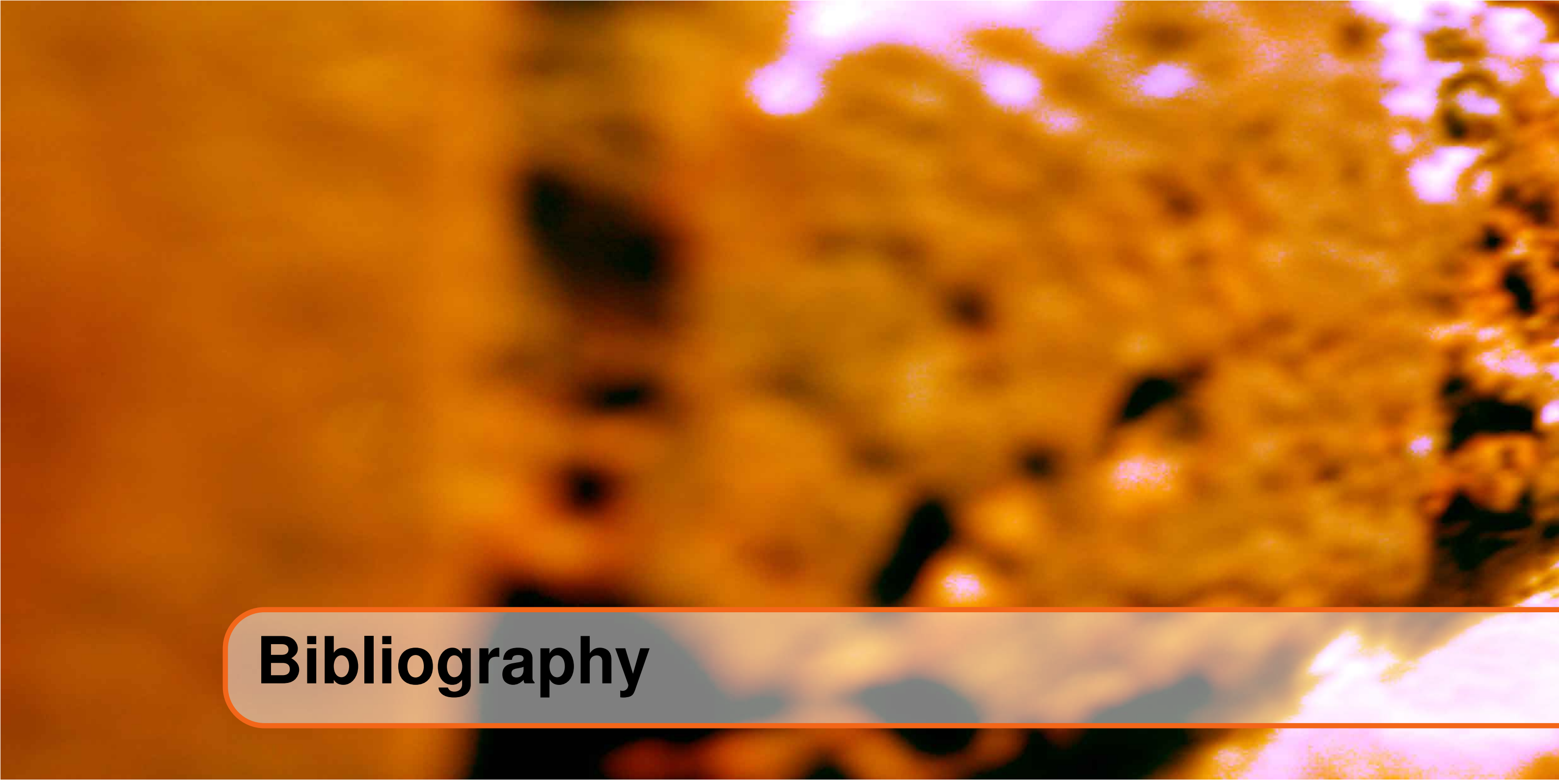
which implies $u = v$ on $[0, \tilde{T}]$. Since $\tilde{T}$ depends only on M, we can repeat the argument on $[\tilde{T}, 2\tilde{T}]$, etc. to obtain by finite induction uniqueness on the full interval $[0,T]$.

Finally, we prove the blow-up alternative by contradiction. Assume that the solution $u \in C([0,T), H^1(\mathbb{R}^2))$ is a maximal solution with $T < \infty$ and

$$M = \|u\|_{L^\infty_T H^1_x} < \infty.$$

By the contraction mapping property Lemma 3.2.3, there exists $T(M)$ such that for any $t \in [0,T)$, since $\|u(t)\|_{H^1} < M$ we can extend the solution u to the interval $[t, t + T(M)]$. Choosing t such that $t + T(M) > T$ gives a contradiction with the supposed maximality of T, and finishes the proof.

The conservation laws can be obtained by explicit calculation which are justified for enough regular solutions, and then can be extended by density arguments. We do not give details and refer to [3] for a complete proof. ■



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